Extremal Eigenvalues of the Laplacian on Euclidean domains and closed surfaces

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ABSTRACT. We investigate properties of the sequences of extremal values that could be achieved by the eigenvalues of the Laplacian on Euclidean domains of unit volume, under Dirichlet and Neumann boundary conditions, respectively. In a second part, we study sequences of extremal eigenvalues of the Laplace-Beltrami operator on closed surfaces of unit area.

1. INTRODUCTION

A classical topic in spectral geometry is to investigate upper and lower bounds of eigenvalues of the Laplacian subject to various boundary conditions and under the fixed volume constraint. Among the most known results in this topic are the Faber-Krahn inequality for the first Dirichlet eigenvalue, the Szegö-Weinberger inequality for the first positive Neumann eigenvalue on bounded Euclidean domains, and Hersch's inequality for the first positive eigenvalue on closed simply connected surfaces.

Just like most of the results one can find in the literature, these sharp inequalities deal with the lowest order positive eigenvalues. Aside from numerical approaches, mainly in dimension 2, the determination of optimal bounds for eigenvalues of higher order is a problem that remains largely open.

In this article our aim will be to show how it is possible, through quite simple considerations, to establish certain intrinsic relationships between the infima (or the suprema) of eigenvalues of different orders. Let us start by fixing some notations.

Given a regular bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, we designate by $\{\lambda_k(\Omega)\}_{k \geq 1}$ (resp. $\{\mu_k(\Omega)\}_{k \geq 0}$) the nondecreasing sequence of eigenvalues of the Laplacian on Ω with Dirichlet (resp. Neumann) boundary conditions,

each repeated according to its multiplicity. We introduce the following universal sequences of real numbers that are attached to the n-dimensional Euclidean space :

$$\lambda_k^*(n) = \inf \{\lambda_k(\Omega) \ : \ \Omega \subset \mathbb{R}^n, \ |\Omega| = 1\}$$

and

$$\mu_k^*(n) = \sup \{\mu_k(\Omega) \ : \ \Omega \subset \mathbb{R}^n, \ |\Omega| = 1\},$$

where $|\Omega|$ stands for the volume of Ω. Notice that thanks to standard continuity results for eigenvalues, the definition of $\lambda_k^*(n)$ (resp. $\mu_k^*(n)$) does not change if the infimum (resp. the supremum) is taken only over connected domains. The famous Faber-Krahn and Szegö-Weinberger isoperimetric inequalities then read respectively as follows:

$$\lambda_1^*(n) = \lambda_1(B^n)|B^n|^{\frac{2}{n}} = j_{\frac{n}{2}-1,1}^2 \omega_n^{\frac{2}{n}}$$

and

$$\mu_1^*(n) = \mu_1(B^n)|B^n|^{\frac{2}{n}} = p_{\frac{n}{2},1}^2 \omega_n^{\frac{2}{n}},$$

where ω_n is the volume of the unit Euclidean ball B^n, $j_{\frac{n}{2}-1,1}$ is the first positive zero of the Bessel function $J_{\frac{n}{2}-1}$ and $p_{\frac{n}{2},1}$ is the first positive zero of the derivative of the Bessel function $J_{\frac{n}{2}}$. It is also well known that (see for instance [13, p. 61])

$$\lambda_2^*(n) = 2^{\frac{2}{n}} \lambda_1^*(n).$$

The same relation is conjectured to hold true between $\mu_2^*(n)$ and $\mu_1^*(n)$ (see [11] for a recent result about this conjecture in the 2-dimensional case). The following inequalities are also expected to be satisfied for every $k \geq 1$ (Pólya's conjecture),

$$\mu_k^*(n) \leq 4\pi^2 \left(\frac{k}{\omega_n}\right)^{\frac{2}{n}} \leq \lambda_k^*(n),$$

where $4\pi^2 \left(\frac{k}{\omega_n}\right)^{\frac{2}{n}}$ is the first term of the Weyl asymptotic expansion of both Dirichlet and Neumann eigenvalues of domains of volume one. Although this conjecture is still open, it was proved by Berezin [3] and Li and Yau [22] that $\lambda_k^*(n) \geq \frac{n}{n+2} 4\pi^2 \left(\frac{k}{\omega_n}\right)^{\frac{2}{n}}$, while Kröger [18, 19] proved that $\mu_k^*(n) \leq \left(1 + \frac{n}{2}\right)^{\frac{2}{n}} 4\pi^2 \left(\frac{k}{\omega_n}\right)^{\frac{2}{n}}$.

The first observation we make in this paper is that the sequence $\lambda_k^*(n)^{n/2}$ is subadditive while $\mu_k^*(n)^{n/2}$ is superadditive. Indeed, we prove (Theorem 2.1) that, for every $k \geq 2$ and any finite family $i_1, \ldots, i_p$ of positive integers such that $i_1 + i_2 + \cdots + i_p = k$,

$$\lambda_k^*(n)^{n/2} \leq \lambda_{i_1}^*(n)^{n/2} + \lambda_{i_2}^*(n)^{n/2} + \cdots + \lambda_{i_p}^*(n)^{n/2} \qquad (1)$$

and

$$\mu_k^*(n)^{n/2} \geq \mu_{i_1}^*(n)^{n/2} + \mu_{i_2}^*(n)^{n/2} + \cdots + \mu_{i_p}^*(n)^{n/2}. \qquad (2)$$

An immediate consequence of Theorem 2.1 and Fekete's Subadditive Lemma is that the sequences $\lambda_k^*(n)/k^{\frac{2}{n}}$ and $\mu_k^*(n)/k^{\frac{2}{n}}$ are convergent and that Pólya's conjecture for Dirichlet (resp. Neumann) eigenvalues is equivalent to the following

$$\lim_k \frac{\lambda_k^*(n)}{k^{\frac{2}{n}}} = 4\pi^2 \omega_n^{-\frac{2}{n}}$$

(resp. $\lim_k \frac{\mu_k^*(n)}{k^{\frac{2}{n}}} = 4\pi^2 \omega_n^{-\frac{2}{n}}$, see Corollary 2.2).

Besides their theoretical interest, the inequalities (1) and (2) provide a "rough test" for the numerical methods used to approximate $\lambda_k^*(n)$ and $\mu_k^*(n)$. For example, we observe that the numerical values for $\lambda_k^*(2)$ obtained by Oudet [25] (see also [13, p. 83]) could be improved since the gap between the approximate values given for some successive $\lambda_k^*(2)$ exceeds $\pi j_{0,1}^2$. Improvements of Oudet's calculations leading to approximate values which are consistent with (1) and (2) have been obtained recently by Antunes and Freitas [2].

Regarding the equality case in (1) we prove that if it holds, then the infimum $\lambda_k^*(n)$ is approximated to any desired accuracy by the λ_k of a disjoint union of p domains A_j, $j = 1, \ldots, p$, each of which being, up to volume normalization, an "almost" minimizing domain for $\lambda_{i_j}^*(n)$ (see Theorem 2.1 for a precise statement). A similar phenomenon occurs for the case of equality in (2).

This result complements that by Wolf and Keller [27] where it is proved that if $\Omega = A \cup B$ is a disconnected minimizer of λ_k, then there exists a positive integer $i < k$ so that, after volume normalizations, A minimizes λ_i and B minimizes λ_{k-i} and, moreover, $\lambda_k^*(n)^{n/2} = \lambda_i^*(n)^{n/2} + \lambda_{k-i}^*(n)^{n/2}$. A Neumann analogue of this result has been recently obtained by Poliquin and Roy-Fortin [26]

Our next observation is that Wolf-Keller's result extends to "almost minimizing" disconnected domains as follows (Theorem 2.2): If a disconnected domain $\Omega = A \cup B$ minimizes λ_k to within some $\varepsilon \geq 0$, then there exists an integer i so that, after volume normalizations, A minimizes $\lambda_i^{n/2}$ to within ε and B minimizes $\lambda_{k-i}^{n/2}$ to within ε, and, moreover,

$$0 \leq \left\{ \lambda_i^*(n)^{n/2} + \lambda_{k-i}^*(n)^{n/2} \right\} - \lambda_k^*(n)^{n/2} \leq \varepsilon.$$

A similar property holds for "almost maximizing" disconnected domains of Neumann eigenvalues (Theorem 2.3).

The second part of the paper is devoted to the case of compact surfaces without boundary. If S is an orientable compact surface of the 3-dimensional space, we denote by $\{\nu_k(S)\}_{k\geq 0}$ the spectrum of the Laplace-Beltrami operator acting on S (here $\nu_0(S) = 0$). The eigenvalue ν_k is not bounded above on the set of compact surfaces of fixed area, as shown in

[4, Theorem 1.4] (which also justifies why we do not consider higher dimensional hypersurfaces). However, according to Korevaar [17], for every integer $\gamma \geq 0$, the k-th eigenvalue ν_k is bounded above on the set $\mathcal{M}(\gamma)$ of compact surfaces of genus γ and fixed area. As before, we introduce the sequence

$$\nu_k^*(\gamma) = \sup\{\nu_k(S) \; : \; S \in \mathcal{M}(\gamma) \text{ and } |S| = 1\} = \sup_{S \in \mathcal{M}(\gamma)} \nu_k(S)|S|.$$

As we will see in Section 3, an equivalent definition of $\nu_k^*(\gamma)$ consists in taking the supremum of the k-th eigenvalue $\nu_k(\Sigma_\gamma, g)$ of the Laplace-Beltrami operator on compact orientable 2-dimensional Riemannian manifolds of genus γ and area one.

For $\gamma = 0$, one has, from the results of Hersch [14] and Nadirashvili [24]

$$\nu_1^*(0) = 8\pi \quad \text{and} \quad \nu_2^*(0) = 16\pi.$$

Results concerning extremal eigenvalues on surfaces of genus 1 and 2 can be found in [8, 7, 9, 10, 15, 16, 21, 23]. On the other hand, we have proved in [5] that the sequence $\nu_k^*(\gamma)$ is non decreasing with respect to γ and that it is bounded below by a linear function of k and γ. A. Hassannezhad [12] has recently proved that $\nu_k^*(\gamma)$ is also bounded from below by such a linear function of k and γ.

In Theorem 3.1 we prove that the double sequence $\nu_k^*(\gamma)$ satisfies the following property (Theorem 3.1): For every $\gamma \geq 0$, $k \geq 1$, if $\gamma_1 \ldots, \gamma_p \in \mathbb{N}$ and $i_1, \ldots, i_p \in \mathbb{N}^*$ are such that $\gamma_1 + \cdots + \gamma_p = \gamma$ and $i_1 + \cdots + i_p = k$, then

$$\nu_k^*(\gamma) \geq \nu_{i_1}^*(\gamma_1) + \cdots + \nu_{i_p}^*(\gamma_p). \tag{3}$$

As before, we investigate the equality case in (3) and establish the following Wolf-Keller's type result (Corollary 3.1) : Assume that the disjoint union $S_1 \sqcup S_2$ of two compact orientable surfaces S_1 and S_2 of genus γ_1, γ_2, respectively, satisfies

$$\nu_k(S_1 \sqcup S_2) = \nu_k^*(\gamma). \tag{4}$$

with $|S_1| + |S_2| = 1$ and $\gamma_1 + \gamma_2 = \gamma$. Then there exists an integer $i \in \{1, \cdots, k-1\}$ such that

$$\nu_k^*(\gamma) = \nu_i^*(\gamma_1) + \nu_{k-i}^*(\gamma_2)$$

$$\nu_i(S_1)|S_1| = \nu_i^*(\gamma_1) \quad \text{and} \quad \nu_{k-i}(S_2)|S_2| = \nu_{k-i}^*(\gamma_2).$$

Actually, we give a more general result where $S_1 \sqcup S_2$ is assumed to maximize ν_k to within a positive ε (Theorem 3.2).

Similar considerations can be made about nonorienrtable surfaces. This is discussed at the end of the paper.

2. DIRICHLET AND NEUMANN EIGENVALUE PROBLEMS ON EUCLIDEAN DOMAINS

To every (sufficiently regular) bounded domain Ω in $\mathbb{R}^n$, $n \geq 2$, we associate two sequences of real numbers

$$0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \cdots \leq \lambda_k(\Omega) \leq \cdots$$

and

$$0 = \mu_0(\Omega) < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \cdots \leq \mu_k(\Omega) \leq \cdots$$

where $\lambda_k(\Omega)$ (resp. $\mu_k(\Omega)$) denotes the k-th eigenvalue of the Laplacian in Ω with Dirichlet (resp. Neumann) boundary conditions on $\partial\Omega$. If t is a positive number, the notation $t\,\Omega$ will designate the image of the domain Ω under the Euclidean dilation of ratio t. One has

$$\lambda_k(t\,\Omega) = t^{-2}\lambda_k(\Omega)\;,\;\;\mu_k(t\,\Omega) = t^{-2}\mu_k(\Omega) \text{ and } |t\,\Omega| = t^n|\Omega|$$

and, then

$$
\begin{aligned}
\lambda_k^*(n) &= \inf\{\lambda_k(\Omega)\;:\;\Omega \subset \mathbb{R}^n,\;|\Omega| = 1\} \\
&= \inf\{\lambda_k(\Omega)\;:\;\Omega \subset \mathbb{R}^n,\;|\Omega| \leq 1\} \\
&= \inf\{\lambda_k(\Omega)|\Omega|^{2/n}\;:\;\Omega \subset \mathbb{R}^n\}
\end{aligned}
\tag{5}
$$

and

$$
\begin{aligned}
\mu_k^*(n) &= \sup\{\mu_k(\Omega)\;:\;\Omega \subset \mathbb{R}^n,\;|\Omega| = 1\} \\
&= \sup\{\mu_k(\Omega)\;:\;\Omega \subset \mathbb{R}^n,\;|\Omega| \geq 1\} \\
&= \sup\{\mu_k(\Omega)|\Omega|^{2/n}\;:\;\Omega \subset \mathbb{R}^n\}.
\end{aligned}
\tag{6}
$$

The sequences $\lambda_k^*(n)$ and $\mu_k^*(n)$ satisfy the following intrinsic properties.

Theorem 2.1. *Let n and k be two positive integers and let $i_1 \leq i_2 \leq \cdots \leq i_p$ be positive integers such that $i_1 + i_2 + \cdots + i_p = k$.*
1) We have,

$$\lambda_k^*(n)^{n/2} \leq \lambda_{i_1}^*(n)^{n/2} + \lambda_{i_2}^*(n)^{n/2} + \cdots + \lambda_{i_p}^*(n)^{n/2} \tag{7}$$

and

$$\mu_k^*(n)^{n/2} \geq \mu_{i_1}^*(n)^{n/2} + \mu_{i_2}^*(n)^{n/2} + \cdots + \mu_{i_p}^*(n)^{n/2}, \tag{8}$$

2) If the equality holds in (7), then, for every $\varepsilon > 0$, there exist p mutually disjoint domains $A_1, A_2, \cdots, A_p$ such that

    i) $\lambda_k(A_1 \cup \cdots \cup A_p) \leq (1 + \varepsilon)\lambda_k^*$;
    ii) $\forall j \leq p, \lambda_{i_j}^* \leq \lambda_{i_j}(A_j)|A_j|^{2/n} \leq (1 + \varepsilon)\lambda_{i_j}^*.$

    iii) $|A_1| + \cdots + |A_p| = 1$ and, $\forall j \leq p, \dfrac{\lambda_{i_j}^*}{(1+\varepsilon)\lambda_k^*} \leq |A_j|^{2/n} \leq \dfrac{(1+\varepsilon)\lambda_{i_j}^*}{\lambda_k^*}$;

where λ_k^* stands for $\lambda_k^*(n)$.

3) If the equality holds in (8), then, for every $\varepsilon > 0$, there exist p mutually disjoint domains $A_1, A_2, \cdots, A_p$ such that

    i) $\mu_k(A_1 \cup \cdots \cup A_p) \geq (1 - \varepsilon)\mu_k^*$;
    ii) $\forall j \leq p, (1 - \varepsilon)\mu_{i_j}^* \leq \mu_{i_j}(A_j)|A_j|^{2/n} \leq \mu_{i_j}^*.$

iii) $|A_1| + \cdots + |A_p| = 1$ and, $\forall j \le p$, $\dfrac{(1-\varepsilon)\mu^*_{i_j}}{\mu^*_k} \le |A_j|^{2/n} \le \dfrac{\mu^*_{i_j}}{(1-\varepsilon)\mu^*_k}$;

where μ^*_k stands for $\mu^*_k(n)$.

Proof. Let ε be any positive real number. For each $j \le p$, let C_j be a domain of volume 1 satisfying

$$\lambda^*_{i_j}(n) \le \lambda_{i_j}(C_j) \le (1 + \varepsilon)\lambda^*_{i_j}(n)$$

and set $B_j = \left(\lambda_{i_j}(C_j)/\lambda^*_k(n)\right)^{\frac{1}{2}} C_j$ so that

$$\lambda_{i_j}(B_j) = \lambda^*_k(n) \quad \text{and} \quad |B_j| = \left(\lambda_{i_j}(C_j)/\lambda^*_k(n)\right)^{\frac{n}{2}}.$$

One can assume w.l.o.g. that the domains $B_1, \cdots, B_p$ are mutually disjoint. Let us introduce the domain $\Omega = B_1 \cup \cdots \cup B_p$. Since for every $j \le p$, $\lambda_{i_j}(B_j) = \lambda^*_k(n)$ and since the spectrum of Ω is the union of the spectra of the B_j's, one has

$$\#\left\{l \in \mathbb{N}^* \ ; \ \lambda_l(\Omega) \le \lambda^*_k(n)\right\} = \sum_{j=1}^{p} \#\left\{l \in \mathbb{N}^* \ ; \ \lambda_l(B_j) \le \lambda^*_k(n)\right\} \ge \sum_{j=1}^{p} i_j = k.$$

Thus, $\lambda_k(\Omega) \le \lambda^*_k(n)$. Since $\lambda^*_k(n) \le \lambda_k(\Omega)|\Omega|^{\frac{2}{n}}$, the volume of Ω should be greater than or equal to 1. Consequently,

$$1 \le |\Omega| = \sum_{j \le p} |B_j| = \frac{1}{\lambda^*_k(n)^{\frac{n}{2}}} \sum_{j \le p} \lambda_{i_j}(C_j)^{\frac{n}{2}} \le \frac{(1 + \varepsilon)^{\frac{n}{2}}}{\lambda^*_k(n)^{\frac{n}{2}}} \sum_{j \le p} \lambda^*_{i_j}(n)^{\frac{n}{2}}. \quad (9)$$

Inequality (7) follows immediately from (9) since ε can be arbitrarily small.

Assume now that the equality holds in (7) and consider for each positive ε, a family $B_1, B_2, \cdots, B_p$ constructed as above. Using (9), one sees that the domain $\Omega = B_1 \cup B_2 \cup \cdots \cup B_p$ satisfies $1 \le |\Omega| \le (1 + \varepsilon)^{\frac{n}{2}}$ and it is easy to check that the domains $A_j := |\Omega|^{-\frac{1}{n}} B_j$, $j \le p$, satisfy the properties (ii) and (iii) of the statement (indeed, $|A_j| = \frac{|B_j|}{|\Omega|}$ with $\left(\lambda^*_{i_j}(n)/\lambda^*_k(n)\right)^{\frac{n}{2}} \le |B_j| \le \left((1 + \varepsilon)\lambda^*_{i_j}(n)/\lambda^*_k(n)\right)^{\frac{n}{2}}$). As for (i), one has for each $j \le p$, $\lambda_{i_j}(A_j) = |\Omega|^{\frac{2}{n}}\lambda^*_k(n)$. Since $k = i_1 + i_2 + \cdots + i_p$, one deduces that $\lambda_k(A_1 \cup A_2 \cup \cdots \cup A_p) = |\Omega|^{\frac{2}{n}}\lambda^*_k(n) \le (1 + \varepsilon)\lambda^*_k(n)$.

The proof in the Neumann case follows the same outline. Indeed, for any positive ε, we consider p mutually disjoint domains $C_1, C_2, \cdots, C_p$ of volume 1 such that, $\forall j \le p$,

$$\mu^*_{i_j}(n) \ge \mu_{i_j}(C_j) \ge (1 - \varepsilon)\mu^*_{i_j}(n)$$

and set $B_j = \left(\mu_{i_j}(C_j)/\mu^*_k(n)\right)^{\frac{1}{2}} C_j$ and $\Omega = B_1 \cup B_2 \cup \cdots \cup B_p$. Since for every $j \le p$, $\mu_{i_j}(B_j) = \mu^*_k(n)$, the number of eigenvalues of B_j that are *strictly* less than $\mu^*_k(n)$ is at most i_j (recall that $\mu_{i_j}(B_j)$ denotes the $(i_j + 1)$-th eigenvalue of B_j). As the spectrum of Ω is the union of the spectra of the B_j's, it is clear that the number of eigenvalues of Ω that are strictly less than $\mu^*_k(n)$ is

at most $k = i_1 + i_2 + \cdots + i_p$. Thus, $\mu_k(\Omega) \geq \mu_k^*(n)$ which implies (since $\mu_k^*(n) \geq \mu_k(\Omega)|\Omega|^{\frac{2}{n}}$) that the volume of Ω is less than or equal to 1. To derive Inequality (8) it suffices to observe that $1 \geq |\Omega| = \sum_{j \leq p} |B_j|$ and that

$$|B_j| = \left(\mu_{i_j}(C_j)/\mu_k^*(n)\right)^{\frac{n}{2}} \geq (1 - \varepsilon)^{\frac{n}{2}} \frac{\mu_{i_j}^*(n)^{\frac{n}{2}}}{\mu_k^*(n)^{\frac{n}{2}}}.$$

Assume now that the equality holds in (8) and consider for each positive ε, a family $B_1, B_2, \cdots, B_p$ constructed as above. The domain $\Omega = B_1 \cup B_2 \cup \cdots \cup B_p$ satisfies $1 \geq |\Omega| \geq (1 - \varepsilon)^{\frac{n}{2}}$ and it is easy to check that the domains $A_j := |\Omega|^{-\frac{1}{n}} B_j$, $j \leq p$, satisfy the properties (ii) and (iii) of the statement (indeed, $|A_j| = \frac{|B_j|}{|\Omega|}$ with $\left((1 - \varepsilon)\mu_{i_j}^*(n)/\mu_k^*(n)\right)^{\frac{n}{2}} \leq |B_j| \leq \left(\mu_{i_j}^*(n)/\mu_k^*(n)\right)^{\frac{n}{2}}$). Moreover, one has for each $j \leq p$, $\mu_{i_j}(A_j) = |\Omega|^{\frac{2}{n}}\mu_k^*(n)$. Thus, $\mu_k(A_1 \cup A_2 \cup \cdots \cup A_p) = |\Omega|^{\frac{2}{n}}\mu_k^*(n) \geq (1 - \varepsilon)\mu_k$ which proves (i).

$\square$

Corollary 2.1. *For every $n \geq 2$ and every $k \geq 1$, we have*

$$\lambda_{k+1}^*(n)^{n/2} - \lambda_k^*(n)^{n/2} \leq \lambda_1^*(n)^{n/2} = j_{\frac{n}{2}-1,1}^n \omega_n$$

and

$$\mu_{k+1}^*(n)^{n/2} - \mu_k^*(n)^{n/2} \geq \mu_1^*(n)^{n/2} = p_{\frac{n}{2},1}^n \omega_n.$$

Remark 2.1. *(i) The first inequality in Corollary 2.1 is sharp for $k = 1$ since we know that $\lambda_2^*(n) = 2^{2/n}\lambda_1^*(n)$.*
(ii) In dimension 2, the inequalities of Corollary 2.1 lead to

$$\lambda_{k+1}^*(2) - \lambda_k^*(2) \leq \pi j_{0,1}^2 \approx 18.168$$

and

$$\mu_{k+1}^*(2) - \mu_k^*(2) \geq \pi p_{1,1}^2 \approx 10.65,$$

which provides a simple tool to test the accuracy of numerical approximations.
(iii) Iterating the inequalities of Corollary 2.1 we get

$$\lambda_k^*(n) \leq j_{\frac{n}{2}-1,1}^2 \omega_n^{2/n} k^{2/n}$$

and

$$\mu_k^*(n) \geq p_{\frac{n}{2},1}^2 \omega_n^{2/n} k^{2/n}.$$

Combining these inequalities with Pólya conjecture, we expect the following estimates

$$p_{\frac{n}{2},1}^2 \omega_n^{2/n} k^{2/n} \leq \mu_k^*(n) \leq 4\pi^2 \left(\frac{k}{\omega_n}\right)^{\frac{2}{n}} \leq \lambda_k^*(n) \leq j_{\frac{n}{2}-1,1}^2 \omega_n^{2/n} k^{2/n}$$

which take the following form in dimension 2 :

$$4\pi k \leq \lambda_k^*(2) \leq 5.784\,\pi k$$

and

$$3.39\,\pi k \leq \mu_k^*(2) \leq 4\,\pi k.$$

(iv) Let $\Omega \subset \mathbb{R}^n$ be the union of k balls of the same radius $r = (k\omega_n)^{-n}$ so that $|\Omega| = 1$. Then

$$\lambda_k(\Omega) = \lambda_1(B^n) = \lambda_1(B^n)(k\omega_n)^{2/n},$$

and

$$\lambda_{k+1}(\Omega) = \lambda_2(B^n) = \lambda_2(B^n)(k\omega_n)^{2/n}.$$

Thus,

$$\lambda_{k+1}(\Omega)^{n/2} - \lambda_k(\Omega)^{n/2} = k\omega_n \left(\lambda_2(B^n)^{n/2} - \lambda_1(B^n)^{n/2}\right).$$

This shows that the gap $\lambda_{k+1}(\Omega)^{n/2} - \lambda_k(\Omega)^{n/2}$ cannot be bounded independently of k (see also Proposition 2.1 below). Corollary 2.1 tells us that such a bound exists when we consider the sequence of infima of λ_k.

Thanks to Fekete's Lemma, the subadditivity of the sequence $\lambda_k^*(n)^{n/2}$ leads immediately to the following corollary.

Corollary 2.2. *For every $n \geq 2$, the sequence $\frac{\lambda_k^*(n)}{k^{2/n}}$ converges to a positive limit with*

$$\lim_k \frac{\lambda_k^*(n)}{k^{2/n}} = \inf_k \frac{\lambda_k^*(n)}{k^{2/n}}.$$

In particular, the two following properties are equivalent :

(1) *(Pólya's conjecture) For every $k \geq 1$ and every domain $\Omega \subset \mathbb{R}^n$,*

$$\lambda_k(\Omega) \geq 4\pi^2(|\Omega|\omega_n)^{-2/n}k^{2/n}$$

(2) $\lim_k \dfrac{\lambda_k^*(n)}{k^{2/n}} = 4\pi^2\omega_n^{-2/n}$.

A similar result holds for the Neumann Laplacian eigenvalues.

The inequality (7) leads to

$$\lambda_k^*(n)^{n/2} \leq \inf_{1 \leq i \leq k-1} \left\{\lambda_i^*(n)^{n/2} + \lambda_{k-i}^*(n)^{n/2}\right\}. \tag{10}$$

Wolf and Keller [27] proved that if λ_k is minimized by a non connected domain, that is $\lambda_k^*(n) = \lambda_k(A \cup B)$ for a couple of disjoint domains A and B with $|A| > 0$, $|B| > 0$ and $|A| + |B| = 1$, then the equality holds in (10) and, moreover, A and B are, up to normalizations, minimizers of λ_i and λ_{k-i}, respectively. The Neumann's analogue of this result has been established by Poliquin and Roy-Fortin [26].

The following theorem shows how Wolf-Keller's result extends to "almost minimizing" disconnected domains.

Theorem 2.2. *Let $k \geq 2$ and assume that there exists a non connected domain $\Omega = A \cup B$ in $\mathbb{R}^n$ with $|A| + |B| = 1$, $|A| > \varepsilon/\lambda_k^*(n)^{n/2}$, $|B| > \varepsilon/\lambda_k^*(n)^{n/2}$ and*

$$\lambda_k(A \cup B)^{n/2} \leq \lambda_k^*(n)^{n/2} + \varepsilon \tag{11}$$

for some $\varepsilon \geq 0$. Then there exists an integer $i \in \{1, \cdots, k-1\}$ such that

$$0 \leq \left\{\lambda_i^*(n)^{n/2} + \lambda_{k-i}^*(n)^{n/2}\right\} - \lambda_k^*(n)^{n/2} \leq \varepsilon,$$

$$0 \leq \lambda_i(A)^{n/2}|A| - \lambda_i^*(n)^{n/2} \leq \varepsilon \quad and \quad 0 \leq \lambda_{k-i}(B)^{n/2}|B| - \lambda_{k-i}^*(n)^{n/2} \leq \varepsilon.$$

Proof. Since the spectrum of $\Omega = A \cup B$ is the re-ordered union of the spectra of A and B, the eigenvalue $\lambda_k(\Omega)$ belongs to the union of the spectra of A and B and, moreover,

$$\#\left\{j \in \mathbb{N}^* \,;\, \lambda_j(A) < \lambda_k(\Omega)\right\} + \#\left\{j \in \mathbb{N}^* \,;\, \lambda_j(B) < \lambda_k(\Omega)\right\} \leq k - 1 \quad (12)$$

and

$$\#\left\{j \in \mathbb{N}^* \,;\, \lambda_j(A) \leq \lambda_k(\Omega)\right\} + \#\left\{j \in \mathbb{N}^* \,;\, \lambda_j(B) \leq \lambda_k(\Omega)\right\} \geq k. \quad (13)$$

Hence, there exists at least one integer $j \in \{1, \ldots, k\}$ such that $\lambda_j(A) = \lambda_k(\Omega)$ or $\lambda_j(B) = \lambda_k(\Omega)$. Assume that the first alternative occurs and let i be the largest integer between 1 and k such that $\lambda_i(A) = \lambda_k(\Omega)$.

Observe first that $i \leq k - 1$. Indeed, if $\lambda_k(A) = \lambda_k(\Omega)$, then

$$\lambda_k^*(n)^{n/2} \leq \lambda_k(A)^{n/2}|A| = \lambda_k(\Omega)^{n/2}|A| \leq \left(\lambda_k^*(n)^{n/2} + \varepsilon\right)|A|$$

which implies $|A| \geq \frac{\lambda_k^*(n)^{n/2}}{\lambda_k^*(n)^{n/2}+\varepsilon}$ and, then $|B| = 1 - |A| \leq \frac{\varepsilon}{\lambda_k^*(n)^{n/2}+\varepsilon} \leq \frac{\varepsilon}{\lambda_k^*(n)^{n/2}}$. This contradicts the volume assumptions of the theorem.

On the other hand, the maximality of i means that

$$\#\left\{j \in \mathbb{N}^* \,;\, \lambda_j(A) \leq \lambda_k(\Omega)\right\} = i$$

which implies, thanks to (13), $\lambda_{k-i}(B) \leq \lambda_k(\Omega)$. Thus,

$$\lambda_k(\Omega)^{n/2} = \lambda_k(\Omega)^{n/2}|A| + \lambda_k(\Omega)^{n/2}|B| \geq \lambda_i(A)^{n/2}|A| + \lambda_{k-i}(B)^{n/2}|B|. \quad (14)$$

Since $\lambda_i(A)^{n/2}|A| \geq \lambda_i^*(n)^{n/2}$ and $\lambda_{k-i}(B)^{n/2}|B| \geq \lambda_{k-i}^*(n)^{n/2}$, we have proved the inequality

$$\lambda_k^*(n)^{n/2} + \varepsilon \geq \lambda_k(\Omega)^{n/2} \geq \lambda_i^*(n)^{n/2} + \lambda_{k-i}^*(n)^{n/2}.$$

Now, we necessarily have the inequality $\lambda_i(A)^{n/2}|A| \leq \lambda_i^*(n)^{n/2} + \varepsilon$. Otherwise, we would have, thanks to (14) and Theorem 2.1,

$$\lambda_k(\Omega)^{n/2} \geq \lambda_i(A)^{n/2}|A| + \lambda_{k-i}(B)^{n/2}|B| \;>\; \lambda_i^*(n)^{n/2} + \varepsilon + \lambda_{k-i}^*(n)^{n/2}$$
$$\geq \; \lambda_k^*(n)^{n/2} + \varepsilon$$

which contradicts the assumption of the theorem. The same argument leads to the inequality $\lambda_{k-i}(B)^{n/2}|B| \leq \lambda_{k-i}^*(n)^{n/2} + \varepsilon$. $\qquad\square$

Remark 2.2. *(i) Taking $\varepsilon = 0$ in Theorem 2.2, all the inequalities of the theorem become equalities and we recover the result of Wolf and Keller. Notice that when $\varepsilon = 0$, it is immediate to see that the integer i is such that $\lambda_i^*(n)^{n/2} + \lambda_{k-i}^*(n)^{n/2}$ is minimal.*

(ii) The assumption that the volume of each of the components A and B of Ω is bounded below in terms of ε is necessary to guarantee that the integer i is different from 0 and k in Theorem 2.2. Indeed, take for A a domain whose volume is almost equal to one and such that $\lambda_k(A)^{n/2} \leq \lambda_k^*(n)^{n/2} + \varepsilon$, and take for B a domain of small volume such that $\lambda_1(B)^{n/2} > \lambda_k^*(n)^{n/2} + \varepsilon$. The domain $\Omega = A \cup B$ would have volume one and $\lambda_k(\Omega) = \lambda_k(A) < \lambda_1(B)$.

Using similar arguments as in the proof of Theorem 2.2 (see also the proof of Theorem 3.2), we obtain the following

Theorem 2.3. *Let* $k \geq 2$ *and assume that there exists a non connected domain* $\Omega = A \cup B$ *in* $\mathbb{R}^n$ *with* $|A| + |B| = 1$ *and*

$$\mu_k(A \cup B)^{n/2} \geq \mu_k^*(n)^{n/2} - \varepsilon \tag{15}$$

for some $\varepsilon \geq 0$. *Then there exists an integer* $i \in \{1, \cdots, k-1\}$ *such that*

$$0 \leq \mu_k^*(n)^{n/2} - \left[\mu_i^*(n)^{n/2} + \mu_{k-i}^*(n)^{n/2}\right] \leq \varepsilon,$$

$$0 \leq \mu_i^*(n)^{n/2} - \mu_i(A)^{n/2}|A| \leq \varepsilon \quad and \quad 0 \leq \mu_{k-i}^*(n)^{n/2} - \mu_{k-i}(B)^{n/2}|B| \leq \varepsilon.$$

Remark 2.3. *(i) Taking* $\varepsilon = 0$ *in Theorem 2.3, all the inequalities of the theorem become equalities and the integer* i *is necessarily such that* $\mu_i^*(n)^{n/2} + \mu_{k-i}^*(n)^{n/2}$ *is maximal.*

(ii) A consequence of Theorem 2.3 is that if for some $\varepsilon > 0$, *there exists a domain* Ω *in* $\mathbb{R}^n$ *with*

$$\mu_k(\Omega)^{n/2} > \sup_{1 \leq i \leq k-1} \left\{\mu_i^*(n)^{n/2} + \mu_{k-i}^*(n)^{n/2}\right\} + \varepsilon,$$

then $\mu_k^*(n)$ *cannot be approached up to* ε *by a non connected domain.*

The following properties are likely well known, we show them here for completeness and comparison with other results in this section.

Proposition 2.1. *For every* $n \geq 2$ *and* $k \geq 1$ *we have*

$$\inf\{\lambda_k(\Omega) - \lambda_1(\Omega) : \Omega \subset \mathbb{R}^n, |\Omega| = 1\} = 0 ; \tag{16}$$

$$\sup\{\lambda_{k+1}(\Omega) - \lambda_k(\Omega) : \Omega \subset \mathbb{R}^n, |\Omega| = 1\} = \infty ; \tag{17}$$

$$\inf\{\mu_k(\Omega) - \mu_1(\Omega) : \Omega \subset \mathbb{R}^n, |\Omega| = 1\} = 0 ; \tag{18}$$

$$\mu_1^*(n)(k+1)^{\frac{2}{n}} \leq \sup\{\mu_{k+1}(\Omega) - \mu_k(\Omega) : \Omega \subset \mathbb{R}^n, |\Omega| = 1\} \leq \mu_{k+1}^*(n). \tag{19}$$

Proof. To see (16) it suffices to consider a domain Ω modeled on the disjoint union of $k+1$ identical balls of volume $\frac{1}{k+1}$. The $k+1$ first Dirichlet eigenvalues of such a domain are almost equal.

Now, take any domain D with $\lambda_{k+1}(D) - \lambda_k(D) > 0$ and observe that $\lambda_{k+1}(tD) - \lambda_k(tD) \to +\infty$ as $t \to 0$. Then attach to the domain tD a sufficiently long and thin domain in order to obtain a volume 1 domain $\Omega(t)$ with $\lambda_k(\Omega(t)) \approx \lambda_k(tD)$ and $\lambda_{k+1}(\Omega(t)) \approx \lambda_{k+1}(tD)$ (recall that the first eigenvalue of a box of volume 1 goes to infinity as the length of one of its sides becomes very small). Thus, $\lambda_{k+1}(\Omega(t)) - \lambda_k(\Omega(t))$ goes to infinity as $t \to 0$ which proves (17).

As for the Neumann eigenvalues of a domain Ω modeled on the disjoint union of $k+1$ identical balls of volume $\frac{1}{k+1}$, one has $\mu_0(\Omega) = 0$ and $\mu_1(\Omega), \cdots, \mu_k(\Omega)$ are almost equal to zero, while $\mu_{k+1}(\Omega)$ is almost equal to the first positive eigenvalue of one of the balls, that is $\mu_{k+1}(\Omega) \approx \mu_1^*(n)(k+1)^{\frac{2}{n}}$. This example proves (18) and (19).

□

3. Eigenvalues of closed surfaces

There are two equivalent approaches to introduce the extremal eigenvalues on closed surfaces.

Let us start with the "embedded" point of view. Indeed, if S is a compact connected surface of the 3-dimensional Euclidean space $\mathbb{R}^3$, we consider on it the Dirichlet's energy functional associated with the tangential gradient, and denote by

$$0 = \nu_0(S) < \nu_1(S) \le \nu_2(S) \le \cdots \le \nu_k(S) \le \cdots.$$

the spectrum of the corresponding Laplacian. According to [4, Theorem 1.4], one has, $\forall k \ge 1$,

$$\sup_{|S|=1} \nu_k(S) = +\infty.$$

However, it is known since the work of Korevaar [17] that for every integer $\gamma \ge 0$, the k-th eigenvalue ν_k is bounded above on the set of compact surfaces of genus γ. Thus, for every integer $\gamma \ge 0$ we denote by $\mathcal{M}(\gamma)$ the set of all compact surfaces of genus γ embedded in $\mathbb{R}^3$ and define the sequence

$$\nu_k^*(\gamma) = \sup\{\nu_k(S) \ ; \ S \in \mathcal{M}(\gamma) \text{ and } |S| = 1\} = \sup_{S \in \mathcal{M}(\gamma)} \nu_k(S)|S|,$$

where $|S|$ stands for the area of S. Regarding the infimum, it is well known that $\inf_{S \in \mathcal{M}(\gamma)} \nu_k(S)|S| = 0$.

Alternatively, let Σ_γ be an abstract closed orientable 2-dimensional smooth manifold of genus γ. To every Riemannian metric g on Σ_γ we associate the sequence of eigenvalues of the Laplace-Beltrami operator Δ_g

$$0 = \nu_0(\Sigma_\gamma, g) < \nu_1(\Sigma_\gamma, g) \le \nu_2(\Sigma_\gamma, g) \le \cdots \le \nu_k(\Sigma_\gamma, g) \le \cdots.$$

Notice that for every positive number t, one has $\nu_k(\Sigma_\gamma, tg) = t^{-1}\nu_k(\Sigma_\gamma, g)$ while the Riemannian area satisfies $|(\Sigma_\gamma, tg)| = t|(\Sigma_\gamma, g)|$ so that the product $\nu_k(\Sigma_\gamma, g)|(\Sigma_\gamma, g)|$ is invariant under scaling of the metric.

Lemma 3.1. *Let Σ_γ be a closed orientable 2-dimensional smooth manifold of genus $\gamma \ge 0$ and denote by $\mathcal{R}(\Sigma_\gamma)$ the set of all Riemannian metrics on Σ_γ. For every positive integer k one has*

$$
\begin{aligned}
\nu_k^*(\gamma) &= \sup\Big\{\nu_k(\Sigma_\gamma, g) \ ; \ g \in \mathcal{R}(\Sigma_\gamma) \text{ and } |(\Sigma_\gamma, g)| = 1\Big\} \\
&= \sup_{g \in \mathcal{R}(\Sigma_\gamma)} \nu_k(\Sigma_\gamma, g)|(\Sigma_\gamma, g)|.
\end{aligned}
$$

Proof. Let us first recall the well-known fact (see e.g. Dodziuk's paper [6]) that if two Riemannian metrics g_1, g_2 on a compact manifold M of dimension m are quasi-isometric with a quasi-isometry ratio close to 1, then the spectra of their Laplacians are close. More precisely, we say that g_1 and g_2 are α-quasi-isometric, with $\alpha \ge 1$, if for each $v \in TM$, $v \ne 0$, we have

$$\frac{1}{\alpha^2} \le \frac{g_1(v,v)}{g_2(v,v)} \le \alpha^2.$$

The spectra of g_1 and g_2 then satisfy, $\forall k \geq 1$,

$$\frac{1}{\alpha^{2(m+1)}} \leq \frac{\nu_k(M, g_1)}{\nu_k(M, g_2)} \leq \alpha^{2(m+1)} \tag{20}$$

while the ratio of their volumes is so that

$$\frac{1}{\alpha^m} \leq \frac{|(M, g_1)|}{|(M, g_2)|} \leq \alpha^m. \tag{21}$$

Now, any surface $S \in \mathcal{M}(\gamma)$ is of the form $S = \phi(\Sigma_\gamma)$, where $\phi : \Sigma_\gamma \to \mathbb{R}^3$ is a smooth embedding. Denoting by g_ϕ the Riemannian metric on Σ_γ defined as the pull back by ϕ of the Euclidean metric of $\mathbb{R}^3$, one clearly has

$$\nu_k(S) = \nu_k(\Sigma_\gamma, g_\phi) \qquad \text{and} \qquad |S| = |(\Sigma_\gamma, g_\phi)|.$$

This immediately shows that $\nu_k^*(\gamma) \leq \sup_{g \in \mathcal{R}(\Sigma_\gamma)} \nu_k(\Sigma_\gamma, g)|(\Sigma_\gamma, g)|$.

Conversely, given any Riemannian metric $g \in \mathcal{R}(\Sigma_\gamma)$, it is well known that there exists a C^1-isometric embedding ϕ from (Σ_γ, g) into $\mathbb{R}^3$ (see [20]). Using standard density results, there exists a sequence $\phi_n : \Sigma_\gamma \to \mathbb{R}^3$ of smooth embeddings that converges to ϕ with respect to the C^1-topology. The metrics $g_n = g_{\phi_n}$ induced by ϕ_n are quasi-isometric to g and the corresponding sequence of quasi-isometry ratios converges to 1. Therefore, using (20) and (21), $\lim_n \nu_k(\Sigma_\gamma, g_n) = \nu_k(\Sigma_\gamma, g)$ and $\lim_n |(\Sigma_\gamma, g_n)| = |(\Sigma_\gamma, g)|$. Hence, the sequence of surfaces $S_n = \phi_n(\Sigma_\gamma) \in \mathcal{M}(\gamma)$ satisfies

$$\lim_n \nu_k(S_n)|S_n| = \lim_n \nu_k(\Sigma_\gamma, g_n)|(\Sigma_\gamma, g_n)| = \nu_k(\Sigma_\gamma, g)|(\Sigma_\gamma, g)|.$$

This completes the proof of the Lemma. $\square$

It is known that $\nu_1^*(0) = \nu_1(\mathbb{S}^2, g_s) = 8\pi$, where g_s is the standard metric of the sphere (see [14]), $\nu_1^*(1) = \nu_1(\mathbb{T}^2, g_{hex}) = \frac{8\pi^2}{\sqrt{3}}$, where g_{hex} is the flat metric on the torus associated with the hexagonal lattice (see [23]), and $\nu_2^*(0) = 16\pi$ (see [24]). Moreover, one has the following inequality (see [21, 7])

$$\nu_1^*(\gamma) \leq 8\pi \left\lfloor \frac{\gamma + 3}{2} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the floor function. Recently, A. Hassannezhad [12] proved that there exist universal constants $A > 0$ and $B > 0$ such that, $\forall (k, \gamma) \in \mathbb{N}^2$,

$$\nu_k^*(\gamma) \leq A\gamma + Bk.$$

On the other hand, $\nu_k^*(\gamma)$ admits also a lower bound in terms of a linear function of γ and k as shown in our previous work [5] where we have also proved that $\nu_k^*(\gamma)$ is **nondecreasing** with respect to γ.

Theorem 3.1. *Let $\gamma \geq 0$ and $k \geq 1$ be two integers and let $\gamma_1 \ldots, \gamma_p \in \mathbb{N}$ and $i_1, \ldots, i_p \in \mathbb{N}^*$ be such that $\gamma_1 + \cdots + \gamma_p = \gamma$ and $i_1 + \cdots + i_p = k$. Then*

$$\nu_k^*(\gamma) \geq \nu_{i_1}^*(\gamma_1) + \cdots + \nu_{i_p}^*(\gamma_p). \tag{22}$$

If the equality holds in (22), then, for every $\varepsilon > 0$, there exist p compact orientable surfaces $S_1, \cdots, S_p$ of genus $\gamma_1 \ldots, \gamma_p$, respectively, such that

i) $v_k(S_1 \sqcup \cdots \sqcup S_p) \geq (1-\varepsilon)v_k^*(\gamma)$;

ii) $\forall j \leq p, (1-\varepsilon)v_{i_j}^*(\gamma_j) \leq v_{i_j}(S_j)|S_j| \leq v_{i_j}^*(\gamma_j)$;

iii) $|S_1| + \cdots + |S_p| = 1$ *and,* $\forall j \leq p, \frac{v_{i_j}^*(\gamma_j)}{(1+\varepsilon)v_k^*(\gamma)} \leq |S_j| \leq \frac{(1+\varepsilon)v_{i_j}^*(\gamma_j)}{v_k^*(\gamma)}$.

Before giving the proof of this theorem we recall that if S_1 and S_2 are two closed orientable surfaces in $\mathbb{R}^3$, then the spectrum $\{v_k(S_1 \sqcup S_2)\}_{k \geq 0}$ of their disjoint union is given by the re-ordered union of the spectra of S_1 and S_2 (in particular, $v_0(S_1 \sqcup S_2) = v_1(S_1 \sqcup S_2) = 0$). The following lemma shows that this spectrum of $S_1 \sqcup S_2$ can be approximated, with arbitrary accuracy, by the spectrum of a closed connected orientable surface of genus $\gamma = \text{genus}(S_1) + \text{genus}(S_2)$.

Lemma 3.2. *Let S_1 and S_2 be two closed surfaces in $\mathbb{R}^3$ of genus γ_1 and γ_2, respectively. There exists a 1-parameter family $S_\delta \in \mathcal{M}(\gamma)$ of closed surfaces of genus $\gamma = \gamma_1 + \gamma_2$ such that, for every $k \geq 0$,*

$$\lim_{\delta \to 0} v_k(S_\delta) = v_k(S_1 \sqcup S_2)$$

and

$$\lim_{\delta \to 0} |S_\delta| = |S_1 \sqcup S_2|.$$

In particular, the definition of $v_k^*(\gamma)$ does not change if we include in $\mathcal{M}(\gamma)$ the disjoint unions of surfaces $S_1 \sqcup \cdots \sqcup S_p$ with $\text{genus}(S_1) + \cdots + \text{genus}(S_p) = \gamma$.

Proof of Lemma 3.2. We denote by g_1 and g_2 the Riemannian metrics induced on S_1 and S_2, respectively. In what follows, we will show how to construct a 1-parameter family g_δ of Riemannian metrics on the connected sum S of S_1 and S_2 so that $\lim_{\delta \to 0} v_k(S, g_\delta) = v_k(S_1 \sqcup S_2)$ and $\lim_{\delta \to 0} |(S, g_\delta)| = |S_1 \sqcup S_2|$. Using arguments as in the proof of Lemma 3.1, we easily see that this family of Riemannian surfaces (S, g_δ) gives rise to a family of embedded surfaces $S_\delta \in \mathcal{M}(\gamma_1 + \gamma_2)$ which satisfies the conditions of the statement. For the sake of clarity we divide the proof into several steps.

Step 1 : Let $x_1 \in S_1$ and $x_2 \in S_2$ be two arbitrary points. For any sufficiently small $\delta > 0$, Lemma 2.3 of [5] tells us that the metrics g_1 and g_2 of S_1 and S_2 are $(1+\delta)$-quasi-isometric to other metrics $g_{1,\delta}$ and $g_{2,\delta}$ which are Euclidean around x_1 and x_2. As in the proof of Lemma 3.1, we use (20) to deduce that $\lim_{\delta \to 0} v_k(S_i, g_{i,\delta}) = v_k(S_i, g_i)$ and, consequently,

$$\lim_{\delta \to 0} v_k\left((S_1, g_{1,\delta}) \sqcup (S_2, g_{2,\delta})\right) = v_k\left((S_1, g_1) \sqcup (S_2, g_2)\right). \tag{23}$$

Step 2 : Let (S, g) be a Riemannian surface which is flat around a point $x \in S$. For every sufficiently small $\varepsilon > 0$, the metric g can be deformed in the complement of the geodesic ball of radius ε into a metric g_ε which is $(1+2\varepsilon)$-quasi-isometric to g and so that the geodesic annulus $\mathcal{A}(x, \varepsilon, \varepsilon + \varepsilon^2)$ centered at x with inner and outer radii ε and $\varepsilon + \varepsilon^2$, is isometric to the cylinder $S_\varepsilon^1 \times (\varepsilon, \varepsilon + \varepsilon^2)$, where S_ε^1 is the circle of radius ε.

Indeed, let us choose ε so that g is flat in the geodesic ball $B(x, 2\varepsilon)$ of radius 2ε centered at x that we identify with the Euclidean ball $B(O, 2\varepsilon) \subset \mathbb{R}^2$. Using polar coordinates, we may write

$$g = dr^2 + r^2 d\theta^2$$

with $r \leq 2\varepsilon$ and $\theta \in [0, 2\pi]$. We consider the family g_ε of metrics on S which coincide with g in the complement of the annulus $\mathcal{A}(x, \varepsilon, 2\varepsilon)$ and whose restriction to this annulus (identified with $\mathcal{A}(0, \varepsilon, 2\varepsilon) \subset \mathbb{R}^2$) is given by

$$g_\varepsilon(r, \theta) = dr^2 + \psi_\varepsilon^2(r) d\theta^2,$$

with $\psi_\varepsilon(r) = \varepsilon$ if $\varepsilon \leq r \leq \varepsilon + \varepsilon^2$, $\psi(r) = r$ if $\varepsilon + 2\varepsilon^2 \leq r \leq 2\varepsilon$, and $\varepsilon \leq \psi_\varepsilon(r) \leq \varepsilon + \varepsilon^2$ if $r \in (\varepsilon + \varepsilon^2, \varepsilon + 2\varepsilon^2)$. Notice that we do not need to define ψ_ε more explicitly since only ψ_ε will be used and not its derivatives.

On the annulus $\mathcal{A}(0, \varepsilon, \varepsilon + \varepsilon^2)$ the metric g_ε coincides with the cylindrical metric $dr^2 + \varepsilon^2 d\theta^2$, that is $\mathcal{A}(x, \varepsilon, \varepsilon + \varepsilon^2)$ is isometric to $S_\varepsilon^1 \times (\varepsilon, \varepsilon + \varepsilon^2)$. On the other hand, the metric g_ε is clearly quasi-isometric to the Euclidean metric $g = dr^2 + r^2 d\theta^2$ on $\mathcal{A}(0, \varepsilon, 2\varepsilon)$ with

$$\min\left(1, \frac{\psi_\varepsilon^2(r)}{r^2}\right) g \leq g_\varepsilon \leq \max\left(1, \frac{\psi_\varepsilon^2(r)}{r^2}\right) g.$$

From the definition of ψ_ε one has, $\forall r \in (\varepsilon, 2\varepsilon)$,

$$\frac{1}{(1 + 2\varepsilon)^2} \leq \frac{\psi_\varepsilon^2(r)}{r^2} \leq (1 + 2\varepsilon)^2.$$

Since g_ε coincides with g in the complement of $\mathcal{A}(x, \varepsilon, 2\varepsilon)$, the metric g_ε is in fact globally $(1 + 2\varepsilon)$-quasi-isometric to g.

Step 3 : Construction of the family of metrics g_δ.

Given a sufficiently small $\delta > 0$, we first apply Step 1 and replace the metrics g_1 and g_2 by $g_{1,\delta}$ and $g_{2,\delta}$ so that, for each $i = 1, 2$, $(S_i, g_{i,\delta})$ is flat around a point $x_i \in S_i$. Thanks to Step 2, for every positive $\varepsilon < \varepsilon_0(\delta)$, we define on S_i a metric $g_{i,\delta,\varepsilon}$ which is $(1 + 2\varepsilon)$-quasi-isometric to g and so that the geodesic annulus $\mathcal{A}(x_i, \varepsilon, \varepsilon + \varepsilon^2)$ is isometric to the cylinder $S_\varepsilon^1 \times (\varepsilon, \varepsilon + \varepsilon^2)$. Thus, one can smoothly glue $(S_1 \setminus B(x_1, \varepsilon), g_{1,\delta,\varepsilon})$ and $(S_2 \setminus B(x_2, \varepsilon), g_{2,\delta,\varepsilon})$ along their boundaries and obtain a smooth Riemannian surface $(S, g_{\delta,\varepsilon})$ of genus $\gamma = \gamma_1 + \gamma_2$.

Let us denote by $\lambda_k(\delta, \varepsilon)$ (resp. $\mu_k(\delta, \varepsilon)$) the eigenvalues of the disjoint union of $(S_1 \setminus B(x_1, \varepsilon), g_{1,\delta})$ and $(S_2 \setminus B(x_2, \varepsilon), g_{2,\delta})$ with Dirichlet (resp. Neumann) boundary conditions. Similarily, we denote by $\bar{\lambda}_k(\delta, \varepsilon)$ (resp. $\bar{\mu}_k(\delta, \varepsilon)$) the eigenvalues of the disjoint union of $(S_1 \setminus B(x_1, \varepsilon), g_{1,\delta,\varepsilon})$ and $(S_2 \setminus B(x_2, \varepsilon), g_{2,\delta,\varepsilon})$ with Dirichlet (resp. Neumann) boundary conditions. From the min-max principle we have the following inequalities:

$$\bar{\mu}_k(\delta, \varepsilon) \leq \nu_k(S, g_{\delta,\varepsilon}) \leq \bar{\lambda}_k(\delta, \varepsilon).$$

Moreover, since $g_{i,\delta,\varepsilon}$ is $(1 + 2\varepsilon)$-quasi-isometric to $g_{i,\delta}$, one has using (20),

$$(1 + 2\varepsilon)^{-6} \mu_k(\delta, \varepsilon) \leq \bar{\mu}_k(\delta, \varepsilon) \quad \text{and} \quad \bar{\lambda}_k(\delta, \varepsilon) \leq (1 + 2\varepsilon)^6 \lambda_k(\delta, \varepsilon).$$

Therefore,

$$(1 + 2\varepsilon)^{-6}\mu_k(\delta, \varepsilon) \le \nu_k(S, g_{\delta,\varepsilon}) \le (1 + 2\varepsilon)^6 \lambda_k(\delta, \varepsilon).$$

On the other hand, according to [1], $\lambda_k(\delta, \varepsilon)$ (resp. $\mu_k(\delta, \varepsilon)$) converges as $\varepsilon \to \infty$, to the k-th eigenvalue of the disjoint union of $(S_1, g_{1,\delta})$ and $(S_2, g_{2,\delta})$. Thus, for every $k \ge 0$,

$$\lim_{\varepsilon \to 0} \nu_k(S, g_{\delta,\varepsilon}) = \nu_k\left((S_1, g_{1,\delta}) \sqcup (S_2, g_{2,\delta})\right).$$

In particular, there exists $\varepsilon(\delta) > 0$ such that, for every $k \le \frac{1}{\delta}$,

$$|\nu_k(S, g_{\delta,\varepsilon(\delta)}) - \nu_k\left((S_1, g_{1,\delta}) \sqcup (S_2, g_{2,\delta})\right)| < \delta.$$

Thus, if we set $g_\delta = g_{\delta,\varepsilon(\delta)}$, then using the last inequality and (23), we will have, for every $k \ge 0$,

$$\lim_{\delta \to 0} \nu_k(S, g_\delta) = \nu_k\left((S_1, g_1) \sqcup (S_2, g_2)\right).$$

As for the area, from the construction of g_δ, it is clear that $|(S, g_\delta)|$ tends to $|S_1| + |S_2|$ as $\delta \to 0$.

$\square$

Proof of Theorem 3.1. Let ε be any positive real number and let $S_1, \cdots, S_p$ be a family of compact orientable surfaces such that, for each positive $j \le p$, genus$(S_j) = \gamma_j$ and

$$\nu_{i_j}(S_j)|S_j| > \nu_{i_j}^*(\gamma_j) - \varepsilon.$$

After rescaling, we may assume that

$$\nu_{i_j}(S_j) = \nu_k^*(\gamma) \qquad \text{and} \qquad |S_j| > \frac{\nu_{i_j}^*(\gamma_j) - \varepsilon}{\nu_k^*(\gamma)}.$$

One has, using arguments as in the proof of Theorem 2.1,

$$\#\left\{l \in \mathbb{N} \; ; \; \nu_l(S_1 \sqcup \cdots \sqcup S_p) < \nu_k^*(\gamma)\right\} = \sum_{j=1}^p \#\left\{l \in \mathbb{N} \; ; \; \nu_l(S_j) < \nu_k^*(\gamma)\right\}$$

$$\le \sum_{j=1}^p i_j = k.$$

Consequently,

$$\nu_k(S_1 \sqcup \cdots \sqcup S_p) \ge \nu_k^*(\gamma).$$

From Lemma 3.2 and the definition of $\nu_k^*(\gamma)$, one then deduces the following:

$$|S_1 \sqcup \cdots \sqcup S_p| = |S_1| + \cdots + |S_p| \le 1.$$

This leads to

$$\sum_{j=1}^p \frac{\nu_{i_j}^*(\gamma_j) - \varepsilon}{\nu_k^*(\gamma)} \le 1,$$

that is,

$$\sum_{j=1}^{p} \nu_{i_j}^*(\gamma_j) \leq \nu_k^*(\gamma) + p\varepsilon.$$

This proves the inequality (22) since ε can be chosen arbitrarily small.

Assume that the equality holds in (22) . We can follow the same arguments as in the proof of Theorem 2.1 and conclude. $\qquad\square$

Remark 3.1. *A direct consequence of Theorem 3.1 is that, for every $\gamma \geq 0$ and every $k \geq 1$, one has*

$$\nu_k^*(\gamma) \geq \sup_{i \leq k-1} \left(\nu_i^*(\gamma) + \nu_{k-i}^*(0) \right).$$

In particular, $\nu_k^*(\gamma) \geq \nu_{k-1}^*(\gamma) + 8\pi$. Therefore, Theorem 3.1 improves our previous results (Theorem C and Corollary 4 of [5]).

The following theorem deals with the situation where $\nu_k^*(\gamma)$ is approached by the k-th eigenvalue of a disjoint union of two surfaces.

Theorem 3.2. *Let $\gamma \geq 0$ and $k \geq 2$ be two integers and assume that there exist two compact orientable surfaces S_1 and S_2 of genus γ_1, γ_2, respectively, such that $|S_1| + |S_2| = 1$, $\gamma_1 + \gamma_2 = \gamma$, and*

$$\nu_k(S_1 \sqcup S_2) \geq \nu_k^*(\gamma) - \varepsilon \tag{24}$$

for some $\varepsilon \geq 0$. Then there exists an integer $i \in \{1, \cdots, k-1\}$ such that

$$0 \leq \nu_k^*(\gamma) - \{\nu_i^*(\gamma_1) + \nu_{k-i}^*(\gamma_2)\} \leq \varepsilon,$$

$$0 \leq \nu_i^*(\gamma_1) - \nu_i(S_1)|S_1| \leq \varepsilon \quad and \quad 0 \leq \nu_{k-i}^*(\gamma_2) - \nu_{k-i}(S_2)|S_2| \leq \varepsilon.$$

Proof. Since the spectrum of $S_1 \sqcup S_2$ is the re-ordrered union of the spectra of S_1 and S_2, the eigenvalue $\nu_k(S_1 \sqcup S_2)$ belongs to this union and, moreover,

$$\#\left\{j \in \mathbb{N} \, ; \, \nu_j(S_1) < \nu_k(S_1 \sqcup S_2)\right\} + \#\left\{j \in \mathbb{N} \, ; \, \nu_j(S_2) < \nu_k(S_1 \sqcup S_2)\right\} \leq k \tag{25}$$

and

$$\#\left\{j \in \mathbb{N} \, ; \, \nu_j(S_1) \leq \nu_k(S_1 \sqcup S_2)\right\} + \#\left\{j \in \mathbb{N} \, ; \, \nu_j(S_2) \leq \nu_k(S_1 \sqcup S_2)\right\} \geq k+1 \tag{26}$$

(recall that the numbering of the eigenvalues start from zero). Hence, there exists at least one integer $j \in \{1, \ldots, k\}$ such that $\nu_j(S_1) = \nu_k(S_1 \sqcup S_2)$ or $\nu_j(S_2) = \nu_k(S_1 \sqcup S_2)$. Assume that the first alternative occurs and let i be the least positive integer such that $\nu_i(S_1) = \nu_k(S_1 \sqcup S_2)$. We necessarily have $\nu_{k-i}(S_2) \geq \nu_k(S_1 \sqcup S_2)$ since, otherwise, the $k+1$ eigenvalues $\nu_0(S_1), \cdots, \nu_{i-1}(S_1)$ and $\nu_0(S_2), \cdots, \nu_{k-i}(S_1)$ would be strictly less than $\nu_k(S_1 \sqcup S_2)$ which contradicts (26). Thus, $i \leq k-1$ and

$$\nu_k(S_1 \sqcup S_2) = \nu_k(S_1 \sqcup S_2)|S_1| + \nu_k(S_1 \sqcup S_2)|S_2| \leq \nu_i(S_1)|S_1| + \nu_{k-i}(S_2)|S_2|. \tag{27}$$

Since $\nu_i(S_1)|S_1| \leq \nu_i^*(\gamma_1)$ and $\nu_{k-i}(S_2)|S_2| \leq \nu_{k-i}^*(\gamma_2)$, we get

$$\nu_k^*(\gamma) - \varepsilon \leq \nu_k(S_1 \sqcup S_2) \leq \nu_i^*(\gamma_1) + \nu_{k-i}^*(\gamma_2).$$

Now, $v_i(S_1)|S_1| \geq v_i^*(\gamma_1) - \varepsilon$. Otherwise, we would have, thanks to (27) and Theorem 3.1,

$$v_k(S_1 \sqcup S_2) \leq v_i(S_1)|S_1| + v_{k-i}(S_2)|S_2| < v_i^*(\gamma_1) - \varepsilon + v_{k-i}^*(\gamma_2) \leq v_k^*(\gamma) - \varepsilon$$

which contradicts the assumption of the theorem. The same argument leads to the inequality $v_{k-i}(S_2)|S_2| \geq v_{k-i}(n) + \varepsilon$.

$\square$

As a consequence of Theorem 3.2, we obtain the following Wolf-Keller type result.

Corollary 3.1. *Let $\gamma \geq 0$ and $k \geq 2$ be two integers and assume that there exist two compact orientable surfaces S_1 and S_2 of genus γ_1, γ_2, respectively, such that $|S_1| + |S_2| = 1$, $\gamma_1 + \gamma_2 = \gamma$, and*

$$v_k(S_1 \sqcup S_2) = v_k^*(\gamma). \tag{28}$$

Then there exists an integer $i \in \{1, \cdots, k-1\}$ such that

$$v_k^*(\gamma) = v_i^*(\gamma_1) + v_{k-i}^*(\gamma_2) = \sup_{j=1,\cdots,k-1} \left\{ v_j^*(\gamma_1) + v_{k-j}^*(\gamma_2) \right\}$$

$$v_i(S_1)|S_1| = v_i^*(\gamma_1) \quad and \quad v_{k-i}(S_2)|S_2| = v_{k-i}^*(\gamma_2).$$

Extremal eigenvalues of nonorientable surfaces.
In the non-orientable case, we can similarly define, for every $\gamma \in \mathbb{N}$ and every $k \in \mathbb{N}$, the number $v_{*,k}(\gamma)$ as the supremum of $v_k(S)|S|$ over compact non-orientable surfaces of genus γ.

We have $v_{*,1}(1) = v_1(\mathbb{R}P^2, g_s) = 12\pi$ where g_s is the standard metric of the projective plane (see [21]), and $v_{*,1}(2) = v_1(\mathbb{K}^2, g_0) = 12\pi E(2\sqrt{2}/3) \simeq 13.365\,\pi$, where g_0 is a non flat metric of revolution on the Klein bottle and $E(2\sqrt{2}/3)$ is the complete elliptic integral of the second kind evaluated at $\frac{2\sqrt{2}}{3}$ (see [8]). Moreover, one has the following inequalities (see [21, 7])

$$v_{*,1}(\gamma) \leq 24\pi \left\lfloor \frac{\gamma+3}{2} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the floor function. The same reasoning as in the orientable case leads to the following results :

Theorem 3.3. *Let $\gamma \geq 0$ and $k \geq 1$ be two integers and let $\gamma_1 \ldots, \gamma_p$ and $i_1, \ldots, i_p$ be such that $\gamma_1 + \cdots + \gamma_p = \gamma$ and $i_1 + \cdots + i_p = k$. Then*

$$v_{*,k}(\gamma) \geq v_{*,i_1}(\gamma_1) + \cdots + v_{*,i_p}(\gamma_p). \tag{29}$$

If the equality holds in (22), then, for every $\varepsilon > 0$, there exist p compact orientable surfaces $S_1, \cdots, S_p$ of genus $\gamma_1 \ldots, \gamma_p$, respectively, such that

  i) $v_k(S_1 \sqcup \cdots \sqcup S_p) \leq (1 + \varepsilon)v_{*,k}(\gamma)$;
  ii) $|S_1| + \cdots + |S_p| = 1$ and, $\forall j \leq p$, $\frac{v_{*,i_j}(\gamma_j)}{(1+\varepsilon)v_{*,k}(\gamma)} \leq |S_j| \leq \frac{(1+\varepsilon)v_{*,i_j}(\gamma_j)}{v_{*,k}(\gamma)}$;
  iii) $\forall j \leq p$, $v_{*,i_j}(\gamma_j) \leq v_{i_j}(S_j)|S_j|^{2/n} \leq (1 + \varepsilon)v_{*,i_j}(\gamma_j)$.

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