Spectrum of Sublaplacians on Strictly Pseudoconvex CR Manifolds

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Le spectre du sous-laplacien sur les variétés CR strictement pseudoconvexes

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Résumé

Le but de cette thèse est d'étudier le spectre du sous-laplacien sur les variétés CR strictement peusdoconvexes. Nous prouvons que le spectre du sous-laplacien Δ_b est discret sur un domaine borné $\Omega \subset M$ d'une variété CR strictement pseudoconvexe qui satisfait l'inégalité de Poincaré, sous les conditions de Dirichlet au bord. Nous étudions le comportement des valeurs propres du sous-laplacien Δ_b sur une variété CR strictement pseudoconvexe compacte M, en tant que fonctionnelle sur l'espace $\mathcal{P}_+$ de formes de contact positivement orientées sur M en dotant $\mathcal{P}_+$ d'une topologie métrique naturelle. Nous établissons des inégalités pour les valeurs propres de Δ_b sur des variétés CR strictement pseudoconvexes (éventuellement à bord non vide). Nos estimations prolongent les résultats obtenus par P-C. Niu & H. Zhang [81] pour les valeurs propres du sous-laplacien avec conditions de Dirichlet au bord sur un domaine borné du groupe de Heisenberg, et sont dans l'esprit des inégalités de Payne-Pólya-Weinberger et Yang. Nous obtenons une nouvelle borne inférieure sur la première valeur propre non nulle $\lambda_1(\theta)$ du sous-laplacien Δ_b sur une variété CR strictement pseudoconvexe compacte M munie d'une forme de contact θ dont la connexion de Tanaka-Webster est à courbure de Ricci minorée.

Mots clés : Sous-laplacien, valeur propre, Structure pseudohermitienne, Forme de contact, Métrique de Webster, Métrique de Fefferman, Variété CR, Groupe de Heisenberg, Espace de type Sobolev sur les variétés CR, Application harmonique sous- elliptique, Application semi-isométrique, Tension de Levi, Formule de Bochner-Lichnerowicz, Inégalité universelle, Inégalité de Reilly.

Abstract

The purpose of this thesis is to study the spectrum of sublaplacians on compact strictly pseudoconvex CR manifolds. We prove the discreteness of the Dirichlet spectrum of the sublaplacian Δ_b on a smoothly bounded domain $\Omega \subset M$ in a strictly pseudoconvex CR manifold M satisfying Poincaré inequality. We study the behavior of the eigenvalues of a sublaplacian Δ_b on a compact strictly pseudoconvex CR manifold M, as functions on the set $\mathcal{P}_+$ of positively oriented contact forms on M by endowing $\mathcal{P}_+$ with a natural metric topology. We establish inequalities for the eigenvalues of Δ_b on compact strictly pseudoconvex CR manifolds (possibly with nonempty boundary) Our estimates extend those obtained by P-C. Niu & H. Zhang [81] for the Dirichlet eigenvalues of the sublaplacian on a bounded domain in the Heisenberg group, in the spirit of Payne-Pólya -Weinberger and Yang inequalities. We establish a new lower bound on the first nonzero eigenvalue $\lambda_1(\theta)$ of the sublaplacian Δ_b on a compact strictly pseudoconvex CR manifold M carrying a contact form θ whose Tanaka-Webster connection has Ricci curvature bounded from below.

Keywords : Sublaplacian, Spectrum, pseudohermitian structure, contact form, Webster metric, Fefferman metric, CR manifold, Heisenberg group, Sobolev type space, subeliptic harmonic map, semi-isometric map, Levi tension field, Bochner-Lichnerowicz formula, universal inequality, Reilly inequality.

Contents

Introduction

The study of spectrae of compact orientable Riemannian manifolds is by now a well defined branch of differential geometry, where differential geometric methods meet with methods from topology and partial differential equations, including aspects of the theory of harmonic maps. The state of the art, at the level of 1971, is described in the monograph by M. Berger & P. Gauduchon & E. Mazet, [71], which is our main model in developing a similar theory within the CR category. The relationship among spectral theory on Riemannian manifolds and harmonic maps starts with the work by R.T. Smith, [86]-[87], and a description of that is already captured in monograph form, cf. H. Urakawa, [52], an exposition of the main facts in the theory of harmonic maps, followed closely by other people (cf. e.g. E. Barletta & S. Dragomir & H. Urakawa, [30]) in building an analogous theory for maps from CR manifolds, as well as by us in the present thesis (cf. Chapter 3). Given a Riemannian manifold (M, g) there is a natural formally self-adjoint, positive, second order differential operator Δ_g, the Laplace-Beltrami operator associated to the metric g. Let $\sigma(\Delta_g)$ be the spectrum of Δ_g i.e. the set of all $\lambda \in \mathbb{R}$ such that $\Delta_g u = \lambda u$ for some $u \in C^\infty(M, \mathbb{R})$. When M is compact and orientable $\sigma(\Delta_g)$ is discrete

$$\sigma(\Delta_g) = \{\lambda_\nu(g) : \nu \geq 0\}, \quad 0 = \lambda_0(g) < \lambda_1(g) < \cdots < \lambda_\nu(g) < \cdots \uparrow +\infty, \tag{1}$$

essentially as a consequence of ellipticity of Δ_g. An array of results, too long to be fully mentioned here, regards properties of the spectrum $\sigma(\Delta_g)$ as implied by the local geometric features of the given Riemannian manifold (M, g), or the way $\sigma(\Delta_g)$ might characterize the Riemannian metric g itself e.g. whether isospectral Riemannian manifolds are isometric. Let us quote the famous result by A. Lichnerowicz, [12], and M. Obata, [72], according to which the first nonzero eigenvalue $\lambda_1(g)$ may be estimated by below as

$$\lambda_1(g) \geq \frac{m}{m-1}\, k \tag{2}$$

provided the Ricci curvature of (M, g) obeys to

$$\mathrm{Ric}_g(X, X) \geq k\, g(X, X), \qquad X \in \mathfrak{X}(M). \tag{3}$$

Here m is the dimension of M. While (2) is due to A. Lichnerowicz, [12], there is a rather spectacular contribution by M. Obata, [72], proving that equality in (2) may only occur when (M, g) is isometric to the sphere S^m with the standard Riemannian metric. The quoted result exerted a great influence on the mathematical community, prompting a series of generalizations in various directions (mentioned later on in this Introduction), including the realm of CR, or rather pseudo-hermitian, geometry, an issue discussed at some length in Chapter 4 of this thesis. Another result, nowadays famous, intertwining differential geometry and PDEs methods, is the existence of an asymptotic development

$$E \sim (4\pi t)^{-m/2 - r^2/(4t)} \left(u_0 + u_1 t + \cdots + u_\nu t^\nu + \cdots\right), \quad t \to 0^+ , \tag{4}$$

of the fundamental solution $E(x, y, t)$ to the heat equation on (M, g), here $r = d(x, y)$. Development (4) is due to S. Minakshisundaram & A. Pleijel, [99] (cf. also [71], 204-205) and the remarkable fact is that $u_\nu \in C^\infty(M \times M)$ are Riemannian invariants. More precisely if

$$Z(M, g; t) = \sum_{\nu=0}^{\infty} m_\nu e^{-\lambda_\mu t}$$

where m_ν is the multiplicity of the eigenvalue λ_ν, then

$$Z(M, g; t) \sim (4\pi t)^{-m/2} \left(a_0 + a_1 t + \cdots + a_\nu t^\nu + \cdots\right), \quad t \to 0^+ , \tag{5}$$

and the coefficients $a_\nu = \int_M u_\nu(x, x)\, dv_g(x)$ may be computed in terms of the curvature of (M, g). For instance

$$a_0 = \mathrm{Vol}(M, g), \tag{6}$$

$$a_1 = \frac{1}{6} \int_M \rho_g \, dv_g , \tag{7}$$

$$a_2 = \frac{1}{360} \int_M \left(2\|R_g\|^2 - 2\|\mathrm{Ric}_g\|^2 + 5\rho_g^2\right) dv_g . \tag{8}$$

Here R_g, Ric_g and ρ_g are respectively the curvature tensor field, the Ricci curvature, and the scalar curvature of the metric g. Finally let us recall that the stability of the identity mapping $1_M : M \to M$, thought of as a harmonic map of (M, g) into itself, is related to the properties of $\sigma(\Delta_g)$ by a result of R.T. Smith, [87]. Precisely if (M, g) is a compact Einstein manifold i.e.

$$\mathrm{Ric}_g(X, Y) = c\, g(X, Y), \quad X, Y \in \mathfrak{X}(M),$$

for some $c \in \mathbb{R}$, then the identity mapping $1_M : M \to M$ is weakly stable if and only if the first nonzero eigenvalue of Δ_g satisfies $\lambda_1(g) \geq 2c$. Also nullity of 1_M is given by

$$\mathrm{null}(1_M) = \dim \mathrm{Iso}(M, g) + \dim\{u \in C^\infty(M, \mathbb{R}) : \Delta_g u = 2cu\} \tag{9}$$

where $\mathrm{Iso}(M, g)$ is the isometry group of (M, g). To close, a particular importance for the themes treated in this thesis present results such as S. Bando & H. Urakawa's (cf. [90]) on the dependence of individual eigenvalues $\lambda_\nu(g)$ on the metric g (i.e. on the behavior of $\lambda_\nu(g)$ as g varies in the space of all Riemannian metrics on M, endowed with an appropriate topology) and the results by A. El Soufi & S. Ilias (cf. [5]-[6]) on variational properties of eigenvalues $\lambda_\nu(t) \equiv \lambda_\nu(g_t)$ under a smooth 1-parameter deformation of the metric. All the mentioned results admit meaningful reformulations on a compact strictly pseudoconvex CR manifold, in the presence of a given positively oriented contact form, and reformulations are either treated in this thesis or indicated as potential research work, to which the author of this thesis will devote further investigations.

The subject of this thesis is, as mentioned above, to start with a compact strictly pseudoconvex CR manifold $(M, T_{1,0}(M))$, of CR dimension n, fix a contact form $\theta \in \mathcal{P}_+$ such that the corresponding Levi form G_θ is positively definite, and study the spectrum $\sigma(\Delta_b)$ of a natural formally self adjoint, positive, second order differential operator Δ_b appearing on a pseudohermitian manifold (M, θ) very much like the Laplacian of a Riemannian manifold. This is the sublaplacian of (M, θ)

$$\Delta_b u = -\mathrm{div}(\nabla^H u), \quad u \in C^2(M). \tag{10}$$

Here $\mathrm{div} : \mathfrak{X}(M) \to C^\infty(M)$ is the divergence operator associated to the volume form $\Psi_\theta = \theta \wedge (d\theta)^n$ and $\nabla^H u$ is the horizontal gradient. Strict pseudoconvexity (actually orientability and nondegeneracy suffice) implies the existence of a unique globally defined nowhere zero, everywhere transverse to the Levi distribution $H(M) = \mathrm{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\}$, tangent vector field $T \in \mathfrak{X}(M)$ (the Reeb vector of (M, θ)) determined by $\theta(T) = 1$ and $T \rfloor d\theta = 0$. The vector field T may then be used to extend the Levi form

$$G_\theta(X, Y) = (d\theta)(X, JY), \quad X, Y \in H(M),$$

to a Riemmannian metric g_θ on M (the Webster metric of (M, θ)) given by

$$g_\theta(X, Y) = G_\theta(X, Y), \quad g_\theta(X, T) = 0, \quad g_\theta(T, T) = 1,$$

for any $X, Y \in H(M)$. The horizontal gradient is then $\nabla^H u = \Pi_H \nabla u$ where $\Pi_H : T(M) \to H(M)$ is the projection associated to the direct sum decomposition $T(M) = H(M) \oplus \mathbb{R}T$ and ∇u is given by $g_\theta(\nabla u, X) = X(u)$ for any $X \in \mathfrak{X}(M)$. So indeed forming $\nabla^H u$ is taking directional derivatives of u only in the *horizontal* directions lying in $H(M)$. Dropping T is then responsible for the degeneration of ellipticity of Δ_b, precisely in the T direction. The sublaplacian Δ_b will be therefore seen to be a degenerate elliptic operator in the sense of M. Bony, [58], this being recognized as the main difficulty in building a theory similar to that for the Laplacian of a Riemannian manifold. Although $\nabla^H u$ rises from omitting a direction in ∇u, the ordinary gradient with respect to the Webster metric, studying the Riemannian geometry of (M, g_θ) doesn't lie within our purposes, for reasons we wish to briefly explain. The CR structure $T_{1,0}(M)$ is but a recast, in the language of complex vector bundles, of the tangential Cauchy-Riemann equations

$$\overline{\partial}_b f = 0, \quad f \in C^1(M, \mathbb{C}), \tag{11}$$

and it is our philosophy, following the line of thought by S. Dragomir & G. Tomassini, [94], that studying various geometric objects associated to θ on M will ultimately unveil local and global properties of solutions to (11). These are related (cf. e.f. A. Boggess, [2]) to the pseudoconvexity properties of M, as understood in complex analysis of functions of several complex variables. On the other hand pseudoconvexity properties aren't captured by the geometry of g_θ but rather are described by (the curvature of) the Tanaka-Webster connection ∇ of (M, θ). The Tanaka-Webster connection ∇ and its curvature R^∇ are among the geometric objects associated to (M, θ), as mentioned above, and are made a preferrenial use with respect to the Levi-Civita connection of (M, g_θ) and its curvature. The source of basic results on CR and pseudohermitian geometry that we closely follow through this thesis is the monograph by S. Dragomir & G. Tomassini, [94]. As recalled previously in this Introduction, the sublaplacian Δ_b is but degenerate elliptic, yet it is subelliptic of order $\epsilon = 1/2$ (cf. e.g. G.B. Folland, [42]). Consequently, by a result of L. Hörmander, [68], Δ_b is hypoelliptic i.e. if u is a distribution solution to $\Delta_b u = f$ with $f \in C^\infty$ then $u \in C^\infty$ as well. A pseudodifferential calculus adapted to hypoelliptic operators, such as developed by A. Menikoff & J. Sjöstrand, [13], shows that Δ_b has a discrete spectrum

$$\sigma(\Delta_b) = \{\lambda_\nu(\theta) : \nu \geq 0\}, \quad 0 = \lambda_0(\theta) < \lambda_1(\theta) < \cdots < \lambda_\nu(\theta) < \cdots \uparrow +\infty, \tag{12}$$

as the Laplacian of a compact Riemannian manifold, to which Δ_b formally resembles, except for the degeneration of ellipticity, as explained above. The crucial property enjoyed by Δ_b, as well as Δ_g, is therefore its hypoellipticity, springing from subellipticity, and the author of this thesis joins the opinion in [94] that subelliptic theory should play within CR geometry the strong, and more consolidated, role played by elliptic theory in Riemannian geometry. Discreteness of $\sigma(\Delta_b)$ also follows easily from the subelliptic estimates

$$\|u\|^2_{1/2} \leq C\left((\Delta_b u\,,\, u)_{L^2(M)} + \|u\|^2_{L^2(M)}\right), \quad u \in C^\infty(M), \tag{13}$$

(where $\|\cdot\|_s$ is the Sobolev norm of order s) together with a Kondrakov type lemma due essentially to L.P. Rothschild & E.M. Stein, [69] (and a general functional analysis description of spectrae of compact operators). Another proof of discreteness of $\sigma(\Delta_b)$, relying on the Poincaré lemma

$$\int_\Omega \varphi^2\, \Psi_\theta \leq C \int_\Omega \|\nabla^H \varphi\|^2\, \Psi_\theta\,, \quad \varphi \in C_0^\infty(\Omega, \mathbb{R}), \tag{14}$$

is given in Chapter 1 of this thesis for the Dirichlet spectrum of Δ_b on a bounded (with respect to the Carnot-Carathéodory distance function d_H of the semi-Riemannian manifold $(M,\, H(M),\, G_\theta)$) domain $\Omega \subset M$ in a complete (again with respect to d_H) pseudohermitian manifold (M, θ).

The exposition is organized as follows. Chapter 1 gathers the preparatory material on tangential Cauchy-Riemann equations (11) and geometric objects naturally associated to them once a positively oriented contact form $\theta \in \mathcal{P}_+$ is fixed, such as the Levi form G_θ, the Webster metric g_θ, the Tanaka-Webster connection ∇, and the Fefferman metric F_θ on $\mathfrak{M}$, the total space of the canonical circle bundle $S^1 \to C(M) \to M$ over M. Especially F_θ, a Lorentzian metric on $\mathfrak{M}$, plays a fundamental role in the derivation of an L^2 Bochner-Lichnerowicz type formula that we derive in Chapter 4. The sublaplacian Δ_b of (M, θ) is then introduced and, following its description, a weak L^2 calculus in appropriate Sobolev type spaces $W^{1,2}_H(\Omega)$ and $\mathring{W}^{1,2}_H(\Omega)$ is presented in some detail, by following essentially E. Barletta & S. Dragomir, [28]. To prove discreetness of Dirichlet spectrum of Δ_b on Ω one needs to solve first the generalized Dirichlet problem

$$\Delta_b u = f \quad \text{in} \;\; \Omega, \qquad u = 0 \quad \text{on} \;\; \partial\Omega, \tag{15}$$

by giving an appropriate L^2 interpretation of the boundary condition in (15) i.e. by looking for a solution $u \in \mathring{W}^{1,2}_H(\Omega)$. When $M = \mathbb{H}^n$ i.e. $\Omega \subset \mathbb{H}^n$ is a bounded domain in the Heisenberg group, the Poincaré lemma (14) readily holds as a consequence of a Sobolev type lemma, while it is our present level of understanding of the theory that for domains in arbitrary complete strictly pseudoconvex manifolds M inequality (14) should be a basic assumption. While the solution to (15) is known when $M = \mathbb{H}^n$ (by work in subelliptic theory, cf. e.g. A. Bonfiglioli & E. Lanconelli & F. Uguzzoni, [3], or by folklore surrounding it), it appears nowhere (in the literature on CR geometry) for domains $\Omega \subset M$ in an arbitrary complete strictly pseudoconvex CR manifold. We therefore give two solutions to the generalized Dirichlet problem, both leading to the variational solution to (15), one as a minimum of the functional

$$F(u) = \frac{1}{2} \int_\Omega \|\nabla^H u\|^2\, \Psi_\theta - (f, u)_{L^2(\Omega)}\,, \quad u \in \mathring{W}^{1,2}_H(\Omega),$$

and another exploiting the Friedrichs extension of the Lagrange sublaplacian $\Delta_{b,0} \equiv \Delta_b\big|_{C^\infty_0(\Omega)}$.

The last two sections in Chapter 1 are devoted to giving a proof to a pseudohermitian analog to Bochner-Lichnerowicz formula due to A. Greenleaf, [9], (and with respect to which our Bochner-Lichnerowicz type formula in Chapter 4 is an alternative) and its use in the proof of the non-negativity of the CR Paneitz operator P_0, due to S-C. Chang & H-L. Chiu, [92]. We repeat the calculations in [92] both because we operate with different quantitative conventions (as to exterior differential calculus in the de Rham algebra of M) and because non-negativity of P_0 is a crucial ingredient in the lower bound on $\lambda_1(\theta)$ that we obtain in Chapter 4, very much as the bound got in [92].

Chapter 2 exposes our results on the behavior of $\sigma(\Delta_b)$ as functions of the given positively oriented contact form. The main results are an extension to the pseudohermitian category of a result by A. El Soufi & S. Ilias, [6]-[7], on the behavior of $\lambda_\nu(t) \equiv \lambda_\nu(\theta_t)$ under a smooth 1-parameter deformation $\{\theta_t\}_{|t|<\delta}$ of the contact form θ, followed by an extension of a result by S. Bando & H. Urakawa, [90]. The result in [90] was that eigenvalues $\lambda_\nu(g)$ of the Laplace-Beltrami operator Δ_g are continuous functions of $g \in \mathcal{M}$, with respect to the natural topology on the space $\mathcal{M}$ of all Riemannian metrics on the given manifold M. We prove a pseudohermitian analog to that, by organizing the space of contact forms $\mathcal{P}$ as a topological space, whose topology is the metric topology of an appropriate distance function on $\mathcal{P}$, and by proving a max-min principle.

Chapter 3 aims to find bounds on the eigenvalues similar Payne-Pólya-Weinberger universal inequalities [66]. These are (as established for the eigenvalues of the Dirichlet Laplacian on a bounded domain in $\mathbb{R}^n$)

$$\lambda_{k+1} - \lambda_k \leq \frac{4}{n}\left\{\frac{1}{k}\sum_{i=1}^{k}\lambda_i\right\}, \quad k \geq 1. \tag{16}$$

Inequalities (16) were improved by several authors (cf. [73], [45], [46]). For instance the following inequality due to H.C. Yang, [46], implies (16)

$$\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}\lambda_i(\lambda_{k+1} - \lambda_i). \tag{17}$$

Extensions of universal inequalities to bounded domains in Riemannian manifolds other than the Euclidean space have also been obtained. Let us mention, for example, the following Yang's type inequality obtained by M.S. Ashbaugh, [73], for domains in the unit sphere $S^n \subset \mathbb{R}^{n+1}$ (cf also [83])

$$\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)(\lambda_i + \frac{n^2}{4}). \tag{18}$$

Equality holds for every k in (18) when λ_i are the eigenvalues of the Laplace-Beltrami operator on the whole sphere, as observed by A. El Soufi & E.M. Harrell & S. Ilias, [8]. There inequality (18) is recovered as a particular case of an inequality satisfied by the eigenvalues of the Laplace-Beltrami operator of any n-dimensional compact Riemannian manifold M (with Dirichlet boundary conditions if $\partial M \neq \emptyset$)

$$\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)(\lambda_i + \frac{1}{4}\|H\|_\infty^2) \tag{19}$$

where H is the mean curvature vector field of an arbitrary isometric immersion of M into Euclidean space $\mathbb{R}^{n+p}$. P-C. Niu & H. Zhang, [81], were the first to address the same issue for subelliptic operators. They obtained Payne-Pólya-Weinberger and Hile-Protter type inequalities for the Dirichlet eigenvalues of the sublaplacian on a bounded domain in the Heisenberg group $\mathbb{H}^n$. The following Yang type inequality was obtained in [8] as an improvement of the results in [81]

$$\sum_{i=1}^{k}(\lambda_{k+1}(\theta) - \lambda_i(\theta))^2 \leq \frac{2}{n}\sum_{i=1}^{k}\lambda_i(\theta)(\lambda_{k+1}(\theta) - \lambda_i(\theta)). \tag{20}$$

Among our results (reported on in Chapter 3, cf. Corollary 3.8), we show that inequality (20) remains valid for any compact strictly pseudoconvex CR manifold M, of CR dimension n, provided it admits a Riemannian submersion over an open set of $\mathbb{R}^{2n}$ which is constant on the characteristic curves of M i.e. on the integral curves of the Reeb vector. The standard projection $\mathbb{H}^n \to \mathbb{R}^{2n}$ satisfies these assumptions. For domains in S^{2n+1} we obtain the following inequality (cf. Corollary 3.4)

$$\sum_{i=1}^{k}(\lambda_{k+1}(\theta) - \lambda_i(\theta))^2 \leq \frac{2}{n}\sum_{i=1}^{k}(\lambda_{k+1}(\theta) - \lambda_i(\theta))\left(\lambda_i(\theta) + n^2\right) \tag{21}$$

which is sharp for $k = 1$. These results are particular cases of our more general Theorem 3.3. We prove that the eigenvalues of the sublaplacian Δ_b in a bounded domain $\Omega \subset M$, with Dirichlet boundary conditions if $\Omega \neq M$, satisfy inequalities of the following form (cf. Theorem 3.3 for a complete statement). For every integer $k \geq 1$ and every $p \in \mathbb{R}$,

$$\sum_{i=1}^{k}(\lambda_{k+1}(\theta) - \lambda_i(\theta))^p \leq \frac{\max\{2,p\}}{n}\sum_{i=1}^{k}(\lambda_{k+1}(\theta) - \lambda_i(\theta))^{p-1}(\lambda_i(\theta) + \frac{1}{4}\|H_b(f)\|_\infty^2), \tag{22}$$

$$\lambda_{k+1}(\theta) \leq (1 + \frac{2}{n})\frac{1}{k}\sum_{i=1}^{k}\lambda_i(\theta) + \frac{1}{2n}\|H_b(f)\|_\infty^2, \tag{23}$$

and

$$\lambda_{k+1}(\theta) \leq (1 + \frac{2}{n})k^{\frac{1}{n}}\lambda_1(\theta) + \frac{1}{4}\left((1 + \frac{2}{n})k^{\frac{1}{n}} - 1\right)\|H_b(f)\|_\infty^2 \tag{24}$$

where f is any C^2 semi-isometric map from (M, θ) to a Euclidean space $\mathbb{R}^m$ and $H_b(f)$ is a vector field similar to the tension field of f in Riemannian geometry. Moreover we show the inequalities (22), (23) and (24) remain true when f is a semi-isometric map from (M, θ) to the Heisenberg group $\mathbb{H}^m$ which maps the Levi distribution of M into that of $\mathbb{H}^m$. For M compact without boundary we establish Reilly type inequalities

$$\lambda_2(\theta) \leq \frac{1}{2nV(M,\theta)}\int_M \|H_b(f)\|_{\mathbb{R}^m}^2, \quad \mathrm{Vol}(M,\theta) \equiv \int_M \Psi_\theta, \tag{25}$$

and show that equality holds in (25) if and only if $f(M)$ is contained in a sphere $S^{m-1}(r)$ of radius $r = \sqrt{2n/\lambda_2(\theta)}$ and $f : M \to S^{m-1}(r)$ is pseudoharmonic (in the sense of E. Barletta & S. Dragomir & H. Urakawa, [31]). Reilly type results are also obtained for maps f from (M, θ) to $\mathbb{H}^m$ which map the Levi distribution of M into that of $\mathbb{H}^m$ (cf. our Theorem 3.16).

The main ingredient in the proof of (2) is the Bochner-Lichnerowicz formula (cf. e.g. (G.IV.5) in [71], p. 131)

$$-\frac{1}{2}\Delta_g\left(\|du\|^2\right) = \|\mathrm{Hess}(u)\|^2 - g\left(Du\,,\, D\Delta_g u\right) + \mathrm{Ric}_g(Du, Du) \tag{26}$$

for any $u \in C^\infty(M, \mathbb{R})$. The great fascination exerted by the Lichnerowicz-Obata theorem on the mathematical community in the last fifty years prompted the many attempts to extend (26) and (2) to other geometric contexts e.g. to Riemannian foliation theory (cf. S-D. Jung & K-R. Lee & K. Richardson, [93], J. Lee & K. Richardson, [56], H-K. Pak & J-H. Park, [47]), to CR and pseudohermitian geometry (cf. E. Barletta & S. Dragomir, [28], E. Barletta, [32], S-C. Chang & H-L. Chiu, [92], H-L. Chiu, [48], A. Greenleaf, [9], S-Y. Li & H-S. Luk, [101]) and to sub-Riemannian geometry (cf. F. Baudoin & N. Garofalo, [38]). Chapter 4 is devoted to a version of the estimate (2) occurring in CR geometry. Given a compact strictly pseudoconvex CR manifold $(M, T_{1,0}(M))$ endowed with a positively oriented contact form θ, the pseudohermitian manifold (M, θ) carries (by a result of N.Tanaka, [79], and S.M. Webster, [100]) (M, θ) carries a natural linear connection ∇ (the Tanaka-Webster connection of (M, θ), cf. also [94], p. 25) whose Ricci tensor field is formally similar to Ricci curvature in Riemannian geometry. It is then a natural problem to look for a lower bound on $\lambda_1(\theta)$ whenever Ric_∇ is bounded from below. As the sublaplacian may be written in divergence form as $\Delta_b u = -\mathrm{div}(\nabla^H u)$, the horizontal gradient $\nabla^H u$ appears to be the pseudohermitian analog to the gradient Du in Riemannian geometry. The first step is then to produce a pseudohermitian version of (26) i.e. compute $\Delta_b(\|\nabla^H u\|^2)$ (for an arbitrary eigenfunction u of Δ_b) in terms of the pseudohermitian Hessian $\nabla^2 u$ and the Ricci curvature Ric_∇ of the Tanaka-Webster connection. The first to realize the difficulties in producing a pseudohermitian analog to (26) was A. Greenleaf, [9]. Indeed his Bochner-Lichnerowicz type formula

$$\Delta_b\left(\|\nabla^{1,0}u\|^2\right) = 2\sum_{\alpha,\beta}\left(u_{\alpha\overline{\beta}}u_{\overline{\alpha}\beta} + u_{\alpha\beta}u_{\overline{\alpha}\overline{\beta}}\right) + 4i\sum_{\alpha}\left(u_{\overline{\alpha}}u_{0\alpha} - u_\alpha u_{0\overline{\alpha}}\right) + \tag{27}$$

$$+2\sum_{\alpha,\beta}R_{\alpha\overline{\beta}}u_{\overline{\alpha}}u_\beta + 2in\sum_{\alpha,\beta}\left(A_{\alpha\beta}u_{\overline{\alpha}}u_{\overline{\beta}} - A_{\overline{\alpha}\overline{\beta}}u_\alpha u_\beta\right) +$$

$$+\sum_{\alpha}\left\{u_{\overline{\alpha}}\left(\Delta_b u\right)_\alpha + u_\alpha\left(\Delta_b u\right)_{\overline{\alpha}}\right\}$$

involves the torsion terms $A_{\alpha\beta}$ (possessing no Riemannian counterpart). Here $\nabla^{1,0}u = \sum_\alpha u_{\overline{\alpha}}T_\alpha$ (notations and conventions as used in (27) are explained in § 2 of Chapter 4). However the attempt to confine oneself to the class of Sasakian manifolds (M, g_θ) (as in [32], since Sasakian metrics g_θ have vanishing pseudohermitian torsion i.e. $A_{\alpha\beta} = 0$) isn't successful either: while torsion terms may be controlled (when exploiting (27) integrated over M) by the L^2 norm of $\nabla^H u$, the main technical difficulties actually arise from the occurrence of terms $\sum_\alpha(u_{\overline{\alpha}}u_{0\alpha} - u_\alpha u_{0\overline{\alpha}})$ containing covariant derivatives of $\nabla^H u$ in the "bad" real direction T transverse to $H(M)$ (the Reeb vector of (M, θ)).

The novelty brought by Chapter 4 is to establish first a version of Bochner-Lichnerowicz formula for a natural Lorentzian metric F_θ (the Fefferman metric of (M, θ), cf. [59], [18]) on the total space of the canonical circle bundle $S^1 \to C(M) \xrightarrow{\pi} M$. Fefferman metric F_θ was discovered by C. Fefferman, [17], in connection with the study of boundary behavior of the Bergman kernel of a strictly pseudoconvex domain in $\mathbb{C}^n$. An array of problems of major interest in CR geometry e.g. the CR Yamabe problem, [24], the study of subelliptic harmonic maps, [54], and Yang-Mills fields on CR manifolds, [31], are closely tied to the geometry of the Lorentzian manifold $(C(M), F_\theta)$. Indeed the aforementioned problems are projections on M via $\pi : C(M) \to M$ of Lorentzian analogs to the corresponding Riemannian problems, as prompted by J.M. Lee's discovery (cf. [59]) that $\pi_* \Box = \Delta_b$, where $\Box$ is the Laplace-Beltrami operator of F_θ (the wave operator on $(C(M), F_\theta)$).

For instance any S^1-invariant harmonic map $\Phi : (C(M), F_\theta) \to N$ into a Riemannian manifold N projects on a subelliptic harmonic map $\phi : M \to N$ (in the sense of [54] and [30]). The arguments in [71] carry over in a straightforward manner (cf. our § 3 in Chapter 4) to Lorenzian geometry and give (cf. (4.21) in Chapter 4)

$$-\frac{1}{2}\,\Box\,(F_\theta(Df, Df)) = F_\theta^*\left(D^2 f\,,\, D^2 f\right) - (Df)(\Box f) + \mathrm{Ric}_D(Df\,,\, Df) \tag{28}$$

and the corresponding integral formula (4.22) there. The projection on M of (28) then leads to another analog (similar to A. Greenleaf's formula (27)) to Bochner-Lichnerowicz formula and then to a new lower bound on $\lambda_1(\theta)$. Precisely we may state

Theorem 0.1. *Let M be a compact, strictly pseudoconvex, CR manifold of CR dimension n. Let $\theta \in \mathcal{P}_+$ be a positively oriented contact form on M and Δ_b the corresponding sublaplacian. Let* Ric_∇ *be the Ricci tensor of the Tanaka-Webster connection ∇ of (M, θ) and $\lambda_1(\theta) \in \sigma(\Delta_b)$ the first nonzero eigenvalue of Δ_b. If*

$$\mathrm{Ric}_\nabla(X, X) \geq k\, G_\theta(X, X) \tag{29}$$

for some constant $k > 0$ and any $X \in H(M)$ then

$$\lambda_1(\theta) \geq \frac{2n}{(n+2)(n+3)}\left\{(n+3)k - (11n+19)\tau_0 - \frac{\rho_0}{2(n+1)}\right\} \tag{30}$$

where $\tau_0 = \sup_{x\in M} \|\tau\|_x$ and $\rho_0 = \sup_{x\in M} \rho(x) \geq nk$, where τ and ρ are respectively the pseudo-hermitian torsion and scalar curvature of (M, θ).

The lower bound (30) is nontrivial only for k sufficiently large (i.e. k must satisfy (4.101) in § 5 of Chapter 4). Let (M, g_θ) be a Sasakian manifold (equivalently $\tau = 0$, cf. e.g. [94]). Then under the same assumption (i.e. (29) in Theorem 0.1) A. Greenleaf established the estimate (cf. [9])

$$\lambda_1(\theta) \geq \frac{nk}{n+1}\,. \tag{31}$$

Lower bound (30) is sharper that (31) when

$$k > \frac{\rho_0}{n(n+3)}\,. \tag{32}$$

If for instance $M = S^{2n+1}$ is the standard sphere in $\mathbb{C}^{n+1}$, endowed with the canonical contact form $\theta = (i/2)\left(\overline{\partial} - \partial\right)|z|^2$, then $\rho_0 = 2n(n+1)$ and $k = 2(n+1)$ hence (32) holds (and (30) is sharper than (31)).

The projection of (28) on M gives

$$\begin{aligned} -\frac{1}{2}\,\Delta_b\left(\|\nabla^H u\|^2\right) = {} & \left\|\Pi_H \nabla^2 u\right\|^2 - (\nabla^H u)(\Delta_b u) + \\ & +4(J\nabla^H u)(u_0) - \frac{3(n+1)}{n+2}\,A(\nabla^H u\,,\, J\nabla^H u) + \\ & +\frac{n+3}{n+2}\,\mathrm{Ric}_\nabla\left(\nabla^H u\,,\, \nabla^H u\right) - \frac{\rho}{2(n+1)(n+2)}\,\|\nabla^H u\|^2 \end{aligned} \tag{33}$$

(the pseudohermitian Bochner-Lichnerowicz formula, cf. (4.91)) and the corresponding integral formula (4.92). The main technical difficulty in the derivation of (33) is to compute the Ricci curvature Ric_D of the Lorentzian manifold $(C(M), F_\theta)$. This is performed by relating the Levi-Civita connection D of $(C(M), F_\theta)$ to the Tanaka-Webster connection ∇ of (M, θ) (cf. (4.23)-(4.27), a result got in [31]) and adapting to $S^1 \to C(M) \to M$ a technique originating in the theory of Riemannian submersions (cf. [14]) and shown to work in spite of the fact that $\pi : (C(M), F_\theta) \to (M, g_\theta)$ isn't a semi-Riemannian submersion (fibres of π are degenerate). The relationship among D and ∇ may then be exploited to compute the full curvature tensor R^D. Only its trace Ric_D is evaluated in [59] and the formula there appears as too involved to be of practical use. Our result (cf. (4.54)-(4.59) in Lemma 4.3 below) is simple, elegant and local frame free. This springs from the decomposition

$$T(C(M)) = \mathrm{Ker}(\sigma) \oplus \mathbb{R}S, \quad \mathrm{Ker}(\sigma) = H(M)^\uparrow \oplus \mathbb{R}T^\uparrow ,$$

itself relying on the discovery (due to C.R. Graham, [18]) that $\sigma \in \Omega^1(C(M))$ (given by (4.17) below) is a connection 1-form in the principal circle bundle $S^1 \to C(M) \to M$. As a byproduct of Lemma 4.3 one reobtains the result by J.M. Lee, [59], that none of the Fefferman metrics $\{F_\theta \in \mathrm{Lor}(C(M)) : \theta \in \mathcal{P}_+\}$ is Einstein. Integration of (33) over M produces (by (4.88) in Lemma 4.5) terms $\|u_0\|_{L^2}$ where $u_0 \equiv T(u)$ and u is an arbitrary eigenfunction of Δ_b, corresponding to a fixed eigenvalue $\lambda \in \sigma(\Delta_b)$. The L^2 norm of the (restriction to the Levi distribution $H(M)$) pseudohermitian Hessian $\Pi_H \nabla^2 u$ is estimated by using (4.94) (a result got in [32]). Torsion terms and Ricci curvature terms are respectively estimated by (4.99) and as a consequence of the assumption (29) in Theorem 0.1 (together with (4.98)). Finally to control $\|u_0\|_{L^2}$ one exploits a fundamental result got in [92], and referred hereafter as the *Chang-Chiu inequality* (cf. (4.118) in § 4.7 of Chapter 4).

The last part contains a work taht is independent from the rest of the thesis. It deals with a new proof of the CR Pohožaev Identity.

Chapter 1

CR and Pseudohermitian Geometry

1.1 Tangential Cauchy-Riemann equations

Let M be a connected C^∞ differentiable manifold, of real dimension $2n+1$. Let $T(M) \otimes \mathbb{C} \to M$ denote the complexified tangent bundle over M. A *CR structure* on M is a complex subbundle $T_{1,0}(M) \subset T(M) \otimes \mathbb{C}$, of complex rank n, such that

$$T_{1,0}(M)_x \cap T_{0,1}(M)_x = \{0_x\}, \quad x \in M, \tag{1.1}$$

$$Z, W \in C^\infty(U, T_{1,0}(M)) \Longrightarrow [Z, W] \in C^\infty(U, T_{1,0}(M)), \tag{1.2}$$

for any open set $U \subset M$. A pair $(M, T_{1,0}(M))$ is a *CR manifold* and the integer n is its *CR dimension*. Here $T_{0,1}(M) = \overline{T_{1,0}(M)}$ and overbars denote complex conjugation. Cf. [94], p. 3-4. Also if $E \to M$ is a vector bundle over M then $C^\infty(U, E)$ denotes the space of all C^∞ sections in E, defined on the open set $U \subset M$. When $U = M$ one writes simply $C^\infty(E) = C^\infty(M, E)$. If $x \in M$ then E_x is the fibre in E over x. The axiom (1.2) is often referred to as the (*Frobenius*) *formal integrability property* (of the CR structure $T_{1,0}(M)$). Standard examples of CR manifolds are real hypersurfaces $M \subset \mathbb{C}^{n+1}$ with the CR structure (induced by the complex structure of the ambient space)

$$T_{1,0}(M)_x = [T_x(M) \otimes_{\mathbb{R}} \mathbb{C}] \cap T^{1,0}(\mathbb{C}^{n+1})_x\,, \quad x \in M.$$

Here $T^{1,0}(\mathbb{C}^{n+1}) \to \mathbb{C}^{n+1}$ denotes the holomorphic tangent bundle over $\mathbb{C}^{n+1}$ (the span of $\{\partial/\partial z^j : 1 \le j \le n+1\}$ where $(z^1, \cdots, z^{n+1})$ are the Cartesian complex coordinates on $\mathbb{C}^{n+1}$).

Let $(M, T_{1,0}(M))$ be a CR manifold, of CR dimension n. The *tangential Cauchy-Riemann operator* is the first order differential operator

$$\overline{\partial}_b : C^\infty(U, \mathbb{C}) \to C^\infty(U, T_{0,1}(M)^*),$$

$$(\overline{\partial}_b f)\overline{Z} = \overline{Z}(f), \quad f \in C^\infty(U, \mathbb{C}), \quad Z \in C^\infty(U, T_{1,0}(M)),$$

with $U \subset M$ open. Next

$$\overline{\partial}_b f = 0 \tag{1.3}$$

are the *tangential Cauchy-Riemann equations*. Clearly $\overline{\partial}_b$ may be defined on C^1 functions, to start with (and then $\overline{\partial}_b f$ is but a continuous section in $T_{0,1}(M)^*$). A C^1 solution to the tangential

Cauchy-Riemann equations (1.3) is a *CR function* on U. The space of CR functions $f : U \to \mathbb{C}$ is denoted by $\mathrm{CR}^1(U, \mathbb{C})$.

CR structures on manifolds appear therefore as a bundle theoretic recast, within the realm of differential geometry, of the tangential Cauchy-Riemann equations, discovered by H. Lewy, [49], in his study of the boundary behavior of holomorphic functions on the Siegel domain. We recall a few details on Lewy's construction, leading to our main example of an open CR manifold, the Heisenberg group.

Let $\Omega = \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : \mathrm{Im}(w) > |z|^2\}$ be the *Siegel domain* in $\mathbb{C}^{n+1}$. Here $|z|^2 = \sum_{\alpha=1}^n z^\alpha \overline{z}^\alpha$ for any $z = (z^1, \cdots, z^n) \in \mathbb{C}^n$. Also $\overline{z}^\alpha = \overline{z^\alpha}$. Let us consider the Dirichlet problem for the ordinary Cauchy-Riemann system

$$\overline{\partial} F = 0 \quad \text{in} \quad \Omega, \tag{1.4}$$

$$F = f \quad \text{on} \quad \partial\Omega. \tag{1.5}$$

Here $f \in C^\infty(\partial\Omega, \mathbb{C})$ and one is interested in the C^∞ regularity up to the boundary of the solution to (1.4)-(1.5) (rather then the existence problem). Let us assume that a C^∞ up to the boundary solution $F \in C^\infty(\overline{\Omega}, \mathbb{C})$ does exist. Let us consider

$$\rho : \mathbb{C}^{n+1} \to \mathbb{R}, \quad \rho(z, w) = \mathrm{Im}(w) - |z|^2, \quad (z, w) \in \mathbb{C}^{n+1},$$

(the defining function of the Siegel domain). For every $a \in \mathbb{R}$ we set

$$M_a = \{(z, w) \in \mathbb{C}^{n+1} : \rho(z, w) = a\}$$

so that $\mathbb{C}^{n+1}$ appears as carrying the foliation $\mathcal{F}$ by level sets of ρ i.e. the leaf space of $\mathcal{F}$ is

$$M/\mathcal{F} = \{M_a : a \in \mathbb{R}\}.$$

For every $\epsilon > 0$ the leaf M_ϵ is contained in the Siegel domain while M_0 is its boundary. Each leaf M_ϵ ($\epsilon \geq 0$) is a real hyperusrface in $\mathbb{C}^{n+1}$ and hence a CR manifold with the induced CR structure

$$T_{1,0}(M_\epsilon) = [T(M_\epsilon) \otimes \mathbb{C}] \cap T^{1,0}(\mathbb{C}^{n+1}).$$

A complex vector field Z of type $(1, 0)$ on $\mathbb{C}^{n+1}$ is tangent to M_ϵ if and only if $Z(\rho_\epsilon) = 0$, where $\rho_\epsilon = \rho - \epsilon$. Hence $T_{1,0}(M_\epsilon)$ is (globally) the span of

$$\left\{ \frac{\partial}{\partial z^\alpha} - 2i\overline{z}^\alpha \frac{\partial}{\partial w} : 1 \leq \alpha \leq n \right\}.$$

For $M_0 = \partial\Omega$ a more precise statement is that $\{L_\alpha : 1 \leq \alpha \leq n\}$ is a (global) frame of $T_{1,0}(\partial\Omega)$, where $L_\alpha \in C^\infty(T(\partial\Omega) \otimes \mathbb{C})$ is the unique complex vector field tangent to $\partial\Omega$ determined by

$$(d_x j) L_{\alpha,x} = \left(\frac{\partial}{\partial z^\alpha} - 2i\overline{z}^\alpha \frac{\partial}{\partial w} \right)_x, \quad x \in \partial\Omega,$$

and $j : \partial\Omega \to \mathbb{C}^{n+1}$ is the inclusion. Let $x \in \Omega$ be an arbitrary point. As F is holomorphic in Ω

$$\left(\frac{\partial F}{\partial \overline{z}^\alpha} + 2iz^\alpha \frac{\partial F}{\partial \overline{w}} \right)(x) = 0. \tag{1.6}$$

As F is smooth up to the boundary we may take $x \to \partial\Omega$ i.e. approach the boundary with x in (1.6) so that to obtain for any $x \in \partial\Omega$

$$0 = \left(\frac{\partial F}{\partial \overline{z}^\alpha} + 2iz^\alpha \frac{\partial F}{\partial \overline{w}}\right)(x) =$$

$$= \left((d_x j)\overline{L}_x\right)(F) = \overline{L}_x(F \circ j) = \overline{L}_x(f) = (\overline{\partial}_b f)_x \overline{L}_x$$

so that the boundary data is a solution to the tangential Cauchy-Riemann equations $\overline{\partial}_b f = 0$ on $\partial\Omega$ i.e. $f \in \mathrm{CR}^\infty(\partial\Omega, \mathbb{C})$.

For further use, we summarize Lewy's construction above, as follows. Let $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ be the *Heisenberg group* i.e. the Lie group with the group law

$$(z,t)\cdot(w,s) = (z + w\,,\, t + s + \mathrm{Im}(z \cdot w)), \quad (z,t),(w,s) \in \mathbb{H}^n\,,$$

where $z\cdot w = \sum_{\alpha=1}^n z^\alpha w^\alpha$. Let us consider the left invariant complex vector fields $T_\alpha \in C^\infty(T(\mathbb{H}^n)\otimes \mathbb{C})$ given by

$$T_\alpha = \frac{\partial}{\partial z^\alpha} + i\overline{z}^\alpha \frac{\partial}{\partial t}\,, \quad 1 \le \alpha \le n.$$

$T_{\overline{\alpha}} = \overline{T_\alpha}$ are referred to as the *Lewy operators*. Then $\left[T_\alpha\,,\,T_\beta\right] = 0$ hence

$$T_{1,0}(\mathbb{H}^n)_x = \mathrm{Span}_{\mathbb{C}}\left\{T_{\alpha,x} : 1 \le \alpha \le n\right\}, \quad x \in \mathbb{H}^n\,,$$

is a (left invariant) CR structure, of CR dimension n, on $\mathbb{H}^n$. Let us consider the map

$$f : \mathbb{H}^n \to \partial\Omega,$$

$$f(z,t) = (z\,,\, t + i|z|^2), \quad (z,t) \in \mathbb{H}^n\,,$$

where $\Omega \subset \mathbb{C}^{n+1}$ is the Siegel domain. Then f is a *CR isomorphism* that is a C^∞ diffeomorphism and a *CR map* i.e. $(d_x f)T_{1,0}(\mathbb{H}^n)_x \subseteq T_{1,0}(\partial\Omega)_{f(x)}$ for any $x \in \mathbb{H}^n$ (and actually equality occurs, as $d_x f$ is a $\mathbb{R}$-linear isomorphism). This follows from

$$(d_x f)T_{\alpha,x} = L_{\alpha,f(x)}\,, \quad x \in \mathbb{H}^n\,, \quad 1 \le \alpha \le n.$$

1.2 Pseudohermitian structures

The *Levi distribution* of the CR manifold $(M, T_{1,0}(M))$ is

$$H(M) = \mathrm{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\}.$$

It carries the complex structure $J : H(M) \to H(M)$ given by

$$J(Z + \overline{Z}) = i(Z - \overline{Z}), \quad Z \in T_{1,0}(M),$$

(with $i = \sqrt{-1}$). A *pseudohermitian structure* is a globally defined, nowhere zero, section $\theta \in C^\infty(H(M)^\perp)$ in the conormal bundle $H(M)^\perp \to M$ defined by

$$H(M)_x^\perp = \{\omega \in T_x^*(M) : \mathrm{Ker}(\omega) \supset H(M)_x\}, \quad x \in M.$$

Under the mere assumption that M is orientable, pseudohermitian structures always exist. Cf. S.M. Webster, [100]. The *Levi form* is

$$L_\theta(Z, \overline{W}) = -i(d\theta)(Z, \overline{W}), \quad Z, W \in T_{1,0}(M).$$

The given CR manifold M is *nondegenerate* (respectively *strictly pseudoconvex*) if L_θ is nondegenerate i.e. $L_\theta(Z, \overline{W}) = 0$ for any $W \in T_{1,0}(M)$ yields $Z = 0$ (respectively positive definite i.e. $L_\theta(Z, \overline{Z}) > 0$ for any $Z \neq 0$) for some θ. Let $\mathcal{P}$ be the set of all pseudohermitian structures. Given a pseudohermitian structure $\theta \in \mathcal{P}$, any other pseudohermitian structure $\hat{\theta} \in \mathcal{P}$ is given by $\hat{\theta} = \lambda\theta$ for some C^∞ function $\lambda : M \to \mathbb{R} \setminus \{0\}$. Thus

$$d\hat{\theta} = (d\lambda) \wedge \theta + \lambda\, d\theta$$

hence (as $\theta(Z) = 0$ for any $Z \in T_{1,0}(M)$)

$$L_{\hat{\theta}}(Z, \overline{W}) = -i(d\hat{\theta})(Z, \overline{W}) = -i\lambda(d\theta)(Z, \overline{W})$$

hence

$$L_{\hat{\theta}} = \lambda\, L_\theta\,. \tag{1.7}$$

Consequently, if L_θ is nondegenerate then so does $L_{\hat{\theta}}$ i.e. nondegeneracy is a CR invariant property. A property will be termed *CR invariant* if it is invariant under a transformation $\hat{\theta} = \lambda\theta$ of the pseudohermitian structure (i.e. that property depends on the CR structure alone, rather than depending on the choice of pseudohermitian structure). The following terminology is also in use. A CR manifold on which a pseudohermitian structure has been fixed is commonly called a *pseudohermitian manifold*. A given pseudohermitian manifold (M, θ) is termed *nondegenerate* (respectively *strictly pseudoconvex*) if L_θ is nondegenerate (respectively positive definite).

If L_θ is positive definite for some $\theta \in \mathcal{P}$ then $L_{-\theta}$ is negative definite, so that strict pseudoconvexity is not a CR invariant property. However the comment shows that $\mathcal{P}$ admits the natural orientation $\mathcal{P}_+$ consisting of all $\theta \in \mathcal{P}$ such that L_θ is positive definite.

We assume from now on that $(M, T_{1,0}(M))$ is a nondegenerate CR manifold, of CR dimension n. If this is the case then each pseudohermitian structure θ is a *contact form* i.e. $\theta \wedge (d\theta)^n$ is a volume form on M. For any contact form $\theta \in \mathcal{P}$ there is (cf. e.g. [94]) a unique globally defined tangent vector field $T \in \mathfrak{X}(M)$, transverse to the Levi distribution, determined by

$$\theta(T) = 1, \quad (d\theta)(T, X) = 0, \quad X \in \mathfrak{X}(M).$$

T is referred to as the *Reeb vector* field of (M, θ). Correspondingly M carries a natural semi-Riemannian metric g_θ (the *Webster metric*) which we proceed to recall. Let $\theta \in \mathcal{P}$ be a contact form and let $T \in \mathfrak{X}(M)$ be the Reeb vector field of (M, θ). Then g_θ is given by

$$g_\theta(X, Y) = (d\theta)(X, JY), \quad g_\theta(X, T) = 0, \quad g_\theta(T, T) = 1,$$

for any $X, Y \in H(M)$. For each $u \in C^1(M, \mathbb{R})$ let ∇u be the gradient of u with respect to g_θ i.e.

$$g_\theta(X, \nabla u) = X(u), \quad X \in \mathfrak{X}(M).$$

The *horizontal gradient* is $\nabla^H u = \Pi_H \nabla u$ where $\Pi_H : T(M) \to H(M)$ is the projection associated to the direct sum decomposition $T(M) = H(M) \oplus \mathbb{R}T$. Let G_θ be given by

$$G_\theta(X, Y) = (d\theta)(X, JY), \quad X, Y \in C^\infty(H(M)),$$

(the *real* Levi form). Clearly L_θ and the $\mathbb{C}$-linear extension of G_θ to $H(M)\otimes\mathbb{C}$ coincide on $T_{1,0}(M)\otimes T_{0,1}(M)$. If the given contact form θ is positively oriented i.e. $\theta \in \mathcal{P}_+$ then the Webster metric g_θ is a Riemannian metric. Also the pair $(H(M), G_\theta)$ is a *sub-Riemannian* structure on M (in the sense of [89]) and the Webster metric g_θ is a *contraction* of G_θ. Precisely, let $d_H(x, y)$ be the Carnot-Carathéodory distance function (cf. [57], [89]) defined as the infimum of lengths (with respect to G_θ) of piecewise C^1 curves tangent to $H(M)$ joining two points $x, y \in M$. If d_θ is the distance function associated to the Riemannian metric g_θ then $d_\theta(x, y) \leq d_H(x, y)$ for any $x, y \in M$.

For any fixed contact form $\theta \in \mathcal{P}$ on M there is (cf. e.g. [94]) a unique linear connection ∇ (the *Tanaka-Webster connection*) on M such that i) the Levi distribution $H(M)$ is parallel with respect to ∇, ii) $\nabla J = 0$, $\nabla g_\theta = 0$, iii) if T_∇ is the torsion tensor field of ∇ then

$$T_\nabla(Z, W) = 0, \quad T_\nabla(Z, \overline{W}) = 2iG_\theta(Z, \overline{W})T, \quad Z, W \in T_{1,0}(M),$$

$$\tau \circ J + J \circ \tau = 0.$$

Here τ (the *pseudohermitian torsion*) is the vector valued 1-form on M given by $\tau(X) = T_\nabla(T, X)$ for any $X \in \mathfrak{X}(M)$. When M is strictly pseudoconvex and $\theta \in \mathcal{P}_+$ it may be shown (cf. e.g. [94]) that $\tau = 0$ if and only if the Webster metric g_θ is Sasakian (in the sense of [22]). By a result of S. Webster, [100], τ is symmetric i.e. $G_\theta(\tau X, Y) = G_\theta(X, \tau Y)$ for any $X, Y \in H(M)$, and traceless i.e. $\mathrm{trace}(\tau) = 0$. By a result in [94] (cf. Lemma 1.3, p. 37) the Levi-Civita connection ∇^{g_θ} of the semi-Riemannian manifold (M, g_θ) and the Tanaka-Webter connection ∇ of (M, θ) are related by

$$\nabla^{g_\theta} = \nabla + (\Omega - A) \otimes T + \tau \otimes \theta + 2\,\theta \odot J. \tag{1.8}$$

Here $\Omega = -d\theta$ and $\odot$ denotes the symmetric tensor product e.g. $\alpha \odot \beta = \frac{1}{2}(\alpha \otimes \beta - \beta \otimes \alpha)$ for any $\alpha, \beta \in \Omega^1(M)$. In particular (as a consequence of (1.8))

$$\nabla^{g_\theta}_X Y = \nabla_X Y + (\Omega(X, Y) - A(X, Y))T, \tag{1.9}$$

$$\nabla^{g_\theta}_X T = JX, \quad \nabla^{g_\theta}_T X = \nabla_T X + JX, \quad \nabla^{g_\theta}_T T = 0 \tag{1.10}$$

for any $X, Y \in C^\infty(H(M))$.

Traces of holomorphic functions on real hypersurfaces $M \subset \mathbb{C}^{n+1}$ (carrying the induced CR structure) are CR functions (of class C^∞) and indeed CR functions enjoy properties similar to those of holomorphic functions. Limitations may occur. For instance any Levi flat (i.e. $G_\theta = 0$) CR manifold admits non trivial real valued CR functions (the local defining submersions of the Levi foliation $\mathcal{F}$ of M such that $T(\mathcal{F}) = H(M)$, cf. [29]) whilst, as well known, real valued holomorphic functions are constants. Nevertheless

Lemma 1.1. *If M is a connected nondegenerate CR manifold then any real valued CR function is a constant.*

Proof. The proof relies on the existence of the Tanaka-Webster connection. Let $\{T_\alpha : 1 \leq \alpha \leq n\}$ be a local frame of $T_{1,0}(M)$, defined on the open set $U \subset M$. We set

$$g_{\alpha\overline{\beta}} = G_\theta(T_\alpha\,, T_{\overline{\beta}}), \quad \nabla_{T_A} T_B = \Gamma^C_{AB} T_C\,, \quad T_{\overline{\alpha}} = \overline{T}_\alpha\,,$$

$$\alpha, \beta, \cdots \in \{1, \cdots, n\}, \quad A, B, \cdots \in \{1, \cdots, n, \overline{1}, \cdots, \overline{n}, 0\}, \quad T_0 = T,$$

for some C^∞ functions $\Gamma^A_{BC} \in C^\infty(U, \mathbb{C})$ (the *Christoffel symbols* of the Tanaka-Webster connection). Axiom (iii) in the description of ∇ yields (for $Z = T_\alpha$ and $W = T_\beta$)

$$\Gamma^{\overline{\gamma}}_{\alpha\overline{\beta}} T_{\overline{\gamma}} - \Gamma^{\gamma}_{\overline{\beta}\alpha} T_\gamma - [T_\alpha \, , \, T_{\overline{\beta}}] = 2i g_{\alpha\overline{\beta}} T. \tag{1.11}$$

Let f be a real valued ($\overline{f} = f$) CR function on M i.e. $T_{\overline{\gamma}}(f) = 0$ on U. By complex conjugation $T_\gamma(f) = 0$ too. Thus (by applying (1.11) to f and exploiting the nondegeneracy of the matrix $\left[g_{\alpha\overline{\beta}}(x)\right]$ at any $x \in U$) $T(f) = 0$ on U. Therefore f is locally constant. Q.e.d.

Let M be a strictly pseudoconvex CR manifold and $\theta \in \mathcal{P}_+$ a positively oriented contact form. Let $d\,\mathrm{vol}(g_\theta)$ be the volume form of the (oriented) Riemannian manifold (M, g_θ) i.e. for any local coordinate neighborhood (U, x^i) on M

$$d\,\mathrm{vol}(g_\theta) = \sqrt{G}\, dx^1 \wedge \cdots \wedge dx^{2n+1} \, ,$$

$$G = \det\left[(g_\theta)_{ij}\right], \quad (g_\theta)_{ij} = g_\theta(\partial/\partial x^i \, , \, \partial/\partial x^j),$$

on U. By a result in [51] there is a constant $C_n > 0$ depending only[1] on the CR dimension n such that

$$d\,\mathrm{vol}(g_\theta) = C_n \, \Psi_\theta \, . \tag{1.12}$$

The precise form of the constant C_n is given in [31]. Let $\mathrm{div} : \mathfrak{X}(M) \to C^\infty(M, \mathbb{R})$ be the divergence operator with respect to the volume form Ψ_θ i.e.

$$\mathcal{L}_X \Psi_\theta = \mathrm{div}(X)\, \Psi_\theta \, , \quad X \in \mathfrak{X}(M),$$

where $\mathcal{L}_X$ denotes the Lie derivative. By (1.12) div is precisely the divergence operator of the Riemannian manifold (M, g_θ) i.e. locally

$$\mathrm{div}(X) = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i} \left(\sqrt{G}\, X^i \right), \quad X = X^i \, \partial/\partial x^i \, .$$

1.3 The Fefferman metric

Let M be a strictly pseudoconvex CR manifold and $\theta \in \mathcal{P}_+$ a positively oriented contact form. A p-form $\omega \in C^\infty(\Lambda^p T^*(M) \otimes \mathbb{C})$ is a $(p, 0)$-*form* (or a form of *type* $(p, 0)$) if $T_{0,1}(M) \rfloor \omega = 0$. Here $\rfloor$ denotes interior product i.e. $X \rfloor \omega = i_X \omega$ for any $X \in \mathfrak{X}(M)$. Unlike the case of complex geometry, top degree $(p, 0)$-forms aren't $(n, 0)$-forms but rather $(n + 1, 0)$-forms, where n denotes the CR dimension. Indeed given a local frame $\{T_\alpha : 1 \leq \alpha \leq n\} \subset C^\infty(U, T_{1,0}(M))$ let $\{\theta^\alpha : 1 \leq \alpha \leq n\}$ be the complex valued 1-forms on U determined by

$$\theta^\alpha(T_\beta) = \delta^\alpha_\beta \, , \quad \theta^\alpha(T_{\overline{\beta}}) = 0, \quad \theta^\alpha(T) = 0.$$

$\{\theta^\alpha : 1 \leq \alpha \leq n\}$ is referred to as an *adapted* local coframe (local frame of $T_{1,0}(M)^*$). Then any $(p, 0)$-form ω on M may be locally represented as sums of exterior monomials of the form

$$\theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_p} \, , \quad \theta \wedge \theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_{p-1}} \, ,$$

[1] Depending on the CR dimension and the signature of the Levi form L_θ, in the nondegenerate case (cf. [31]).

with $C^\infty(U,\mathbb{C})$-coefficients. A top degree $(p,0)$-form ω is therefore locally represented as

$$\omega = \lambda\,\theta \wedge \theta^1 \wedge \cdots \wedge \theta^n$$

for some $\lambda \in C^\infty(U,\mathbb{C})$. We denote by $K(M) \to M$ the complex line bundle whose sections are the $(n+1,0)$-forms on M (the *canonical* line bundle). The multiplicative group $\mathrm{GL}^+(1,\mathbb{R}) = (0,+\infty)$ of the positive reals acts on $K_0(M) = K(M) \setminus \{\text{zero section}\}$ in a natural manner. Let $C(M) = K_0(M)/\mathrm{GL}^+(1,\mathbb{R})$ and $\pi : C(M) \to M$ be the quotient space and projection. The synthetic object $(C(M),\pi,M,S^1)$ is a principal bundle (the *canonical circle bundle* over M, cf. e.g. Definition 2.9 in [94], p. 119). We set $\mathfrak{M} = C(M)$ for simplicity. By a remarkable finding of C. Fefferman, [17], the total space $\mathfrak{M}$ of the canonical circle bundle carries a natural Lorentzian metric (the *Fefferman metric*) associated to a choice of $\theta \in \mathcal{P}_+$. The original construction in [17] is related to the investigations in [16] (on the boundary behavior of the Bergman kernel of a domain $\Omega \subset \mathbb{C}^{n+1}$) and produces a Lorentzian metric on $\partial\Omega \times S^1$ for each smoothly bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^{n+1}$. Here we recall the successive construction due to J.M. Lee, [59], producing the Fefferman metric on $\mathfrak{M}$ for an arbitrary strictly pseudoconvex manifold (abstract i.e. not necessarily embedded as a real hypersurface in $\mathbb{C}^{n+1}$). When M is the boundary of a domain in $\mathbb{C}^{n+1}$, or merely a real hypersurface in $\mathbb{C}^{n+1}$, the canonical circle bundle is trivial ($C(M) \approx M \times S^1$) and the Lorentzian metrics on $\mathfrak{M}$ (as in [59]) and $M \times S^1$ (as in [17]) are related by a conformal diffeomorphism.

Let $\theta \in \mathcal{P}_+$ be a positively oriented contact form on M. The *Fefferman metric* is the Lorentzian metric F_θ on $\mathfrak{M}$ given by

$$F_\theta = \pi^*\tilde{G}_\theta + 2(\pi^*\theta) \odot \sigma, \tag{1.13}$$

$$\sigma = \frac{1}{n+2}\left\{d\gamma + \pi^*\left(i\,\omega_\alpha{}^\alpha - \frac{i}{2}\,g^{\alpha\bar\beta}\,dg_{\alpha\bar\beta} - \frac{\rho}{4(n+1)}\,\theta\right)\right\}. \tag{1.14}$$

Cf. Definition 2.15 and Theorem 2.4 in [94], p. 128-129. As to the notations in (1.13)-(1.14) we define $\tilde{G}_\theta$ by $\tilde{G}_\theta = G_\theta$ on $H(M) \otimes H(M)$ and $\tilde{G}_\theta(T,W) = 0$ for any $W \in \mathfrak{X}(M)$. Moreover γ is a local fibre coordinate on $\mathfrak{M}$. Precisely if $\{T_\alpha : 1 \le \alpha \le n\} \subset C^\infty(U, T_{1,0}(M))$ is a local frame of $T_{1,0}(M)$ and $\{\theta^\alpha : 1 \le \alpha \le n\}$ is the corresponding adapted coframe then each class $z \in \mathfrak{M}$ admits a representative $\omega \in K_0(M)_x$ i.e.

$$z = [\omega] \in C(M)_x\,, \quad x \in M, \quad \omega = \lambda\left(\theta \wedge \theta^1 \wedge \cdots \wedge \theta^n\right)_x\,,$$

and the fibre coordinate in (1.14) is defined by

$$\gamma(z) = \arg\frac{\lambda}{|\lambda|}$$

where $\arg : S^1 \to [0,2\pi)$. Moreover $\omega_\beta{}^\alpha$ are the (local) connection 1-forms of the Tanaka-Webster connection, relative to the local frame $\{T_\alpha : 1 \le \alpha \le n\}$ i.e.

$$\nabla_{T_\beta} T_\alpha = \omega_\beta{}^\alpha \otimes T_\alpha\,.$$

Also $[g^{\alpha\bar\beta}] = [g_{\alpha\bar\beta}]^{-1}$ i.e. $g_{\alpha\bar\beta}g^{\bar\beta\gamma} = \delta_\alpha^\gamma$. Finally if R^∇ is the curvature tensor field of ∇ and

$$R_{\alpha\bar\beta} = \mathrm{Ric}(T_\alpha\,,\,T_{\bar\beta}),$$

$$\mathrm{Ric}(X, Y) = \mathrm{trace}\left\{Z \in \mathfrak{X}(M) \longmapsto R^{\nabla}(Z, Y)X\right\}, \quad X, Y \in \mathfrak{X}(M),$$

($R_{\alpha\overline{\beta}}$ is the *pseudohermitian Ricci curvature*, cf. Definition 1.29 in [94], p. 50) then $\rho = g^{\alpha\overline{\beta}} R_{\alpha\overline{\beta}}$ (the *pseudohermitian scalar curvature*). The Fefferman metric F_θ is a Lorentz metric on $\mathfrak{M}$ (a semi-Riemannian metric of signature $(- + \cdots +)$) and its *restricted* conformal class $\{e^{u \circ \pi} F_\theta : u \in C^\infty(M, \mathbb{R})\}$ is a CR invariant (cf. e.g. [59]).

1.4 Sublaplacians

Let M be a strictly pseudoconvex CR manifold, of CR dimension n, and $\theta \in \mathcal{P}_+$. The *sublaplacian* of (M, θ) is the second order differential operator Δ_b given by

$$\Delta_b u = -\mathrm{div}\left(\nabla^H u\right), \quad u \in C^2(M, \mathbb{R}). \tag{1.15}$$

Definitions together with Green's identity yield the useful identity

$$\int_M u \, \Delta_b u \, \Psi_\theta = \int_M \|\nabla^H u\|^2 \, \Psi_\theta \, , \quad u \in C_0^\infty(M, \mathbb{R}).$$

In particular Δ_b is a positive operator. Let Δ_{g_θ} be the Laplace-Beltrami operator (on functions) of the Riemannian manifold (M, g_θ)

$$\Delta_{g_\theta} u = -\mathrm{div}(\nabla u), \quad u \in C^2(M, \mathbb{R}).$$

Then (*Greenleaf's formula*, cf. [94])

$$\Delta_b = \Delta_{g_\theta} + T^2 \tag{1.16}$$

on functions, where T is the Reeb vector field of (M, θ). As a consequence of (1.15) the sublaplacian is locally given by

$$\Delta_b u = -\sum_{a=1}^{2n} \{X_a(X_a u) - (\nabla_{X_a} X_a)(u)\} \tag{1.17}$$

where $\{X_a : 1 \leq a \leq 2n\}$ is a local G_θ-orthonormal frame in $H(M)$. Indeed the volume form $\Psi_\theta = \theta \wedge (d\theta)^n$ is parallel with respect to the Tanaka-Webster connection ($\nabla \Psi_\theta = 0$) hence the divergence of a tangent vector field $X \in \mathfrak{X}(M)$ may be computed as

$$\mathrm{div}(X) = \mathrm{trace}\,\{Y \in \mathfrak{X}(M) \mapsto \nabla_Y X\}$$

hence locally

$$\mathrm{div}(X) = \sum_{a=1}^{2n} g_\theta(\nabla_{X_a} X, X_a) + g_\theta(\nabla_T X, T).$$

In the case of interest $X = \nabla^H u \in H(M)$ and $H(M)$ is ∇-parallel hence $g_\theta(\nabla_T \nabla^H u, T) = \theta(\nabla_T \nabla^H u) = 0$.

Let (U, x^i) be a local coordinate system on M and $\{X_a : 1 \leq a \leq 2n\}$ a G_θ-orthonormal frame of $H(M)$ defined on the same open set $U \subset M$. Moreover let us set

$$X_a = b_a^i \, \frac{\partial}{\partial x^i} \, , \quad 1 \leq a \leq 2n,$$

$$\nabla_{\partial/\partial x^j} \frac{\partial}{\partial x^k} = \Gamma^i_{jk} \frac{\partial}{\partial x^i} ,$$

$$b^i_a , \ \Gamma^i_{jk} \in C^\infty(U, \mathbb{R}), \quad 1 \leq i, j, k \leq 2n + 1.$$

Then

$$\Delta_b u = - \sum_{ij=1}^{2n+1} \frac{\partial}{\partial x^i} \left(a^{ij} \frac{\partial u}{\partial x^j} \right) + \sum_{j=1}^{2n+1} a^j \frac{\partial u}{\partial x^j} , \tag{1.18}$$

$$a^{ij} \equiv \sum_{a=1}^{2n} b^i_a b^j_a , \quad a^j \equiv \frac{\partial a^{ij}}{\partial x^i} + a^{ik} \Gamma^j_{ik} .$$

It should be observed that the matrix $\left[a^{ij}(x)\right]$ is but positive semi-definite for any $x \in U$, which is to say that Δ_b is degenerate elliptic. Indeed $\left[a^{ij}(x)\right]$ is not positive definite, for θ_x is a characteristic direction. Let X^*_a be the formal adjoint of X_a i.e.

$$X^*_a f = -\frac{\partial}{\partial x^i} \left(b^i_a f \right) - b^i_a \Gamma^i_{ij} f, \quad f \in C^1_0(U).$$

We shall make use of the *Hörmander operator* H_X (associated to the system of vector fields $X = \{X_a : 1 \leq a \leq 2n\}$) given by

$$H_X u = \sum_{a=1}^{2n} X^*_a X_a u. \tag{1.19}$$

It is straightforward (cf. e.g. [94], p. 113) that locally $\Delta_b = H_X$. Through this thesis, by a distribution on (M, θ) one means a continuous linear functional on $C^\infty_0(M)$. This is not the ordinary approach on an arbitrary C^∞ manifold (cf. [67], p. 142-145) for in that case given $u \in L^1_{\text{loc}}(M)$ and $\varphi \in C^\infty_0(M)$ there is no invariant manner of integrating $u\varphi$ (so that to identify f with a continuous linear functional on $C^\infty_0(M)$). In the case at hand however, one integrates with respect to the volume form Ψ_θ i.e. $T_u(\varphi) = \int_M u\varphi \Psi_\theta$. Let L be a differential operator and T a distribution on M. Then LT is the distribution given by $(LT)\varphi = T(L^*\varphi)$ where L^* is the formal adjoint of L. The differential operator L is *hypoelliptic* if given $f \in C^\infty(M)$ any distribution solution T to $LT = f$ is C^∞ i.e. there is $u \in C^\infty(M)$ such that $T = T_u$. We recall that a formally selfadjoint second order differential operator $L : C^\infty(M) \to C^\infty(M)$ is *subelliptic* of order ϵ (with $0 < \epsilon \leq 1$) at a point $x \in M$ if there is an open neighborhood $U \subset M$ of x such that

$$\|u\|^2_\epsilon \leq C \left(\left| (Lu, u)_{L^2} \right| + \|u\|^2_{L^2} \right), \quad u \in C^\infty_0(U). \tag{1.20}$$

Here $\| \cdot \|_s$ is the Sobolev norm of order s (cf. e.g. [104], p. 216-217). The Sobolev norms $\| \cdot \|_s$ are recalled explicitly in § 1.5 where we also prove a version of (1.20) for a compact manifold. The sublaplacian Δ_b is known (cf. Theorem 2.1 in [94], p. 114) to be subelliptic of order $\epsilon = 1/2$ at any $x \in M$

$$\|u\|^2_{1/2} \leq C \left((\Delta_b u, u)_{L^2} + \|u\|^2_{L^2} \right), \quad u \in C^\infty_0(U).$$

As such Δ_b is (by a result due to J.J. Kohn & L. Nirenberg, [55]) hypoelliptic and satisfies the *a priori* estimates

$$\|u\|^2_{s+1} \leq C_s \left(\|\Delta_b u\|^2_s + \|u\|^2_{L^2} \right), \quad u \in C^\infty_0(U), \quad s \geq 0.$$

Let $\Omega \subset \mathbb{R}^N$ be a domain and let L be a second order differential operator with real valued C^∞ coefficients defined in Ω

$$Lu(x) = -\sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2 u}{\partial x^i\,\partial x^j} + \sum_{i=1}^{N} a^i(x)\frac{\partial u}{\partial x^i} + a(x)u. \tag{1.21}$$

We adopt the following terminology (due to J.M. Bony, [58]). The differential operator L is *degenerate elliptic* (*in the sense of Bony*) if i) the matrix $\left[a^{ij}(x)\right]$ is positive semi-definite, but not positive definite, at each $x \in \Omega$ i.e.

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geq 0, \quad \xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n\,,$$

ii) $a(x) \geq 0$ for any $x \in \Omega$, and iii) L may be written as

$$Lu = \sum_{a=1}^{r} X_a(X_a u) + Y(u) + au$$

for some C^∞ vector fields $X_a, Y \in \mathfrak{X}(\Omega)$. The sublaplacian Δ_b is a degenerate elliptic (in the sense of J.M. Bony) second order differential operator on M (cf. the discussion above or [94], p. 111-119). Degenerate elliptic operators satisfy a useful weak form of the maximum principle (cf. Theorem 3.28 in [94], p. 209). Precisely if a C^2 function u achieves at x_0 a nonpositive local maximum then $(Lu)(x_0) \geq 0$. If additionally this maximum if < 0 and $a(x_0) > 0$ then $(Lu)(x_0) > 0$.

This thesis is mostly concerned with the study of spectrae of sublaplacians on strictly pseudoconvex manifolds, so we review the basic terminology (in general spectral theory, cf. e.g. [104], p. 365) for the specific case of $\Delta_b : \mathcal{D}(\Delta_b) \subset L^2(M) \to L^2(M)$. The *resolvent* set $\rho(\Delta_b) \subset \mathbb{C}$ consists of all complex numbers $\lambda \in \mathbb{C}$ such that $\Delta_b - \lambda I : \mathcal{D}(\Delta_b) \to L^2(M)$ is an invertible map such that $(\Delta_b - \lambda I)^{-1} \in \mathcal{B}(L^2(M))$. Here $\mathcal{B}(L^2(M))$ is the Banach algebra of all bounded linear operators $A : L^2(M) \to L^2(M)$. The operator $R(\lambda; \Delta_b) = (\lambda I - \Delta_b)^{-1}$ is known as the *resolvent* of Δ_b. The *spectrum* of Δ_b is the set $\sigma(\Delta_b) = \mathbb{C} \setminus \rho(\Delta_b)$.

1.5 Sobolev type spaces on CR manifolds

Let M be a strictly pseudoconvex CR manifold. Abstract CR manifolds with boundary were considered in [95]. Through this section we only deal with bounded (with respect to the Carnot-Carathéodory distance function d_H) domains $\Omega \subset M$ with C^2 boundary. Let θ be a fixed contact form on M and set $\Psi_\theta = \theta \wedge (d\theta)^n$ for simplicity. Let $\pi : E \to M$ be a Riemannian vector bundle with the Riemannian bundle metric h. We denote by $L^2(E_\Omega)$ the space of all L^2 sections in $E_\Omega = \pi^{-1}(\Omega)$ (the portion of E over Ω) that is $s \in L^2(E_\Omega)$ if $h(s,s) \in L^1(\Omega)$ i.e. $\int_\Omega h(s,s)\,\Psi_\theta < \infty$. If $\Omega \times \mathbb{R}$ is the trivial vector bundle over Ω we write briefly $L^2(\Omega) = L^2(\Omega \times \mathbb{R})$. If $u \in C^1(\Omega, \mathbb{R})$ and $X \in C_0^\infty(\Omega, H(M))$ then (by Green's lemma)

$$\int_\Omega g_\theta(\nabla^H u, X)\,\Psi_\theta = \int_\Omega X(u)\,\Psi_\theta = \tag{1.22}$$

$$= \int_{\partial\Omega} u\, g_\theta(X, \nu)\, da - \int_\Omega u\,\mathrm{div}(X)\,\Psi_\theta = -\int_\Omega u\,\mathrm{div}(X)\,\Psi_\theta.$$

Here ν is the outward unit normal on $\partial\Omega$ and the divergence of X is computed with respect to the volume form Ψ_θ i.e. $\mathcal{L}_X \Psi_\theta = \operatorname{div}(X)\Psi_\theta$ ($\mathcal{L}_X$ denotes the Lie derivative). The simple calculation (1.22) suggests a calculus with functions which are but weakly differentiable along the Levi distribution, cf. [28].

A function $u \in L^1_{\mathrm{loc}}(\Omega)$ is *weakly differentiable* along the Levi distribution if there is a section Y_u in $H(\Omega)$ such that $\|Y_u\| = g_\theta(Y_u, Y_u)^{1/2} \in L^1_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega g_\theta(Y_u\,, X)\,\Psi_\theta = -\int_\Omega u\operatorname{div}(X)\,\Psi_\theta, \quad X \in C_0^\infty(H(\Omega)).$$

Such Y_u is unique up to a set of measure zero and is denoted by $Y_u = \nabla^H u$ (the *weak horizontal gradient* of u). Let $\mathcal{D}(\nabla^H) = W_H^{1,2}(\Omega)$ be the space consisting of all $u \in L^2(\Omega)$ such that u is weakly differentiable along the Levi distribution and $\nabla^H u \in L^2(H(\Omega))$. Therefore the weak horizontal gradient may be regarded as a linear operator $\nabla^H : \mathcal{D}(\nabla^H) \subset L^2(\Omega) \to L^2(H(\Omega))$ of Hilbert spaces (densely defined, as $C_0^\infty(\Omega) \subset \mathcal{D}(\nabla^H)$). Moreover $W_H^{1,2}(\Omega)$ is a Hilbert space with the inner product

$$(f, g)_{W_H^{1,2}} = \int_\Omega fg\,\Psi_\theta + \int_\Omega g_\theta(\nabla^H f, \nabla^H g)\,\Psi_\theta$$

(cf. Proposition 3 in [28], p. 7). In particular $W_H^{1,2}(\Omega)$ is reflexive. For further use let $\mathring{W}_H^{1,2}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W_H^{1,2}(\Omega)$.

Lemma 1.2. *Let $\Omega \subset M$ be a domain satisfying the Poincaré inequality*

$$\int_\Omega \varphi^2\,\Psi_\theta \le C\int_\Omega \|\nabla^H\varphi\|^2\,\Psi_\theta \tag{1.23}$$

for some constant $C > 0$ and any $\varphi \in C_0^\infty(\Omega, \mathbb{R})$. Then i)

$$\|\varphi\|_{\mathring{W}_H^{1,2}} = \left(\int_\Omega \|\nabla^H\varphi\|^2\,\Psi_\theta\right)^{\frac{1}{2}}$$

is a norm on $C_0^\infty(\Omega, \mathbb{R})$. Also ii)

$$\|\varphi\|_{\mathring{W}_H^{1,2}} \le \|\varphi\|_{W_H^{1,2}} \le (1 + C)^{\frac{1}{2}}\,\|\varphi\|_{\mathring{W}_H^{1,2}}$$

for any $\varphi \in C_0^\infty(\Omega, \mathbb{R})$ i.e. $\|\cdot\|_{\mathring{W}_H^{1,2}}$ and $\|\cdot\|_{W_H^{1,2}}$ are equivalent norms on $C_0^\infty(\Omega, \mathbb{R})$. In particular iii) *$\mathring{W}_H^{1,2}(\Omega)$ is a Hilbert space with the inner product*

$$a_b(f, g) = \int_\Omega g_\theta(\nabla^H f, \nabla^H g)\,\Psi_\theta.$$

Proof. i) If $\varphi \in C_0^\infty(\Omega)$ is a test function such that $\|\varphi\|_{\mathring{W}_H^{1,2}} = 0$ then $\|\nabla^H\varphi\| = 0$ a.e. in Ω. Yet $\|\nabla^H\varphi\|$ is continuous and the measure associated to the volume form Ψ_θ is Borelian, hence $\|\nabla^H\varphi\| = 0$ everywhere in Ω. Thus $\nabla^H\varphi = 0$ so that φ is a real valued CR function, and then a constant $c \in \mathbb{R}$. Yet φ is zero at the boundary so $c = 0$.

ii) For every $\varphi \in C_0^\infty(\Omega, \mathbb{R})$

$$\|\varphi\|^2_{W_H^{1,2}} = \|\varphi\|^2_{L^2} + \|\nabla^H\varphi\|^2_{L^2} \ge \|\nabla^H\varphi\|^2_{L^2} = \|\varphi\|^2_{\mathring{W}_H^{1,2}}\,,$$

$$\|\varphi\|^2_{W^{1,2}_H} = \|\varphi\|^2_{L^2} + \|\nabla^H\varphi\|^2_{L^2} \le (C+1)\|\nabla^H\varphi\|^2_{L^2} = (C+1)\|\varphi\|^2_{\mathring{W}^{1,2}_H},$$

so that

$$\|\varphi\|_{\mathring{W}^{1,2}_H} \le \|\varphi\|_{W^{1,2}_H} \le (1+C)^{1/2}\,\|\varphi\|_{\mathring{W}^{1,2}_H}, \quad \varphi \in C^\infty_0(\Omega,\mathbb{R}).$$

Q.e.d.

Let $(\nabla^H)^* : \mathcal{D}[(\nabla^H)^*] \subset L^2(H(\Omega)) \to L^2(\Omega)$ be the adjoint of ∇^H i.e. i) $\mathcal{D}[(\nabla^H)^*]$ consists of all $X \in L^2(H(\Omega))$ such that

$$\int_\Omega g_\theta(\nabla^H u, X)\,\Psi_\theta = \int_\Omega u X^*\,\Psi_\theta$$

for some $X^* \in L^2(\Omega)$ and any $u \in \mathcal{D}(\nabla^H)$, and ii) $(\nabla^H)^* X = X^*$. Then $C^\infty_0(H(\Omega)) \subset \mathcal{D}[(\nabla^H)^*]$ and the restriction of $(\nabla^H)^*$ to $C^\infty_0(H(\Omega))$ is $-\mathrm{div}$. It is customary to set $\mathcal{D}(\Delta_b) = \{u \in \mathcal{D}(\nabla^H) : \nabla^H u \in \mathcal{D}[(\nabla^H)^*]\}$ and refer to the linear operator $(\nabla^H)^* \circ \nabla^H : \mathcal{D}(\Delta_b) \subset L^2(\Omega) \to L^2(\Omega)$ as the sublaplacian of (M,θ), as well. Then

$$\Delta_b u = \left((\nabla^H)^* \circ \nabla^H\right) u, \quad u \in C^\infty_0(\Omega). \tag{1.24}$$

Let $N = 2n+1$ and let $\hat{u}$ denote the Fourier transform of a function[2] $u \in C^\infty_0(\mathbb{R}^N)$. For every $s \in \mathbb{R}$ we consider the Sobolev norm

$$\|u\|_s = \left((2\pi)^{-N/2}\int_{\mathbb{R}^N}\left(1+|\xi|^2\right)^s \left|\hat{u}(\xi)^2\right|^2\,d\xi\right)^{1/2},$$

and the inner product

$$(u,v)_s = (2\pi)^{-N/2}\int_{\mathbb{R}^N}\left(1+|\xi|^2\right)^s \hat{u}(\xi)\overline{\hat{v}(\xi)}\,d\xi,$$

for any $u, v \in C^\infty_0(\mathbb{R}^N)$. Let $H_s(\mathbb{R}^N)$ be the Hilbert space got as the completion of $C^\infty_0(\mathbb{R}^N)$ with respect to the norm $\|\cdot\|_s$. Next let us consider a compact N-dimensional manifold M without boundary ($\partial M = \emptyset$). Let $\mathcal{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ be a finite open covering of M with domains of local coordinate systems $\chi_\lambda : U_\lambda \to \mathbb{R}^N$ such that $\chi_\lambda(U_\lambda) = \mathbb{R}^N$. Moreover let $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ be a C^∞ partition of unity subordinated to the open covering $\mathcal{U}$

$$\varphi_\lambda \in C^\infty(M), \quad \mathrm{Supp}(\varphi_\lambda) \subset U_\lambda, \quad 0 \le \varphi_\lambda \le 1, \quad \sum_{\lambda\in\Lambda}\varphi_\lambda = 1.$$

Let us consider the Sobolev norms

$$\|u\|^{\mathcal{S}}_s = \left(\sum_{\lambda\in\Lambda}\|(u\varphi_\lambda)\circ\chi_\lambda^{-1}\|^2_s\right)^{1/2},$$

and the inner products

$$(u,v)^{\mathcal{S}}_s = \sum_{\lambda\in\Lambda}\left((u\varphi_\lambda)\circ\chi_\lambda^{-1},\,(v\varphi_\lambda)\circ\chi_\lambda^{-1}\right)_s,$$

for every $u, v \in C^\infty(M)$, where $\mathcal{S} \equiv \{(U_\lambda, \chi_\lambda, \varphi_\lambda) : \lambda \in \Lambda\}$. Definitions clearly depend on the choice of the system $\mathcal{S}$ (and this is captured in the notation). The map

$$u \in C^\infty(M) \longmapsto \left((u\varphi_\lambda)\circ\chi_\lambda^{-1}\right)_{\lambda\in\Lambda} \in C^\infty_0(\mathbb{R}^N)^{|\Lambda|}$$

[2]Unless otherwise specified functions are assumed to be complex valued.

is a linear injective operator $C^\infty(M) \to C_0^\infty(\mathbb{R}^N)^{|\Lambda|}$. Here $|\Lambda|$ is the cardinality of the (finite) set Λ. The composition of this operator with the inclusion $C_0^\infty(\mathbb{R}^N)^{|\Lambda|} \to H_s(\mathbb{R}^N)^{|\Lambda|}$ furnishes a linear injective map

$$\eta_s^{\mathcal{S}} : C^\infty(M) \to H_s(\mathbb{R}^N)^{|\Lambda|}\,.$$

Let us denote by $H_s^{\mathcal{S}}(M)$ the closure of the image of $\eta_s^{\mathcal{S}}$ in the Hilbert space $H_s(\mathbb{R}^N)^{|\Lambda|}$. Also let us identity $C^\infty(M)$ with its canonical image in $H_s^{\mathcal{S}}(M)$. $H_s^{\mathcal{S}}(M)$ is a closed subspace of $H_s(\mathbb{R}^N)^{|\Lambda|}$ hence a Hilbert space itself. From now on, for fixed $\mathcal{S}$ we write merely $\|\cdot\|_s$ and $H_s(M)$. Let M be a compact strictly pseudoconvex CR manifold and $\theta \in \mathcal{P}_+$. Let Δ_b be the sublaplacian of (M, θ). The estimates (1.20) imply

$$\|u\|_{1/2}^2 \le C\left\{(\Delta_b u, u)_{L^2} + \|u\|_{L^2}^2\right\}, \quad u \in C^\infty(M), \tag{1.25}$$

for some constant $C > 0$ depending only on M. Indeed (by (1.20))

$$\left(\|u\|_{1/2}^{\mathcal{S}}\right)^2 = \sum_{\lambda\in\Lambda} \|(u\varphi_\lambda)\circ\chi_\lambda^{-1}\|_{1/2}^2 \le$$

$$\le \sum_{\lambda\in\Lambda} C_\lambda \left\{(\Delta_b(u\varphi_\lambda)\,,\, u\varphi_\lambda)_{L^2} + \|u\varphi_\lambda\|_{L^2}^2\right\}$$

for some $C_\lambda > 0$ and

$$\Delta_b(u\varphi_\lambda) = u\,\Delta_b\varphi_\lambda + \varphi_\lambda\Delta_b u + 2\,G_\theta(\nabla^H u\,,\,\nabla^H\varphi_\lambda),$$

$$\sum_{\lambda\in\Lambda}(\Delta_b(u\varphi_\lambda)\,,\,u\varphi_\lambda)_{L^2} = \sum_\lambda \varphi_\lambda\;(\Delta_b(u\varphi_\lambda)\,,\,u)_{L^2} \le$$

(as $(\Delta_b(u\varphi_\lambda), u\varphi_\lambda)_{L^2} \ge 0$ and $\varphi_\lambda \ge 0$ yield $(\Delta_b(u\varphi_\lambda), u)_{L^2} \ge 0$ and then one may exploit $\varphi_\lambda \le 1$)

$$\le \sum_\lambda \int_M \left\{u\,\Delta_b\varphi_\lambda + \varphi_\lambda\,\Delta_b u + 2\,G_\theta(\nabla^H u\,,\,\nabla^H\varphi_\lambda)\right\}\overline{u}\,\Psi_\theta =$$

$$= \int_M \left\{u\,\Delta_b(\sum_\lambda\varphi_\lambda) + (\sum_\lambda\varphi_\lambda)\,\Delta_b u + 2\,G_\theta\left(\nabla^H u\,,\,\nabla^H(\sum_\lambda\varphi_\lambda)\right)\right\}\overline{u}\,\Psi_\theta =$$

$$= \int_M (\Delta_b u)\,\overline{u}\,\Psi_\theta = (\Delta_b u\,,\,u)_{L^2}\;,$$

$$\sum_{\lambda\in\Lambda}\|u\varphi_\lambda\|_{L^2}^2 = \sum_\lambda \int_M |u|^2\varphi_\lambda^2\,\Psi_\theta \le$$

(as $\varphi_\lambda \le 1$)

$$\le \sum_\lambda \int_M |u|^2\varphi_\lambda\,\Psi_\theta = \|u\|_{L^2}^2$$

so that (1.25) holds with $C = \max\{C_\lambda : \lambda \in \Lambda\}$.

We shall make use of the ordinary Sobolev spaces $W^{s,p}(\Omega)$ with $s \in \mathbb{R}$ and $1 < p < \infty$ and an arbitrary domain $\Omega \subset \mathbb{R}^N$, as built in [85], p. 204. Another method of constructing fractional order spaces (in terms of Fourier transforms of tempered distributions) furnishes the spaces $H^{s,p}(\Omega)$, cf. [85], p. 219. The spaces $W^{s,p}(\Omega)$ and $H^{p,s}(\Omega)$ are known to coincide when $s \in \mathbb{Z}$ and $1 < p < \infty$

or when $s \in \mathbb{R}$ and $p = 2$. Our considerations so far (related to the subelliptic estimates (1.20)) only required the spaces $H_s(\mathbb{R}^N) = H^{s,2}(\mathbb{R}^N)$. A general embedding result we shall make use of is Theorem 7.58 in [85], p. 218-219. This is stated for $\Omega = \mathbb{R}^N$ yet holds for domains $\Omega \subset \mathbb{R}^N$ possessing the regularity properties requested in Theorem 7.41, [85], p. 207. These requirements are satisfied by the unit ball $\Omega = \mathbb{B}^N = \{x \in \mathbb{R}^N : |x| < 1\}$ so that for any $s > 0$, $1 < p \le q < \infty$ and $1 \le K \le N$ the following embedding holds

$$W^{s,p}(\mathbb{B}^N) \longrightarrow W^{\chi,q}(\mathbb{B}^K), \quad \chi \equiv s - \frac{N}{p} + \frac{K}{q}\,, \tag{1.26}$$

provided that either i) $\chi \ge 0$ and $p < q$, or ii) $\chi > 0$ and $\chi \in \mathbb{R} \setminus \mathbb{Z}$, or iii) $\chi \ge 0$ and $1 < p \le 2$. We wish to specialize (1.26) to the case

$$s = \frac{1}{2}\,, \quad p = 2, \quad K = N,$$

that is

$$W^{1/2\,,2}(\mathbb{B}^N) \longrightarrow W^{N/q-(N-1)/2\,,q}(\mathbb{B}^N), \quad 2 \le q < \infty, \tag{1.27}$$

holding when

$$\frac{N}{q} \ge \frac{N-1}{2}\,. \tag{1.28}$$

On the other hand we need the Kondrakov lemma (cf. e.g. Theorem 2.33 in [102], p. 53). Let $k \in \mathbb{Z}$, $k \ge 0$, and $p, q \in \mathbb{R}$ such that

$$1 \ge \frac{1}{p} > \frac{1}{q} - \frac{k}{N} > 0. \tag{1.29}$$

Moreover let $\Omega \subset \mathbb{R}^N$ be a bounded open set whose boundary $\partial\Omega$ is C^1 (Lipschitzian actually suffices). Then the embedding

$$W^{k,q}(\Omega) \longrightarrow L^p(\Omega) \tag{1.30}$$

is compact[3]. We wish to specialize (1.30) to

$$\Omega = \mathbb{B}^N\,, \quad p = 2, \quad k = \frac{N}{q} - \frac{N-1}{2}\,, \quad 2 \le q < \infty, \tag{1.31}$$

with the requirements (1.28)-(1.29). Solving for q in (1.31) gives

$$q = \frac{N}{k + \dfrac{N-1}{2}}\,, \quad k \in \mathbb{Z}, \quad k \ge 0. \tag{1.32}$$

It is straightforward that the numbers $q \in \mathbb{R}$ given by (1.32) satisfy Kondrakov lemma's requirement (1.29) with $p = 2$ hence for any $k \in \mathbb{Z}$, $k \ge 0$, the embedding

$$W^{k,\,N/(k+(N-1)/2)}(\mathbb{B}^N) \longrightarrow L^2(\mathbb{B}^N) \tag{1.33}$$

is compact. Let us set $k = 1$ in (1.33) so that the embedding

$$W^{1,\,2N/(N+1)}(\mathbb{B}^N) \longrightarrow L^2(\mathbb{B}^N)$$

[3]The image by (1.30) of any bounded set in $W^{k,q}(\Omega)$ is compact in $L^p(\Omega)$.

is compact, as well. The composition with (1.27) then gives the compact embedding

$$W^{1/2,2}(\mathbb{B}^N) \longrightarrow L^2(\mathbb{B}^N). \tag{1.34}$$

Embedding (1.34) yields

Lemma 1.3. *Let M be a compact strictly pseudoconvex CR manifold without boundary. Then $H_{1/2}(M) = W^{1/2,2}(M)$ admits a compact embedding into $L^2(M)$.*

Proof. We may cover M with a finite number of open sets which are domains of local coordinate charts whose image is the unit ball $\mathbb{B}^{2n+1}$. The proof of Lemma 1.3 is then a *verbatim* repetition of the arguments in the proof of Theorem 2.34 in [102], p. 55 (replacing the use of Theorem 2.33 in [102], p. 53, by that of (1.34) above).

Lemma 1.4. *On any compact strictly pseudoconvex CR manifold the operator $(\Delta_b+I)^{-1} : \mathcal{D}((\Delta_b+I)^{-1}) \subset L^2(M) \to L^2(M)$ is compact.*

Proof. The estimate (1.25) may be written

$$\|u\|^2_{1/2} \leq C\,((\Delta_b + I)u\,,\,u)_{L^2}\;,\quad u \in C^\infty(M), \tag{1.35}$$

hence $\mathrm{Ker}(\Delta_b + I) = (0)$. Consequently we may consider the inverse

$$\Delta_b + I : C^\infty(M) \to \mathcal{R}(\Delta_b + I) \subset C^\infty(M)$$

is invertible, where $\mathcal{R}(A)$ denotes the range of the operator A. Therefore we may consider the inverse

$$(\Delta_b + I)^{-1} : \mathcal{D}\Big((\Delta_b + I)^{-1}\Big) = \mathcal{R}(\Delta_b + I) \subset L^2(M) \to H_{1/2}(M).$$

Let $v \in \mathcal{D}\Big((\Delta_b + I)^{-1}\Big)$ and let us apply (1.35) to the function $u = (\Delta_b + I)^{-1}(v)$ followed by the Cauchy-Schwartz inequality

$$\|(\Delta_b + I)^{-1}v\|^2_{1/2} \leq C\left(v\,,\,(\Delta_b + I)^{-1}v\right)_{L^2} \leq C\|v\|_{L^2}\,\|(\Delta_b + I)^{-1}v\|_{L^2}\,.$$

Moreover, there is a continuous embedding $H_{1/2}(M) \to L^2(M)$ so that

$$\|u\|_{L^2} \leq C'\|u\|_{1/2}\,,\quad u \in H_{1/2}(M),$$

for some constant $C' > 0$ independent of u. Thus

$$\|(\Delta_b + I)^{-1}v\|^2_{1/2} \leq C''\|v\|_{L^2}\|(\Delta_b + I)^{-1}v\|_{1/2}$$

(with $C'' = CC'$) or

$$\|(\Delta_b + I)^{-1}v\|_{1/2} \leq C''\|v\|_{L^2}$$

so that $(\Delta_b + I)^{-1}$ is a continuous operator. Finally (by Lemma 1.3) the embedding $H_{1/2}(M) \to L^2(M)$ is compact hence $(\Delta_b + I)^{-1} : \mathcal{D}((\Delta_b + I)^{-1}) \subset L^2(M) \to L^2(M)$ is compact (as the composition of a compact operator with a continuous operator). Q.e.d.

Corollary 1.5. *The spectrum $\sigma(\Delta_b)$ of the sublaplacian on any compact strictly pseudoconvex pseudohermitian manifold is discrete.*

Proof. Follows from Lemma 1.4 together with the general result in functional analysis that completely continuous linear operators (here $(\Delta_b + I)^{-1}$) have discrete spectrae.

1.6 Dirichlet Spectrum of a Sublaplacian

Let M be a strictly pseudoconvex CR manifold and $\Omega \subset M$ a smoothly bounded (with respect to the Carnot-Carthéodory metric) domain. Let θ be a contact form on M, such that the Levi form L_θ is positive definite, and let Δ_b be the sublaplacian of the pseudohermitian manifold (M, θ). The scope of this section is to study the Dirichlet problem

$$\Delta_b u = \lambda u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.36}$$

where $\lambda \in \mathbb{R}$ is a parameter. A number $\lambda \in \mathbb{R}$ is an *eigenvalue* of (1.36) if there is a function $u \in \mathring{W}^{1,2}_H(\Omega) \setminus \{0\}$ satisfying the functional equation

$$a_b(u, \varphi) = \lambda (u, \varphi)_{L^2}, \quad \varphi \in \mathring{W}^{1,2}_H(\Omega). \tag{1.37}$$

We shall show that

Theorem 1.6. *Let (M, θ) be a strictly pseudoconvex pseudohermitian manifold and $\Omega \subset M$ a bounded (with respect to the Carnot-Carathéodory metric d_H) domain satisfying Poincaré inequality. If the metric space (M, d_H) is complete then the Dirichlet problem (1.36) admits an infinite sequence of eigenvalues* $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_\nu \leq \cdots$ *and an infinite sequence of eigenfunctions $\{u_\nu\}_{\nu \geq 1} \subset \mathring{W}^{1,2}_H(\Omega)$ corresponding to the eigenvalues $\{\lambda_\nu\}_{\nu \geq 1}$ such that* $\lim_{\nu\to\infty} \lambda_\nu = +\infty$ *and* $(u_\mu, u_\nu)_{L^2} = \delta_{\mu\nu}$.

"By Poincare' inequality we mean

$$\int_\Omega \varphi^2 \, \Psi_\theta \leq C \int_\Omega \|\nabla^H \varphi\|^2 \, \Psi_\theta , \quad \varphi \in C^\infty_0(\Omega, \mathbb{R}). \tag{1.38}$$

Besides from (1.38) proof of Theorem 1.6 relies on the compactness of the inclusion $\mathring{W}^{1,2}_H(\Omega) \to L^2(\Omega)$.

1.7 Generalized Dirichlet problem

Let $\Omega \subset M$ be a bounded domain in a strictly pseudoconvex CR manifold and let θ be a contact form on M with L_θ positive definite. Let Δ_b be the sublaplacian of (M, θ). We shall solve the homogeneous Dirichlet problem

$$\Delta_b u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{1.39}$$

A function $u_0 \in \mathring{W}^{1,2}_H(\Omega)$ is a *generalized solution* to the Dirichlet problem (1.39) if $a_b(u_0, \varphi) = (f, \varphi)$ for any $\varphi \in \mathring{W}^{1,2}_H(\Omega)$. Let

$$(\nabla^H)^* : \mathcal{D}[(\nabla^H)^*] \subset L^2(H(\Omega)) \to L^2(\Omega)$$

be the adjoint of ∇^H i.e. i) $\mathcal{D}[(\nabla^H)^*]$ consists of all $X \in L^2(H(\Omega))$ such that

$$\int_\Omega g_\theta(\nabla^H u, X) \, \Psi_\theta = \int_\Omega u X^* \, \Psi_\theta$$

for some $X^* \in L^2(\Omega)$ and any $u \in \mathcal{D}(\nabla^H)$, and ii) $(\nabla^H)^* X = X^*$. Then $C_0^\infty(H(\Omega)) \subset \mathcal{D}[(\nabla^H)^*]$ and the restriction of $(\nabla^H)^*$ to $C_0^\infty(H(\Omega))$ is $-\mathrm{div}$. It is customary to set $\mathcal{D}(\Delta_b) = \{u \in \mathcal{D}(\nabla^H) : \nabla^H u \in \mathcal{D}[(\nabla^H)^*]\}$ and refer to the linear operator $(\nabla^H)^* \circ \nabla^H : \mathcal{D}(\Delta_b) \subset L^2(\Omega) \to L^2(\Omega)$ as the sublaplacian of (M, θ), as well. Then $\Delta_b = (\nabla^H)^* \circ \nabla^H$ on $C_0^\infty(\Omega)$. Consequently, any strong solution $u_0 \in C^2(\Omega) \cap C(\overline{\Omega})$ to (1.39) is also a generalized solution and, viceversa, any generalized solution $u_0 \in \mathring{W}_H^{1,2}(\Omega) \cap C^2(\Omega)$ is a strong solution to (1.39). We shall establish the following

Theorem 1.7. *Let M be a strictly pseudoconvex CR manifold and θ a contact form on M. Let $\Omega \subset M$ be a bounded domain on which the Poincaré inequality* (1.38) *holds. Then for any $f \in L^2(\Omega)$ the Dirichlet problem* (1.39) *admits a unique generalized solution.*

To prove Theorem 1.7 we set

$$E_b(u) = \frac{1}{2}\, a_b(u, u), \quad F(u) = E_b(u) - (f, u), \quad u \in \mathring{W}_H^{1,2}(\Omega).$$

Lemma 1.8. i) *For each $u \in \mathring{W}_H^{1,2}(\Omega)$ the functional $\varphi \mapsto a_b(u, \varphi)$ is continuous on $\mathring{W}_H^{1,2}(\Omega)$.* ii) *For each $f \in L^2(\Omega)$ the functional $\varphi \mapsto (f, \varphi)_{L^2}$ is continuous on $\mathring{W}_H^{1,2}(\Omega)$.* iii) *$F$ is differentiable at any $u \in \mathring{W}_H^{1,2}(\Omega)$ and its Gateaux derivative is given by*

$$F'(u)\varphi = a_b(u, \varphi) - (f, \varphi)_{L^2}, \quad \varphi \in \mathring{W}_H^{1,2}(\Omega). \tag{1.40}$$

iv) *F is strictly convex and*

$$\lim_{E_b(u) \to \infty} F(u) = +\infty. \tag{1.41}$$

Proof. i) For any $u, \varphi \in \mathring{W}_H^{1,2}(\Omega)$ (by Cauchy's inequality, both pointwise on (H, g_θ) and L^2)

$$|a_b(u, \varphi)| \leq \int_\Omega \|\nabla^H u\| \, \|\nabla^H \varphi\| \, \Psi_\theta \leq$$

$$\leq \|\nabla^H u\|_{L^2} \|\nabla^H \varphi\|_{L^2} = 2 E_b(u)^{1/2} E_b(\varphi)^{1/2}$$

and $E_b^{1/2}$ is a norm on $\mathring{W}_H^{1,2}(\Omega)$.

ii) By Poincaré's inequality (1.38)

$$\left|(f, \varphi)_{L^2}\right| \leq \|f\|_{L^2} \, \|\varphi\|_{L^2} \leq \sqrt{2C} \|f\|_{L^2} E_b(\varphi)^{1/2}. \tag{1.42}$$

Besides from implying (ii) the simple estimate (1.42) is essential in establishing property (1.41) of F.

iii) We start by recalling a few standard notions familiar within the variational treatment of elliptic partial differential equations (cf. e.g. [23]). Most of the underlying methods are sufficiently general to apply to Δ_b or admit *ad hoc* adaptations to the case of interest, as shown below. Given a real Hilbert space $\mathcal{H}$ a functional $A : \mathcal{H} \to \mathbb{R}$ is differentiable at the point $u \in \mathcal{H}$ in the direction $v \in \mathcal{H}$ if the limit $\lim_{\lambda \to 0} \lambda^{-1} [A(u + \lambda v) - A(u)]$ exists and then the limit is denoted by $A'(u; v) \in \mathbb{R}$. If the limit $A'(u; v)$ exists for any $v \in \mathcal{H}$ then A is differentiable at u and the functional $v \in \mathcal{H} \longmapsto A'(u; v) \in \mathbb{R}$ is the differential of A at u. If the differential of A at u is linear and continuous then A is commonly said to be *Gateaux differentiable* and the functional $v \in \mathcal{H} \longmapsto A'(u; v) \in \mathbb{R}$ is denoted by $A'(u)$ and referred to as the *Gateaux derivative* of A at u.

The bilinear form a_b is symmetric, hence $q = 2E_b$ is a quadratic[4] form. Also (by (i) in Lemma 1.8) $a_b(u, \cdot)$ is continuous for any fixed u, hence q is differentiable and its Gateaux derivative at $u \in \mathring{W}_H^{1,2}(\Omega)$ is $2a_b(u, \cdot)$. Finally (by (ii) in Lemma 1.8) $(f, \cdot)_{L^2}$ is differentiable at u and coincides with its Gateaux derivative at any u. Thus F is differentiable and (1.40) holds.

iv) A functional $A : \mathcal{H} \to \mathbb{R}$ is convex if $A(\lambda u + (1-\lambda)v) \leq \lambda A(u) + (1-\lambda)A(v)$ for any $u, v \in \mathcal{H}$ and any $\lambda \in [0, 1]$. If the above inequality is strict for any $u \neq v$ then A is strictly convex. A general result, that we are going to use in the sequel, is that any positive quadratic form is a convex functional and any positive definite quadratic form is a strictly convex functional.

The quadratic form $q(u) = a_b(u, u)$ is positive definite. This situation should be compared to that of an arbitrary uniformly elliptic operator in divergence form (where uniform ellipticity implies coercivity of the associated quadratic form cf. e.g. [23]). In the case at hand, the sublaplacian Δ_b isn't elliptic yet already

Lemma 1.9. *$E_b^{1/2}$ is a norm on $\mathring{W}_H^{1,2}(\Omega)$.*

This follows from Lemma 1.2 . Strict convexity of q then yields strict convexity of F. Finally (by (1.42))

$$F(u) = E_b(u) - (f, u)_{L^2} \geq E_b(u) - \sqrt{2C}\|f\|_{L^2} E_b(u)^{1/2}$$

for any $u \in \mathring{W}_H^{1,2}(\Omega)$, and $t^2 - \sqrt{2C}\|f\|_{L^2} t \to +\infty$ as $t \to +\infty$ thus proving (1.41). Q.e.d.

To proceed we need to recall a few standard results from the calculus of variations. Let $\mathcal{H}$ be a real Hilbert space and $A : \mathcal{H} \to \mathbb{R}$ an arbitrary functional. Then $u_0 \in \mathcal{H}$ is a global minimum point of A if $A(u_0) \leq A(u)$ for any $u \in \mathcal{H}$. The number $A(u_0) \in \mathbb{R}$ is the global minimum of A. Here we shall only be interested in global minima of certain functionals. We remind however that $u_0 \in \mathcal{H}$ is a local minimum point if there is a neighborhood of u_0 such that the inequality $A(u_0) \leq A(u)$ holds on that neighborhood. Let $A : \mathcal{H} \to \mathbb{R}$ be a Gateaux differentiable functional. By a standard result, if $u_0 \in \mathcal{H}$ is a local minimum point of A then $A'(u_0) = 0$ (i.e. $A'(u_0)v = 0$ for any $v \in \mathcal{H}$). Also if the functional $A : \mathcal{H} \to \mathbb{R}$ is convex and Gateaux differentiable then $A'(u_0) = 0$ is a necessary and sufficient condition for $u_0 \in \mathcal{H}$ to be a global minimum point of A. Finally, we shall make use of the following results. If the convex and Gateaux differentiable functional $A : \mathcal{H} \to \mathbb{R}$ satisfies the condition $\lim_{\|u\|_{\mathcal{H}} \to \infty} A(u) = +\infty$ then A has at least a global minimum point. Also if $A : \mathcal{H} \to \mathbb{R}$ is a strictly convex functional then A admits at most one global minimum point.

At this point we may end the proof of Theorem 1.6. Strict convexity together with (1.41) imply the existence of a global minimum point $u_0 \in \mathring{W}_H^{1,2}(\Omega)$ for F. Consequently $F'(u_0)\varphi = 0$ for any $\varphi \in \mathring{W}_H^{1,2}(\Omega)$ or $a_b(u_0, \varphi) = (f, \varphi)_{L^2}$ i.e. u_0 is a generalized solution to the Dirichlet problem (1.39). Uniqueness of the solution is again a standard consequence of strict convexity. Indeed if $u_1 \in \mathring{W}_H^{1,2}(\Omega)$ satisfies $a_b(u_1, \varphi) = (f, \varphi)_{L^2}$ then $F'(u_1) = 0$ on $\mathring{W}_H^{1,2}(\Omega)$ so that (as F is convex and differentiable) u_1 is a global minimum point of F. Let us set $d = F(u_0) = F(u_1)$. Finally if $u_0 \neq u_1$

[4]Let $\mathcal{H}$ be a Hilbert space and $b : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ a $\mathbb{R}$-bilinear form. b is symmetric if $b(u, v) = b(v, u)$ for any $u, v \in \mathcal{H}$. A functional $q : \mathcal{H} \to \mathbb{R}$ is a quadratic form if $q(u) = b(u, u)$ for some symmetric bilinear form $b : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$. The quadratic form $q : \mathcal{H} \to \mathbb{R}$ is *positive* if $q(u) \geq 0$ for any $u \in \mathcal{H}$. A quadratic form $q : \mathcal{H} \to \mathbb{R}$ is *positive definite* if $q(u) > 0$ for any $u \in \mathcal{H} \setminus \{0\}$. A quadratic form $q : \mathcal{H} \to \mathbb{R}$ is *coercive* if there is a constant $\gamma > 0$ such that $q(u) \geq \gamma \|u\|_{\mathcal{H}}^2$ for any $u \in \mathcal{H}$.

then (by strict convexity)

$$F((1-t)u_0 + tu_1) < (1-t)F(u_0) + tF(u_1) = d,$$

for any $0 < t < 1$, a contradiction.

1.8 Generalized Dirichlet eigenvalue problem

To start with, we need a brief preparation of functional analysis (cf. e.g. [104]). Let $\mathcal{H}$ be a real Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ a continuous linear map. A number $\lambda \in \mathbb{R}$ is an eigenvalue of T if there is $u \in \mathcal{H} \setminus \{0\}$ such that $Tu = \lambda u$ (and then u is an eigenvector of T corresponding to the eigenvalue λ). Standard functional analysis methods apply to the study of eigenvalues and eigenvectors of selfadjoint completely continuous operators. T is selfadjoint if $(Tu, v)_{\mathcal{H}} = (u, Tv)_{\mathcal{H}}$ for any $u, v \in \mathcal{H}$. Also T is compact if it maps bounded sets in compact sets. A continuous compact operator is *completely continuous*. Completely continuous operators map weakly convergent sequences in strongly convergent sequences. By the norm of T one understands $\|T\| = \sup_{\|u\|_{\mathcal{H}}=1} \|Tu\|_{\mathcal{H}} = \sup_{\|u\|_{\mathcal{H}}=1} = |(Tu, u)_{\mathcal{H}}|$ (the last equality is a consequence of the fact that T is selfadjoint). A general result we rely on is that a selfadjoint completely continuous operator $T : \mathcal{H} \to \mathcal{H}$ has at least one eigenvalue and one eigenvector. Moreover, if $T : \mathcal{H} \to \mathcal{H}$ is selfadjoint then eigenvectors corresponding to distinct eigenvalues are orthogonal. Also if $T : \mathcal{H} \to \mathcal{H}$ is selfadjoint and completely continuous then to any eigenvalue $\lambda \in \mathbb{R} \setminus \{0\}$ there corresponds a finite number of linearly independent eigenvectors. Finally, the crucial results from functional analysis that we shall use in the sequel, may be stated as follows. Let $T : \mathcal{H} \to \mathcal{H}$ be selfadjoint and completely continuous. Then i) T admits at most an infinite sequence of eigenvalues and ii) the only accumulation point of the sequence of eigenvalues is 0. Also iii) if $\{u_\nu\}_{\nu \geq 1}$ is the orthonormal system consisting of the eigenvectors of T corresponding to the eigenvalues of T then for any $u \in \mathcal{H}$ one has $Tu = \sum_{\nu=1}^{\infty} (Tu, u_\nu)_{\mathcal{H}} u_\nu$.

Under the assumptions of Theorem 1.6, for each $f \in L^2(\Omega)$ there is a unique $u \in \mathring{W}_H^{1,2}(\Omega)$ such that $a_b(u, \varphi) = (f, \varphi)_{L^2}$ for any $\varphi \in \mathring{W}_H^{1,2}(\Omega)$. We may then consider the map $G_D : L^2(\Omega) \to L^2(\Omega)$ given by $G_D(f) = u$, hence the tautology $a_b(G_D f, \varphi) = (f, \varphi)_{L^2}$. We shall show that G_D is linear, continuous, self-adjoint and compact, so that the functional analysis result recalled above applies to its spectrum $\sigma(G_D)$. The usefulness of G_D is due to the relationship among the spectrae of G_D and the Dirichlet problem (1.36): if λ is an eigenvalue of (1.36) and u an eigenfunction corresponding to λ then $\mu = 1/\lambda \in \sigma(G_D)$ and $u \in \text{Eigen}(G_D; \mu)$, and conversely. For instance if $a_b(u, \varphi) = \lambda(u, \varphi)_{L^2}$ then $G_D(\lambda u) = u$ hence, once linearity of G_D is proved, $1/\lambda \in \sigma(G_D)$.

Lemma 1.10. i) *G_D is linear,* ii) *G_D is continuous,* iii) *G_D is self-adjoint, and* iv) *G_D is compact.*

Proof. i) By the very definition of G_D

$$a_b(G_D(\alpha f + \beta g), \varphi) = (\alpha f + \beta g, \varphi)_{L^2} =$$

$$= a_b(\alpha G_D(f) + \beta G_D(g), \varphi)_{L^2}, \quad \varphi \in \mathring{W}_H^{1,2}(\Omega),$$

for any $f, g \in L^2(\Omega)$, $\alpha, \beta \in \mathbb{R}$, hence $a_b(G_D(\alpha f + \beta g) - (\alpha G_D(f) + \beta G_D(g)), \varphi) = 0$. Since G_D is $\mathring{W}_H^{1,2}(\Omega)$-valued, one may use the previous identity for $\varphi \equiv G_D(\alpha f + \beta g) - (\alpha G_D(f) + \beta G_D(g))$ so that $E_b(\varphi) = 0$ yielding $\varphi = 0$.

ii) For each $f \in L^2(\Omega)$ (by the Poincaré inequality)

$$\|G_D f\|^2_{L^2} = \int_\Omega (G_D f)^2\, \Psi_\theta \le C \int_\Omega \|\nabla^H G_D f\|^2\, \Psi_\theta =$$

$$= C\, a_b(G_D f, G_D f) = C\,(f, G_D f)_{L^2} \le C\, \|f\|_{L^2} \|G_D f\|_{L^2}$$

hence

$$\|G_D f\|_{L^2} \le C\, \|f\|_{L^2} \tag{1.43}$$

i.e. G_D is bounded.

iii) If $G_D f = 0$ then $(f, \varphi)_{L^2} = a_b(G_D f, \varphi) = 0$ hence (as $\mathring{W}^{1,2}_H(\Omega)$ is dense in $L^2(\Omega)$) $f = 0$ i.e. G is injective. Let G_D^{-1} be the inverse of $G_D : L^2(\Omega) \to R(G_D)$. Then

$$(G_D^{-1} u, \varphi)_{L^2} = (u, G_D^{-1}\varphi)_{L^2}\,, \quad u, \varphi \in R(G_D). \tag{1.44}$$

Indeed

$$(G_D^{-1} u, \varphi)_{L^2} = a_b(G_D G_D^{-1} u, \varphi) = a_b(u, \varphi) = a_b(\varphi, u) =$$

$$= a_b(G_D G_D^{-1}\varphi, u) = (G_D^{-1}\varphi, u)_{L^2} = (u, G_D^{-1}\varphi)_{L^2}\,.$$

Finally for any $f, g \in L^2(\Omega)$ (by (1.44) with $u = G_D f$ and $\varphi = G_D g$)

$$(G_D f, g)_{L^2} = (G_D f, G_D^{-1} G_D g)_{L^2} = (G_D^{-1} G_D f, G_D g)_{L^2} = (f, G_D g)_{L^2}\,.$$

iv) Let $B \subset L^2(\Omega)$ be a bounded subset i.e. $\|f\|_{L^2} \le C_1$ for any $f \in B$ and some constant $C_1 > 0$. Then (by (1.43))

$$E_b(G_D f) = \frac{1}{2}\, a_b(G_D f, G_D f) = \frac{1}{2}\,(f, G_D f)_{L^2} \le$$

$$\le \frac{1}{2}\, \|f\|_{L^2} \|G_D f\|_{L^2} \le \frac{C}{2}\, \|f\|^2_{L^2} \le \frac{C C_1^2}{2}$$

so that $G_D(B)$ is a bounded subset of $\mathring{W}^{1,2}_H(\Omega)$. Finally, the inclusion $\mathring{W}^{1,2}_H(\Omega) \to L^2(\Omega)$ is compact, hence $G_D(B)$ is a compact subset of $L^2(\Omega)$. Q.e.d.

At this point we may prove Theorem 1.6. By Lemma 1.10 the map $G_D : L^2(\Omega) \to L^2(\Omega)$ admits at most an infinite sequence of eigenvalues and the only accumulation point of $\sigma(G_D)$ is 0. If $\sigma(G_D) = \{\mu_\nu : \nu \ge 1\}$ we set $\lambda_\nu = 1/\mu_\nu$. Let $\{u_\nu : \nu \ge 1\}$ be eigenfunctions of G_D corresponding to $\{\mu_\nu : \nu \ge 1\}$. We assume the elements of $\sigma(G_D)$ are labeled such that $|\mu_1| \ge |\mu_2| \ge \cdots \ge |\mu_\nu| \ge \cdots$ and the system $\{u_\nu : \nu \ge 1\}$ is orthonormal. By the comment preceding Lemma 1.10 the spectrum of the Dirichlet eigenvalue problem (1.36) is at most countable $|\lambda_1| \le |\lambda_2| \le \cdots \le |\lambda_\nu| \le \cdots$ and one may easily check that $\lambda_\nu > 0$ for any $\nu \ge 1$. Indeed $G_D u_\nu = \mu_\nu u_\nu$ yields

$$2\mu_\nu E_b(u_\nu) = \mu_\nu\, a_b(u_\nu\,,\, u_\nu) = a_b(G_D u_\nu\,,\, u_\nu) = \|u_\nu\|^2_{L^2}$$

hence $\mu_\nu > 0$. Once again one should observe that in the known case where Δ_b is replaced by an uniformly elliptic operator, positivity of eigenvalues follows from the coercivity of the associated quadratic form (while the failure of Δ_b to be elliptic is immaterial due to the fact that $E_b^{1/2}$ is already a norm).

To see that (1.36) admits an infinite sequence of eigenvalues, one starts by showing that the range $R(G_D)$ is infinite dimensional. Let $u \in C_0^\infty(\Omega)$ and let us set $f = \Delta_b u$ so that u is a strong solution to the Dirichlet problem (1.39). In particular $a_b(u, \varphi) = (f, \varphi)_{L^2}$ for any $\varphi \in \mathring{W}_H^{1,2}(\Omega)$. Thus $G_D f = u$, proving that $R(G_D) \supset C_0^\infty(\Omega)$. Yet $C_0^\infty(\Omega)$ is infinite dimensional, hence so does $R(G_D)$. Let us assume now that G_D has but a finite number of eigenvalues $\sigma(G_D) = \{\mu_1, \cdots, \mu_k\}$. Then $G_D f = \sum_{\nu=1}^k (G_D f, u_\nu)_{L^2} u_\nu$ for any $f \in L^2(\Omega)$, hence $\{u_1, \cdots, u_k\}$ is a linear basis of $R(G_D)$ i.e. $\dim_{\mathbb{R}} R(G_D) < \infty$, a contradiction.

Finally as $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_\nu \geq \cdots$ and $\mu_\nu > 0$ it follows that $\{\mu_\nu\}_{\nu\geq 1}$ is convergent to some $\mu \in \mathbb{R}$. Thus μ is an accumulation point of $\sigma(G_D)$ hence $\mu = 0$ and we may conclude that $\lim_{\nu\to\infty} \lambda_\nu = +\infty$. Q.e.d.

1.9 An energy space approach

Let (M, θ) be a strictly pseudoconvex pseudohermitian manifold and $\Omega \subset M$ a smoothly bounded domain. Let us consider the sublaplacian $\Delta_b \equiv (\nabla^H)^* \circ \nabla^H : \mathcal{D}(\Delta_b) = \{u \in \mathcal{D}(\nabla^H) : \nabla^H u \in \mathcal{D}((\nabla^H u)^*)\} \subset L^2(\Omega) \to L^2(\Omega)$ of (M, θ). Unlike previous sections we work with complex valued functions $u : \Omega \to \mathbb{C}$. In this section we consider the problem of the existence of solutions to

$$\Delta_b u = f, \quad f \in L^2(\Omega), \tag{1.45}$$

by making use of the Freidrichs extension of $\Delta_{b,0}$ where

$$\Delta_{b,0} = \Delta_b\big|_{C_0^\infty(\Omega)}$$

i.e. $\Delta_{b,0}$ is the Lagrange sublaplacian. Precisely we prove

Theorem 1.11. *Let $\Omega \subset M$ be a smoothly bounded domain satisfying the Poincaré inequality*

$$\int_\Omega |u|^2 \,\Psi_\theta \leq C \int_\Omega \|\nabla^H u\|^2 \,\Psi_\theta\,, \quad u \in C_0^\infty(\Omega).$$

Then for any $f \in L^2(\Omega)$ the Poisson equation for the sublaplacian (1.45) *admits a weak solution $u_f \in L^2(\Omega)$. The weak solution u_f is weakly differentiable along the Levi distribution $H(\Omega)$ and $u_f = 0$ on $\partial\Omega$ in the variational sense i.e. $u_f \in \mathring{W}_H^{1,2}(\Omega)$. In particular u_f is a solution to the generalized Dirichlet problem* (1.39)*. If $f \in C^\infty(\Omega)$ then $u_f \in C^\infty(\Omega)$.*

We start by noticing that for every $u \in C_0^\infty(\Omega)$

$$(\Delta_{b,0}\, u\,,\, u)_{L^2} = \|\nabla^H u\|_{L^2}^2 \geq C\|u\|_{L^2}^2 \tag{1.46}$$

by $\Delta_{b,0}\, u = (\nabla^H u)^* \,\nabla^H u$ and the Poincaré inequality. Hence the Lagrange sublaplacian is positive definite as an operator $\Delta_{b,0} : \mathcal{D}(\Delta_{b,0}) = C_0^\infty(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$. Therefore we may apply Friedrichs' extension theorem for positive definite linear operators, which we proceed to recall. Let $\mathcal{H}$ be a Hilbert space and $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ a linear operator $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$, assumed to be positive definite i.e.

$$(Au, u)_{\mathcal{H}} \geq \gamma^2 \|u\|_{\mathcal{H}}^2\,, \quad u \in \mathcal{D}(A),$$

with γ^2 = const. > 0. Friedrichs' theorem (cf. e.g. Theorem 1.32 in [15], p. 97) is that, under these assumptions, there is a linear operator $\tilde{A} : \mathcal{D}(\tilde{A}) \subset \mathcal{H} \to \mathcal{H}$ such that 1) $\tilde{A}$ is an extension of A i.e.

$$\mathcal{D}(A) \subset \mathcal{D}(\tilde{A}), \quad \tilde{A}\big|_{\mathcal{D}(A)} = A,$$

2) $\tilde{A}$ is selfadjoint[5] and surjective i.e. $R(\tilde{A}) = \mathcal{H}$. Here $R(\tilde{A})$ is the range of $\tilde{A}$. Finally 3) $\tilde{A}$ is positive definite with the same constant as A i.e.

$$\left(\tilde{A}u\,,\,u\right)_{\mathcal{H}} \geq \gamma^2\,\|u\|^2_{\mathcal{H}}\,, \quad u \in \mathcal{D}(\tilde{A}).$$

By Friedrichs' theorem (with $A = \Delta_{b,0}$) there is a linear operator

$$\tilde{\Delta}_{b,0} : \mathcal{D}(\tilde{\Delta}_{b,0}) \subset L^2(\Omega) \to L^2(\Omega)$$

such that 1) $\Delta_{b,0} \subset \tilde{\Delta}_{b,0}$ i.e.

$$\mathcal{D}(\Delta_{b,0}) \subset \mathcal{D}(\tilde{\Delta}_{b,0}), \quad \tilde{\Delta}_{b,0}\big|_{\mathcal{D}(\Delta_{b,0})} = \Delta_{b,0}\,,$$

and 2) $\tilde{\Delta}^*_{b,0} = \tilde{\Delta}_{b,0}$ and $R(\tilde{\Delta}_{b,0}) = L^2(\Omega)$, and 3) $\tilde{\Delta}_{b,0}$ is positive definite i.e.

$$\left(\tilde{\Delta}_{b,0}u\,,\,u\right)_{L^2} \geq C\,\|u\|^2_{L^2}\,, \quad u \in \mathcal{D}(\tilde{\Delta}_{b,0}).$$

A crucial point in the so called energy space approach is to consider on $\mathcal{D}(\Delta_{b,0})$, besides from the inner product $(\ ,\ )_{L^2}$ induced from $L^2(\Omega)$, a new inner product given by

$$(u, v)_{\mathcal{H}(\Delta_{b,0})} = (\Delta_{b,0}u\,,\,v)_{L^2}\,, \quad u, v \in \mathcal{D}(\Delta_{b,0}).$$

The properties of the operator $\Delta_{b,0}$ and of the L^2 inner product $(\ ,\ )_{L^2}$ allow one to show that $(\ ,\ )_{\mathcal{H}(\Delta_{b,0})}$ is indeed an inner product on $\mathcal{D}(\Delta_{b,0})$. For instance let us check that

$$(u, v)_{\mathcal{H}(\Delta_{b,0})} = \overline{(v, u)_{\mathcal{H}(\Delta_{b,0})}}\,, \quad u, v \in \mathcal{D}(\Delta_{b,0}),$$

$$(u, u)_{\mathcal{H}(\Delta_{b,0})} \geq 0, \quad (u, u)_{\mathcal{H}(\Delta_{b,0})} = 0 \Longrightarrow u = 0.$$

The operator $\Delta_{b,0}$ is symmetric[6] (as a consequence of (1.46) and Proposition[7] 1.12 in [15], p. 54). Hence

$$(u, v)_{\mathcal{H}(\Delta_{b,0})} = (\Delta_{b,0}u\,,\,v)_{L^2} = (u, \Delta_{b,0}v)_{L^2} = \overline{(\Delta_{b,0}v, u)_{L^2}} = \overline{(v, u)_{\mathcal{H}(\Delta_{b,0})}}$$

for any $u, v \in \mathcal{D}(\Delta_{b,0})$. Next for any $u \in \mathcal{D}(\Delta_{b,0})$

$$(u, u)_{\mathcal{H}(\Delta_{b,0})} = (\Delta_{b,0}u, u)_{L^2} \geq C\,\|u\|^2_{L^2} \geq 0, \tag{1.47}$$

$$(u, u)_{\mathcal{H}(\Delta_{b,0})} = 0 \Longrightarrow \|u\|_{L^2} = 0 \Longrightarrow u = 0.$$

[5]That is $\tilde{A}$ coincides with its adjoint ($\tilde{A}^* = \tilde{A}$).

[6]Let $\mathcal{H}$ be a Hilbert space and $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ a linear operator. A is *symmetric* if $\mathcal{D}(A)$ is a dense subspace of $\mathcal{H}$ and $A \subset A^*$. As well known (cf. e.g. [15], p. 54) this is equivalent to $\mathcal{D}(A)$ being a dense subspace together with $(Au, v)_{\mathcal{H}} = (u, Av)_{\mathcal{H}}$ for any $u, v \in \mathcal{D}(A)$.

[7]If $\mathcal{H}$ is a complex Hilbert space and $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ is a densely defined linear operator then A is symmetric if and only if $(Au, u)_{\mathcal{H}} \in \mathbb{R}$ for any $u \in \mathcal{H}$.

Let us consider the norm associated to the inner product $(\ ,\ )_{\mathcal{H}(\Delta_{b,0})}$ i.e.

$$\|u\|_{\mathcal{H}(\Delta_{b,0})} = \sqrt{(u,u)_{\mathcal{H}(\Delta_{b,0})}}\,, \quad u \in \mathcal{D}(\Delta_{b,0}).$$

As a consequence of (1.47)

$$\|u\|_{L^2} \le \frac{1}{\sqrt{C}}\,\|u\|_{\mathcal{H}(\Delta_{b,0})}\,, \quad u \in \mathcal{D}(\Delta_{b,0}). \tag{1.48}$$

In general $\mathcal{D}(\Delta_{b,0})$ is not complete with respect to the norm $\|\cdot\|_{\mathcal{H}(\Delta_{b,0})}$. The energy space $\mathcal{H}(\Delta_{b,0})$ of $\Delta_{b,0}$ is by definition the completion of $\mathcal{D}(\Delta_{b,0})$ with respect to $\|\cdot\|_{\mathcal{H}(\Delta_{b,0})}$. Then

Lemma 1.12. *The energy space $\mathcal{H}(\Delta_{b,0})$ admits a continuous linear injection into $L^2(\Omega)$. Precisely there is a continuous, injective, linear map $\varphi : \mathcal{H}(\Delta_{b,0}) \to L^2(\Omega)$ such that*

$$\varphi(u) = u, \quad u \in \mathcal{D}(\Delta_{b,0}), \tag{1.49}$$

$$\|\varphi(u)\|_{L^2} \le \frac{1}{\sqrt{C}}\,\|u\|_{\mathcal{H}(\Delta_{b,0})}\,, \quad u \in \mathcal{H}(\Delta_{b,0}). \tag{1.50}$$

This is again a general result (holding for any positive definite linear operator $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$, cf. e.g. Theorem 1.33 in [15], p. 98) and we only indicate the construction of φ. Let $u \in \mathcal{H}(\Delta_{b,0})$. As $\mathcal{H}(\Delta_{b,0})$ is the completion of $\mathcal{D}(\Delta_{b,0})$ in the norm $\|\cdot\|_{\mathcal{H}(\Delta_{b,0})}$, there is a Cauchy sequence $\{u_\nu\}_{\nu\ge 1} \subset \mathcal{D}(\Delta_{b,0})$ representing u i.e. for any $\epsilon > 0$ there is $\nu_\epsilon \ge 1$ such that

$$\|u_\nu - u_\mu\|_{\mathcal{H}(\Delta_{b,0})} < \epsilon, \quad \forall\ \nu,\mu \ge \nu_\epsilon.$$

Then (by (1.48))

$$\|u_\nu - u_\mu\|_{L^2} < \frac{\epsilon}{\sqrt{C}}\,, \quad \forall\ \nu,\mu \ge \nu_\epsilon,$$

i.e. $\{u_\nu\}_{\nu\ge 1}$ is a Cauchy sequence in $L^2(\Omega)$ as well. Yet $L^2(\Omega)$ is complete hence there is $u_0 \in L^2(\Omega)$ such that $u_\nu \to u_0$ in $L^2(\Omega)$ as $\nu \to \infty$. One then sets by definition $\varphi(u) = u_0$. It may be easily checked (cf. e.g. [15], p. 99) that $\varphi(u)$ is well defined (i.e. the definition doesn't depend upon the choice of the representative $\{u_\nu\}_{\nu\ge 1}$ of u), linear, injective and continuous. Moreover, an inspection of the proof of Friedrichs' theorem (cf. e.g. [15], p. 102) shows that the domain of the yielded selfadjoint extension $\tilde{\Delta}_{b,0}$ is

$$\mathcal{D}(\tilde{\Delta}_{b,0}) = \mathcal{D}(\Delta^*_{b,0}) \cap \varphi(\mathcal{H}(\Delta_{b,0})) \subset L^2(\Omega) \tag{1.51}$$

while $\tilde{\Delta}_{b,0}$ itself is given by

$$\tilde{\Delta}_{b,0} = \Delta^*_{b,0}\Big|_{\mathcal{D}(\Delta^*_{b,0})\cap\varphi(\mathcal{H}(\Delta_{b,0}))}\,. \tag{1.52}$$

Since $R(\tilde{\Delta}_{b,0}) = L^2(\Omega)$ there is $u_f \in \mathcal{D}(\tilde{\Delta}_{b,0})$ such that $\tilde{\Delta}_{b,0}u_f = f$. As a positive definite operator $\tilde{\Delta}_{b,0}$ is already injective, so such $u_f \in \mathcal{D}(\tilde{\Delta}_{b,0})$ is unique. Finally (by (1.51)-(1.52))

$$f = \tilde{\Delta}_{b,0}u_f = \Delta^*_{b,0}u_f$$

so that for any $\psi \in C^\infty_0(\Omega)$

$$\left(\Delta_{b,0}\psi\,,\ u_f\right)_{L^2} = \left(\psi\,,\ \Delta^*_{b,0}u_f\right)_{L^2} = (\psi, f)_{L^2}$$

i.e. u_f is a weak solution to $\Delta_b u = f$. Let us look at the boundary conditions satisfied by u_f. Since $u_f \in \mathcal{D}(\tilde{\Delta}_{b,0})$ it follows that $u_f \in \mathcal{H}(\Delta_{b,0})$, the energy space of the Lagrange sublaplacian $\Delta_{b,0}$. Hence for any $u, v \in C_0^\infty(\Omega)$

$$(u,v)_{\mathcal{H}(\Delta_{b,0})} = (\Delta_{b,0}u\,,\,v)_{L^2} = \int_\Omega (\Delta_b u) v\,\Psi_\theta =$$

$$= -\int_\Omega \operatorname{div}(\nabla^H u)\,v\,\Psi_\theta = \int_\Omega G_\theta(\nabla^h u\,,\,\nabla^H v)\,\Psi_\theta$$

by Green's lemma. So

$$(u,v)_{\mathcal{H}(\Delta_{b,0})} = \int_\Omega G_\theta(\nabla^H u\,,\,\nabla^H v)\,\Psi_\theta\,, \quad u,v \in C_0^\infty(\Omega), \tag{1.53}$$

and the norm associated to the inner product (1.53) on $C_0^\infty(\Omega)$ is

$$\|u\|^2_{\mathcal{H}(\Delta_{b,0})} = \int_\Omega \|\nabla^H u\|^2\,\Psi_\theta = 2E_b(u). \tag{1.54}$$

The norm $\|\cdot\|_{\mathcal{H}(\Delta_{b,0})}$ and the norm induced by $\|\cdot\|_{W_H^{1,2}}$ on $C_0^\infty(\Omega)$ are equivalent. This is actually a consequence of the Poincaré inequality

$$\int_\Omega |u|^2\,\Psi_\theta \le C\int_\Omega \|\nabla^H u\|^2\,\Psi_\theta\,, \quad u \in C_0^\infty(\Omega), \tag{1.55}$$

as follows. First for every $u \in C_0^\infty(\Omega)$

$$\|u\|^2_{W_H^{1,2}} = \|u\|^2_{L^2} + \|\nabla^H u\|^2_{L^2} \ge \|\nabla^H u\|^2_{L^2} = \|u\|^2_{\mathcal{H}(\Delta_{b,0})}\,,$$

$$\|u\|^2_{W_H^{1,2}} = \|u\|^2_{L^2} + \|\nabla^H u\|^2_{L^2} \le (C+1)\|\nabla^H u\|^2_{L^2} = (C+1)\|u\|^2_{\mathcal{H}(\Delta_{b,0})}\,,$$

so that

$$\|u\|_{\mathcal{H}(\Delta_{b,0})} \le \|u\|_{W_H^{1,2}} \le (1+C)^{1/2}\,\|u\|_{\mathcal{H}(\Delta_{b,0})}\,, \quad u \in C_0^\infty(\Omega). \tag{1.56}$$

Since the norms $\|\cdot\|_{\mathcal{H}(\Delta_{b,0})}$ and $\|\cdot\|_{W_H^{1,2}}$ are equivalent on $C_0^\infty(\Omega)$, the spaces $\mathcal{H}(\Delta_{b,0})$ and $\mathring{W}_H^{1,2}(\Omega)$ may be identified, algebraically and topologically. Under this identification φ is the natural imbedding of $\mathring{W}_H^{1,2}(\Omega)$ so that

$$u_f \in \mathcal{D}(\Delta^*_{b,0}) \cap \mathring{W}_H^{1,2}(\Omega).$$

Consequently u_f is weakly differentiable along $H(\Omega)$ and $u_f = 0$ on $\partial\Omega$ in the sense of variational calculus.

1.10 Bochner-Lichnerowicz formula after A. Greenleaf

Let M be a compact strictly pseudoconvex CR manifold, of CR dimension n. Let $\theta \in \mathcal{P}_+$ and let ∇ be the Tanaka-Webster connection of (M,θ). Let $x_0 \in M$ be an arbitrary point. As $H(M)$ and g_θ are parallel with respect to ∇ we may build a local g_θ-orthonormal frame $\{E_a : 1 \le a \le 2n\}$ of $H(M)$, defined on an open neighborhood $U \subset M$ of x_0, such that

$$(\nabla_{E_a} E_b)\,(x_0) = 0, \quad 1 \le a,b \le 2n. \tag{1.57}$$

Indeed E_a is got by parallel displacement (with respect to ∇) of a given g_{θ,x_0}- orthonormal frame $\{v_1, \cdots, v_{2n}\} \subset H(M)_{x_0}$, along the geodesics of ∇ issuing at x_0. As $\nabla J = 0$ we may also assume that $E_{n+\alpha} = JE_\alpha$ for any $1 \le \alpha \le n$. Then (by $\Delta_b u = -\sum_{a=1}^{2n} \left\{E_a^2(u) - (\nabla_{E_a} E_a)(u)\right\}$ and $\nabla g_\theta = 0$)

$$\Delta_b \left(\|\nabla^H u\|^2\right)(x_0) = -\sum_a E_a^2 \left(\|\nabla^H u\|^2\right)(x_0) =$$

$$= -2 \sum_a E_a \left(g_\theta(\nabla_{E_a} \nabla^H u\,,\, \nabla^H u)\right)_{x_0} =$$

$$= -2 \sum_a \left\{g_\theta \left(\nabla_{E_a} \nabla_{E_a} \nabla^H u\,,\, \nabla^H u\right) + g_\theta \left(\nabla_{E_a} \nabla^H u\,,\, \nabla_{E_a} \nabla^H u\right)\right\}_{x_0} .$$

As $\{E_a : 1 \le a \le 2n\}$ is g_θ-orthonormal, the first term in the above sum is

$$\sum_{a,b} g_\theta \left(\nabla_{E_a} \nabla_{E_a} \nabla^H u\,,\, E_b\right) E_b(u).$$

Moreover (by (1.57))

$$g_\theta \left(\nabla_{E_a} \nabla_{E_a} \nabla^H u\,,\, X_b\right)_{x_0} = \left\{E_a \left(g_\theta(\nabla_{E_a} \nabla^H u\,,\, E_b)\right) - g_\theta \left(\nabla_{E_a} \nabla^H u\,,\, \nabla_{E_a} E_b\right)\right\}_{x_0} =$$

$$= E_a \left(E_a \left(g_\theta(\nabla^H u\,,\, E_b)\right) - g_\theta(\nabla^H u\,,\, \nabla_{E_a} E_b)\right)_{x_0} =$$

$$= E_a \left(E_a E_b u - (\nabla_{E_a} E_b)(u)\right)_{x_0} = E_a \left((\nabla^2 u)(E_a\,,\, E_b)\right)_{x_0}$$

where $\nabla^2 u$ is the Hessian of u with respect to the Tanaka-Webster connection i.e.

$$(\nabla^2 u)(X, Y) = (\nabla_X du)Y = X(Y(u)) - (\nabla_X Y)(u), \quad X, Y \in \mathfrak{X}(M).$$

We emphasize that, unlike the Hessian in Riemannian geometry, $\nabla^2 u$ is never symmetric

$$(\nabla^2 u)(X, Y) = (\nabla^2 u)(Y, x) - T_\nabla(X, Y)(u). \tag{1.58}$$

On the other hand T_∇ is pure hence

$$T_\nabla(X, Y) = -2\,\Omega(X, Y)T, \quad X, Y \in H(M). \tag{1.59}$$

Here $\Omega = -d\theta$. Then (by (1.58)-(1.59))

$$g_\theta(\nabla_{E_a} \nabla_{E_a} \nabla^H u\,,\, X_b)_{x_0} = X_a \left((\nabla^2 u)(E_a, E_b)\right)_{x_0} =$$

$$= E_a \left((\nabla^2 u)(E_b, E_a) + 2\Omega(E_a, E_b)Tu\right)_{x_0} =$$

$$= g_\theta(\nabla_{E_a} \nabla_{E_b} \nabla^H u\,,\, E_a)_{x_0} + 2\Omega(E_a, E_b)_{x_0} E_a(Tu)_{x_0}$$

so that

$$-\frac{1}{2}\, \Delta_b \left(\|\nabla^H u\|^2\right)(x_0) = \sum_a \|\nabla_{E_a} \nabla^H u\|^2_{x_0} + \sum_{a,b} \left\{g_\theta(\nabla_{E_a} \nabla_{E_b} \nabla^H u\,,\, E_a) + \right. \tag{1.60}$$

$$\left. + 2\Omega(E_a, E_b) E_a(Tu)\right\}_{x_0} E_b(u)_{x_0} .$$

For each bilinear form B on $T(M)$ we indicate as customary with $\Pi_H B$ the restriction of B to $H(M) \otimes H(M)$. The norm of $\Pi_H B$ is given by $\|\Pi_H B\|^2 = \sum_{a,b} B(E_a, E_b)^2$. Then

$$\|\Pi_H \nabla^2 u\|^2 = \sum_{a,b} (\nabla^2 u)(E_a, E_b)^2 = \sum_{a,b} [E_a(E_b(u)) - (\nabla_{E_a} E_b)(u)]^2 =$$

$$= \sum_{a,b} g_\theta(\nabla_{E_a} \nabla^H u\,,\, E_b)^2 = \sum_a g_\theta(\nabla_{E_a} \nabla^H u\,,\, \nabla_{E_a} \nabla^H u)$$

so that

$$\|\Pi_H \nabla^2 u\|^2 = \sum_a \|\nabla_{E_a} \nabla^H u\|^2. \tag{1.61}$$

Next

$$[E_a, E_b] = \nabla_{E_a} E_b - \nabla_{E_b} E_a - T_\nabla(E_a, E_b)$$

hence (by (1.57) and (1.59))

$$[E_a, E_b]_{x_0} = 2\Omega(E_a, E_b)_{x_0} T_{x_0}$$

and taking into account

$$\nabla_X \nabla_Y = \nabla_Y \nabla_X + R^\nabla(X, Y) + \nabla_{[X,Y]}\,, \quad X, Y \in \mathfrak{X}(M),$$

where R^∇ is the curvature tensor field of ∇) one obtains

$$\nabla_{E_a} \nabla_{E_b} \nabla^H u = \nabla_{E_b} \nabla_{E_a} \nabla^H u + R^\nabla(E_a, E_b) \nabla^H u + 2\Omega(E_a, E_b) \nabla_T \nabla^H u \tag{1.62}$$

at x_0. Moreover

$$g_\theta(\nabla_{E_b} \nabla_{E_a} \nabla^H u, X_a)_{x_0} = \left\{ E_b \left(g_\theta(\nabla_{E_a} \nabla^H u\,,\, E_a) \right) - g_\theta(\nabla_{E_a} \nabla^H u, \nabla_{E_b} E_a) \right\}_{x_0} =$$

$$= E_b \left(E_a^2(u) - (\nabla_{E_a} E_a)(u) \right)_{x_0}$$

that is

$$\sum_a g_\theta(\nabla_{E_b} \nabla E_a \nabla^H u\,,\, E_a)_{x_0} = -E_b\,(\Delta_b u)_{x_0}\ . \tag{1.63}$$

Therefore (by (1.62)-(1.63))

$$\sum_{a,b} g_\theta(\nabla_{E_a} \nabla_{E_b} \nabla^H u\,,\, E_a)_{x_0} E_b(f)_{x_0} =$$

$$= -\sum_c \{E_c(\Delta_b u)\, E_c(u)\}_{x_0} + \sum_{a,c} \Big\{ g_\theta(R^\nabla(E_a, E_c) \nabla^H u\,,\, E_a) E_c(u) +$$

$$+ 2\Omega(E_a, E_c)\, g_\theta(\nabla_T \nabla^H u\,,\, E_a) E_c(u) \Big\}_{x_0} =$$

$$= -(\nabla^H u)(\Delta_b u)_{x_0} + \sum_a \Big\{ g_\theta(R^\nabla(E_a, \nabla^H u) \nabla^H u\,,\, E_a) +$$

$$+ 2 g_\theta(E_a, J \nabla^H u) g_\theta(\nabla_T \nabla^H u, E_a) \Big\}_{x_0} =$$

$$= -(\nabla^H u)(\Delta_b u)_{x_0} + \mathrm{Ric}_\nabla(\nabla^H u\,,\, \nabla^H u)_{x_0} + 2 g_\theta(\nabla_T \nabla^H u\,,\, J \nabla^H u)_{x_0}$$

where $\mathrm{Ric}_\nabla(X,Y) = \mathrm{trace}\{Z \mapsto R^\nabla(Z,Y)X\}$ as customary. Then (by (1.61)) the identity (1.60) becomes

$$-\frac{1}{2}\Delta_b\left(\|\nabla^H u\|^2\right) = \|\Pi_H \nabla^H u\|^2 - (\nabla^H u)(\Delta_b u) + \mathrm{Ric}_\nabla(\nabla^H u, \nabla^H u) +$$

$$+2g_\theta(\nabla_T \nabla^H u, J\nabla^H u) + 2g_\theta(\nabla^H T u, J\nabla^H u)$$

yielding the following pseudohermitian version of *Bochner-Lichnerowicz formula*

$$-\frac{1}{2}\,\Delta_b\left(\left\|\nabla^H u\right\|^2\right) = \left\|\Pi_H \nabla^2 u\right\|^2 - (\nabla^H u)\,(\Delta_b u) + \tag{1.64}$$

$$+\mathrm{Ric}_\nabla\left(\nabla^h u\,,\,\nabla^H u\right) + 2\,Lu,$$

for any $u \in C^\infty(M,\mathbb{R})$. Here the differential operator L is given by

$$Lu \equiv \left(J\nabla^H u\right)(Tu) - (J\nabla_T \nabla^H u)(u) \tag{1.65}$$

and its presence in (1.64) is of course the main bias from the Riemannian case. Formula (1.64) was derived by E. Barletta (cf. equations (6)-(7) in [32], p. 79). However only the formalism is new (the local calculation in [9] is replaced by a local frame free $\nabla_X Y$ calculation) and (1.64) is qualitatively that obtained by A. Greenleaf, [9]. Indeed let $\{T_\alpha : 1 \le \alpha \le n\}$ be a local G_θ-orthonormal (i.e. $G_\theta(T_\alpha, T_{\overline{\beta}}) = \delta_{\alpha\beta}$) frame of the CR structure $T_{1,0}(M)$. Then

$$\nabla^H u = \sum_{\alpha=1}^n \left(u_{\overline{\alpha}} T_\alpha + u_\alpha T_{\overline{\alpha}}\right), \quad u_\alpha \equiv T_\alpha(u), \quad u \in C^1(M,\mathbb{R}).$$

Let us compute the terms in (1.64) with respect to the local frame $\{T_\alpha : 1 \le \alpha \le n\}$. Using (4.32)-(4.33) in Chapter 4 of this thesis one obtains

$$\mathrm{Ric}_\nabla\left(\nabla^H u\,,\,\nabla^H u\right) = \sum_{\alpha,\beta=1}^n \left\{2R_{\alpha\overline{\beta}} u_{\overline{\alpha}} u_\beta + \right. \tag{1.66}$$

$$\left. + \,i(n-1)\left(A_{\alpha\beta} u_{\overline{\alpha}} u_{\overline{\beta}} - A_{\overline{\alpha}\overline{\beta}} u_\alpha u_\beta\right)\right\},$$

$$\left\|\nabla^H u\right\|^2 = 2\left\|\nabla^{1,0} u\right\|^2 = 2\sum_{\alpha=1}^n u_\alpha u_{\overline{\alpha}}\,, \tag{1.67}$$

$$\left(\nabla^H u\right)(\Delta_b u) = \sum_{\alpha=1}^n \left\{u_{\overline{\alpha}}\,(\Delta_b u)_\alpha + u_\alpha\,(\Delta_b u)_{\overline{\alpha}}\right\}, \tag{1.68}$$

for any $u \in C^3(M,\mathbb{R})$. The calculation of $\left\|\Pi_H \nabla^2 u\right\|^2$ is more involved. We start by setting

$$E_\alpha = \frac{1}{\sqrt{2}}\,(T_\alpha + T_{\overline{\alpha}})\,, \quad E_{n+\alpha} = JE_\alpha = \frac{i}{\sqrt{2}}\,(T_\alpha - T_{\overline{\alpha}})\,,$$

so that $G_\theta(E_a\,,\,E_b) = \delta_{ab}$ for any $1 \le a, b \le 2n$. Then

$$\left\|\Pi_H \nabla^2 u\right\|^2 = \sum_{a,b=1}^{2n} \left(\nabla^2 u\right)(E_a\,,\,E_b)^2 = \tag{1.69}$$

$$= 2\sum_{\alpha,\beta=1}^{n}\left\{\left(\nabla_\alpha u_\beta\right)\left(\nabla_{\overline{\alpha}} u_{\overline{\beta}}\right) + \left(\nabla_\alpha u_{\overline{\beta}}\right)\left(\nabla_{\overline{\alpha}} u_\beta\right)\right\}.$$

Finally

$$J\nabla^H u = i\sum_{\alpha=1}^{n}\left(u_{\overline{\alpha}} T_\alpha - u_\alpha T_{\overline{\alpha}}\right),$$

$$\left(J\nabla^H u\right)(u_0) = i\sum_{\alpha=1}^{n}\left\{u_{\overline{\alpha}} T_\alpha(u_0) - u_\alpha T_{\overline{\alpha}}(u_0)\right\}, \quad u_0 \equiv T(u),$$

$$T_A(u_0) = T_A(T(u)) - (\nabla_{T_A} T)(u) = (\nabla_{T_A} du)T = \nabla_A u_0\,, \quad A \in \{1, \cdots, n, \overline{1}, \cdots, \overline{n}\},$$

$$J\nabla_T \nabla^H u = i\sum_{\alpha=1}^{n}\left\{T(u_{\overline{\alpha}}) T_\alpha + u_{\overline{\alpha}} \nabla_T T_\alpha - T(u_\alpha) T_{\overline{\alpha}} - u_\alpha \nabla_T T_{\overline{\alpha}}\right\},$$

$$\left(J\nabla_T\nabla^H u\right)(u) = i\sum_{\alpha=1}^{n}\left\{T(u_{\overline{\alpha}}) u_\alpha + u_{\overline{\alpha}} \Gamma^{\beta}_{0\alpha} u_\beta - T(u_\alpha) u_{\overline{\alpha}} - u_\alpha \Gamma^{\overline{\beta}}_{0\overline{\alpha}} u_{\overline{\beta}}\right\} =$$

$$= -i\sum_{\alpha}\left\{u_{\overline{\alpha}}\left[T(u_\alpha) - \Gamma^{\beta}_{0\alpha} u_\beta\right] - u_\alpha\left[T(u_{\overline{\alpha}}) - \Gamma^{\overline{\beta}}_{0\overline{\alpha}}\right]\right\} =$$

$$= -i\sum_{\alpha}\left\{u_{\overline{\alpha}} \nabla_0 u_\alpha - u_\alpha \nabla_0 u_{\overline{\alpha}}\right\},$$

hence (by (1.65) and the commutation formula $\nabla_0 u_\beta = \nabla_\beta u_0 - u_{\overline{\alpha}} A^{\overline{\alpha}}_{\beta}$)

$$Lu = 2i\sum_{\alpha=1}^{n}\left(u_{\overline{\alpha}}\, \nabla_0 u_\alpha - u_\alpha\, \nabla_0 u_{\overline{\alpha}}\right) + \tag{1.70}$$

$$+i\sum_{\alpha,\beta=1}^{n}\left(A_{\alpha\beta} u_{\overline{\alpha}} u_{\overline{\beta}} - A_{\overline{\alpha}\overline{\beta}} u_\alpha u_\beta\right).$$

Substitution from (1.66)-(1.70) into (1.64) leads to

$$-\Delta_b\left(\left\|\nabla^H u\right\|^2\right) = 2\sum_{\alpha,\beta=1}^{n}\left\{(\nabla_\alpha u_\beta)(\nabla_{\overline{\alpha}} u_{\overline{\beta}}) + (\nabla_\alpha u_{\overline{\beta}})(\nabla_{\overline{\alpha}} u_\beta)\right\} + \tag{1.71}$$

$$+4i\sum_{\alpha=1}^{n}\left\{u_{\overline{\alpha}}\, \nabla_0 u_\alpha - u_\alpha\, \nabla_0 u_{\overline{\alpha}}\right\} +$$

$$+2\sum_{\alpha,\beta=1}^{n} R_{\alpha\overline{\beta}} u_{\overline{\alpha}} u_\beta - \sum_{\alpha=1}^{n}\left\{u_{\overline{\alpha}}\,(\Delta_b u)_\alpha + u_\alpha\,(\Delta_b u)_{\overline{\alpha}}\right\} +$$

$$+i(n+1)\sum_{\alpha,\beta=1}^{n}\left\{A_{\overline{\alpha}\overline{\beta}} u_\alpha u_\beta - A_{\alpha\beta} u_{\overline{\alpha}} u_{\overline{\beta}}\right\}$$

which is the pseudohermitian analog to Bochner-Lichnerowicz formula as got by A. Greenleaf (cf. [9]) except for the coefficient[8] $n + 1$ in the last row of (1.71).

[8] Said coefficient is $2n$ in [9] and the difference is perhaps due to distinct exterior calculus conventions.

Our next purpose, in this section, is to derive an alternative version of Greenleaf's formula (1.64) or (1.71) written in terms of the so called *CR Paneitz operator* as introduced by S-C. Chang & H-L. Chiu, [92] (and used by us in Chapter 4 of this thesis). S-C. Chang & H-L. Chiu's operator P is locally[9] given by

$$P_\alpha f \equiv f_\alpha{}^{\overline{\beta}}{}_{\overline{\beta}} + 2niA_{\alpha\beta}f^\beta\,,$$

$$f^\alpha = g^{\alpha\overline{\beta}} f_{\overline{\beta}}\,, \quad f_{\overline{\alpha}} = T_{\overline{\alpha}}(f), \quad f \in C^1(M, \mathbb{C}).$$

Through the reminder of this section we work with a local G_θ-orthonormal frame $\{T_\alpha : 1 \le \alpha \le n\} \subset C^\infty(U, T_{1,0}(M))$. Hence the operator P_α, as well as the two commutation formulae we shall use, may be written

$$P_\alpha u = \sum_{\beta=1}^{n} \left\{ u_{\alpha\beta\overline{\beta}} + 2niA_{\alpha\beta}u_{\overline{\beta}} \right\}, \tag{1.72}$$

$$\nabla_0 u_\beta = \nabla_\beta u_0 - \sum_{\alpha=1}^{n} A_{\alpha\beta}u_{\overline{\alpha}}\,, \tag{1.73}$$

$$\nabla_\alpha u_{\overline{\beta}} = \nabla_{\overline{\beta}} u_\alpha - 2i\delta_{\alpha\beta}u_0\,, \tag{1.74}$$

for any $u \in C^1(M, \mathbb{R})$. We shall also need a commutation formulae for third order covariant derivatives, that we proceed to derive. One has

$$u_{\alpha\overline{\beta}\gamma} = (\nabla^3 u)\left(T_\alpha\,, T_{\overline{\beta}}\,, T_\gamma\right) = \left(\nabla_{T_\alpha}\nabla^2 u\right)\left(T_{\overline{\beta}}\,, T_\gamma\right) =$$

$$= T_\alpha\left((\nabla^2 u)(T_{\overline{\beta}}\,, T_\gamma)\right) - (\nabla^2 u)\left(\nabla_{T_\alpha}T_{\overline{\beta}}\,, T_\gamma\right) - (\nabla^2 u)\left(T_{\overline{\beta}}\,, \nabla_{T_\alpha}T_\gamma\right) =$$

$$= T_\alpha\left(\nabla_{\overline{\beta}} u_\gamma\right) - \Gamma^{\overline{\mu}}_{\alpha\overline{\beta}}\nabla_{\overline{\mu}}u_\gamma - \Gamma^{\mu}_{\alpha\gamma}\nabla_{\overline{\beta}}u_\mu =$$

(by using (1.74) three times)

$$= T_\alpha\left(\nabla_\gamma u_{\overline{\beta}} + 2i\delta_{\beta\gamma}u_0\right) - \Gamma^{\overline{\mu}}_{\alpha\overline{\beta}}\left(\nabla_\gamma u_{\overline{\mu}} + 2i\delta_{\mu\gamma}u_0\right) - \Gamma^{\mu}_{\alpha\gamma}\left(\nabla_\mu u_{\overline{\beta}} + 2i\delta_{\beta\mu}u_0\right) =$$

$$= T_\alpha\left((\nabla^2 u)(T_\gamma\,, T_{\overline{\beta}})\right) - (\nabla^2 u)\left(\nabla_{T_\alpha}T_\gamma\,, T_{\overline{\beta}}\right) - (\nabla^2 u)\left(T_\gamma\,, \nabla_{T_\alpha}T_{\overline{\beta}}\right) +$$

$$+2i\left\{\delta_{\beta\gamma}T_\alpha(u_0) - \Gamma^{\beta}_{\alpha\gamma}u_0 - \Gamma^{\overline{\gamma}}_{\alpha\overline{\beta}}u_0\right\} =$$

$$= \left(\nabla_{T_\alpha}\nabla^2 u\right)(T_\gamma\,, T_{\overline{\beta}}) + 2i\delta_{\beta\gamma}T_\alpha(u_0)$$

because of

$$\Gamma^{\overline{\gamma}}_{\alpha\overline{\beta}} = -\Gamma^{\beta}_{\alpha\gamma} \tag{1.75}$$

as a peculiarity of the fact that we make use of orthonormal frames of $T_{1,0}(M)$. Indeed $\nabla g_\theta = 0$ may be written locally

$$T_\alpha\left(g_{\beta\overline{\gamma}}\right) = \Gamma^{\mu}_{\alpha\beta}g_{\mu\overline{\gamma}} + g_{\beta\overline{\mu}}\Gamma^{\overline{\mu}}_{\alpha\overline{\gamma}}$$

an identity which for $g_{\alpha\overline{\beta}} = \delta_{\alpha\beta}$ is easily seen to yield (1.75). Summing up we have proved

$$u_{\alpha\overline{\beta}\gamma} = u_{\alpha\gamma\overline{\beta}} + 2i\delta_{\beta\gamma}\nabla_\alpha u_0 \tag{1.76}$$

[9]The global expression of the operator P is given in Chapter 4.

because of $\nabla T = 0$ (implying that $T_\alpha(u_0) = \nabla_\alpha u_0$). Let us contract β and γ in (1.76) so that to derive

$$\sum_{\beta=1}^{n}\left(u_{\alpha\bar\beta\beta} - u_{\alpha\beta\bar\beta}\right) = 2ni\,\nabla_\alpha u_0\,. \tag{1.77}$$

The next step is to compute $4i\sum_{\alpha=1}^{n}\{u_{\overline{\alpha}}\,\nabla_0 u_\alpha - u_\alpha\,\nabla_0 u_{\overline{\alpha}}\}$ in terms of third order covariant derivatives (by making use of (1.77)). This brings into picture the operator P, as claimed. Indeed (by (1.73) and (1.77))

$$2i\sum_\alpha u_{\overline{\alpha}}\nabla_0 u_\alpha = 2i\sum_\alpha u_{\overline{\alpha}}\left(\nabla_\alpha u_0 - \sum_\beta A_{\alpha\beta}u_{\overline{\beta}}\right) =$$

$$= \frac{1}{n}\sum_{\alpha,\beta} u_{\overline{\alpha}}\left(u_{\alpha\bar\beta\beta} - u_{\alpha\beta\bar\beta}\right) - 2i\sum_{\alpha,\beta} A_{\alpha\beta}u_{\overline{\alpha}}u_{\overline{\beta}} =$$

(by (1.72))

$$= \frac{1}{n}\sum_\alpha u_{\overline{\alpha}}\left[\sum_\beta u_{\alpha\bar\beta\beta} - P_\alpha u + 2ni\sum_\beta A_{\alpha\beta}u_{\overline{\beta}}\right] - 2i\sum_{\alpha,\beta} A_{\alpha\beta}u_{\overline{\alpha}}u_{\overline{\beta}} =$$

$$= -\frac{1}{n}\sum_\alpha u_{\overline{\alpha}}\,P_\alpha u + \frac{1}{n}\sum_{\alpha,\beta} u_{\overline{\alpha}}u_{\alpha\bar\beta\beta}\,.$$

At this point we may add the complex conjugate so that to obtain

$$4i\sum_{\alpha=1}^{n}\{u_{\overline{\alpha}}\,\nabla_0 u_\alpha - u_\alpha\,\nabla_0 u_{\overline{\alpha}}\} = \tag{1.78}$$

$$= -\frac{2}{n}\sum_{\alpha=1}^{n}\{u_{\overline{\alpha}}\,P_\alpha u + u_\alpha\,P_{\overline{\alpha}}u\} + \frac{2}{n}\sum_{\alpha,\beta=1}^{n}\left\{u_{\overline{\alpha}}u_{\alpha\bar\beta\beta} + u_\alpha u_{\bar\alpha\beta\bar\beta}\right\}.$$

To deal with the third order covariant derivatives in (1.78) we shall compute $G_\theta\left(\nabla^H u\,,\,\nabla^H(\Delta_b u)\right)$. To this end we need the following local formula for the sublaplacian

$$\Delta_b u = -\sum_{\alpha=1}^{n}(\nabla_\alpha u_{\overline{\alpha}} + \nabla_{\overline{\alpha}}u_\alpha)\,. \tag{1.79}$$

Formula (1.79) is an easy consequence of definitions. Indeed

$$\Delta_b u = -\mathrm{trace}\left(\nabla^H u\right) = -\mathrm{trace}\left\{Z \longmapsto \nabla_Z\nabla^H u\right\} =$$

$$= -\mathrm{trace}\left\{\sum_{\alpha,\beta}\left[Z(u_{\overline{\alpha}}) + u_{\overline{\beta}}\omega_\beta^\alpha(Z)\right]T_\alpha + \sum_{\alpha,\beta}\left[Z(u_\alpha) + u_\beta\omega_{\overline{\beta}}^{\overline{\alpha}}(Z)\right]T_{\overline{\alpha}}\right\} =$$

$$= -\mathrm{trace}\begin{pmatrix} T_\gamma(u_{\overline{\alpha}}) + \sum_\beta u_{\overline{\beta}}\Gamma^\alpha_{\gamma\beta} & T_\gamma(u_\alpha) + \sum_\beta u_\beta\Gamma^{\overline{\alpha}}_{\gamma\overline{\beta}} \\ T_{\overline{\gamma}}(u_{\overline{\alpha}}) + \sum_\beta u_{\overline{\beta}}\Gamma^\alpha_{\overline{\gamma}\beta} & T_{\overline{\gamma}}(u_\alpha) + \sum_\beta u_\beta\Gamma^{\overline{\alpha}}_{\overline{\gamma}\overline{\beta}}\end{pmatrix}_{1\le\gamma,\alpha\le n} =$$

(by contracting the indices α and γ)

$$= -\sum_{\alpha,\beta}\left\{T_\alpha(u_{\overline{\alpha}}) + \Gamma^\alpha_{\alpha\beta}u_{\overline{\beta}} + T_{\overline{\alpha}}(u_\alpha) + \Gamma^{\overline{\alpha}}_{\overline{\alpha}\overline{\beta}}u_\beta\right\}$$

as the trace of an endomorphism of a real vector space V coincides with the trace of the $\mathbb{C}$-linear extension to $V \otimes_{\mathbb{R}} \mathbb{C}$ of that endomorphism. On the other hand

$$\nabla_\alpha u_{\overline{\beta}} = (\nabla_{T_\alpha} du)T_{\overline{\beta}} = T_\alpha(u_{\overline{\beta}}) - \Gamma^{\overline{\gamma}}_{\alpha\overline{\beta}}u_{\overline{\gamma}} =$$

(by (1.75))

$$= T_\alpha(u_{\overline{\beta}}) + \sum_\gamma \Gamma^\beta_{\alpha\gamma}u_{\overline{\gamma}}$$

and (1.79) is proved. We may then perform the calculation (by (1.79))

$$G_\theta(\nabla^H u\,,\ \nabla^H\Delta_b u) = \sum_\alpha \{u_{\overline{\alpha}}(\Delta_b u)_\alpha + u_\alpha(\Delta_b u)_{\overline{\alpha}}\} = \tag{1.80}$$

$$= -\sum_{\alpha,\beta}\left\{u_{\overline{\alpha}}T_\alpha\left(\nabla_\beta u_{\overline{\beta}} + \nabla_{\overline{\beta}}u_\beta\right) + u_\alpha T_{\overline{\alpha}}\left(\nabla_\beta u_{\overline{\beta}} + \nabla_{\overline{\beta}}u_\beta\right)\right\}.$$

We shall replace the ordinary derivatives in (1.80) by covariant derivatives. To this end note that

$$u_{A\beta\overline{\gamma}} = (\nabla^3 u)(T_A\,,\ T_\beta\,,\ T_{\overline{\gamma}}) = (\nabla_{T_A}\nabla^2 u)(T_\beta\,,\ T_{\overline{\gamma}}) =$$

$$= T_A\left(\nabla_\beta u_{\overline{\gamma}}\right) - \Gamma^\sigma_{A\beta}\nabla_\sigma u_{\overline{\gamma}} - \Gamma^{\overline{\sigma}}_{A\overline{\gamma}}\nabla_\beta u_{\overline{\sigma}}$$

i.e.

$$T_A\left(\nabla_\beta u_{\overline{\gamma}}\right) = u_{A\beta\overline{\gamma}} + \Gamma^\sigma_{A\beta}\nabla_\sigma u_{\overline{\gamma}} + \Gamma^{\overline{\sigma}}_{A\overline{\gamma}}\nabla_\beta u_{\overline{\sigma}}\,. \tag{1.81}$$

Let us substitute from (1.81) and its complex conjugate into (1.80) and observe the cancellation (by (1.75)) of Christoffel symbols. We obtain (by (1.72))

$$G_\theta\left(\nabla^H u\,,\ \nabla^H\Delta_b u\right) = -\sum_{\alpha,\beta}\left\{u_{\overline{\alpha}}\left(u_{\alpha\beta\overline{\beta}} + u_{\alpha\overline{\beta}\beta}\right) + u_\alpha\left(u_{\overline{\alpha}\beta\overline{\beta}} + u_{\overline{\alpha}\overline{\beta}\beta}\right)\right\} =$$

$$= -\sum_{\alpha,\beta}u_{\overline{\alpha}}u_{\alpha\overline{\beta}\beta} - \sum_\alpha u_{\overline{\alpha}}\left(P_\alpha u - 2ni\sum_\beta A_{\alpha\beta}u_{\overline{\beta}}\right) -$$

$$-\sum_{\alpha,\beta}u_\alpha u_{\overline{\alpha}\beta\overline{\beta}} - \sum_\alpha u_\alpha\left(P_{\overline{\alpha}}u + 2ni\sum_\beta A_{\overline{\alpha}\overline{\beta}}u_\beta\right)$$

hence

$$\sum_{\alpha,\beta}\left(u_{\overline{\alpha}}u_{\alpha\overline{\beta}\beta} + u_\alpha u_{\overline{\alpha}\beta\overline{\beta}}\right) = -G_\theta\left(\nabla^H u\,,\ \nabla^H\Delta_b u\right) - \tag{1.82}$$

$$-\sum_\alpha\left(u_{\overline{\alpha}}P_\alpha u + u_\alpha P_{\overline{\alpha}}u\right) + 2ni\sum_{\alpha,\beta}\left(A_{\alpha\beta}u_{\overline{\alpha}}u_{\overline{\beta}} - A_{\overline{\alpha}\overline{\beta}}u_\alpha u_\beta\right).$$

Substitution from (1.82) into (1.78) now gives

$$4i\sum_{\alpha}(u_{\overline{\alpha}}\nabla_0 u_\alpha - u_\alpha\nabla_0 u_{\overline{\alpha}}) = -\frac{4}{n}\sum_{\alpha}(u_{\overline{\alpha}}P_\alpha u + u_\alpha P_{\overline{\alpha}}u) + \tag{1.83}$$

$$+4i\sum_{\alpha,\beta}\left(A_{\alpha\beta}u_{\overline{\alpha}}u_{\overline{\beta}} - A_{\overline{\alpha}\overline{\beta}}u_\alpha u_\beta\right) - \frac{2}{n}G_\theta\left(\nabla^H u\,,\,\nabla^H\Delta_b u\right).$$

Finally substitution from (1.83) into Greenleaf's formula (1.71) gives

$$-\frac{1}{2}\Delta_b\left(\left\|\nabla^H u\right\|^2\right) = \left\|\Pi_H\nabla^2 u\right\|^2 - \left(1+\frac{2}{n}\right)G_\theta\left(\nabla^H\,,\,\nabla^H\Delta_b u\right) + \tag{1.84}$$

$$+2\sum_{\alpha,\beta=1}^{n}R_{\alpha\overline{\beta}}u_{\overline{\alpha}}u_\beta + i(n-3)\sum_{\alpha,\beta=1}^{n}\left(A_{\overline{\alpha}\overline{\beta}}u_\alpha u_\beta - A_{\alpha\beta}u_{\overline{\alpha}}u_{\overline{\beta}}\right) -$$

$$-\frac{4}{n}\sum_{\alpha=1}^{n}(u_{\overline{\alpha}}P_\alpha u + u_\alpha P_{\overline{\alpha}}u)\,.$$

This is formula (2.1) in S-C. Chang & H-L. Chiu work (cf. [92]), except for the $n-3$ factor[10]. Greenleaf's formula in Chang-Chiu version replaces the "non-Riemannian" term Lu (cf. (1.70) above) in terms of the operator P_α. In view of the non-nengativity of the CR Paneitz operator P_0 (cf. [92], p. 269-271, and exploited by us in Chapter 4 of this thesis) re-writing Greenleaf's formula as in (1.84) proves to be a crucial ingredient.

Identity (4.121) in Chapter 4, re-written in terms of a local G_θ-orthonormal frame $\{T_\alpha : 1 \le \alpha \le n\} \subset C^\infty(U, T_{1,0}(M))$ (rather than an arbitrary local frame of $T_{1,0}(M)$), reads

$$i\sum_{\beta=1}^{n}\left(u_\beta\nabla_0 u_{\overline{\beta}} - u_{\overline{\beta}}\nabla_0 u_\beta\right) = 2nu_0^2 + \tag{1.85}$$

$$+i\sum_{\alpha,\beta=1}^{n}\left(A_{\alpha\beta}u_{\overline{\alpha}}u_{\overline{\beta}} - A_{\overline{\alpha}\overline{\beta}}u_\alpha u_\beta\right) - \mathrm{div}\left(u_0 J\nabla^H u\right).$$

Of course functions appearing in the left and right hand side of (1.85) are local expressions on U of globally defined functions on M. By a common language abuse we shall use the same symbols to denote both the given global function and its local expression with respect to $\{T_\alpha : 1 \le \alpha \le n\}$. For instance $i\sum_{\beta=1}^{n}\left(u_\beta\nabla_0 u_{\overline{\beta}} - u_{\overline{\beta}}\nabla_0 u_\beta\right)$ will denote both the underlying element of $C^\infty(M,\mathbb{R})$ and its restriction to U. Then we may integrate over M in (1.85) and use Green's lemma to obtain

$$i\int_M\sum_{\beta=1}^{n}\left(u_\beta\nabla_0 u_{\overline{\beta}} - u_{\overline{\beta}}\nabla_0 u_\beta\right)\Psi_\theta = 2n\int_M u_0^2\,\Psi_\theta + \tag{1.86}$$

$$+i\int_M\sum_{\alpha,\beta=1}^{n}\left(A_{\alpha\beta}u_{\overline{\alpha}}u_{\overline{\beta}} - A_{\overline{\alpha}\overline{\beta}}u_\alpha u_\beta\right)\Psi_\theta\,.$$

[10] Which is $2n$ in (2.1) of [92], p. 265, again due to different exterior calculus conventions.

This is essentially[11] formula (2.4) in Lemma 2.2 of [92], p. 268. We close this section by proving the identity

$$i\int_M \sum_{\alpha=1}^{n} (u_\alpha \nabla_0 u_{\overline{\alpha}} - u_{\overline{\alpha}} \nabla_0 u_\alpha) = \tag{1.87}$$

$$= \frac{1}{n}\int_M \sum_{\alpha,\beta=1}^{n} \left\{ (\nabla_\beta u_\alpha)(\nabla_{\overline{\beta}} u_{\overline{\alpha}}) - (\nabla_{\overline{\beta}} u_\alpha)(\nabla_\beta u_{\overline{\alpha}}) + R_{\alpha\overline{\beta}} u_{\overline{\alpha}} u_\beta \right\}.$$

This[12] is (2.5) in Lemma 2.3 of [92], p. 268. The rather involved proof of (1.87) makes use of a commutation formula for third order covariant derivatives

$$-u_{\beta\overline{\gamma}\,\overline{\alpha}} + u_{\overline{\gamma}\beta\overline{\alpha}} = 2ig_{\beta\overline{\gamma}}\nabla_0 u_{\overline{\alpha}} - R_{\overline{\alpha}}{}^{\overline{\mu}}{}_{\overline{\gamma}\beta}\, u_{\overline{\mu}}\,. \tag{1.88}$$

that we proceed to derive. We first compute

$$T_\beta\left(\nabla_{\overline{\gamma}} u_{\overline{\alpha}}\right) = T_\beta\left((\nabla^2 u)(T_{\overline{\gamma}}\,,\, T_{\overline{\alpha}})\right) = T_\beta\left((\nabla_{T_{\overline{\gamma}}} du)T_{\overline{\alpha}}\right) =$$

$$= T_\beta\left(T_{\overline{\gamma}}(u_{\overline{\alpha}}) - \Gamma^{\overline{\mu}}_{\overline{\gamma}\,\overline{\alpha}}\, u_{\overline{\mu}}\right) =$$

$$= T_\beta T_{\overline{\gamma}} u_{\overline{\alpha}} - T_\beta\left(\Gamma^{\overline{\mu}}_{\overline{\gamma}\,\overline{\alpha}}\right) u_{\overline{\mu}} - \Gamma^{\overline{\mu}}_{\overline{\gamma}\,\overline{\alpha}}\, T_\beta\left(u_{\overline{\mu}}\right)$$

or (by introducing the Lie bracket of T_β and $T_{\overline{\gamma}}$ and replacing ordinary derivatives $T_\beta\left(u_{\overline{\mu}}\right)$ by covariant derivatives)

$$T_\beta\left(\nabla_{\overline{\gamma}} u_{\overline{\alpha}}\right) = \left[T_\beta\,,\, T_{\overline{\gamma}}\right](u_{\overline{\alpha}}) + \tag{1.89}$$

$$+ T_{\overline{\gamma}} T_\beta u_{\overline{\alpha}} - T_\beta\left(\Gamma^{\overline{\mu}}_{\overline{\gamma}\,\overline{\alpha}}\right) u_{\overline{\mu}} - \Gamma^{\overline{\mu}}_{\overline{\gamma}\,\overline{\alpha}}\left(\nabla_\beta u_{\overline{\mu}} + \Gamma^{\overline{\sigma}}_{\beta\overline{\mu}} u_{\overline{\sigma}}\right).$$

The point is that one may express the Lie bracket $\left[T_\beta\,,\, T_{\overline{\gamma}}\right]$ in terms of the Tanaka-Webster connection ∇ of (M, θ), as a consequence of the purity of its torsion T_∇

$$2ig_{\beta\overline{\gamma}}\, T = T_\nabla(T_\beta\,,\, T_{\overline{\gamma}}) = \Gamma^{\overline{\mu}}_{\beta\overline{\gamma}} T_{\overline{\mu}} - \Gamma^{\mu}_{\overline{\gamma}\beta} T_\mu - \left[T_\beta\,,\, T_{\overline{\gamma}}\right]$$

i.e.

$$\left[T_\beta\,,\, T_{\overline{\gamma}}\right] = \Gamma^{\overline{\mu}}_{\beta\overline{\gamma}} T_{\overline{\mu}} - \Gamma^{\mu}_{\overline{\gamma}\beta} T_\mu - 2i\delta_{\beta\gamma} T. \tag{1.90}$$

Let us substitute from (1.90) into (1.89). We obtain (by also replacing ordinary derivatives $T_\beta(u_{\overline{\alpha}})$ in terms of covariant derivatives)

$$T_\beta\left(\nabla_{\overline{\gamma}} u_{\overline{\alpha}}\right) = \Gamma^{\overline{\mu}}_{\beta\overline{\gamma}} T_{\overline{\mu}}(u_{\overline{\alpha}}) - \Gamma^{\mu}_{\overline{\gamma}\beta} T_\mu(u_{\overline{\alpha}}) - 2i\delta_{\beta\gamma} T(u_{\overline{\alpha}}) +$$

$$+ T_{\overline{\gamma}}\left(\nabla_\beta u_{\overline{\alpha}} + \Gamma^{\overline{\mu}}_{\beta\overline{\alpha}} u_{\overline{\mu}}\right) - T_\beta\left(\Gamma^{\overline{\mu}}_{\overline{\gamma}\,\overline{\alpha}}\right) u_{\overline{\mu}} - \Gamma^{\overline{\mu}}_{\overline{\gamma\alpha}} \nabla_\beta u_{\overline{\mu}} - \Gamma^{\overline{\mu}}_{\overline{\gamma}\,\overline{\alpha}} \Gamma^{\overline{\sigma}}_{\beta\overline{\mu}} u_{\overline{\sigma}} =$$

$$= \Gamma^{\overline{\mu}}_{\beta\overline{\gamma}}\left(\nabla_{\overline{\mu}} u_{\overline{\alpha}} + \Gamma^{\overline{\sigma}}_{\overline{\mu\alpha}} u_{\overline{\sigma}}\right) - \Gamma^{\mu}_{\overline{\gamma}\beta}\left(\nabla_\mu u_{\overline{\alpha}} + \Gamma^{\overline{\sigma}}_{\mu\overline{\alpha}} u_{\overline{\sigma}}\right) +$$

$$+ T_{\overline{\gamma}}\left(\nabla_\beta u_{\overline{\alpha}}\right) + T_{\overline{\gamma}}\left(\Gamma^{\overline{\mu}}_{\beta\overline{\alpha}}\right) u_{\overline{\mu}} - T_\beta\left(\Gamma^{\overline{\mu}}_{\overline{\gamma}\,\overline{\alpha}}\right) u_{\overline{\mu}} - \Gamma^{\overline{\mu}}_{\overline{\gamma}\,\overline{\alpha}} \nabla_\beta u_{\overline{\mu}} - \Gamma^{\overline{\mu}}_{\overline{\gamma\alpha}} \Gamma^{\overline{\sigma}}_{\beta\overline{\mu}} u_{\overline{\sigma}} +$$

[11] Second integral in (2.4) of [92], p. 268, bears a n factor (rather than a $2n$ factor as in our (1.86)).

[12] Once again, as compared to our formula (1.87), identity (2.5) in Lemma 2.3 of [92], p. 268, has an extra 2 factor in its right hand member.

$$+\Gamma^{\overline{\mu}}_{\beta\overline{\alpha}}\Gamma^{\overline{\sigma}}_{\overline{\gamma}\,\overline{\mu}}u_{\overline{\sigma}} + \Gamma^{\overline{\mu}}_{\beta\overline{\alpha}}\nabla_{\overline{\gamma}}u_{\overline{\mu}} - 2i\delta_{\beta\gamma}\nabla_0 u_{\overline{\alpha}} - 2i\delta_{\beta\gamma}\Gamma^{\overline{\mu}}_{0\overline{\alpha}}u_{\overline{\mu}}$$

or

$$T_\beta\left(\nabla_{\overline{\gamma}}u_{\overline{\alpha}}\right) = T_{\overline{\gamma}}\left(\nabla_\beta u_{\overline{\alpha}}\right) + \tag{1.91}$$

$$+T_{\overline{\gamma}}\left(\Gamma^{\overline{\mu}}_{\beta\overline{\alpha}}\right)u_{\overline{\mu}} - T_\beta\left(\Gamma^{\overline{\mu}}_{\overline{\gamma}\,\overline{\alpha}}\right)u_{\overline{\mu}} + \Gamma^{\overline{\mu}}_{\beta\overline{\gamma}}\Gamma^{\overline{\sigma}}_{\overline{\mu}\overline{\alpha}}u_{\overline{\sigma}} - \Gamma^{\mu}_{\overline{\gamma}\beta}\Gamma^{\overline{\sigma}}_{\mu\overline{\alpha}}u_{\overline{\sigma}}+$$

$$-\Gamma^{\overline{\mu}}_{\overline{\gamma}\overline{\alpha}}\Gamma^{\overline{\sigma}}_{\beta\overline{\mu}}u_{\overline{\sigma}} + \Gamma^{\overline{\mu}}_{\beta\overline{\alpha}}\Gamma^{\overline{\sigma}}_{\overline{\gamma}\,\overline{\mu}}u_{\overline{\sigma}} - 2i\delta_{\beta\gamma}\Gamma^{\overline{\mu}}_{0\overline{\alpha}}u_{\overline{\mu}} -$$

$$-\Gamma^{\overline{\mu}}_{\overline{\gamma}\,\overline{\alpha}}\nabla_\beta u_{\overline{\mu}} + \Gamma^{\overline{\mu}}_{\beta\overline{\alpha}}\nabla_{\overline{\gamma}}u_{\overline{\mu}} - 2i\delta_{\beta\gamma}\nabla_0 u_{\overline{\alpha}} + \Gamma^{\overline{\mu}}_{\beta\overline{\gamma}}\nabla_{\overline{\mu}}u_{\overline{\alpha}} - \Gamma^{\mu}_{\overline{\gamma}\beta}\nabla_\mu u_{\overline{\alpha}}\,.$$

To understand the meaning of identity (1.91) let us observe that $T_{\overline{\gamma}}\left(\nabla_\beta u_{\overline{\alpha}}\right)$ is the term looked for (switching the derivatives in the directions T_β and $T_{\overline{\gamma}}$). The next two rows in (1.91) will be recognized as a curvature term (of the Tanaka-Webster connection). The remaining terms will be shortly seen to be a third order covariant derivative of the function u. To recognize curvature we need to conduct the following calculation

$$R_{\overline{\alpha}}{}^{\overline{\mu}}{}_{\overline{\gamma}\beta}T_{\overline{\mu}} = R^\nabla(T_{\overline{\gamma}}, T_\beta)T_{\overline{\alpha}} = \nabla_{T_{\overline{\gamma}}}\nabla_{T_\beta}T_{\overline{\alpha}} - \nabla_{T_\beta}\nabla_{T_{\overline{\gamma}}}T_{\overline{\alpha}} - \nabla_{[T_{\overline{\gamma}},T_\beta]}T_{\overline{\alpha}}$$

leading to (by (1.90))

$$R_{\overline{\alpha}}{}^{\mu}{}_{\overline{\gamma}\beta} = T_{\overline{\gamma}}\left(\Gamma^{\overline{\mu}}_{\beta\overline{\alpha}}\right) - T_\beta\left(\Gamma^{\overline{\mu}}_{\overline{\gamma}\,\overline{\alpha}}\right) - 2ig_{\beta\overline{\gamma}}\Gamma^{\overline{\mu}}_{0\overline{\alpha}} + \tag{1.92}$$

$$+\Gamma^{\overline{\sigma}}_{\beta\overline{\alpha}}\Gamma^{\overline{\mu}}_{\overline{\gamma}\,\overline{\sigma}} - \Gamma^{\overline{\sigma}}_{\overline{\gamma}\,\overline{\alpha}}\Gamma^{\overline{\mu}}_{\beta\overline{\sigma}} + \Gamma^{\overline{\sigma}}_{\beta\overline{\gamma}}\Gamma^{\overline{\mu}}_{\overline{\sigma}\,\overline{\alpha}} - \Gamma^{\sigma}_{\overline{\gamma}\beta}\gamma^{\overline{\mu}}_{\sigma\overline{\alpha}}\,.$$

Let us substitute from (1.92) into (1.91). We obtain

$$T_\beta\left(\nabla_{\overline{\gamma}}u_{\overline{\alpha}}\right) = T_{\overline{\gamma}}\left(\nabla_\beta u_{\overline{\alpha}}\right) + R_{\overline{\alpha}}{}^{\overline{\mu}}{}_{\overline{\gamma}\beta}u_{\overline{\mu}} - \tag{1.93}$$

$$-\Gamma^{\overline{\mu}}_{\overline{\gamma}\,\overline{\alpha}}\nabla_\beta u_{\overline{\mu}} + \Gamma^{\overline{\mu}}_{\beta\overline{\alpha}}\nabla_{\overline{\gamma}}u_{\overline{\mu}} - 2i\delta_{\beta\gamma}\nabla_0 u_{\overline{\alpha}} + \Gamma^{\overline{\mu}}_{\beta\overline{\gamma}}\nabla_{\overline{\mu}}u_{\overline{\alpha}} - \Gamma^{\mu}_{\overline{\gamma}\beta}\nabla_\mu u_{\overline{\alpha}}\,.$$

Finally we may compute $-u_{\beta\overline{\gamma}\,\overline{\alpha}} + u_{\overline{\gamma}\beta\overline{\alpha}}$ (by making use of (1.93)) and observe the cancellation of Christoffel symbols. This leads to the commutation formula (1.88). Identity (1.88) actually holds for an arbitrary local frame $\{T_\alpha : 1 \le \alpha \le n\}$ of $T_{1,0}(M)$ (as emphasized by the presence of the metric components $g_{\beta\overline{\gamma}}$ there) yet it will be only used for orthonormal frames ($g_{\beta\overline{\gamma}} = \delta_{\beta\gamma}$). If this is the case let us contract β and γ in (1.88) so that to derive

$$\sum_{\beta=1}^{n}\left(-u_{\beta\overline{\beta}\,\overline{\alpha}} + u_{\overline{\beta}\beta\overline{\alpha}}\right) = 2in\,\nabla_0 u_{\overline{\alpha}} - \sum_{\beta,\mu=1}^{n} R_{\overline{\alpha}\mu\overline{\beta}\beta}u_{\overline{\mu}}\,. \tag{1.94}$$

Let us go back to the proof of (1.87). Using (1.94) we may replace terms of the form $\nabla_0 u_A$ in terms of third order covariant derivatives plus curvature. Precisely

$$2i\sum_\alpha\left(u_\alpha\nabla_0 u_{\overline{\alpha}} - u_{\overline{\alpha}}\nabla_0 u_\alpha\right) = \tag{1.95}$$

$$= \frac{1}{n}\sum_{\alpha,\beta}u_\alpha\left(-u_{\beta\overline{\beta}\,\overline{\alpha}} + u_{\overline{\beta}\beta\overline{\alpha}} + \sum_\mu R_{\overline{\alpha}\mu\overline{\beta}\beta}u_{\overline{\mu}}\right) +$$

$$+\frac{1}{n}\sum_{\alpha,\beta}u_{\overline{\alpha}}\left(-u_{\overline{\beta}\beta\alpha} + u_{\beta\overline{\beta}\alpha} + \sum_\mu R_{\alpha\overline{\mu}\beta\overline{\beta}}u_\mu\right)$$

ad let us integrate over M to get a candidate to (1.87). Here we shall integrate by parts so that to replace the third order covariant derivatives by second order covariant derivatives. For instance

$$u_\alpha u_{\beta\overline{\beta}\,\overline{\alpha}} = u_\alpha \left(\nabla^2_{T_\beta}\nabla^2 u\right)(T_{\overline{\beta}}\,,\,T_{\overline{\alpha}}) =$$

$$= u_\alpha \left\{T_\beta\left(\nabla_{\overline{\beta}} u_{\overline{\alpha}}\right) - \Gamma^{\mu}_{\beta\overline{\beta}}\nabla_{\overline{\mu}} u_{\overline{\alpha}} - \Gamma^{\overline{\mu}}_{\beta\overline{\alpha}}\nabla_{\overline{\beta}} u_{\overline{\mu}}\right\} =$$

(by exploiting the derivative of the product $u_\alpha\nabla_{\overline{\beta}}u_{\overline{\alpha}}$)

$$= T_\beta\left(u_\alpha\nabla_{\overline{\beta}}u_{\overline{\alpha}}\right) - T_\beta(u_\alpha)\nabla_{\overline{\beta}}u_{\overline{\alpha}} - u_\alpha\Gamma^{\overline{\mu}}_{\beta\overline{\beta}}\nabla_{\overline{\mu}}u_{\overline{\alpha}} - u_\alpha\Gamma^{\overline{\mu}}_{\beta\overline{\alpha}}\nabla_{\overline{\beta}}u_{\overline{\mu}} =$$

(by replacing the ordinary derivative T_β in terms of covariant derivative ∇_β)

$$= \nabla_\beta\left(u_\alpha\nabla_{\overline{\beta}}u_{\overline{\alpha}}\right) + \Gamma^{\overline{\mu}}_{\beta\overline{\beta}}u_\alpha\nabla_{\overline{\mu}}u_{\overline{\alpha}} - \left(\nabla_\beta u_\alpha + \Gamma^{\mu}_{\beta\alpha}u_\mu\right)\nabla_{\overline{\beta}}u_{\overline{\alpha}} -$$

$$-u_\alpha\Gamma^{\overline{\mu}}_{\beta\overline{\beta}}\nabla_{\overline{\mu}}u_{\overline{\alpha}} - u_\alpha\Gamma^{\overline{\mu}}_{\beta\overline{\alpha}}\nabla_{\overline{\beta}}u_{\overline{\mu}} =$$

(by observing the cancellation of $\Gamma^{\overline{\mu}}_{\beta\overline{\beta}}$ and by using identity (1.75))

$$= \operatorname{div}\left(u_\alpha\,(\nabla^\beta u_{\overline{\alpha}})T_\beta\right) -$$

$$-(\nabla_\beta u_\alpha)(\nabla_{\overline{\beta}}u_{\overline{\alpha}}) + \sum_\mu \Gamma^{\overline{\alpha}}_{\beta\overline{\mu}}u_\mu\nabla_{\overline{\beta}}u_{\overline{\alpha}} - u_\alpha\Gamma^{\overline{\mu}}_{\beta\overline{\alpha}}\nabla_{\overline{\beta}}u_{\overline{\mu}}$$

so that (by observing the cancellation of $\Gamma^{\overline{\mu}}_{\beta\overline{\alpha}}$)

$$u_\alpha u_{\beta\overline{\beta}\,\overline{\alpha}} = -(\nabla_\beta u_\alpha)(\nabla_{\overline{\beta}}u_{\overline{\alpha}}) + \operatorname{div}\left(u_\alpha\,(\nabla^\beta u_{\overline{\alpha}})T_\beta\right). \tag{1.96}$$

Similarly

$$u_\alpha u_{\overline{\beta}\beta\overline{\alpha}} = -(\nabla_{\overline{\beta}}u_\alpha)(\nabla_\beta u_{\overline{\alpha}}) + \operatorname{div}\left(u_\alpha(\nabla^{\overline{\beta}}u_{\overline{\alpha}})T_{\overline{\beta}}\right). \tag{1.97}$$

Identities (1.96)-(1.97) then lead to

$$\sum_{\alpha,\beta} u_\alpha\left(-u_{\beta\overline{\beta}\,\overline{\alpha}} + u_{\overline{\beta}\beta\overline{\alpha}}\right) \equiv \sum_{\alpha,\beta}\left[(\nabla_\beta u_\alpha)(\nabla_{\overline{\beta}}u_{\overline{\alpha}}) - (\nabla_{\overline{\beta}}u_\alpha)(\nabla_\beta u_{\overline{\alpha}})\right], \quad \text{mod div}$$

hence (by integrating (1.95) and using Green's lemma)

$$2ni\int_M \sum_\alpha (u_\alpha\nabla_0 u_{\overline{\alpha}} - u_{\overline{\alpha}}\nabla_0 u_\alpha)\,\Psi_\theta = \tag{1.98}$$

$$= \int_M \sum_{\alpha,\beta}\left[(\nabla_\beta u_\alpha)(\nabla_{\overline{\beta}}u_{\overline{\alpha}}) - (\nabla_{\overline{\beta}}u_\alpha)(\nabla_\beta u_{\overline{\alpha}})\right]\Psi_\theta +$$

$$+ \int_M \sum_{\alpha,\beta,\gamma}\left(R_{\overline{\alpha}\gamma\overline{\beta}\beta}u_\alpha u_{\overline{\gamma}} + R_{\alpha\overline{\gamma}\beta\overline{\beta}}u_{\overline{\alpha}}u_\gamma\right)\Psi_\theta.$$

The last step (in the proof of (1.87)) is to recognize pseudohermitian Ricci curvature $R_{\alpha\overline{\beta}}$ in the contracted curvature terms appearing in (1.98). This is a rather involved calculation, based on curvature theory (for the Tanaka-Webster connection) as built in [94]. We start with Theorem 1.6 in [94] according to which

$$R^{\nabla^\theta}(X,Y)Z = R^\nabla(X,Y)Z+ \tag{1.99}$$

$$+(LX \wedge LY)Z - 2\Omega(X,Y)JZ - g_\theta(S(X,Y),Z)T$$

for any $X, Y, Z \in H(M)$. Here

$$L = \tau + J, \quad S(X,Y) = (\nabla_X \tau)Y - (\nabla_Y \tau)X.$$

Also R^{∇^θ} is the curvature tensor field of the Levi-Civita connection ∇^θ of the Riemannian manifold (M, g_θ). Taking the inner product of (1.99) with $W \in H(M)$ gives (by $g_\theta(W,T) = 0$)

$$K^{\nabla^\theta}(W,Z,X,Y) = K^\nabla(W,Z,X,Y)+ \tag{1.100}$$

$$+g_\theta((LX \wedge LY)Z, W) - 2\Omega(X,Y)g_\theta(JZ,W).$$

Here K^{∇^θ} and K^∇ are respectively the Riemann-Christoffel tensor of (M, g_θ) and its pseudohermitian analog. For instance

$$K^\nabla(W,Z,X,Y) = g_\theta(R^\nabla(X,Y)Z, W).$$

Moreover (by recalling the meaning of wedge product of two vector fields $(X \wedge Y)Z = g_\theta(X,Z)Y - g_\theta(Y,Z)X$)

$$g_\theta((LX \wedge LY)Z, W) = g_\theta(LX,Z)g_\theta(LY,W) - g_\theta(LY,Z)g_\theta(LX,W)$$

so that (1.100) becomes

$$K^{\nabla^\theta}(W,Z,X,Y) = K^\nabla(W,Z,X,Y) + 2\Omega(X,Y)\Omega(Z,W)+ \tag{1.101}$$

$$+g_\theta(LX,Z)g_\theta(LY,W) - g_\theta(LY,Z)g_\theta(LX,W).$$

using (1.101) and the known symmetry

$$K^{\nabla^\theta}(W,Z,X,Y) = K^{\nabla^\theta}(X,Y,W,Z)$$

of the Riemann-Christoffel tensor (a symmetry which K^∇ fails to enjoy, as one of the known obstacles in pseudohermitian geometry) one obtains

$$K^\nabla(W,Z,X,Y) = K^\nabla(X,Y,W,Z)+ \tag{1.102}$$

$$+g_\theta(LW,Y)g_\theta(LZ,X) - g_\theta(LZ,Y)g_\theta(LW,X)+$$

$$+g_\theta(LY,Z)g_\theta(LX,W) - g_\theta(LX,Z)g_\theta(LY,W).$$

Finally the terms of the form $g_\theta(LX,Y)$ may be explicitly calculated (by $L = \tau + J$) so that (1.102) may be written

$$K^\nabla(W,Z,X,Y) = K^\nabla(X,Y,W,Z)+ \tag{1.103}$$

$$+2\{A(X,Z)\Omega(Y,W) - A(Y,Z)\Omega(X,W)+$$

$$+ A(Y,W)\Omega(X,Z) - A(X,W)\Omega(Y,Z)\}$$

for any $X, Y, Z, W \in H(M)$. Next

$$R_{\overline{\alpha}\gamma\overline{\beta}\beta} = g_\theta\left(R^\nabla(T_{\overline{\beta}}, T_\beta)T_{\overline{\alpha}}, T_\gamma\right) =$$

$$= K^{\nabla}\left(T_\gamma\,,\,T_{\overline{\alpha}}\,,\,T_{\overline{\beta}}\,,\,T_\beta\right) =$$

(by the symmetry property (1.103))

$$= K^{\nabla}(T_{\overline{\beta}}\,,\,T_\beta\,,\,T_\gamma\,,\,T_{\overline{\alpha}}) +$$

$$+2\left\{A(T_{\overline{\beta}}\,,\,T_{\overline{\alpha}})\Omega(T_\beta, T_\gamma) - A(T_\beta\,,\,T_{\overline{\alpha}})\Omega(T_{\overline{\beta}}, T_\gamma) +\right.$$

$$\left.+A(T_\beta\,,\,T_{\overline{\alpha}})\Omega(T_{\overline{\beta}}, T_{\overline{\alpha}}) - A(T_{\overline{\beta}}, T_\gamma)\Omega(T_\beta\,,\,T_{\overline{\alpha}})\right\} =$$

(as A vanishes on complex vector fields of distinct types, while Ω vanishes on complex vector fields of the same type)

$$= K^{\nabla}(T_{\overline{\beta}}\,,\,T_\beta\,,\,T_\gamma\,,\,T_{\overline{\alpha}}) = R_{\beta\overline{\beta}\gamma\overline{\alpha}} =$$

(by the symmetry property in Theorem 1.8 of [94])

$$= R_{\gamma\overline{\beta}\beta\overline{\alpha}}\,.$$

Yet

$$R_{\alpha\overline{\beta}} = \mathrm{Ric}_\nabla(T_\alpha, T_{\overline{\beta}}) = \mathrm{trace}\left\{Z \mapsto R^{\nabla}(Z, T_{\overline{\beta}})T_\alpha\right\} =$$

(as $R^{\nabla}(X, Y)$ maps $T_{1,0}(M)$ into $T_{1,0}(M)$)

$$= \mathrm{trace}\left\{T_\gamma \mapsto R^{\nabla}(T_\gamma, T_{\overline{\beta}})T_\alpha = R_\alpha{}^\mu{}_{\gamma\overline{\beta}}T_\mu\right\} =$$

$$= R_\alpha{}^\gamma{}_{\gamma\overline{\beta}} = \sum_\gamma R_{\alpha\overline{\gamma}\gamma\overline{\beta}}$$

and we may conclude that

$$\sum_\beta R_{\overline{\alpha}\gamma\overline{\beta}\beta} = \sum_\beta R_{\gamma\overline{\beta}\beta\overline{\alpha}} = R_{\gamma\overline{\alpha}}\,. \tag{1.104}$$

Finally substitution from (1.104) into (1.98) leads to (1.87). Q.e.d.

As an immediate consequence of the above

$$\int_M u_0^2 \Psi_\theta = \frac{1}{4n^2}\int_M (\Delta_b u)^2\ \Psi_\theta + \tag{1.105}$$

$$+\frac{i}{n}\int_M \sum_{\alpha,\beta=1}^n \left(A_{\overline{\alpha}\overline{\beta}}u_\alpha u_\beta - A_{\alpha\beta}u_{\overline{\alpha}}u_{\overline{\beta}}\right)\Psi_\theta - \frac{1}{2n^2}\int_M u\,P_0 u\,\Psi_\theta\,.$$

This is essentially[13] (2.6) in Corollary 2.4 of [92], p. 269. To check (1.105) we start by integrating (1.83) over M

$$2i\int_M \sum_\alpha (u_{\overline{\alpha}}\nabla_0 u_\alpha - u_\alpha \nabla_0 u_{\overline{\alpha}})\ \Psi_\theta =$$

$$= -\frac{2}{n}\sum_\alpha (u_{\overline{\alpha}}P_\alpha u + u_\alpha P_{\overline{\alpha}} u)\ \Psi_\theta +$$

[13] With respect to our identity (1.105), the relation (2.6) in [92], p. 269, bears an extra 2 factor in its right hand side.

$$+2i\int_M \sum_{\alpha,\beta}\left(A_{\alpha\beta}u_{\overline{\alpha}}u_{\overline{\beta}} - A_{\overline{\alpha}\,\overline{\beta}}u_\alpha u_\beta\right)\Psi_\theta - \frac{1}{n}\int_M \left(\nabla^H u\right)(\Delta_b u)\,\Psi_\theta\,,$$

substitute $\int_M \sum_\alpha (u_{\overline{\alpha}}\nabla_0 u_\alpha - u_\alpha \nabla_0 u_{\overline{\alpha}})\,\Psi_\theta$ from (1.83)

$$-4n\int_M u_0^2\Psi_\theta - 2i\int_M \sum_{\alpha,\beta}\left(A_{\alpha\beta}u_{\overline{\alpha}}u_{\overline{\beta}} - A_{\overline{\alpha}\,\overline{\beta}}u_\alpha u_\beta\right)\Psi_\theta =$$

$$= -\frac{2}{n}\int_M g_\theta^*\left((P_\alpha u)\theta^\alpha + (P_{\overline{\alpha}}u)\theta^{\overline{\alpha}}\,,\ u_\beta\theta^\beta + u_{\overline{\beta}}\theta^{\overline{\beta}}\right)\Psi_\theta +$$

$$+2i\int_M \sum_{\alpha,\beta}\left(A_{\alpha\beta}u_{\overline{\alpha}}u_{\overline{\beta}} - A_{\overline{\alpha}\,\overline{\beta}}u_\alpha u_\beta\right)\Psi_\theta,$$

simplify torsion terms and use the identities

$$\int_M g_\theta^*\left((P+\overline{P})u\,,\ d_b u\right)\Psi_\theta = -\int_M u\,P_0 u\,\Psi_\theta$$

(a consequence of our calculations in Chapter 4) and

$$\int_M (\nabla^H u)(\Delta_b u)\,\Psi_\theta =$$

$$= \int_M \left\{\operatorname{div}\left((\Delta_b u)\,\nabla^H u\right) - (\Delta_b u)\operatorname{div}(\nabla^H u)\right\}\Psi_\theta = \int_M (\Delta_b u)^2\ \Psi_\theta$$

(by Green's lemma). The proof of (1.105) is complete.

1.11 Non-negativity of CR Paneitz operator

We close Chapter 1 by giving a proof of

$$\int_M u\,P_0 u\ \Psi_\theta \geq 0, \quad u \in C^\infty(M,\mathbb{R}), \tag{1.106}$$

i.e. the CR Paneitz operator P_0 is non-negative. This has been shown in [92], p. 269-270. Our proof follows the ideas in [92], transposed under the conventions adopted in this thesis. The result is used in Chapter 4 and leads to a new lower bound on the first nonzero eigenvalue of the sublaplacian Δ_b. To prove (1.106) we start from the observation that $i\int_M \sum_\alpha (u_\alpha \nabla_0 u_{\overline{\alpha}} - u_{\overline{\alpha}}\nabla_0 u_\alpha)$ has been previously calculated in two different manners, the outcome being that in formulae (1.86)-(1.87). Hence, for any $c \in \mathbb{R}$, we may write

$$i\int_M \sum_\alpha (u_\alpha \nabla_0 u_{\overline{\alpha}} - u_{\overline{\alpha}}\nabla_0 u_\alpha) =$$

$$= c \times (\text{RHS of } (1.86)) + (1-c)\,(\text{RHS of } (1.87)) =$$

$$= c\left\{2n\int_M u_0^2\,\Psi_\theta + i\int_M \sum_{\alpha,\beta}\left(A_{\alpha\beta}u_{\overline{\alpha}}u_{\overline{\beta}} - A_{\overline{\alpha}\overline{\beta}}u_\alpha u_\beta\right)\Psi_\theta\right\} +$$

$$+\frac{1-c}{n}\sum_{\alpha,\beta}\left[(\nabla_\beta u_\alpha)(\nabla_{\overline{\beta}} u_{\overline{\alpha}}) - (\nabla_{\overline{\beta}} u_\alpha)(\nabla_\beta u_{\overline{\alpha}}) + R_{\alpha\overline{\beta}} u_{\overline{\alpha}} u_{\overline{\alpha}} u_\beta\right] \Psi_\theta$$

and substitution into Greenleaf's formula (1.71) integrated over M leads to

$$\frac{1}{2}\int_M (\Delta_b u)^2\,\Psi_\theta + 4cn\int_M u_0^2\,\Psi_\theta = \tag{1.107}$$

$$= \left(1 + \frac{2(1-c)}{n}\right)\int_M \sum_{\alpha,\beta}(\nabla_\alpha u_{\overline{\beta}})(\nabla_{\overline{\alpha}} u_\beta)\,\Psi_\theta +$$

$$+\left(1 - \frac{2(1-c)}{n}\right)\int_M \sum_{\alpha,\beta}\left\{(\nabla_\alpha u_\beta)(\nabla_{\overline{\alpha}} u_{\overline{\beta}}) + R_{\alpha\overline{\beta}} u_{\overline{\alpha}} u_\beta\right\}\Psi_\theta +$$

$$+i\left(2c + \frac{n+1}{2}\right)\int_M \sum_{\alpha,\beta}\left(A_{\overline{\alpha}\overline{\beta}} u_\alpha u_\beta - A_{\alpha\beta} u_{\overline{\alpha}} u_{\overline{\beta}}\right).$$

Let $c \in \mathbb{R}$ such that $n - 2(1-c) \neq 0$ so that (by (1.107))

$$-\int_M \sum_{\alpha,\beta}\left\{(\nabla_\alpha u_\beta)(\nabla_{\overline{\alpha}} u_{\overline{\beta}}) + R_{\alpha\overline{\beta}}\right\}\Psi_\theta = \tag{1.108}$$

$$= \frac{n+2(1-c)}{n-2(1-c)}\int_M \sum_{\alpha,\beta}(\nabla_\alpha u_{\overline{\beta}})(\nabla_{\overline{\alpha}} u_\beta)\,\Psi_\theta +$$

$$+\frac{n(4c+n+1)}{2[n-2(1-c)]}\int_M i\sum_{\alpha,\beta}\left(A_{\overline{\alpha}\overline{\beta}} u_\alpha u_\beta - A_{\alpha\beta} u_{\overline{\alpha}} u_{\overline{\beta}}\right)\Psi_\theta -$$

$$-\frac{n}{n-2(1-c)}\left\{\frac{1}{2}\int_M (\Delta_b u)^2\,\Psi_\theta + 4cn\int_M u_0^2\,\Psi_\theta\right\}.$$

On the other hand (by (1.105))

$$\int_M u P_0 u\,\Psi_\theta = \frac{1}{2}\int_M \left\{(\Delta_b u)^2 - 4n^2 u_0^2\right\}\Psi_\theta + \tag{1.109}$$

$$+2n\int_M i\sum_{\alpha,\beta}\left(A_{\overline{\alpha}\overline{\beta}} u_\alpha u_\beta - A_{\alpha\beta} u_{\overline{\alpha}\overline{\beta}}\right)\Psi_\theta .$$

Equation (1.107) for $c = -n/2$ becomes

$$\frac{1}{2}\int_M \left\{(\Delta_b u)^2 - 4n^2 u_0^2\right\}\Psi_\theta = \frac{2(n+1)}{n}\int_M \sum_{\alpha,\beta}(\nabla_\alpha u_{\overline{\beta}})(\nabla_{\overline{\alpha}} u_\beta)\,\Psi_\theta - \tag{1.110}$$

$$-\frac{2}{n}\int_M \sum_{\alpha,\beta}\left\{(\nabla_\alpha u_\beta)(\nabla_{\overline{\alpha}} u_{\overline{\beta}}) + R_{\alpha\overline{\beta}} u_{\overline{\alpha}} u_\beta\right\}\Psi_\theta +$$

$$+\frac{n-1}{2}\int_M i\sum_{\alpha,\beta}\left(A_{\alpha\beta} u_{\overline{\alpha}} u_{\overline{\beta}} - A_{\overline{\alpha}\overline{\beta}} u_\alpha u_\beta\right).$$

Let us substitute $\frac{1}{2}\int_M \left\{(\Delta_b u)^2 - 4n^2 u_0^2\right\}\Psi_\theta$ from (1.110) into (1.109). We obtain

$$\int_M uP_0u\,\Psi_\theta = \frac{2(n+1)}{n}\int_M \sum_{\alpha,\beta}(\nabla_\alpha u_{\overline{\beta}})(\nabla_{\overline{\alpha}} u_\beta)\,\Psi_\theta - \tag{1.111}$$

$$-\frac{2}{n}\int_M \sum_{\alpha,\beta}\left\{(\nabla_\alpha u_\beta)(\nabla_{\overline{\alpha}} u_{\overline{\beta}}) + R_{\alpha\overline{\beta}}u_{\overline{\alpha}}u_\beta\right\}\Psi_\theta +$$

$$+\left(2n - \frac{n-1}{2}\right)\int_M i\sum_{\alpha,\beta}\left(A_{\overline{\alpha}\overline{\beta}}u_\alpha u_\beta - A_{\alpha\beta}u_{\overline{\alpha}\overline{\beta}}\right)\Psi_\theta\,.$$

Let us substitute $-\int_M \sum_{\alpha,\beta}\left\{(\nabla_\alpha u_\beta)(\nabla_{\overline{\alpha}} u_{\overline{\beta}}) + R_{\alpha\overline{\beta}}\right\}\Psi_\theta$ from (1.108) into (1.111). We obtain

$$\int_M uP_0u\,\Psi_\theta = \frac{2}{n}\left[n+1+\frac{n+2(1-c)}{n-2(1-c)}\right]\int_M \sum_{\alpha,\beta}(\nabla_\alpha u_{\overline{\beta}})(\nabla_{\overline{\alpha}} u_\beta)\,\Psi_\theta + \tag{1.112}$$

$$+\left[2n - \frac{n-1}{2} + \frac{4c+n+1}{n[n-2(1-c)]}\right]\int_M i\sum_{\alpha,\beta}\left(A_{\overline{\alpha}\overline{\beta}}u_\alpha u_\beta - A_{\alpha\beta}u_{\overline{\alpha}\overline{\beta}}\right)\Psi_\theta -$$

$$-\frac{2}{n-2(1-c)}\left\{\frac{1}{2}\int_M (\Delta_b u)^2\,\Psi_\theta + 4cn\int_M u_0^2\,\Psi_\theta\right\}.$$

Let $c_0 \in \mathbb{R}$ be the solution to

$$2n - \frac{n-1}{2} + \frac{4c_0+n+1}{n[n-2(1-c_0)]} = 0. \tag{1.113}$$

In particular for $c = c_0$ equation (1.112) becomes

$$\int_M uP_0u\,\Psi_\theta = \tag{1.114}$$

$$= \frac{2}{n}\left[n+1+\frac{n+2(1-c_0)}{n-2(1-c_0)}\right]\int_M \sum_{\alpha,\beta}(\nabla_\alpha u_{\overline{\beta}})(\nabla_{\overline{\alpha}} u_\beta)\,\Psi_\theta -$$

$$-\frac{2}{n-2(1-c_0)}\left\{\frac{1}{2}\int_M (\Delta_b u)^2\,\Psi_\theta + 4c_0 n\int_M u_0^2\,\Psi_\theta\right\}.$$

We shall need the identity

$$4\left|\sum_{\alpha=1}^{n}\nabla_\alpha u_{\overline{\alpha}}\right|^2 = (\Delta_b u)^2 + 4n^2u_0^2\,. \tag{1.115}$$

This follows easily from $2inu_0 = \sum_\alpha(\nabla_{\overline{\alpha}} u_\alpha - \nabla_\alpha u_{\overline{\alpha}})$. Indeed

$$-4n^2u_0^2 = \left(\sum_\alpha \nabla_{\overline{\alpha}} u_\alpha - \nabla_\alpha u_{\overline{\alpha}}\right)^2 =$$

$$= \left(\sum_\alpha \nabla_{\overline{\alpha}} u_\alpha\right)^2 - 2\left|\sum_\alpha \nabla_\alpha u_{\overline{\alpha}}\right|^2 + \left(\sum_\alpha \nabla_\alpha u_{\overline{\alpha}}\right)^2 =$$

$$= \left(\sum_{\alpha} \nabla_{\overline{\alpha}} u_{\alpha} + \sum_{\alpha} \nabla_{\alpha} u_{\overline{\alpha}} \right)^2 - 4 \left| \sum_{\alpha} \nabla_{\alpha} u_{\overline{\alpha}} \right|^2 =$$

$$= (\Delta_b u)^2 - 4 \left| \sum_{\alpha} \nabla_{\alpha} u_{\overline{\alpha}} \right|^2 .$$

Q.e.d. Next (by (1.115))

$$\sum_{\alpha,\beta} \left| \nabla_{\alpha} u_{\overline{\beta}} \right|^2 \geq \frac{1}{n} \left| \sum_{\alpha} \nabla_{\alpha} u_{\overline{\alpha}} \right|^2 = \frac{1}{4n} (\Delta_b u)^2 + n u_0^2$$

hence (by (1.114))

$$\int_M u \, P_0 u \, \Psi_\theta \geq$$

$$\geq \left\{ \frac{1}{2n^2} \left[n + 1 + \frac{n + 2(1 - c_0)}{n - 2(1 - c_0)} \right] - \frac{1}{n - 2(1 - c_0)} \right\} \int_M (\Delta_b u)^2 \, \Psi_\theta +$$

$$+ \left\{ 2 \left[n + 1 + \frac{n + 2(1 - c_0)}{n - 2(1 - c_0)} \right] - \frac{8 n c_0}{n - 2(1 - c_0)} \right\} \int_M u_0^2 \, \Psi_\theta \geq 0$$

as both coefficients are non-negative (as a consequence of (1.113)).

Chapter 2

Eigenvalues as functions of the contact structure

2.1 1-Parameter variations of the contact form

We start by recalling the needed notions of functional analysis, cf. e.g. A. Kriegl & P.W. Michor, [10]. Let $\mathcal{H}$ be a Hilbert space and $\{A(t)\}_{t\in\mathbb{R}}$ a family of linear operators $A(t) : \mathcal{D}(A(t)) \subset \mathcal{H} \to \mathcal{H}$. We say $A(t)$ is *real analytic* (respectively C^∞, or $C^{k,\alpha}$) with respect to the parameter t if there is a dense subspace $V \subset \mathcal{H}$ such that i) $\mathcal{D}(A(t)) = V$ and $A(t)$ is selfadjoint for any $t \in \mathbb{R}$ and ii) the function $t \in \mathbb{R} \longmapsto (A(t)u\,,\, v)_{\mathcal{H}} \in \mathbb{C}$ is real analytic (respectively C^∞, or $C^{k,\alpha}$) for every $u \in V$ and $v \in \mathcal{H}$. If this is the case then (by a result in [11]) the (vector valued) function

$$\mathbb{R} \to \mathcal{H}, \quad t \in \mathbb{R} \longmapsto A(t)u \in \mathcal{H},$$

is of the same class for every $u \in V$. Also it is customary to call $t \in \mathbb{R} \mapsto A(t)$ an analytic curve (respectively a curve of class C^∞, or $C^{k,\alpha}$). A function $f : \mathbb{R} \to \mathcal{H}$ is of class $C^{k,\alpha}$ if the set $\{|t-s|^{-\alpha}[f^{(k)}(t) - f^{(k)}(s)] : t \neq s\}$ is locally bounded.

A sequence $\{\lambda_\nu\}_{\nu\geq 1}$ of scalar functions $\lambda_\nu : \mathbb{R} \to \mathbb{C}$ is said to *parametrize the eigenvalues* of $\{A(t)\}_{t\in\mathbb{R}}$ if for any $t \in \mathbb{R}$ and any $\lambda \in \sigma(A(t))$ the cardinality of the set $\{\nu \geq 1 : \lambda_\nu(t) = \lambda\}$ equals the multiplicity of λ.

We shall make use of the following result, which is referred hereafter as the Rellich-Alekseevsky-Kriegl-Losik-Michor theorem (cf. F. Rellich, [41], for statement (i), D. Alekseevski & A. Kriegl & M. Losik & P.W. Michor, [20], for statement (ii), and A. Kriegl & P.W. Michor, [10], for statements (iii)-(iv))

Theorem 2.1. *Let $t \in \mathbb{R} \mapsto A(t)$ be a curve of unbounded selfadjoint operators in a Hilbert space $\mathcal{H}$, with common domain of definition and compact resolvent. Then*

i) *If $A(t)$ is real analytic in $t \in \mathbb{R}$ then the eigenvalues and the eigenvectors of $A(t)$ may be parameterized real analytically in t.*

ii) *If $A(t)$ is C^∞ in $t \in \mathbb{R}$ and if no two unequal continuously parameterized eigenvalues meet of infinite order at any $t \in \mathbb{R}$, then the eigenvalues and eigenvectors can be parameterized C^∞ in t on the whole parameter domain.*

iii) *If A is C^∞ then the eigenvalues of $A(t)$ may be parameterized C^2 in t.*

iv) *If $A(t)$ is $C^{k,\alpha}$ in $t \in \mathbb{R}$ for some $\alpha > 0$ then the eigenvalues of $A(t)$ may be parameterized C^1 in t.*

More is actually proved in [10] and statements (iii)-(iv) in Theorem 2.1 follow from the stronger result (cf. [10], p. 2)

Theorem 2.2. *Under the assumptions of Theorem* 2.1

iii.1) *If $A(t)$ is $C^{3n,\alpha}$ in t and if the multiplicity of an eigenvalue never exceeds n, then the eigenvalues of A may be parameterized C^2.*

iii.2) *If the multiplicity of any eigenvalue never exceeds n, and if the resolvent $(A(t) - \lambda I)^{-1}$ is C^{3n} into $L(\mathcal{H}, \mathcal{H})$ in t and λ jointly , then the eigenvalues of $A(t)$ may be parameterized C^2 in t.*

iv.1) *If the resolvent $(A(t) - \lambda I)^{-1}$ is C^1 into $L(\mathcal{H}, \mathcal{H})$ jointly in t and λ then the eigenvalues of $A(t)$ may be parameterized C^1 in t.*

iv.2) *Under the hypothesis of statements* (iv) *or* (iv.1)*, for any continuous parametrization $\lambda_\nu(t)$ of $\sigma(A(t))$, every function λ_ν has a right sided derivative $\lambda_\nu^{(+)}(t)$ and a left sided derivative $\lambda_\nu^{(-)}(t)$ at each t, and $\{\lambda_\nu^{(+)}(t) : \lambda_\nu(t) = \lambda\}$ equals $\{\lambda_\nu^{(-)}(t) : \lambda_\nu(t) = \lambda\}$ with correct multiplicities.*

Among the applications to statement (iii) in Theorem 2.1 as proposed in [10] one may consider a compact manifold M and a smooth curve $t \mapsto g_t$ of smooth Riemannian metrics on M. If moreover $t \mapsto \Delta_{g_t}$ is the corresponding smooth curve of Laplace-Beltrami operators on $L^2(M)$ then (by (iii) in Theorem 2.1) the eigenvalues may be parameterized C^2 in t. This was exploited by A. El Soufi & S. Ilias, [5]-[7], who discussed an array of related questions such as critical points of the functional $g \in \mathcal{M} \mapsto \lambda_k(g)$, or suitable deformations of $g \in \mathcal{M}$ producing quantitative variations of λ_k. Here $\mathcal{M}$ is the set of all Riemannian metrics on M.

Let (M, θ) be a compact strictly pseudoconvex pseudohermitian manifold, of CR dimension n. Let

$$\theta(t) = e^{u_t}\, \theta, \quad t \in \mathbb{R},$$

be an *analytic deformation* of θ i.e. $\{u_t\}_{t\in\mathbb{R}}$ is a family of real valued C^∞ functions which is analytic with respect to t and $u_0 = 0$. Here $C^\infty(M, \mathbb{R})$ is thought of as organized as a real Fréchet space and the vector valued function

$$u : \mathbb{R} \to C^\infty(M, \mathbb{R}), \quad u(t) = u_t\,, \quad t \in \mathbb{R},$$

is assumed to be of class C^ω. For a theory of power series in Fréchet spaces we shall use Appendix B in [27]. Let $\Delta_{b,t}$ be the sublaplacian of $(M, \theta(t))$.

Theorem 2.3. *If $\theta(t) = e^{u_t}\, \theta$ is an analytic deformation of θ then there is $\epsilon > 0$ and a family of real analytic functions $\{\lambda_\nu\}_{\nu\geq 1} \subset C^\omega((-\epsilon, \epsilon), \mathbb{R})$ such that for any $|t| < \epsilon$ and for any eigenvalue $\lambda \in \sigma(\Delta_{b,t})$ of multiplicity m there exist m families of C^∞ functions*

$$\{u_i(t)\}_{|t|<\epsilon} \in C^\infty(M, \mathbb{R}), \quad 1 \leq i \leq m,$$

such that each $u_i : (-\epsilon, \epsilon) \to C^\infty(M, \mathbb{R})$ is real analytic in t and

1) *$\lambda_i(t) = \lambda$, $1 \leq i \leq m$,*

2) *$\Delta_{b,t} u_i(t) = \lambda\, u_i(t)$, $1 \leq i \leq m$,*

3) *$\{u_i(t) : 1 \leq i \leq m\}$ is orthonormal in $L^2(M, \Psi_{\theta(t)})$.*

Proof. The proof relies on the Rellich-Alekseevici-Kriegl-Losik-Michor theorem (cf. Theorem 2.1 above). To this end we introduce the family of operators

$$U_t : L^2(M, \Psi_\theta) \to L^2(M, \Psi_{\theta(t)}), \quad U_t u = e^{-(n+1)u_t/2}\, u, \quad u \in L^2(M, \Psi_\theta).$$

$\{U_t\}_{t\in\mathbb{R}}$ is a real analytic family of unitary i.e.

$$\|U_t\, u\|_{L^2(M,\Psi_{\theta(t)})} = \|u\|_{L^2(M,\Psi_\theta)}$$

operators among the Hilbert spaces $L^2(M, \Psi_\theta)$ and $L^2(M, \Psi_{\theta(t)})$ and $U_t^{-1}\, u = e^{(n+1)u_t/2}\, u$. Moreover let $A(t)$ be the family of operators

$$A(t) = U_t^{-1} \circ \Delta_{b,t} \circ U_t : L^2(M, \Psi_\theta) \to L^2(M, \Psi_\theta).$$

Then

$$\Delta_{b,t} u_i(t) = \lambda\, u_i(t) \Longleftrightarrow A(t)\left(U_t^{-1}\, u_i(t)\right) = \lambda\, U_t^{-1}\, u_i(t).$$

Let us show that the family $\{A(t)\}_{t\in\mathbb{R}}$ is analytic in t. Indeed the dense subspace $\mathcal{D}(\Delta_b) = C^\infty(M) \subset L^2(M, \Psi_\theta)$ is the domain of $A(t)$ and, as we shall check in a moment, $A(t) \subset A(t)^*$. By a result of E. Barletta & S. Dragomir (cf. Proposition 5 in [28], p. 11) if $\theta(t) = e^{u_t}\,\theta$ then the sublaplacians $\Delta_{b,\theta}$ and $\Delta_{b,t} = \Delta_{b,\theta(t)}$ are related by

$$\Delta_{b,t} v = e^{-u_t}\left(\Delta_b v - n(\nabla^H v)(u_t)\right), \quad v \in C^2(M). \tag{2.1}$$

Then for each $v \in \mathcal{D}(\Delta_b)$

$$\begin{aligned}
A(t)v &= (U_t^{-1} \circ \Delta_{b,t} \circ U_t)v \\
&= e^{\frac{n+1}{2}u_t}\,\Delta_{b,t}\left(e^{-\frac{n+1}{2}u_t}\, v\right) \\
&= e^{\frac{n+1}{2}u_t}\, e^{-u_t}\left(\Delta_b(e^{-\frac{n+1}{2}u_t}\, v) - n(\nabla^H e^{-\frac{n+1}{2}u_t}\, v)(u_t)\right) \\
&= e^{\frac{n+1}{2}u_t}\, e^{-u_t}\left(\Delta_b(e^{-\frac{n+1}{2}u_t}\, v)\right) - n\, e^{\frac{n+1}{2}u_t}\, e^{-u_t}\,(\nabla^H e^{-\frac{n+1}{2}u_t}\, v)(u_t) \\
&= e^{-u_t}\left(\Delta_b v + v\, e^{\frac{n+1}{2}u_t}\,\Delta_b\, e^{-\frac{n+1}{2}u_t} - 2\, e^{\frac{n+1}{2}u_t}\, G_\theta(\nabla^H v\,,\, \nabla^H e^{-\frac{n+1}{2}u_t})\right) \\
&- n\, e^{\frac{n+1}{2}u_t}\, e^{-u_t}\,(-v\,\frac{n+1}{2}\, e^{-\frac{n+1}{2}u_t}\,\nabla^H\, u_t + e^{-\frac{n+1}{2}u_t}\,\nabla^H\, v)(u_t) \\
&= e^{-u_t}\left[\Delta_b v - v\frac{n+1}{2}(\Delta_b u_t + \frac{n+1}{2}\left|\nabla^H u_t\right|^2) + (n+1)\, G_\theta(\nabla^H v\,,\, \nabla^H u_t)\right] \\
&- n\, e^{-u_t}\,(-v\,\frac{n+1}{2}\,\nabla^H\, u_t + \nabla^H\, v)(u_t) \\
&= e^{-u_t}\,\Delta_b v - v\, e^{-u_t}\frac{(n+1)}{2}\left[\Delta_b u_t + \frac{(n+1)}{2}\left|\nabla^H u_t\right|^2 - n(\nabla^H u_t)(u_t)\right] \\
&+ e^{-u_t}\, G_\theta(\nabla^H v\,,\, \nabla^H u_t) \\
&= e^{-u_t}\left[\Delta_b v + G_\theta(\nabla^H u_t, \nabla^H v) - \frac{n+1}{2}\left(\Delta_b u_t - \frac{(n-1)}{2}|\nabla^H u_t|^2\right) v\right].
\end{aligned}$$

Therefore for any $v \in \mathcal{D}(\Delta_b)$ and $w \in L^2(M, \Psi_\theta)$

$$\langle A(t)v\,,\, w\rangle_{L^2(M,\Psi_\theta)} =$$
$$= \langle e^{-u_t}\left(\Delta_b v + G_\theta(\nabla^H u_t, \nabla^H v)\right), w\rangle_{L^2(M,\Psi_\theta)}$$
$$-\langle e^{-u_t}\frac{(n+1)}{2}\left(\Delta_b u_t - \frac{(n-1)}{2}|\nabla^H u_t|^2\right)v\,,\, w\rangle_{L^2(M,\Psi_\theta)}$$

Finally the family $\{A(t)\}_{t\in\mathbb{R}}$ satisfies the Krigel-Michor theorem, [10]: be a self-adjoint operator in $L^2(M, \Psi_\theta)$ with common domain of definition and with compact resolvent (see Lemma 1.4), then we have the eigenvalues and the eigenvectors of $A(t)$ are analytically in t i.e there exists m analytic families of vectors $u_i(t)$ and m real analytic valued functions $\Lambda_i(t)$ in t satisfying $1, 2$ et 3 of Theorem 2.3. □

2.2 Critical contact forms

We adopt the notations and conventions in [4]. We start by discussing derivatives of eigenvalues with respect to deformations of contact forms. Let M be a compact strictly pseudoconvex CR manifold. For any positively oriented contact form $\theta \in \mathcal{P}_+$ let $0 < \lambda_1(\theta) \le \lambda_2(\theta) \le \cdots \le \lambda_k(\theta) \le \cdots$ be the spectrum of the sublaplacian $\Delta_b = \Delta_{b,\theta}$ of (M, θ). For every $k \in \mathbb{N}$ let

$$E_k(\theta) = \text{Ker } (\Delta_b - \lambda_k(\theta) I)$$

be the eigenspace Δ_b corresponding to the eigenvalue to $\lambda_k(\theta)$. Also let $\pi_k : L^2(M, \Psi_\theta) \to E_k(\theta)$ be the orthogonal projection on $E_k(\theta)$. Let us fix $k \in \mathbb{N}$ and consider the functional $\theta \in \mathcal{P}_+ \longmapsto \lambda_k(\theta) \in \mathbb{R}$. This functional is continuous (with respect to an appropriate metric topology on $\mathcal{P}_+$, as shown in § 2.7) but not differentiable in general. However, by perturbation theory λ_k is left and right differentiable along any analytic curve in $\mathcal{P}_+$. The main purpose of this section is to express the derivatives of λ_k (with respect to analytic deformations of contact structures) in terms of the eigenvalues of an explicit quadratic form on $E_k(\theta)$.

Theorem 2.4. *Let M be a compact strictly pseudoconvex CR manifold. For every $\theta \in \mathcal{P}_+$ on M let $\{\theta(t)\}_{|t|<\epsilon} \subset \mathcal{P}_+$ be a complex analytic family of contact forms such that $\theta(0) = \theta$. Then*

1) *The function $t \in (-\epsilon, \epsilon) \longmapsto \lambda_k(\theta(t))$ admits left and right derivatives at $t = 0$.*

2) *The derivatives*

$$\frac{d}{dt}\{\lambda_k(\theta(t))\}_{t=0^-}\ ,\ \frac{d}{dt}\{\lambda_k(\theta(t))\}_{t=0^+} \in \mathbb{R}$$

are eigenvalues of the operator

$$\pi_k \circ \Delta_b' : E_k(\theta) \longrightarrow E_k(\theta), \quad \Delta_b' \equiv \frac{d}{dt}\{\Delta_{b,t}\}_{t=0}\,.$$

3) *If $\lambda_k(\theta) > \lambda_{k-1}(\theta)$, then $\frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^-}$ and $\frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^+}$ are the greatest and the least eigenvalues of $\pi_k \circ \Delta_b'$ on $E_k(\theta)$, respectively.*

4) *If $\lambda_k(\theta) < \lambda_{k+1}(\theta)$ then*

$$\frac{d}{dt}\{\lambda_k(\theta(t))\}_{t=0^-}\ ,\ \frac{d}{dt}\{\lambda_k(\theta(t))\}_{t=0^+} \in \mathbb{R}$$

are the smallest and the greatest eigenvalue of $\pi_k \circ \Delta_b'$ on $E_k(\theta)$ respectively.

Proof. 1. By Theorem 3.1, for $|t| < \epsilon$, there exist $\Lambda_i(t) \in \mathbb{R}$ and $u_i(t) \in C^\infty(M),\ i = 1, ..., m$ depending real analytically on t where m is the dimension of $E_k(\theta)$ and $\Lambda_1(0) = ... = \Lambda_m(0) = \lambda_k(\theta)$. Since $t \mapsto \lambda_k(\theta(t))$ is continuous and, $\forall i \leq m,\ t \mapsto \Lambda_i(t)$ is analytic with $\Lambda_i(0) = \lambda_k(\theta)$, there exist $\delta > 0$ and two integers $p,\ q \leq m$ such that

$$\lambda_k(\theta(t)) = \begin{cases} \Lambda_p(t) & \text{for } t \in (-\delta, 0) \\ \Lambda_q(t) & \text{for } t \in (0, \delta). \end{cases}$$

Then the function $t \longmapsto \lambda_k(\theta(t))$ admits left and right derivatives at $t = 0$. Moreover, one has

$$\frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^-} = \Lambda'_p(0) \ \textit{and}\ \frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^+} = \Lambda'_q(0).$$

2. For $i \leq m$, let $\Delta_{b,t}u_i(t) = \Lambda_i(t)u_i(t)$ by deriving at $t = 0$, we get

$$\Delta'_b u_i(0) + \Delta_b u_i(0) = \Lambda'_i(0)u_i(0) + \lambda_k(\theta)u'_i(0) \tag{2.2}$$

where $u'_i(0) = \frac{d}{dt}u_i(t)\big|_{t=0}$, we obtain after multiplying (4.1) by u_j and integrating by parts

$$\int_M u_j \Delta'_b u_i \Psi_\theta = \begin{cases} \Lambda'_i(0) & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases}$$

Since $\{u_1, \cdots, u_m\}$ is an orthonormal basis of $E_k(\theta)$ with respect to the $L^2(M, \theta)$, we deduce that

$$(\pi_k \circ \Delta'_b)u_i = \Lambda'_i(0)u_i.$$

In particular, $\Lambda'_p(0)$ and $\Lambda'_q(0)$ are eigenvalues of $\pi_k \circ \Delta'_b$.

3. Assume now $\lambda_k(\theta) > \lambda_{k-1}(\theta)$ and for any $i \leq m$, one has $\Lambda_i(0) = \lambda_k(\theta) > \lambda_{k-1}(\theta)$. Then by continuity, we have $\Lambda_i(t) > \lambda_{k-1}(\theta(t))$ for sufficiently small t. Hence, there exists $\eta > 0$ such that, $\forall\, |t| < \eta$ and $\forall i \leq m$, $\Lambda_i(t) \geq \lambda_k(\theta(t))$, which means that $\lambda_k(\theta(t)) = \min\{\Lambda_1(t), \cdots, \Lambda_m(t)\}$. This implies that

$$\frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^-} = \max\left\{\Lambda'_1(0), \cdots, \Lambda'_m(0)\right\}.$$

and

$$\frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^+} = \min\left\{\Lambda'_1(0), \cdots, \Lambda'_m(0)\right\}.$$

4. The proof is similar to the previous one. If $\lambda_k(\theta) < \lambda_{k+1}(\theta)$, one has, for sufficiently small t, $\Lambda_i(t) \leq \lambda_k(\theta(t))$ which means that $\lambda_k(\theta(t)) = \max\{\Lambda_1(t), \cdots, \Lambda_m(t)\}$ and, then,

$$\frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^+} = \max\left\{\Lambda'_1(0), \cdots, \Lambda'_m(0)\right\}$$

and

$$\frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^-} = \min\left\{\Lambda'_1(0), \cdots, \Lambda'_m(0)\right\}.$$

□

Let M be a compact strictly pseudoconvex CR manifold. For each $\theta \in \mathcal{P}_+$ we set

$$C(\theta) = \left\{e^f\theta\,;\ f \in C^\infty(M) \text{ and } vol(e^f\theta) = vol(\theta)\right\}$$

where $\mathrm{Vol}(\theta) = \int_M \Psi_\theta$ is the volume of (M, θ). In the following, we study critical pseudohermitian structure of the functional λ_k restricted to a conformal class $C(\theta)$ for any positive integer k.

Definition 2.5. *A pseudohermitian structure θ is said to be critical for the functional λ_k restricted to $C(\theta)$ if for any analytic deformation $\{\theta(t) = e^{u_t}\theta\} \subset C(\theta)$ with $\theta(0) = \theta$, we have*

$$\frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^-} \times \frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^+} \leq 0.$$

We denote by $\mathcal{A}_0(M, \theta)$ the set of regular functions f with zero mean on M, that is, $\int_M f\,\Psi_\theta = 0$.

Theorem 2.6. *Let θ be a pseudohermitian structure on a compact strictly pseudoconvex CR manifolds M.*

1. *If θ is a critical pseudohermitian structure of the functional λ_k restricted to $C(\theta)$, then, $\forall f \in \mathcal{A}_0(M, \theta)$, the quadratic form*

$$Q_f(u) = (n+1)\int_M \left(\lambda_k(\theta)u^2 - \frac{n}{n+1}\left\|\nabla^H u\right\|^2\right) f\,\Psi_\theta \tag{2.3}$$

   is indefinite on $E_k(\theta)$.

2. *Assume that $\lambda_k(\theta) > \lambda_{k-1}(\theta)$ or $\lambda_k(\theta) < \lambda_{k+1}(\theta)$. The pseudohermitian structure θ is critical for the functional λ_k restricted to $C(\theta)$ if and only if, $\forall f \in \mathcal{A}_0(M, \theta)$, the quadratic form Q_f is indefinite on $E_k(\theta)$.*

Proof. 1. $\forall f \in \mathcal{A}_0(M, \theta)$, the conformal deformation of θ given by

$$\theta(t) = \left[\frac{\mathrm{vol}(\theta)}{\mathrm{vol}(e^{tf}\theta)}\right]^{\frac{1}{n+1}} e^{tf}\theta$$

belongs to $C(\theta)$ and depends analytically on t with $\frac{d}{dt}\theta(t)\big|_{t=0} = f\theta$, for $f \in \mathcal{A}_0(M, \theta)$. The sub-Laplacian $\Delta_{b,t}$ associated with $\theta(t)$ is given by

$$\Delta_{b,t}u = \left[\frac{\mathrm{vol}(e^{tf}\theta)}{\mathrm{vol}(\theta)}\right]^{\frac{1}{n+1}} e^{-tf}\left(\Delta_b u - nt\langle\nabla^H u, \nabla f\rangle_{G_\theta}\right).$$

Therefore, since $\int_M f\Psi_\theta = 0$, we have

$$\begin{aligned}\frac{d}{dt}\mathrm{vol}(e^{tf}\theta(t))\Big|_{t=0} &= \frac{d}{dt}\int_M e^{(n+1)ft}\Psi_\theta\Big|_{t=0}\\ &= (n+1)\int_M f\Psi_\theta = 0,\end{aligned}$$

and, then,

$$\begin{aligned}\Delta_b' u &= \frac{d}{dt}\Delta_{b,t}\Big|_{t=0}\\ &= -f\Delta_b u - n\langle \nabla^H u, \nabla f\rangle_{G_\theta}.\end{aligned}$$

Consequently, $\forall u \in E_k(\theta)$,

$$\begin{aligned}\int_M u(\pi_k \circ \Delta_b')u\Psi_\theta &= \int_M u\Delta_b' u\Psi_\theta\\ &= \int_M \big(- fu\Delta_b u - nu\langle \nabla^H u, \nabla f\rangle\big)\Psi_\theta\\ &= \int_M \Big(- fu\Delta_b u - \frac{n}{2}\langle \nabla^H u^2, \nabla f\rangle\Big)\Psi_\theta\\ &= \int_M \Big(f\lambda_k(\theta)u^2 - \frac{n}{2}f\Delta_b u^2\Big)\Psi_\theta\\ &= \int_M \Big(\lambda_k(\theta)u^2 - nu\Delta_b u - n\left\|\nabla^H u\right\|^2\Big)f\Psi_\theta\\ &= \int_M \Big((n+1)\lambda_k(\theta)u^2 - n\left\|\nabla^H u\right\|^2\Big)f\Psi_\theta.\end{aligned}$$

Thus,

$$\int_M u(\pi_k \circ \Delta_b')u\Psi_\theta = Q_f(u) \tag{2.4}$$

Since θ is critical, we apply (2) of Theorem 2.4 to deduce that the eigenvalues of the operator $\pi_k \circ \Delta_b'$ restricted to $E_k(\theta)$ are not all positive or all negative. From (2.4), it follows that the quadratic form Q_f is indefinite on $E_k(\theta)$.

2. Let $\theta(t) = e^{u_t}\theta \in C(\theta)$ be an analytic deformation of θ. Since $\mathrm{vol}(\theta(t))$ is constant with respect to t, the function $f = \frac{d}{dt}u_t\big|_{t=0} \in \mathcal{A}_0(M,\theta)$. Indeed,

$$\frac{d}{dt}\mathrm{vol}(\theta(t))\Big|_{t=0} = \frac{d}{dt}\int_M e^{(n+1)u_t}\Psi_\theta\Big|_{t=0} = (n+1)\int_M f\Psi_\theta.$$

Using (2.4) and (3), (4) of Theorem 4.1, we get the result.

□

Proposition 2.7. *Let θ be a pseudohermitian structure on a compact strictly pseudoconvex CR manifold M. The two following conditions are equivalent:*

1. *For all $f \in \mathcal{A}_0(M,\theta)$, the quadratic form Q_f is indefinite on $E_k(\theta)$.*
2. *There exists a finite family $\{u_1, \cdots, u_d\} \subset E_k(\theta)$ of eigenfunctions associated with $\lambda_k(\theta)$ such that $\sum_i^d u_i^2 = 1$.*

Proof. 1. Let K be the convex hull :

$$K = \left\{\sum_{i\in J}(n+1)\left[\lambda_k(\theta)u_i^2 - \frac{n}{n+1}\left\|\nabla^H u_i\right\|^2\right]; u_i \in E_k(\theta),\ J \subset \mathbb{N},\ J \text{ finite}\right\} \subset L^2(M,\theta).$$

We show that the constant function 1 belongs to K. Indeed, if $1 \notin K$, then, applying classical separation theorem in the finite dimensional subspace of $L^2(M,\theta)$ generated by K and θ, we deduce the existence of $v \in L^2(M,\theta)$ such that $\int_M v\Psi_\theta > 0$ and, $\forall\, w \in K$, $\int_M vw\Psi_\theta \leq 0$. Let $f_0 = v - \frac{1}{\mathrm{vol}(\theta(t))}\int_M v\Psi_\theta \in \mathcal{A}_0(M,\theta)$. Then , $\forall u \in E_k(\theta)$

$$\begin{aligned}
Q_{f_0}(u) &= \int_M (\lambda_k(\theta)u^2 - \frac{n}{n+1}\left\|\nabla^H u\right\|^2) f_0\Psi_\theta \\
&= \int_M (\lambda_k(\theta)u^2 - \frac{n}{n+1}\left\|\nabla^H u\right\|^2) v\Psi_\theta \\
&- \frac{\int_M v\Psi_\theta}{\mathrm{vol}(\theta(t))}\int_M (\lambda_k(\theta)u^2 - \frac{n}{n+1}\left\|\nabla^H u\right\|^2)\Psi_\theta \\
&= \int_M (\lambda_k(\theta)u^2 - \frac{n}{n+1}\left\|\nabla^H u\right\|^2) v\Psi_\theta \\
&- \frac{\lambda_k(\theta)\int_M v\Psi_\theta}{(n+1)\mathrm{vol}(\theta(t))}\int_M u^2\Psi_\theta.
\end{aligned}$$

Since $\int_M (\lambda_k(\theta)u^2 - \frac{n}{n+1}\left\|\nabla^H u\right\|^2) v\Psi_\theta \leq 0$, the quadratic form Q_{f_0} is negative definite, which contradicts the assumtion (1). Hence, there exist $u_1, \cdots, u_d \in E_k(\theta)$ such that

$$\sum_i^d \left(\lambda_k(\theta)u_i^2 - \frac{n}{n+1}\left\|\nabla^H u_i\right\|^2\right) = \frac{1}{(n+1)}\lambda_k(\theta). \tag{2.5}$$

We set $g = \sum_{i\leq d} u_i^2 - 1$. From (2.5) we get

$$\begin{aligned}
\frac{n}{2}\Delta_b g &= n\Big(\lambda_k(\theta)\sum_i^d u_i^2 + \sum_i^d \left\|\nabla^H u_i\right\|^2\Big) \\
&= \lambda_k(\theta) g.
\end{aligned}$$

This implies that $g = 0$, since the sub-Laplacian admits no negative eigenvalues. Therefore $\sum_i^d u_i^2 = 1$.

2. Let $u_1, \cdots, u_d \in E_k(\theta)$ such that $\sum_i^d u_i^2 = 1$. One has

$$\begin{aligned}
\sum_i^d \left\|\nabla^H u_i\right\|^2 &= \frac{1}{2}\Delta_b \sum_i^d u_i^2 + \lambda_k(\theta)\sum_i^d u_i^2 \\
&= \lambda_k(\theta).
\end{aligned}$$

Therefore,

$$\int_M \sum_i^d \left[\lambda_k(\theta)u_i^2 - \frac{n}{n+1}\left\|\nabla^H u_i\right\|^2\right] f\Psi_\theta = \frac{\lambda_k(\theta)}{n+1}\int_M f\Psi_\theta = 0$$

$\forall f \in \mathcal{A}_0(M,\theta)$. This implies that Q_f is indefinite on $E_k(\theta)$.

□

Theorem 2.6 and Proposition 2.7 lead to the following

Theorem 2.8. *Let θ be a pseudohermitian structure on a compact strictly pseudoconvex CR manifolds M.*

1. *If θ is a critical pseudohermitian structure of the functional λ_k restricted to $C(\theta)$, then there exists a finite family $\{u_1, \cdots, u_d\} \subset E_k(\theta)$ of eigenfunctions associated with λ_k such that $\sum_i^d u_i^2 = 1$.*
2. *Assume that $\lambda_k(\theta) > \lambda_{k-1}(\theta)$ or $\lambda_k(\theta) < \lambda_{k+1}(\theta)$. Then, θ is critical for the functional λ_k restricted to $C(\theta)$ if and only if, there exists a finite family $\{u_1, \cdots, u_d\} \subset E_k(\theta)$ of eigenfunctions associated with $\lambda_k(\theta)$ such that $\sum_i^d u_i^2 = 1$.*

An immediate consequence is the following:

Corollary 2.9. *If θ is a critical metric of the functional λ_k restricted to $C(\theta)$, then $\lambda_k(\theta)$ is a degenerate eigenvalue, that is*

$$\dim E_k(\theta) \geq 2.$$

This last condition means that at least one of the following holds: $\lambda_k(\theta) = \lambda_{k-1}(\theta)$ or $\lambda_k(\theta) = \lambda_{k+1}(\theta)$. In the case when θ is a local maximizer or a local minimizer, we have the following more precise result

Proposition 2.10.

1. *If θ is a local minimizer of the functional λ_k restricted to $C(g)$, then $\lambda_k(\theta) = \lambda_{k-1}(\theta)$.*
2. *If θ is a local maximizer of the functional λ_k restricted to $C(g)$, then $\lambda_k(\theta) = \lambda_{k+1}(\theta)$.*

Proof. Assume that θ is a local minimizer and that $\lambda_k(\theta) > \lambda_{k-1}(\theta)$. Let $f \in \mathcal{A}_0(M, \theta)$ and let $\theta(t) = e^{\alpha_t}\theta \in C(\theta)$ be a volume-preserving analytic deformation of θ such that $\frac{d}{dt}\theta(t)\big|_{t=0} = f\theta$. Denote by $\Lambda_1(t), \cdots, \Lambda_m(t)$, the associated family of eigenvalues of $\Delta_{b,t}$, depending analytically on t and such that $\Lambda_1(0) = \cdots = \Lambda_m(0) = \lambda_k(\theta)$ with $m = \dim E_k(\theta)$ (see the proof of Theorem 2.4). For continuity reasons, we have, for sufficiently small t and all $i \leq m$,

$$\Lambda_i(t) > \lambda_{k-1}(\theta(t)).$$

Hence, $\forall i \leq m$ and $\forall t$ sufficiently small,

$$\Lambda_i(t) \geq \lambda_k(\theta(t)) \geq \lambda_k(\theta) = \Lambda_i(0).$$

Consequently for all $i \leq m$, $\Lambda_i'(0) = 0$. Since $\Lambda_1'(0), \cdots, \Lambda_m'(0)$ are eigenvalues of the operator $\pi_k \circ \Delta_b'$ (by Theorem 2.4) and $(\pi_k \circ \Delta_b')u = 0$, $\forall u \in E_k(\theta)$. Applying (2.4), we deduce that, $\forall f \in \mathcal{A}_0(M, \theta)$,

$$Q_f(u) = 0.$$

$\forall u \in E_k(\theta)$. Thus, there exists a constant β so that

$$(n+1)\Big(\lambda_k(\theta)u^2 - \frac{n}{n+1}\left\|\nabla^H u\right\|^2\Big) = \beta$$

Integrating, we get

$$\beta = \frac{\lambda_k(\theta)}{\mathrm{vol}(\theta)} \int_M u^2 \Psi_\theta.$$

Then, we obtain

$$(n+1)u^2 - \frac{n}{\lambda_k(\theta)} \left\|\nabla^H u\right\|^2 = \frac{1}{\mathrm{vol}(\theta)} \int_M u^2 \Psi_\theta.$$

Let $x \in M$ be a point where u^2 achieves its minimum. At x, we have

$$\left\|\nabla^H u(x)\right\|^2 = 0$$

and

$$(n+1)u^2(x) = \frac{1}{\mathrm{vol}(\theta)} \int_M u^2 \Psi_\theta$$

which leads to a contradiction (since u is not constant).
A similar proof works for (2). □

2.3 Eigenvalues ratio functionals

Let (M, θ) be a compact strictly pseudoconvex CR manifold of CR dimension n. This section deals with the functional $\theta \longmapsto \dfrac{\lambda_{k+1}(\theta)}{\lambda_k(\theta)}$. This functional is invariant under scaling, so it is not necessary to fix the volume of pseudohermitian structure form under consideration. If $\theta(t)$ is any analytic deformation of a pseudohermitian structure form θ, then $t \longmapsto \dfrac{\lambda_{k+1}(\theta(t))}{\lambda_k(\theta(t))}$ admits left and right derivatives at $t = 0$ (Theorem 2.4).

Definition 2.11. *1. A pseudohermitian structure form θ is said to be critical for the ratio $\dfrac{\lambda_{k+1}}{\lambda_k}$ if for any analytic deformation $\theta(t)$ of θ, the left and right derivatives of $\dfrac{\lambda_{k+1}(\theta(t))}{\lambda_k(\theta(t))}$ at $t = 0$ have opposite signs.*

2. The pseudohermitian structure form θ is said to be critical for the ratio functional $\dfrac{\lambda_{k+1}}{\lambda_k}$ restricted to the conformal class $C(\theta)$ if the condition above holds for any conformal analytic deformation $\theta(t) = e^{\alpha_t}\theta$ of θ.

Let θ be a pseudohermitian structure form on M. We introduce the following operator

$$P_k : E_k(\theta) \otimes E_{k+1}(\theta) \longrightarrow E_k(\theta) \otimes E_{k+1}(\theta)$$

defined by

$$P_k = \lambda_{k+1}(\theta)(\pi_k \circ \Delta_b') \otimes Id_{E_{k+1}(\theta)} - \lambda_k(\theta) Id_{E_k(\theta)} \otimes (\pi_{k+1} \circ \Delta_b')$$

where $\pi_k : L^2(M, \Psi_\theta) \to E_k(\theta)$. The quadratic form naturally associated with P_k is denoted by $\tilde{Q}_f$ and is given by, $\forall u \in E_k(\theta)$ and $\forall v \in E_{k+1}(\theta)$,

$$\tilde{Q}_f(u \otimes v) = \lambda_{k+1}(\theta) \|v\|^2_{L^2(\theta)} Q_f(u) - \lambda_k(\theta) \|u\|^2_{L^2(\theta)} Q_f(v),$$

where

$$Q_f(u) = (n+1) \int_M \left(\lambda_k(\theta) u^2 - \frac{n}{n+1} \left\|\nabla^H u\right\|^2\right) f\, \Psi_\theta.$$

Theorem 2.12. *A pseudohermitian structure θ on M is critical for the functional $\frac{\lambda_{k+1}}{\lambda_k}$ if and only if the quadratic form $\tilde{Q}_f$ is indefinite on $E_k(\theta) \otimes E_{k+1}(\theta)$.*

Proof. The case where $\lambda_{k+1}(g) = \lambda_k(g)$ is obvious $\tilde{Q}_f(u \otimes u) = 0$. Assume that $\lambda_{k+1}(\theta) > \lambda_k(\theta)$ and let $\theta(t)$ be an analytic deformation of θ. From Theorem (2.4)

$$\frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^-} \quad and \quad \frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^+}$$

are the least and the greatest eigenvalues of $(\pi_k \circ \Delta'_b)$ on $E_k(\theta)$ respectively.
Similarly, $\frac{d}{dt}\lambda_k(g_t)\Big|_{t=0^-}$ and $\frac{d}{dt}\lambda_k(g_t)\Big|_{t=0^+}$ are the greatest and the least eigenvalues of $(\pi_{k+1} \circ \Delta'_b)$ on $E_k(\theta)$. Therefore,

$$\lambda_k(\theta)^2 \frac{d}{dt}\frac{\lambda_{k+1}(\theta(t))}{\lambda_k(\theta(t))}\Big|_{t=0^-} = \left[\lambda_k(\theta)\frac{d}{dt}\lambda_{k+1}(\theta(t))\Big|_{t=0^-} - \lambda_{k+1}(\theta)\frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^-}\right]$$

is the greatest eigenvalue of P_k on $E_k(\theta) \otimes E_{k+1}(\theta)$, and

$$\lambda_k(\theta)^2 \frac{d}{dt}\frac{\lambda_{k+1}(\theta(t))}{\lambda_k(\theta(t))}\Big|_{t=0^+} = \left[\lambda_k(\theta)\frac{d}{dt}\lambda_{k+1}(\theta(t))\Big|_{t=0^+} - \lambda_{k+1}(\theta(t))\frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^+}\right]$$

is the least eigenvalue of P_k on $E_k(\theta) \otimes E_{k+1}(\theta)$. Hence, the criticality of θ for $\frac{\lambda_{k+1}}{\lambda_k}$ is equivalent to the fact that P_k admits eigenvalues of both signs, which is equivalent to the indefiniteness of $\tilde{Q}_f$. □

Proposition 2.13. *Let M be a compact strictly pseudoconvex CR manifold. For any pseudohermitian structure θ on M, the two following conditions are equivalent:*

1. *$\forall f \in \mathcal{A}_0(M,\theta)$, the quadratic form $\tilde{Q}_f$ is indefinite on $E_k(\theta) \otimes E_{k+1}(\theta)$.*
2. *There exist two finite families $\{u_1, \cdots, u_p\} \subset E_k(\theta)$ and $\{v_1, \cdots, v_q\} \subset E_{k+1}(\theta)$ of eigenfunctions associated with $\lambda_k(\theta)$ and $\lambda_{k+1}(\theta)$ respectively, such that*

$$\sum_i^p \left(\lambda_k(\theta)u_i^2 - \frac{n}{n+1}\left\|\nabla^H u_i\right\|^2\right) = \sum_j^q \left(\lambda_{k+1}(\theta)v_i^2 - \frac{n}{n+1}\left\|\nabla^H v_i\right\|^2\right) \tag{2.6}$$

Proof. 1. $\Rightarrow$ (2) : Let us introduce the two following convex cones

$$K_1 = \left\{\sum_{i\in I}\left(\lambda_k(\theta)u_i^2 - \frac{n}{n+1}\left\|\nabla^H u_i\right\|^2\right);\ u_i \in E_k(\theta),\ I \subset \mathbb{N},\ I \text{ finite}\right\} \subset L^2(M,\theta)$$

and

$$K_2 = \left\{\sum_{i\in I}\left(\lambda_{k+1}(\theta)v_i^2 - \frac{n}{n+1}\left\|\nabla^H v_i\right\|^2\right);\ v_i \in E_{k+1}(\theta),\ I \subset \mathbb{N},\ I \text{ finite}\right\} \subset L^2(M,\theta)$$

It suffices to prove that K_1 and K_2 have a nontrivial intersection. Indeed, otherwise, applying classical separation theorems, we show the existence of $h \in L^2(M,\Psi_\theta)$ such that, $\forall w_1 \in K_1, w_1 \neq 0$,

$$\int_M w_1 h > 0$$

and $\forall w_2 \in K_2$,

$$\int_M w_1 h \leq 0$$

Therefore, $\forall u \in E_k(\theta)$ and $\forall v \in E_{k+1}(\theta)$, with $u \neq 0$ and $v \neq 0$, one has $Q_f(u) < 0$, $Q_f(v) \geq 0$ and

$$\begin{aligned}\tilde{Q}_f(u \otimes v) &= \lambda_{k+1}(\theta)\, \|v\|^2_{L^2(\theta)}\, Q_f(u) - \lambda_k(\theta)\, \|u\|^2_{L^2(\theta)}\, Q_f(v) \\ &\leq \lambda_{k+1}(\theta)\, \|v\|^2_{L^2(\theta)}\, Q_f(u) < 0,\end{aligned}$$

which implies that $\tilde{Q}_f$ is negative definite on $E_k(\theta) \otimes E_{k+1}(\theta)$.

2. $\Rightarrow (1)$: Let $\left\{u_1, \cdots, u_p\right\} \subset E_k(\theta)$ and $\left\{v_1, \cdots, v_q\right\} \subset E_{k+1}(\theta)$. From the identity (2.6), we get, after taking the trace and integrating

$$\sum_i^p \int_M \left\|\nabla^H u_i\right\|^2 \Psi_\theta = \sum_j^q \int_M \left\|\nabla^H v_i\right\|^2 \Psi_\theta,$$

which gives,

$$\lambda_k(\theta) \sum_i^p \|u_i\|^2_{L^2(\theta)} = \lambda_{k+1}(\theta) \sum_j^q \left\|v_j\right\|^2_{L^2(\theta)}.$$

Therefore,

$$\sum_{i,j} \tilde{Q}_f(u_i \otimes v_j) = \sum_{i,j} \lambda_{k+1}(\theta) \left\|v_j\right\|^2_{L^2(\theta)} Q_f(u_i) - \lambda_k(\theta)\, \|u_i\|^2_{L^2(\theta)}\, Q_f(v_j).$$

Then (2.6) implies that

$$\sum_i^p Q_f(u_i) = \sum_j^q Q_f(v_j).$$

Therefore,

$$\sum_{i,j} \tilde{Q}_f(u_i \otimes v_j) = \left(\sum_j^q \lambda_{k+1}(\theta) \left\|v_j\right\|^2_{L^2(\theta)} - \sum_i^p \lambda_k(\theta)\, \|u_i\|^2_{L^2(\theta)}\right) \sum_i^p Q_f(u_i) = 0.$$

Consequently, $\tilde{Q}_f$ is indefinite on $E_k(\theta) \otimes E_{k+1}(\theta)$. □

Theorem 2.14. *Let M be a compact strictly pseudoconvex CR manifold. A pseudohermitian structure θ on M is critical for the functional $\frac{\lambda_{k+1}}{\lambda_k}$ restricted to $C(\theta)$ if and only if, there exist two families $\left\{u_1, \cdots, u_p\right\} \subset E_k(\theta)$ and $\left\{v_1, \cdots, v_q\right\} \subset E_{k+1}(\theta)$ of eigenfunctions associated with $\lambda_k(\theta)$ and $\lambda_{k+1}(\theta)$, respectively, such that*

$$\lambda_k(\theta) \sum_i^p u_i^2 - \lambda_{k+1}(\theta) \sum_j^q v_j^2 = \frac{n}{n+1}\Big(\sum_i^p \left\|\nabla^H u_i\right\|^2 - \sum_j^q \left\|\nabla^H v_j\right\|^2\Big). \tag{2.7}$$

Proof. A straightforward calculation shows that the equation (2.6) are equivalent to the condition (2) of Proposition 2.13. □

2.4 A topology on the space of oriented contact forms

We study the behavior of the eigenvalues of a sublaplacian Δ_b on a compact strictly pseudoconvex CR manifold M, as functions on the set $\mathcal{P}_+$ of positively oriented contact forms on M by endowing $\mathcal{P}_+$ with a natural metric topology.

Let M be a compact strictly pseudoconvex CR manifold of CR dimension n, without boundary. Let $\mathcal{P}$ be the set of all C^∞ pseudohermitian structures on M. Every $\theta \in \mathcal{P}$ is a contact form on M i.e. $\theta \wedge (d\theta)^n$ is a volume form. Let $\mathcal{P}_\pm$ be the sets of $\theta \in \mathcal{P}$ such that the Levi form G_θ is positive definite (respectively negative definite). For $\theta \in \mathcal{P}_+$ let Δ_b be the sublaplacian

$$\Delta_b u = -\mathrm{div}(\nabla^H u) \tag{2.8}$$

of (M, θ) acting on smooth real valued functions $u \in C^\infty(M, \mathbb{R})$. As Δ_b is a subelliptic operator (of order $1/2$) it has a discrete spectrum

$$0 = \lambda_0(\theta) < \lambda_1(\theta) \leq \lambda_2(\theta) \leq \cdots \uparrow +\infty \tag{2.9}$$

(the eigenvalues of Δ_b are counted with their multiplicities). Each eigenvalue $\lambda_\nu(\theta)$, $\nu = 0, 1, 2, \cdots$, is thought of as a function of $\theta \in \mathcal{P}_+$. We shall deal mainly with the following problem: *Is there a natural topology on $\mathcal{P}_+$ such that each eigenvalue function $\lambda_\nu : \mathcal{P}_+ \to \mathbb{R}$ is continuous*? The analogous problem for the spectrum of the Laplace-Beltrami operator on a compact Riemannian manifold was solved by S. Bando & H. Urakawa, [90], and our main result is imitative of their Theorem 2.2 (cf. *op. cit.*, p. 155). We shall establish

Corollary 2.15. *For every compact strictly pseudoconvex CR manifold M the space of positively oriented contact forms $\mathcal{P}_+$ admits a natural complete distance function $d : \mathcal{P}_+ \times \mathcal{P}_+ \to [0, +\infty)$ such that each eigenvalue function $\lambda_k : \mathcal{P}_+ \to \mathbb{R}$ is continuous relative to the d-topology.*

By a result of J.M. Lee, [59], for every $\theta \in \mathcal{P}_+$ there is a Lorentzian metric $F_\theta \in \mathrm{Lor}(C(M))$ (the Fefferman metric) on the total space $\mathfrak{M}$ of the canonical circle bundle $S^1 \to C(M) \xrightarrow{\pi} M$. Also if $\Box$ is the Laplace-Beltrami operator of F_θ (the wave operator) then $\mathrm{Spec}(\Delta_b) \subset \mathrm{Spec}(\Box)$. Therefore the eigenvalues λ_k may be thought of as functions $\lambda_k^\uparrow : C \to \mathbb{R}$ on the set $C = \{F_\theta \in \mathrm{Lor}(C(M)) : \theta \in \mathcal{P}_+\}$ of all Fefferman metrics on $\mathfrak{M}$. On the other hand $\mathrm{Lor}(C(M))$ may be endowed with the distance function d_g^∞ considered by P. Mounoud, [80] (associated to a fixed Riemannian metric g on $\mathfrak{M}$) and hence (C, d_g^∞) is itself a metric space. It is then a natural question whether $\lambda_k^\uparrow$ are continuous functions relative to the d_g^∞-topology.

This section is organized as follows. The distance function d (in Corollary 2.15) is built in the following. In § 2.5 we establish a Max-Mini principle (cf. Proposition 2.21) for the eigenvalues of a sublaplacian. Then Corollary 2.15 follows from Theorem 2.22 in § 2.6. In § 2.7 we prove the continuity of the eigenvalues with respect to the Fefferman metric (cf. Corollary 2.23) though only as functions on $C_+ = \{e^{u \circ \pi} F_{\theta_0} : u \in C^\infty(M, \mathbb{R}), \ \ u > 0\}$.

Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be a finite open covering of M such that the closure of each U_λ is contained in a larger open set V_λ which is both the domain of a local frame $\{X_a : 1 \leq a \leq 2n\} \subset C^\infty(V_\lambda, H(M))$ with $X_{\alpha+n} = JX_\alpha$ for any $1 \leq \alpha \leq n$, and a coordinate neighborhood with the local coordinates $(x^1, \cdots, x^{2n+1})$. For each point $x \in M$ let P_x (respectively S_x) be the set of all symmetric positive definite (respectively merely symmetric) bilinear forms on $T_x(M)$. If $\varphi, \psi \in S_x$ then we consider the anti-reflexive partial order relation $\varphi < \psi \iff \psi - \varphi \in P_x$. Next let $\rho''_x : P_x \times P_x \to [0, +\infty)$ be the distance function given by

$$\rho''_x(\varphi, \psi) = \inf\{\delta > 0 : \exp(-\delta)\,\varphi < \psi < \exp(\delta)\,\varphi\}$$

for any $\varphi, \psi \in P_x$. Then $(P_x\,, \rho''_x)$ is a complete metric space (by (iii) of Lemma 1.1 in [90], p. 158).

Let $\mathcal{M}$ be the set of all Riemannian metrics on M, so that $g_\theta \in \mathcal{M}$ for every $\theta \in \mathcal{P}_+$. Following [90] one may endow $\mathcal{M}$ with a complete distance function ρ. Indeed as M is compact one may set

$$\rho''(g_1\,, g_2) = \sup_{x \in M} \rho''_x(g_{1,x}\,, g_{2,x}), \quad g_1\,, g_2 \in \mathcal{M}.$$

Also let $S(M)$ be the space of all C^∞ symmetric $(0,2)$-tensor fields on M, organized as a Fréchet space by the family of seminorms $\{|\cdot|_k : k \in \mathbb{N} \cup \{0\}\}$ where

$$|g|_k = \sum_{\lambda \in \Lambda} |g|_{\lambda,k}\,, \quad |g|_{\lambda,k} = \sup_{x \in \overline{U}_\lambda} \sum_{|\alpha| \le k} \left| D^\alpha g_{ij}(x) \right|,$$

where $D^\alpha = \partial^{|\alpha|}/\partial(x^1)^{\alpha_1} \cdots \partial(x^{2n+1})^{\alpha_{2n+1}}$ and $g_{ij} = g(\partial/\partial x^i\,, \partial/\partial x^j) \in C^\infty(V_\lambda, \mathbb{R})$ for any $g \in S(M)$. The topology of $S(M)$ as a locally convex space is compatible to the distance function

$$\rho'(g_1\,, g_2) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{|g_1 - g_2|_k}{1 + |g_1 - g_2|_k}\,, \quad g_1\,, g_2 \in S(M).$$

In particular $(S(M), \rho')$ is a complete metric space. If $\rho(g_1\,, g_2) = \rho'(g_1\,, g_2) + \rho''(g_1\,, g_2)$ then $(\mathcal{M}, \rho)$ is a complete metric space (cf. Proposition 2 in [90], p. 158). Each metric $g \in \mathcal{M}$ determines a Laplace-Beltrami operator Δ_g hence the eigenvalues of Δ_g may be though of as functions of g and as such the eigenvalues are (by Theorem 2.2 in [90], p. 161) continuous functions on $(\mathcal{M}, \rho)$. To deal with the similar problem for the spectrum of a sublaplacian, we start by observing that the natural counterpart of $\mathcal{M}$ in the category of strictly pseudoconvex CR manifolds is the set $\mathcal{M}_H$ of all sub-Riemannian metrics on $(M, H(M))$. Nevertheless only a particular sort of sub-Riemannian metric gives rise to a sublaplacian i.e. Δ_b is associated to $G_\theta \in \mathcal{M}_H$ for some positively oriented contact form $\theta \in \mathcal{P}_+$. Of course $\mathcal{P}_+ \subset \Omega^1(M)$ and one may endow $\Omega^1(M)$ with the C^∞ topology. One may then attempt to repeat the arguments in [90] (by replacing $S(M)$ with $\Omega^1(M)$). The situation at hand is however much simpler since, once a contact form $\theta_0 \in \mathcal{P}_+$ is fixed, all others are parameterized by $C^\infty(M, \mathbb{R})$ i.e. for any $\theta \in \mathcal{P}_+$ there is a unique $u \in C^\infty(M, \mathbb{R})$ such that $\theta = e^u \theta_0$. We may then use the canonical Fréchet space structure (and corresponding complete distance function) of $C^\infty(M, \mathbb{R})$. Precisely, for every $u \in C^\infty(M, \mathbb{R})$, $\lambda \in \Lambda$ and $k \in \mathbb{N} \cup \{0\}$ we set

$$p_{\lambda,k}(u) = \sup_{x \in \overline{U}_k} \sum_{|\alpha| \le k} \left| D^\alpha u(x) \right|,$$

$$p_k(u) = \sum_{\lambda \in \Lambda} p_{\lambda,k}(u)\,, \quad |u|_{C^\infty} = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{p_k(u)}{1 + p_k(u)}\,.$$

If $\theta_0 \in \mathcal{P}_+$ is a fixed contact form then we set

$$d'(\theta_1\,, \theta_2) = |u_1 - u_2|_{C^\infty}\,, \quad \theta_1\,, \theta_2 \in \mathcal{P}_+\,,$$

where $u_i \in C^\infty(M, \mathbb{R})$ are given by $\theta_i = e^{u_i}\theta_0$ for any $i \in \{1, 2\}$. The definition of d' doesn't depend upon the choice of $\theta_0 \in \mathcal{P}_+$.

Lemma 2.16. *$(\mathcal{P}_+ , d')$ is a complete metric space.*

Proof. Let $\{\theta_\nu\}_{\nu\geq 1}$ be a Cauchy sequence in $(\mathcal{P}_+ , d')$. If $u_\nu \in C^\infty(M, \mathbb{R})$ is the function determined by $\theta_\nu = e^{u_\nu}\theta_0$ then (by the very definition of d') $\{u_\nu\}_{\nu\geq 1}$ is a Cauchy sequence in $C^\infty(M, \mathbb{R})$. Here $C^\infty(M, \mathbb{R})$ is organized as a Fréchet space by the (countable, separating) family of seminorms $\{p_k : k \in \mathbb{N} \cup \{0\}\}$. Hence there is $u \in C^\infty(M, \mathbb{R})$ such that $|u_\nu - u|_{C^\infty} \to 0$ as $\nu \to \infty$. Finally if $\theta = e^u\theta_0 \in \mathcal{P}_+$ then $d'(\theta_\nu , \theta) \to 0$ as $\nu \to \infty$. Q.e.d.

Let $S(H) \subset H(M)^* \otimes H(M)^*$ be the subbundle of all bilinear symmetric forms on $H(M)$. For every $G \in C^\infty(S(H))$, $k \in \mathbb{Z}$, $k \geq 0$, and $\lambda \in \Lambda$ we set

$$|G|_{\lambda,k} = \sup_{x\in\overline{U}_\lambda} \sum_{|\alpha|\leq k} \sum_{a,b=1}^{2n} \left|D^\alpha G_{ab}(x)\right| ,$$

$$|G|_k = \sum_{\lambda\in\Lambda} |G|_{\lambda,k} , \quad |G|_{C^\infty} = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{|G|_k}{1 + |G|_k} ,$$

where $G_{ab} = G(X_a, X_b) \in C^\infty(V_\lambda, \mathbb{R})$. Moreover we set

$$\rho'_H(G_1 , G_2) = |G_1 - G_2|_{C^\infty} , \quad G_1, G_2 \in C^\infty(S(H)).$$

Lemma 2.17. *$\{|\cdot|_k : k \in \mathbb{N} \cup \{0\}\}$ is a countable separating family of seminorms organizing $\mathfrak{X} = C^\infty(S(H))$ as a Fréchet space. In particular $(\mathfrak{X}, \rho'_H)$ is a complete metric space.*

Proof. For each $k \in \mathbb{N} \cup \{0\}$ and $N \in \mathbb{N}$ we set

$$V(k, N) = \left\{G \in \mathfrak{X} : |G|_k < \frac{1}{N}\right\}. \tag{2.10}$$

Let $\mathcal{B}$ be the collection of all finite intersections of sets (2.10). Then $\mathcal{B}$ is (cf. e.g. Theorem 1.37 in [104], p. 27) a convex balanced local base for a topology τ on $\mathfrak{X}$ which makes $\mathfrak{X}$ into a locally convex space such that every seminorm $|\cdot|_k$ is continuous and a set $E \subset \mathfrak{X}$ is bounded if and only if every $|\cdot|_k$ is bounded on E. τ is compatible with the distance function ρ'_H. Let $\{G_m\}_{m\geq 1} \subset \mathfrak{X}$ be a Cauchy sequence relative to ρ'_H. Thus for every fixed $k \in \mathbb{N} \cup \{0\}$ and $N \in \mathbb{N}$ one has $G_m - G_p \in V(k, N)$ for m, p sufficiently large. Consequently

$$\left|D^\alpha(G_m)_{ab}(x) - D^\alpha(G_p)_{ab}(x)\right| < \frac{1}{N} ,$$

$$x \in \overline{U}_\lambda , \quad \lambda \in \Lambda, \quad |\alpha| \leq k, \quad 1 \leq a, b \leq 2n.$$

It follows that each sequence $\{D^\alpha(G_m)_{ab}\}_{m\geq 1}$ converges uniformly on $\overline{U}_\lambda$ to a function G^α_{ab}. In particular for $\alpha = \mathbf{0}$ one has $(G_m)_{ab}(x) \to G^{\mathbf{0}}_{ab}(x)$ as $m \to \infty$, uniformly in $x \in \overline{U}_\lambda$. If $\lambda, \lambda' \in \Lambda$ are such that $U_\lambda \cap U_{\lambda'} \neq \emptyset$ and

$$X'_b = A^a_b X_a , \quad A \equiv \left[A^a_b\right] : U_\lambda \cap U_{\lambda'} \to \mathrm{GL}(2n, \mathbb{R}),$$

is a local transformation of the frame in $H(M)$ then

$$(G_m)'_{ab} = A^c_a A^d_b (G_m)_{cd} \quad \text{on} \quad U_\lambda \cap U_{\lambda'}$$

so that (for $m \to \infty$) $G'^{\mathbf{0}}_{ab} = A^c_a A^d_b G^{\mathbf{0}}_{cd}$ on $U_\lambda \cap U_{\lambda'}$. Thus $G^{\mathbf{0}}_{ab} \in C^\infty(U_\lambda)$ glue up to a (globally defined) bilinear symmetric form $G^{\mathbf{0}}$ on $H(M)$ and $G_m \to G^{\mathbf{0}}$ in $\mathfrak{X}$ as $m \to \infty$. Q.e.d.

For each point $x \in M$ let $P(H)_x$ be the set of all symmetric positive definite bilinear forms on $H(M)_x$. If $\varphi, \psi \in S(H)_x$ then we consider the anti-reflexive partial order relation

$$\varphi < \psi \Longleftrightarrow \psi - \varphi \in P(H)_x \, .$$

Next let $\rho''_x : P(H)_x \times P(H)_x \to [0, +\infty)$ be given by

$$\rho''_x(\varphi, \psi) = \inf \{\delta > 0 : \exp(-\delta)\, \varphi < \psi < \exp(\delta)\, \varphi\}$$

for any $\varphi, \psi \in P(H)_x$.

Lemma 2.18. *ρ''_x is a distance function on $P(H)_x$.*

Proof. As $e^{-\delta}\varphi < \psi < e^{\delta}\varphi$ is equivalent to $e^{-\delta}\psi < \varphi < e^{\delta}\psi$, it follows that ρ''_x is symmetric. To prove the triangle inequality we assume that $\rho''_x(\varphi, \psi) > \rho''_x(\varphi, \chi) + \rho''(\chi, \psi)$ for some $\varphi, \psi, \chi \in P(H)_x$. Then

$$\rho''_x(\varphi, \psi) - \rho''_x(\varphi, \chi) > \inf\{\delta > 0 : \exp(-\delta)\chi < \psi < \exp(\delta)\chi\}$$

hence there is $\delta_2 > 0$ such that $e^{-\delta_2}\chi < \psi < e^{\delta_2}\chi$ and $\rho''_x(\varphi, \psi) - \rho''_x(\varphi, \chi) > \delta_2$. Similarly

$$\rho''_x(\varphi, \psi) - \delta_2 > \inf\{\delta > 0 : \exp(-\delta)\varphi < \chi < \exp(\delta)\varphi\}$$

yields the existence of a number $\delta_1 > 0$ such that $e^{-\delta_1}\varphi < \chi < e^{\delta_1}\varphi$ and $\rho''_x(\varphi, \psi) - \delta_2 > \delta_1$. Let us set $\delta \equiv \delta_1 + \delta_2$. The inequalities written so far show that $e^{-\delta}\varphi < \psi < e^{\delta}\varphi$ and $\rho''_x(\varphi, \psi) > \delta$, a contradiction. Finally, let us assume that $\rho''_x(\varphi, \psi) = 0$ so that for any $k \in \mathbb{N}$

$$\inf\{\delta > 0 : \exp(-\delta)\varphi < \psi < \exp(\delta)\varphi\} < \frac{1}{k}$$

i.e. there is $\delta_k > 0$ such that $e^{-\delta_k}\varphi < \psi < e^{\delta_k}\varphi$ and $\delta_k < 1/k$. Thus $\lim_{k\to\infty} \delta_k = 0$ and $\psi - e^{-\delta_k}\varphi \in P(H)_x$ shows (by passing to the limit with $k \to \infty$ in $\psi(v, v) - e^{-\delta_k}\varphi(v, v) > 0$, $v \in H(M)_x \setminus \{0\}$) that $\varphi < \psi$. Similarly $e^{\delta_k}\varphi - \psi \in P(H)_x$ yields in the limit $\psi < \varphi$, and we may conclude that $\varphi = \psi$. Viceversa, if $\varphi \in P(H)_x$ then

$$\{\delta > 0 : (1 - e^{-\delta})\varphi, \ (e^{\delta} - 1)\varphi \in P(H)_x\} = (0, +\infty)$$

hence $\rho''_x(\varphi, \varphi) = 0$. Q.e.d.

Lemma 2.19. i) *$(P(H)_x\, , \rho''_x)$ is a complete metric space.*

ii) *Let $\{\varphi_j\}_{j\in\mathbb{N}} \subset P(H)_x$ such that* $\lim_{j\to\infty} \varphi_j = \varphi \in P(H)_x$ *in the ρ''_x-topology. Then* $\lim_{j\to\infty} \varphi_j(v, w) = \varphi(v, w)$ *for any $v, w \in H(M)_x$.*

Proof. i) Let $\{\varphi_j\}_{j\in\mathbb{N}} \subset P(H)_x$ be a Cauchy sequence in the ρ''_x-topology i.e. for any $\epsilon > 0$ there is $j_\epsilon \in \mathbb{N}$ such that $\rho''_x(\varphi_{j+p}, \varphi_j) > \epsilon$ for any $j \geq j_\epsilon$ and any $p = 1, 2, \cdots$. Hence there is $\delta_\epsilon > 0$ such that $e^{-\delta_\epsilon}\varphi_j < \varphi_{j+p} < e^{\delta_\epsilon}\varphi_j$ and $\delta_\epsilon < \epsilon$. Consequently

$$\left|\log \varphi_{j+p}(v, v) - \log \varphi_j(v, v)\right| < \delta_\epsilon < \epsilon$$

for any $v \in H(M)_x \setminus \{0\}$. Therefore if

$$\xi_j \equiv (\log \varphi_j(v,v), \cdots, \log \varphi_j(v,v)) \in \mathbb{R}^{2n}$$

then $\{\xi_j\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}^{2n}$. Let then $\xi = \lim_{j\to\infty} \xi_j$ and let $\varphi : H(M)_x \times H(M)_x \to \mathbb{R}$ be the bilinear form given by $\varphi(v,v) = \exp(\xi^a)$ for any $v \in H(M)_x \setminus \{0\}$ followed by polarization. Here $\xi = (\xi^1, \cdots, \xi^{2n})$. Then $\varphi \in P(H)_x$ and $\lim_{j\to\infty} \varphi_j = \varphi$ in the ρ''_x-topology.

ii) If $\varphi_j \to \varphi$ as $j \to \infty$ then $\log \varphi_j(v,v) \to \log \varphi(v,v)$ as $j \to \infty$, for any $v \in H(M)_x \setminus \{0\}$. Then $\lim_{j\to\infty} \varphi_j(v,v) = \varphi(v,v)$ uniformly in v and statement (ii) follows by polarization. Q.e.d.

As M is compact we may set

$$\rho''_H(G_1, G_2) = \sup_{x\in M} \rho''_x(G_{1,x}\,,\, G_{2,x}),$$

$$\rho_H(G_1, G_2) = \rho'_H(G_1, G_2) + \rho''_H(G_1, G_2), \quad G_1, G_2 \in \mathcal{M}_H\,.$$

Also let d be the distance function on $\mathcal{P}_+$ given by

$$d(\theta_1\,,\,\theta_2) = d'(\theta_1\,,\,\theta_2) + \rho''_H(G_{\theta_1}\,,\,G_{\theta_2}), \quad \theta_1\,,\,\theta_2 \in \mathcal{P}_+\,.$$

Proposition 2.20. i) *$(\mathcal{M}_H\,,\,\rho_H)$ is a complete metric space.*

ii) *The map $\theta \in \mathcal{P}_+ \mapsto G_\theta \in \mathcal{M}_H$ of $(\mathcal{P}_+\,,\,d)$ into $(\mathcal{M}_H, \rho_H)$ is continuous.*

iii) *$(\mathcal{P}_+\,,\,d)$ is a complete metric space.*

iv) *Two fixed contact forms $\theta_0\,,\,\tilde{\theta}_0$ define equivalent distance functions d, $\tilde{d}$ on $\mathcal{P}_+$.*

Proof. i) Let $\{G_j\}_{j\geq 1}$ be a Cauchy sequence in $(\mathcal{M}_H\,,\,\rho_H)$. Then $\{G_j\}_{j\geq 1}$ is a Cauchy sequence in both $(\mathfrak{X}\,,\,\rho'_H)$ and $(\mathcal{M}_H\,,\,\rho''_H)$. Yet $(\mathfrak{X}, \rho'_H)$ is complete (by Lemma 4.2). Thus $\rho'_H(G_j\,,\,G) \to 0$ as $j \to \infty$ for some $G \in \mathfrak{X}$. In particular

$$\lim_{j\to\infty} G_{j,x}(v,w) = G_x(v,w) \tag{2.11}$$

for every $x \in M$ and $v, w \in H(M)_x$. On the other hand, as $\{G_j\}_{j\geq 1}$ is Cauchy in $(\mathcal{M}_H\,,\,\rho''_H)$, for every $\epsilon > 0$ there is $N_\epsilon \geq 1$ such that

$$\rho''_x(G_{i,x}\,,\,G_{j,x}) \leq \rho''_H(G_i\,,\,G_j) < \epsilon \tag{2.12}$$

for every $i, j \geq N_\epsilon$ and $x \in M$. Thus $\{G_{j,x}\}_{j\geq 1}$ is Cauchy in the complete (by Lemma 2.19) metric space $(P(H)_x\,,\,\rho''_x)$ so that $\rho''_x(G_{j,x}\,,\,\varphi) \to 0$ as $j \to \infty$ for some $\varphi \in P(H)_x$. Then (by (iii) in Lemma 2.19) $\lim_{j\to\infty} G_{j,x}(v,w) = \varphi(v,w)$ for every $v, w \in H(M)_x$ hence $G_x = \varphi$ yielding $G \in \mathcal{M}_H$.

ii) Let $\{\theta_\nu\}_{\nu\geq 1} \subset \mathcal{P}_+$ such that $d(\theta_\nu, \theta) \to 0$ for $\nu \to \infty$ for some $\theta \in \mathcal{P}_+$. If $\theta_\nu = e^{u_\nu}\theta_0$ and $\theta = e^u\theta_0$ then $|u_\nu - u|_{C^\infty} \to 0$ as $\nu \to \infty$. Then $G_{\theta_\nu} = e^{u_\nu} G_{\theta_0}$ and $G_\theta = e^u G_{\theta_0}$. Since $D^\alpha u_\nu \to D^\alpha u$ as $\nu \to \infty$, uniformly on $\overline{U}_\lambda$, for any $\lambda \in \Lambda$, $|\alpha| \leq k$ and $k \in \mathbb{N} \cup \{0\}$, it follows that $D^\alpha(G_{\theta_\nu})_{ab} \to D^\alpha(G_\theta)_{ab}$ as $\nu \to \infty$ uniformly on $\overline{U}_\lambda$ for any $1 \leq a, b \leq 2n$. Hence $G_{\theta_\nu} \to G_\theta$ in $\mathfrak{X}$ so that (by the very definition of d and ρ_H) $\rho_H(G_{\theta_\nu}\,,\,G_\theta) \to 0$. Q.e.d.

iii) If $\{\theta_\nu\}_{\nu\geq 1}$ is a Cauchy sequence in $(\mathcal{P}_+\,,\,d)$ then $\{u_\nu\}_{\nu\geq 1}$ is Cauchy in $(\mathcal{P}_+\,,\,d')$ as well. Yet (by Lemma 2.16) $(\mathcal{P}_+\,,\,d')$ is complete hence $d'(\theta_\nu\,,\,\theta) \to 0$ for some $\theta \in \mathcal{P}_+$. Then, as a byproduct of the proof of statement (ii), one has $G_{\theta_\nu} \to G_\theta$ in $\mathfrak{X}$. Finally, the *verbatim* repetition of the arguments in the proof of statement (i) yields $\rho''_H(G_{\theta_\nu}\,,\,G_\theta) \to 0$ so that $d(\theta_\nu\,,\,\theta) \to 0$. Q.e.d.

2.5 A max-mini principle

For each $k \in \mathbb{N} \cup \{0\}$ we consider a $(k+1)$-dimensional real subspace $L_{k+1} \subset C^\infty(M, \mathbb{R})$ and set

$$\Lambda_\theta(L_{k+1}) = \sup \left\{ \frac{\|\nabla^H f\|_{L^2}^2}{\|f\|_{L^2}^2} : f \in L_{k+1} \setminus \{0\} \right\}.$$

Here

$$\|f\|_{L^2} = \left(\int_M f^2 \ \Psi_\theta \right)^{\frac{1}{2}}, \quad \|X\|_{L^2} = \left(\int_M g_\theta(X, X) \ \Psi_\theta \right)^{\frac{1}{2}},$$

for any $f \in C^\infty(M, \mathbb{R})$ and any $X \in \mathfrak{X}(M)$. Let $\{u_\nu\}_{\nu \geq 0} \subset C^\infty(M, \mathbb{R})$ be a complete orthonormal system relative to the L^2 inner product $(f, g)_{L^2} = \int_M fg \ \Psi_\theta$ such that $u_\nu \in \mathrm{Eigen}(\Delta_b\,;\, \lambda_\nu(\theta))$ for every $\nu \geq 0$. If $f \in C^\infty(M, \mathbb{R})$ then $f = \sum_{\nu=0}^\infty a_\nu(f)\, u_\nu$ (L^2 convergence) for some $a_\nu(f) \in \mathbb{R}$. Let L_{k+1}^0 be the subspace of $C^\infty(M, \mathbb{R})$ spanned by $\{u_\nu : 0 \leq \nu \leq k\}$. Let $(\nabla^H)^*$ be the formal adjoint of ∇^H i.e.

$$(\nabla^H f\,,\, X)_{L^2} = (f\,,\, (\nabla^H)^* X)_{L^2}$$

for any $f \in C^\infty(M, \mathbb{R})$ and $X \in C^\infty(H(M))$. Mere integration by parts shows that

$$(\nabla^H)^* X = -\mathrm{div}(X), \quad X \in C^\infty(H(M)),$$

implying (by (2.8)) the useful identity

$$\|\nabla^H f\|_{L^2}^2 = (f\,,\, \Delta_b f)_{L^2}\,, \quad f \in C^\infty(M, \mathbb{R}). \tag{2.13}$$

Let $f \in L_{k+1}^0 \setminus \{0\}$ so that $f = \sum_{\nu=0}^k a_\nu u_\nu$ for some $a_\nu \in \mathbb{R}$. Then (by (2.13))

$$\left\|\nabla^H f\right\|_{L^2}^2 = \sum_{\nu=0}^k a_\nu^2 \, \lambda_\nu(\theta) \leq \lambda_k(\theta) \sum_{\nu=0}^k a_\nu^2 = \lambda_k(\theta)\, \|f\|_{L^2}^2$$

hence

$$\Lambda_\theta(L_{k+1}^0) \leq \lambda_k(\theta). \tag{2.14}$$

Our purpose in this section is to establish

Proposition 2.21. *Let M be a compact strictly pseudoconvex CR manifold and $\theta \in \mathcal{P}_+$ a positively oriented contact form. Then*

$$\lambda_k(\theta) = \inf_{L_{k+1}} \Lambda_\theta(L_{k+1}) \tag{2.15}$$

where the g.l.b. is taken over all subspaces $L_{k+1} \subset C^\infty(M, \mathbb{R})$ with $\dim_{\mathbb{R}} L_{k+1} = k+1$.

So far (by (2.14)) $\lambda_k(\theta) \geq \Lambda_\theta(L_{k+1}^0) \geq \inf_{L_{k+1}} \Lambda_\theta(L_{k+1})$. The proof of Proposition 2.21 is by contradiction. We assume that $\lambda_k(\theta) > \inf_{L_{k+1}} \Lambda_\theta(L_{k+1})$ i.e. there is a $(k+1)$-dimensional subspace $L_{k+1} \subset C^\infty(M, \mathbb{R})$ such that $\Lambda_\theta(L_{k+1}) < \lambda_k(\theta)$. Then $\Lambda_\theta(L_{k+1})$ is finite and

$$\|f\|_{L^2}^2 \Lambda_\theta(L_{k+1}) \geq \|\nabla^H f\|_{L^2}^2\,, \quad f \in L_{k+1}\,.$$

Then (by (2.13))

$$\sum_{\nu=0}^\infty a_\nu(f)^2 \Lambda_\theta(L_{k+1}) \geq \sum_{\nu=0}^\infty \lambda_\nu(\theta) a_\nu(f)^2$$

so that

$$\sum_{\Lambda_\theta(L_{k+1})\geq\Lambda_\nu(\theta)} a_\nu(f)^2\,[\Lambda_\theta(L_{k+1})-\lambda_\nu(\theta)] \geq \tag{2.16}$$
$$\geq \sum_{\Lambda_\theta(L_{k+1})<\lambda_\nu(\theta)} a_\nu(f)^2\,[\lambda_\nu(\theta)-\Lambda_\theta(L_{k+1})]\,.$$

Let $\Phi : L_{k+1} \to C^\infty(M,\mathbb{R})$ be the linear map given by

$$\Phi(f) = \sum_{\nu=0}^{m} a_\nu(f)\,u_\nu\,, \quad f \in L_{k+1}\,,$$

where $m = \max\{\nu \geq 0 : \lambda_\nu(\theta) \leq \Lambda_\theta(L_{k+1})\}$. Note that $0 \leq m \leq k-1$ (by the contradiction assumption). We claim that

$$\mathrm{Ker}(\Phi) \neq (0). \tag{2.17}$$

Of course (2.17) is only true within the contradiction loop. The statement follows from $\dim_{\mathbb{R}} \Phi(L_{k+1}) \leq m+1 \leq k < k+1$ (hence Φ cannot be injective). Let (by (2.17)) $f_0 \in L_{k+1}$ such that $\Phi(f_0) = 0$ and $f_0 \neq 0$. Then $a_\nu(f_0) = 0$ for any $0 \leq \nu \leq m$ i.e. whenever $\Lambda_\theta(L_{k+1}) \geq \lambda_\nu(\theta)$. Applying (2.16) to $f = f_0$ yields $a_\nu(f_0) = 0$ whenever $\Lambda_\theta(L_{k+1}) < \lambda_\nu(\theta)$. Thus $f_0 = 0$, a contradiction.

2.6 Continuity of eigenvalues

The scope of § 2.6 is to establish

Theorem 2.22. *Let M be a compact strictly pseudoconvex CR manifold. If $\delta > 0$ and $\theta, \hat\theta \in \mathcal{P}_+$ are two contact forms on M such that $d(\theta\,,\,\hat\theta) < \delta$ then $e^{-\delta}\,\lambda_k(\theta) \leq \lambda_k(\hat\theta) \leq e^{\delta}\,\lambda_k(\theta)$ for any $k \geq 0$.*

Proof. For any $x \in M$

$$\delta > \inf\left\{\epsilon > 0 : e^{-\epsilon} G_{\theta,x} < G_{\hat\theta,x} < e^{\epsilon} G_{\theta,x}\right\}$$

i.e. there is $0 < \epsilon < \delta$ such that $G_{\hat\theta,x} - e^{-\epsilon}G_{\theta,x} \in P(H)_x$ and $e^{\epsilon}G_{\theta,x} - G_{\hat\theta,x} \in P(H)_x$. There is a unique $u \in C^\infty(M,\mathbb{R})$ such that $\hat\theta = e^u\theta$. Consequently

$$\hat\theta \wedge (d\hat\theta)^n = e^{(n+1)u}\,\theta \wedge (d\theta)^n\,. \tag{2.18}$$

On the other hand $e^{-\delta}G_{\theta,x}(v,v) < G_{\hat\theta,x}(v,v) < e^{\delta}G_{\theta,x}(v,v)$ for any $v \in H(M)_x \setminus \{0\}$ implies $|u| < \delta$. Then for every $f \in C^\infty(M)$ (by (2.18))

$$e^{-(n+1)\delta}\int_M f^2\,\Psi_\theta \leq \int_M f^2\,\Psi_{\hat\theta} \leq e^{(n+1)\delta}\int_M f^2\,\Psi_\theta. \tag{2.19}$$

Moreover

$$\hat\nabla^H f = e^{-u}\,\nabla^H f \tag{2.20}$$

where $\hat\nabla^H f$ is the horizontal gradient of f with respect to $\hat\theta$. Thus (by (2.20)) $\|\hat\nabla^H f\|^2_{\hat\theta} = e^{-u}\|\nabla^H f\|^2_\theta < e^{\delta}\|\nabla^H f\|^2_\theta$ so that (by (2.18))

$$e^{-(n+2)\delta}\int_M \|\nabla^H f\|^2_\theta\,\Psi_\theta \leq \int_M \|\hat\nabla^H f\|^2_{\hat\theta}\,\Psi_{\hat\theta} \leq \tag{2.21}$$

$$\leq e^{(n+2)\delta} \int_M \|\nabla^H f\|^2_\theta \, \Psi_\theta .$$

Finally (by (2.19)-(2.20))

$$e^{-\delta} \, \frac{\|\nabla^H f\|^2_{L^2}}{\|f\|^2_{L^2}} \leq \frac{\int_M \|\hat{\nabla}^H f\|^2_{\hat{\theta}} \, \Psi_{\hat{\theta}}}{\int_M f^2 \, \Psi_{\hat{\theta}}} \leq e^{\delta} \, \frac{\|\nabla^H f\|^2_{L^2}}{\|f\|^2_{L^2}}$$

so that (by the Max-Mini principle)

$$e^{-\delta} \, \lambda_k(\theta) \leq \lambda_k(\hat{\theta}) \leq e^{\delta} \, \lambda_k(\theta). \tag{2.22}$$

Theorem 2.22 is proved. Corollary 2.15 follows from (2.22).

2.7 Spectra of Δ_b and $\Box$

Let F_θ be the Fefferman metric of (M, θ) and $\Box$ the corresponding wave operator (the Laplace-Beltrami operator of $(C(M), F_\theta)$). We set $\mathfrak{M} = C(M)$ for simplicity. Let g be a fixed Riemannian metric on $\mathfrak{M}$. The space $S(\mathfrak{M})$ of all symmetric tensor fields may be identified with the space of all fields of endomorphisms of $T(\mathfrak{M})$ which are symmetric with respect to g i.e. for each $h \in S(\mathfrak{M})$ let $\tilde{h} \in C^\infty(\mathrm{End}(T(\mathfrak{M})))$ be given by

$$g(\tilde{h}X, Y) = h(X, Y), \quad X, Y \in \mathfrak{X}(\mathfrak{M}).$$

From now on we assume that M is compact. Then $\mathfrak{M}$ is compact as well (as $\mathcal{M}$ is the total space of a principal bundle with compact base and compact fibres) and we endow $S(\mathfrak{M})$ with the distance function

$$d^\infty_g(h_1 \, , \, h_2) = \sup_{z \in \mathfrak{M}} \Big[\mathrm{trace} \left(\varphi^2_z \right) \Big]^{1/2} \, , \quad h_1, h_2 \in S(\mathfrak{M}),$$

where $\varphi = \tilde{h}_1 - \tilde{h}_2$ and $\varphi^2_z = \varphi_z \circ \varphi_z$. The set $\mathrm{Lor}(\mathfrak{M})$ of all Lorentz metrics on $\mathfrak{M}$ is an open set of $(S(\mathfrak{M}), d^\infty_g)$ and for any pair g_1, g_2 of Riemannian metrics on $\mathfrak{M}$ the distance functions d_{g_1} and d_{g_2} are uniformly equivalent (cf. e.g. [80], p. 49). We shall use the topology induced by d^∞_g on $\mathrm{Lor}(\mathfrak{M})$ (and therefore on $C \subset \mathrm{Lor}(\mathfrak{M})$). By a result of J.M. Lee, [59], the sublaplacian Δ_b of (M, θ) is the pushforward of the wave operator i.e. $\pi_* \Box = \Delta_b$. In particular $\mathrm{Spec}(\Delta_b) \subset \mathrm{Spec}(\Box)$. Thus each $\lambda_k : \mathcal{P}_+ \to \mathbb{R}$ may be thought of as a function $\lambda^\uparrow_k : C \to \mathbb{R}$ such that $\lambda^\uparrow_k \circ F = \lambda_k$ for every $k \geq 0$, where $F : \mathcal{P}_+ \to C$ is the map given by $F(\theta) = F_\theta$ for every $\theta \in \mathcal{P}_+$. As another consequence of Theorem 2.22 we establish

Corollary 2.23. *Let M be a compact strictly pseudoconvex CR manifold and let g be an arbitrary Riemannian metric on $\mathfrak{M} = C(M)$. Let $\theta_0 \in \mathcal{P}_+$ be a fixed contact form and $\mathcal{P}_{++} = \{e^u \theta_0 : u \in C^\infty(M, \mathbb{R}), \;\; u > 0\}$. If $C_+ = \{F_\theta : \theta \in \mathcal{P}_{++}\}$ then for every $k \in \mathbb{N} \cup \{0\}$ the function $\lambda^\uparrow_k : C_+ \to \mathbb{R}$ is continuous relative to the d^∞_g-topology.*

Proof. Let $\theta_i \in \mathcal{P}_+$, $i \in \{1, 2\}$, and let us set $\varphi = \tilde{F}_{\theta_1} - \tilde{F}_{\theta_2}$. Let $\{E_p : 1 \leq p \leq 2n+2\}$ be a local g-orthonormal frame on $T(\mathfrak{M})$, defined on the open set $\mathcal{U} \subset \mathfrak{M}$. Then

$$\mathrm{trace} \left(\varphi^2 \right) = \sum_{p=1}^{2n+2} g(\varphi^2 E_p \, , \, E_p) = \sum_p \left\{ F_{\theta_1}(\varphi E_p \, , \, E_p) - F_{\theta_2}(\varphi E_p \, , \, E_p) \right\}$$

on $\mathcal{U}$. On the other hand if $\varphi E_p = \varphi_p^q E_q$ then $\varphi_p^q = F(\theta_1)(E_p\,,\,E_q) - F(\theta_2)(E_p\,,\,E_q)$ hence

$$\operatorname{trace}\left(\varphi^2\right) = (e^{u_1\circ\pi} - e^{u_2\circ\pi})^2\,\|F_{\theta_0}\|_g^2 \tag{2.23}$$

where $u_i \in C^\infty(M,\mathbb{R})$ is given by $\theta_i = e^{u_i}\theta_0$ and $\|F_{\theta_0}\|_g$ is the norm of F_{θ_0} as a $(0,2)$-tensor field on $\mathfrak{M}$ with respect to g. Then (by (2.23))

$$d_g^\infty\left(F_{\theta_1}\,,\,F_{\theta_2}\right) = \sup_{\mathfrak{M}}\left|e^{u_1\circ\pi} - e^{u_2\circ\pi}\right|\,\|F_{\theta_0}\|_g\,. \tag{2.24}$$

As $\mathfrak{M}$ is compact $a = \inf_{z\in\mathfrak{M}}\|F_{\theta_0}\|_{g,z} > 0$. Indeed (by compactness) $a = \|F_{\theta_0}\|_{g,z_0}$ for some $z_0 \in \mathfrak{M}$. If $a = 0$ then $F_{\theta_0\,,z_0} = 0$, a contradiction (as F_{θ_0} is Lorentzian, and hence nondegenerate). Let $\epsilon > 0$ such that $d_g^\infty(F_{\theta_1}\,,\,F_{\theta_2}) < \epsilon$. Then $|e^{u_1} - e^{u_2}| < \epsilon/a$ everywhere on M. As both $u_1 > 0$ and $u_2 > 0$ it follows that $|u_1 - u_2| < \log(1 + \epsilon/a)$. Indeed $e^{u_1} - e^{u_2} < \epsilon/a$ is equivalent to $e^{u_1-u_2} < 1 + (\epsilon/a)e^{-u_2}$ hence (as $u_2 > 0$)

$$u_1 - u_2 < \log[1 + (\epsilon/a)\,e^{-u_2}] < \log(1 + \epsilon/a).$$

Therefore

$$\left(1 + \frac{\epsilon}{a}\right)^{-1} G_{\theta_1\,,x}(v,v) < G_{\theta_2\,,x}(v,v) < \left(1 + \frac{\epsilon}{a}\right) G_{\theta_1\,,x}(v,v)$$

for any $v \in H(M)_x\setminus\{0\}$ and any $x \in M$. Consequently $\rho_H''(G_{\theta_1}\,,\,G_{\theta_2}) < \log(1+\epsilon/a)$. The arguments in § 5 then yield

$$\left(1 + \frac{\epsilon}{a}\right)^{-1} \lambda_k^\uparrow(F_{\theta_1}) \le \lambda_k^\uparrow(F_{\theta_2}) \le \left(1 + \frac{\epsilon}{a}\right) \lambda_k^\uparrow(F_{\theta_1})$$

and Corollary 2.23 follows. The problem of the behavior of $\lambda_k^\uparrow : C \to \mathbb{R}$ is open. So does the more general problem of the behavior of the spectrum of the wave operator on $\mathfrak{M}$ with respect to a change of $F \in \mathrm{Lor}(\mathfrak{M})$.

Chapter 3

Subelliptic Harmonic Maps and Spectrum of CR Manifolds

3.1 Levi tension field

Let (M, θ) be a strictly pseudoconvex CR manifold, of CR dimension n, and let (N, h) be a Riemannian manifold, where h is its Riemannian metric. The concept of energy density of a smooth map $f : M \longrightarrow N$ was adapted to the CR case by E. Barletta & S. Dragomir & H. Urakawa, [30], as follows. Let $f^{-1}T(N) \to M$ be the pullback bundle i.e. $(f^{-1}T(N))_x = T_{f(x)}(N)$ for any $x \in M$. For every $X \in \mathfrak{X}(M)$ we consider the section $f_*X \in C^\infty(f^{-1}T(N))$ defined by

$$(f_*X)(x) = (d_x f)X_x \,, \quad x \in M.$$

The *natural lift* $Y^f \in C^\infty(f^{-1}T(N))$ of $Y \in \mathfrak{X}(N)$ is given by

$$Y^f(x) = Y_{f(x)} \,, \quad x \in M.$$

In particular if (V, y^i) is a local coordinate system on N and $s_i = (\partial/\partial y^i)^f \in C^\infty(U,\ f^{-1}T(N))$ is the natural lift of the local vector field $\partial/\partial y^i$ then $\{s_i : 1 \le i \le \nu\}$ is a local frame in $f^{-1}T(N) \to M$ defined on the open set $U = f^{-1}(V)$. Here $\nu = \dim(N)$. Let $h^f = f^{-1}h$ be the pullback of h by f i.e. the Riemannian bundle metric on $f^{-1}T(N) \to M$ locally given by

$$h^f(s_i\,,\ s_j) = h\left(\frac{\partial}{\partial y^i}\,,\ \frac{\partial}{\partial y^j}\right) \circ f, \quad 1 \le i, j \le \nu.$$

For further use we denote by $\mathcal{C}(f^{-1}T(N))$ and $\mathcal{C}(f^{-1}T(N), h^f)$ the affine space of all connections in the vector bundle $f^{-1}T(N) \to M$, respectively the affine subspace of all $D \in \mathcal{C}(f^{-1}T(N))$ such that $Dh^f = 0$. Let $e_b(f) : M \to \mathbb{R}$ be defined by

$$e_b(f) = \frac{1}{2}\,\mathrm{trace}_{G_\theta}(\Pi_H f^* h). \tag{3.1}$$

Here $\Pi_H f^*h$ is the restriction of f^*h to $H(M) \otimes H(M)$. Let $x \in M$ be an arbitrary point and $\{X_a : 1 \le a \le 2n\}$ a local frame of the Levi distribution $H(M)$, defined on an open neighborhood $U \subset M$ of x. Then

$$e_b(f)_x = \frac{1}{2} \sum_{a=1}^{2n} h_{f(x)}\left((d_x f)X_{a,x}\,,\ (d_x f)X_{a,x}\right). \tag{3.2}$$

By a result in [30] (cf. Theorem 3.1 there) the first variation of the energy functional

$$E_b(f) = \int_M e_b(f)\,\Psi_\theta \tag{3.3}$$

is

$$\frac{d}{dt}\{E_b(f_t)\}_{t=0} = -\int_M h^f(H_b(f)\,,\,V)\,\Psi_\theta$$

where $H_b(f) \in C^\infty(f^{-1}T(N))$ is given by

$$H_b(f) = \mathrm{trace}_{G_\theta}\left(\Pi_H \beta_f\right). \tag{3.4}$$

The section $H_b(f)$ in $f^{-1}T(N) \to M$ is referred to as the *Levi tension field* of f. Here β_f is the vector valued bilinear form on $H(M)$ given by

$$\beta_f(X,Y) = \nabla^f_X f_* Y - f_* \nabla_X Y, \quad X,Y \in \mathfrak{X}(M),$$

and $\Pi_H \beta_f$ denotes the restriction of β_f to $H(M)\otimes H(M)$. Also $\nabla^f = f^{-1}\nabla^h \in C(f^{-1}T(N))$ is the pullback by f of the Levi-Civita connection to ∇^h of (N,h). Moreover ∇ is the Tanaka-Webster connection of (M,θ). Locally

$$H_b(f) = \sum_{a=1}^{2n} \nabla^f_{X_a} f_* X_a - f_* \nabla_{X_a} X_a\,. \tag{3.5}$$

Mappings with $H_b(f) = 0$ are called *pseudo-harmonic* by E. Barletta & S. Dragomir & H. Urakawa [30]. In the case where (N,h) is the standard $\mathbb{R}^m$, it is clear that

$$H_b(f) = (\Delta_b f_1, ..., \Delta_b f_m). \tag{3.6}$$

For the natural inclusion $j : \mathbb{S}^{2n+1} \hookrightarrow \mathbb{C}^{n+1}$ of $\mathbb{S}^{2n+1}$, the form β_j is given by, $\beta_j(X,Y) = -\langle X,Y\rangle_{\mathbb{C}^{n+1}}\,\vec{x} + \frac{1}{2}\langle JX,Y\rangle_{\mathbb{C}^{n+1}}\,J\vec{x}$, where $\vec{x}$ is the position vector field. Thus,

$$H_b(j) = -2n\,\vec{x}. \tag{3.7}$$

In the particular case where f is an isometric immersion from (M,g_θ) to (N,h), one has (see [30, p. 740])

$$H_b(f) = H(f) - B_f(T,T),$$

where B_f is the second fundamental form and $H(f) = trace_{g_\theta} B_f$ is the mean curvature vector of f.

In the sequel we will focus on maps $f : (M,\theta) \longrightarrow (N,h)$ that preserve lengths in the horizontal directions as well as the orthogonality between $H(M)$ and T, that is, $\forall X \in H(M)$,

$$|df(X)|_h = |X|_{G_\theta} \qquad \text{and} \qquad \langle df(X), df(T)\rangle_h = 0,$$

which also amounts to $f^*h = g_\theta + (\mu-1)\theta^2$ for some nonnegative function μ on M. For convenience, such a map will be termed *semi-isometric*. Notice that the dimension of the target manifold N should be at least $2n$. When the dimension of N is $2n$, then a semi-isometric map $f : (M,\theta) \longrightarrow (N,h)$ is noting but a Riemannian submersion satisfying $df(T) = 0$. Important examples are given by the standard projection from the Heisenberg group $\mathbb{H}^n$ to $\mathbb{R}^{2n}$ and the Hopf fibration $\mathbb{S}^{2n+1} \to \mathbb{C}P^n$.

Lemma 3.1. *Let (M,θ) be a strictly pseudoconvex CR manifold and let (N,h) be a Riemannian manifold. If $f : (M,\theta) \longrightarrow (N,h)$ is a C^2 semi-isometric map, then the form β_f takes its values in the orthogonal complement of $df(H(M))$. In particular, the vector $H_b(f)$ is orthogonal to $df(H(M))$.*

Proof. Let X, Y and Z be three horizontal vector fields. Since the Levi-Civita connection of (N,h) is torsionless, one has $\nabla^f_X df(Y) - \nabla^f_Y df(X) = df([X,Y])$. From the properties of the torsion of the Tanaka-Webster connection ∇, one has $\nabla_X Y - \nabla_Y X = [X,Y]^H$. Thus,

$$\beta_f(X,Y) - \beta_f(Y,X) = \theta([X,Y])df(T).$$

Since $df(T)$ is orthogonal to $df(H(M))$, we deduce the following symmetry property:

$$\langle \beta_f(X,Y), df(Z)\rangle_h = \langle \beta_f(Y,X), df(Z)\rangle_h. \tag{3.8}$$

On the other hand, we have,

$$Z \cdot \langle df(X), df(Y)\rangle_h = Z \cdot \langle X, Y\rangle_{G_\theta}. \tag{3.9}$$

Since G_θ is parallel with respect to the Tanaka-Webster connection ∇ and h is parallel with respect to the Levi-Civita connection ∇^h, one gets

$$Z \cdot \langle df(X), df(Y)\rangle_h = \langle \nabla^f_Z df(X), df(Y)\rangle_h + \langle df(X), \nabla^f_Z df(Y)\rangle_h$$

and

$$\begin{aligned} Z \cdot \langle X, Y\rangle_{G_\theta} &= \langle \nabla_Z X, Y\rangle_{G_\theta} + \langle X, \nabla_Z Y\rangle_{G_\theta} \\ &= \langle df(\nabla_Z X), df(Y)\rangle_h + \langle df(X), df(\nabla_Z Y)\rangle_h \end{aligned}$$

where the last equality comes from the fact that $\nabla_Z X$ and $\nabla_Z Y$ are horizontal. Replacing into (3.9) we obtain

$$\langle \nabla^f_Z df(X) - df(\nabla_Z X), df(Y)\rangle_h + \langle \nabla^f_Z df(Y) - df(\nabla_Z Y), df(X)\rangle_h = 0.$$

Therefore, $\forall\ X, Y, Z \in H(M)$,

$$\langle \beta_f(Z,X), df(Y)\rangle_h + \langle \beta_f(Z,Y), df(X)\rangle_h = 0. \tag{3.10}$$

Taking $X = Y$ in (3.10) we obtain, $\forall\ X, Z \in H(M)$,

$$\langle \beta_f(Z,X), df(X)\rangle_h = 0. \tag{3.11}$$

Now, taking $Z = X$ in (3.10) and using (3.8) and (3.11), we get, $\forall\ X, Y \in H(M)$,

$$\langle \beta_f(X,X), df(Y)\rangle_h = 0.$$

The symmetry property (3.8) enables us to conclude. □

A direct consequence of Lemma 3.1 is the following

Corollary 3.2. *If $f : (M,\theta) \longrightarrow (N,h)$ is a Riemannian submersion from a strictly pseudoconvex CR manifold (M,θ) to a Riemannian manifold (N,h) with $df(T) = 0$, then $\beta_f = 0$ and $H_b(f) = 0$.*

3.2 Semi-isometric maps into Euclidean space

Let (M, θ) be a strictly pseudoconvex CR manifold and let Ω be a bounded (relatively compact) domain of M. In the case where M is a closed manifold, we allow Ω to be equal to the whole of M. We are interested in Schrödinger-type operator $-\Delta_b + V$ where V is a function on Ω. We assume in all the sequel that the spectrum of $-\Delta_b + V$ in Ω, with Dirichlet boundary conditions if $\partial\Omega \neq \emptyset$, is discrete and bounded from below. We will always denote by $\{\lambda_j(\theta)\}_{j\geq 1}$ the non decreasing sequence of eigenvalues of $-\Delta_b + V$ and by $\{u_j\}_{j\geq 1}$ a complete orthonormal family of eigenfunctions in Ω with $(-\Delta_b + V)u_j = \lambda_j(\theta)u_j$.

Theorem 3.3. *Let (M, θ) be a strictly pseudoconvex CR manifold of real dimension $2n + 1$ and let $f : (M, \theta) \longrightarrow \mathbb{R}^m$ be a semi-isometric C^2 map. The sequence of eigenvalues $\{\lambda_j(\theta)\}_{j\geq 1}$ of the Schrödinger-type operator $-\Delta_b + V$ in a bounded domain $\Omega \subset M$, with Dirichlet boundary conditions if $\Omega \neq M$, satisfies for every $k \geq 1$ and $p \in \mathbb{R}$,*

$$\sum_{i=1}^{k} (\lambda_{k+1}(\theta) - \lambda_i(\theta))^p \leq \frac{\max\{2, p\}}{n} \sum_{i=1}^{k} (\lambda_{k+1}(\theta) - \lambda_i(\theta))^{p-1} (\lambda_i(\theta) + \frac{1}{4} D_i) \tag{3.12}$$

with

$$D_i = \int_\Omega \Big(|H_b(f)|^2_{\mathbb{R}^m} - 4V\Big) u_i^2 \, \Psi_\theta.$$

Moreover, if V is bounded below on Ω, then for every $k \geq 1$,

$$\lambda_{k+1}(\theta) \leq (1 + \frac{2}{n}) \frac{1}{k} \sum_{i=1}^{k} \lambda_i(\theta) + \frac{1}{2n} D_\infty \tag{3.13}$$

and

$$\lambda_{k+1}(\theta) \leq (1 + \frac{2}{n}) k^{\frac{1}{n}} \lambda_1(\theta) + \frac{1}{4} \left((1 + \frac{2}{n}) k^{\frac{1}{n}} - 1 \right) D_\infty \tag{3.14}$$

with $D_\infty = \sup_\Omega \Big(|H_b(f)|^2_{\mathbb{R}^m} - 4V\Big)$.

Applying this result to the standard CR sphere whose standard embedding $j : \mathbb{S}^{2n+1} \to \mathbb{C}^{n+1}$ satisfies $|H_b(j)|^2_{\mathbb{C}^{n+1}} = 4n^2$ (see (3.7)), we get the following

Corollary 3.4. *Let Ω be a domain in the standard CR sphere $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$. The eigenvalues of the operator $-\Delta_b + V$ in Ω, with Dirichlet boundary conditions if $\Omega \neq \mathbb{S}^{2n+1}$, satisfy, for every $k \geq 1$ and $p \in \mathbb{R}$,*

$$\sum_{i=1}^{k} (\lambda_{k+1}(\theta) - \lambda_i(\theta))^p \leq \frac{\max\{2, p\}}{n} \sum_{i=1}^{k} (\lambda_{k+1}(\theta) - \lambda_i(\theta))^{p-1} (\lambda_i(\theta) + n^2 - T_i)$$

with $T_i = \int_\Omega V u_i^2 \Psi_\theta$. Moreover, if V is bounded below on Ω, then, for every $k \geq 1$,

$$\lambda_{k+1}(\theta) \leq (1 + \frac{2}{n}) \frac{1}{k} \sum_{i=1}^{k} \lambda_i(\theta) + 2n - \frac{2}{n} \inf_\Omega V$$

and

$$\lambda_{k+1}(\theta) \leq (1+\frac{2}{n})k^{\frac{1}{n}}\lambda_1(\theta) + C(n,k,V)$$

with $C(n,k,V) = \left((1+\frac{2}{n})k^{\frac{1}{n}} - 1\right)\left(n^2 - \inf_\Omega V\right)$.

Theorem 3.3 also applies to the Heisenberg group $\mathbb{H}^n$ endowed with its standard CR structure. The corresponding sub-Laplacian is nothing but the operator $\Delta_{\mathbb{H}^n} = \frac{1}{4}\sum_{j\leq n}(X_j^2 + Y_j^2)$ (see section 3.4 for details). Since the standard projection $\mathbb{H}^n \to \mathbb{R}^{2n}$ is semi-isometric (up to a dilation) with zero Levi-tension (see Corollary 3.2), Theorem 3.3 leads to the following corollary which improves the results by Niu-Zhang [81] and El Soufi-Harrell-Ilias [8].

Corollary 3.5. *Let Ω be a domain in the Heisenberg group $\mathbb{H}^n$. The eigenvalues of the operator $-\Delta_b + V$ in Ω, with Dirichlet boundary conditions, satisfy, for every $k \geq 1$ and $p \in \mathbb{R}$,*

$$\sum_{i=1}^{k}(\lambda_{k+1}(\theta) - \lambda_i(\theta))^p \leq \frac{\max\{2,p\}}{n}\sum_{i=1}^{k}(\lambda_{k+1}(\theta) - \lambda_i(\theta))^{p-1}(\lambda_i(\theta) - T_i)$$

with $T_i = \int_\Omega V u_i^2 \Psi_\theta$. Moreover, if V is bounded below on Ω, then, for every $k \geq 1$,

$$\lambda_{k+1}(\theta) \leq (1+\frac{2}{n})\frac{1}{k}\sum_{i=1}^{k}\lambda_i(\theta) - \frac{2}{n}\inf_\Omega V$$

and

$$\lambda_{k+1}(\theta) \leq (1+\frac{2}{n})k^{\frac{1}{n}}\lambda_1(\theta) - \left((1+\frac{2}{n})k^{\frac{1}{n}} - 1\right)\inf_\Omega V.$$

The proof of Theorem 3.3 relies on a general result of algebraic nature using commutators. The use of this approach in obtaining bounds for eigenvalues is now fairly prevalent. Pioneering works in this direction are due to Harrell, alone or with collaborators (see [8, 35, 36]). For our purpose, we will use the following version that can be found in a recent paper by Ashbaugh and Hermi [74] (see inequality (26) of Corollary 3 and inequality (46) of Corollary 8 in [74]).

Lemma 3.6. *Let $A : \mathcal{D} \subset \mathcal{H} \to \mathcal{H}$ be a self-adjoint operator defined on a dense domain $\mathcal{D}$ which is semibounded below and has a discrete spectrum $\lambda_1(\theta) \leq \lambda_2(\theta) \cdots \leq \lambda_i(\theta) \leq \cdots$. Let $B : A(\mathcal{D}) \to \mathcal{H}$ be a symmetric operator which leaves $\mathcal{D}$ invariant. Denoting by $\{u_i\}_{i\geq 1}$ a complete orthonormal family of eigenvectors of A with $Au_i = \lambda_i(\theta)u_i$, we have, for every $k \geq 1$ and $p \in \mathbb{R}$,*

$$\sum_{i=1}^{k}(\lambda_{k+1}(\theta) - \lambda_i(\theta))^p\langle[A,B]u_i, Bu_i\rangle \leq \max\{1,\frac{p}{2}\}\sum_{i=1}^{k}(\lambda_{k+1}(\theta) - \lambda_i(\theta))^{p-1}\|[A,B]u_i\|^2.$$

Proof of Theorem 3.3. Let $f : (M,\theta) \to \mathbb{R}^m$ be a semi-isometric map and let $f_1, ..., f_m$ be its Euclidean components. For each $\alpha = 1, ..., m$, we denote by f_α the multiplication operator naturally associated with f_α. Let us start by the calculation of $\langle[-\Delta_b + V, f_\alpha]u_i, f_\alpha u_i\rangle_{L^2}$ and $\|[-\Delta_b + V, f_\alpha]u_i\|^2_{L^2}$. One has,

$$\begin{aligned}
[-\Delta_b + V, f_\alpha]u_i &= -\Delta_b(f_\alpha u_i) + f_\alpha(\Delta_b u_i) \\
&= -(\Delta_b f_\alpha)u_i - 2\langle\nabla^H f_\alpha, \nabla^H u_i\rangle_{G_\theta}.
\end{aligned}$$

Thus,

$$\langle[-\Delta_b + V, f_\alpha]u_i, f_\alpha u_i\rangle_{L^2} = -\int_\Omega f_\alpha(\Delta_b f_\alpha)u_i^2 - \frac{1}{2}\int_\Omega \langle\nabla^H f_\alpha^2, \nabla^H u_i^2\rangle_{G_\theta}. \tag{3.15}$$

Here and in the sequel, all the integrals over M are calculated with respect to the volume form Ψ_θ or, equivalently, the Riemannian volume element induced by the Webster metric g_θ. The integration over the eventual boundary is calculated with respect to the Riemannian metric induced on $\partial\Omega$ by the Webster metric g_θ. Integration by parts leads to (see (1.15))

$$\int_\Omega \langle\nabla^H f_\alpha^2, \nabla^H u_i^2\rangle_{G_\theta} = -\int_\Omega (\Delta_b f_\alpha^2)u_i^2 + \int_{\partial M} u_i^2\langle\nabla^H f_\alpha^2, \nu\rangle_{g_\theta}$$

where ν is the unit normal vector to the boundary with respect to the Webster metric g_θ. Since u_i vanishes on $\partial\Omega$ when $\partial\Omega \neq \emptyset$, we get

$$\begin{aligned}\int_\Omega \langle\nabla^H f_\alpha^2, \nabla^H u_i^2\rangle_{G_\theta} &= -\int_\Omega (\Delta_b f_\alpha^2)u_i^2 \\ &= -2\left[\int_\Omega f_\alpha(\Delta_b f_\alpha)u_i^2 + \int_\Omega |\nabla^H f_\alpha|^2_{G_\theta} u_i^2\right].\end{aligned}$$

Substituting in (3.15) we obtain

$$\langle[-\Delta_b + V, f_\alpha]u_i, f_\alpha u_i\rangle_{L^2} = \int_\Omega |\nabla^H f_\alpha|^2_{G_\theta} u_i^2.$$

Thus

$$\sum_{\alpha=1}^m \langle[-\Delta_b + V, f_\alpha]u_i, f_\alpha u_i\rangle_{L^2} = \sum_{\alpha=1}^m \int_\Omega |\nabla^H f_\alpha|^2_{G_\theta} u_i^2.$$

Now, since f preserves the Levi-form, one has with respect to a G_θ-orthonormal frame $\{e_i\}$ of $H_p(M)$,

$$\begin{aligned}\sum_{\alpha=1}^m |\nabla^H f_\alpha|^2_{G_\theta} &= \sum_{\alpha=1}^m \sum_{i=1}^{2n} \langle\nabla^H f_\alpha, e_i\rangle^2_{G_\theta} = \sum_{i=1}^{2n}\sum_{\alpha=1}^m \langle\nabla f_\alpha, e_i\rangle^2_{G_\theta} \\ &= \sum_{i=1}^{2n} |df(e_i)|^2_{\mathbb{R}^m} = \sum_{i=1}^{2n} |e_i|^2_{G_\theta} = 2n.\end{aligned}$$

Therefore,

$$\sum_{\alpha=1}^m \langle[-\Delta_b + V, f_\alpha]u_i, f_\alpha u_i\rangle_{L^2} = 2n\int_\Omega u_i^2 = 2n. \tag{3.16}$$

On the other hand, we have

$$\begin{aligned}\|[-\Delta_b + V, f_\alpha]u_i\|^2_{L^2} &= \int_\Omega \Big((\Delta_b f_\alpha)u_i + 2\langle\nabla^H f_\alpha, \nabla^H u_i\rangle_{G_\theta}\Big)^2 \\ &= \int_\Omega (\Delta_b f_\alpha)^2 u_i^2 + 4\int_\Omega \langle\nabla^H f_\alpha, \nabla^H u_i\rangle^2_{G_\theta} \\ &+ 2\int_\Omega (\Delta_b f_\alpha)\langle\nabla^H f_\alpha, \nabla^H u_i^2\rangle_{G_\theta}.\end{aligned}$$

Using (3.6), we get

$$\sum_{\alpha=1}^{m}\int_\Omega (\Delta_b f_\alpha)^2 u_i^2 = \int_\Omega |H_b(f)|^2_{\mathbb{R}^m} u_i^2.$$

Using the isometry property of f with respect to horizontal directions, we get

$$\begin{aligned}\sum_{\alpha=1}^{m}\langle\nabla^H f_\alpha,\nabla^H u_i\rangle^2_{G_\theta} &= \sum_{\alpha=1}^{m}\langle\nabla f_\alpha,\nabla^H u_i\rangle^2_{G_\theta} = \sum_{\alpha=1}^{m}|df_\alpha(\nabla^H u_i)|^2_{\mathbb{R}^m}\\ &= |df(\nabla^H u_i)|^2_{\mathbb{R}^m} = |\nabla^H u_i|^2_{G_\theta}.\end{aligned}$$

Thus,

$$\sum_{\alpha=1}^{m}\int_\Omega\langle\nabla^H f_\alpha,\nabla^H u_i\rangle^2_{G_\theta} = \int_\Omega|\nabla^H u_i|^2_{G_\theta} = \lambda_i(\theta) - \int_\Omega V u_i^2.$$

Finally, denoting by $\{E_\alpha\}$ the standard basis of $\mathbb{R}^m$ and using Lemma 3.1, we get,

$$\begin{aligned}\sum_{\alpha}^{m}\int_\Omega \Delta_b f_\alpha\langle\nabla^H f_\alpha,\nabla^H u_i^2\rangle_{G_\theta} &= \langle\sum_{\alpha}^{m}\Delta_b f_\alpha E_\alpha,\sum_{\alpha}^{m}\langle\nabla f_\alpha,\nabla^H u_i^2\rangle_{G_\theta}E_\alpha\rangle_{\mathbb{R}^m}\\ &= \langle H_b(f), df(\nabla^H u_i^2)\rangle_{\mathbb{R}^m} = 0.\end{aligned}$$

Using all these facts, we get

$$\sum_{\alpha=1}^{m}\|[-\Delta_b + V, f_\alpha]u_i\|^2_{L^2} = 4\left(\lambda_i(\theta) - \int_\Omega V u_i^2\right) + \int_\Omega |H_b(f)|^2_{\mathbb{R}^m}u_i^2. \tag{3.17}$$

Applying Lemma 3.6 with $A = -\Delta_b + V$ and $B = f_\alpha$, summing up with respect to $\alpha = 1, ..., m$, and using (3.16) and (3.17), we get the inequality (3.12).

To prove the inequality (3.13), we consider the quadratic relation that we derive from (3.12) after replacing p by 2 and D_i by D_∞, that is, $\forall\ k \geq 1$,

$$\sum_{i=1}^{k}(\lambda_{k+1}(\theta)-\lambda_i(\theta))^2 \leq \frac{2}{n}\sum_{i=1}^{k}(\lambda_{k+1}(\theta)-\lambda_i(\theta))(\lambda_i(\theta)+\frac{D_\infty}{4}) \tag{3.18}$$

which leads to

$$\lambda^2_{k+1}(\theta) - \lambda_{k+1}(\theta)\Big((2+\frac{2}{n})M_k + \frac{1}{2n}D_\infty\Big) + (1+\frac{2}{n})Q_k + \frac{1}{2n}D_\infty M_k \leq 0$$

with $M_k = \frac{1}{k}\sum_{i=1}^{k}\lambda_i(\theta)$ and $Q_k = \frac{1}{k}\sum_{i=1}^{k}\lambda_i^2(\theta)$. Using Cauchy-Schwarz inequality $M_k^2 \leq Q_k$, we get

$$\lambda^2_{k+1}(\theta) - \lambda_{k+1}(\theta)\Big((2+\frac{2}{n})M_k + \frac{1}{2n}D_\infty\Big) + (1+\frac{2}{n})M_k^2 + \frac{1}{2n}D_\infty M_k \leq 0$$

which can also be written as follows:

$$(\lambda_{k+1}(\theta) - M_k)\left(\lambda_{k+1}(\theta) - (1+\frac{2}{n})M_k - \frac{1}{2n}D_\infty\right) \leq 0.$$

Since $\lambda_{k+1}(\theta) - M_k$ is clearly nonnegative, we get $\lambda_{k+1}(\theta) \leq (1 + \frac{2}{n})M_k + \frac{1}{2n}D_\infty$ which proves (3.13).

Now, if we set $\overline{\lambda}_i(\theta) := \lambda_i(\theta) + \frac{1}{4}D_\infty$, then the inequality (3.18) reads

$$\sum_1^k (\overline{\lambda}_{k+1}(\theta) - \overline{\lambda}_i(\theta))^2 \leq \frac{2}{n} \sum_1^k (\overline{\lambda}_{k+1}(\theta) - \overline{\lambda}_i(\theta))\overline{\lambda}_i(\theta).$$

Following Cheng and Yang's argument [83, Theorem 2.1 and Corollary 2.1], we obtain

$$\overline{\lambda}_{k+1}(\theta) \leq \left(1 + \frac{2}{n}\right)\overline{\lambda}_1(\theta)k^{\frac{1}{n}}$$

which gives immediately the last inequality of the theorem. □

3.3 Riemannian submersions

Let (M, θ) be a strictly pseudoconvex CR manifold and let $f : (M, \theta) \to N$ be a Riemannian submersion over a Riemannian manifold N of dimension $2n$. The manifold N admits infinitely many isometric immersions into Euclidean spaces. For every integer $m \geq 2n$ we denote by $\mathcal{I}(N, \mathbb{R}^m)$ the set of all C^2 isometric immersions from N to the m-dimensional Euclidean space $\mathbb{R}^m$. Thanks to the Nash embedding theorem, the set $\cup_{m\in\mathbb{N}}\mathcal{I}(N, \mathbb{R}^m)$ is never empty, which motivates the introduction of the following invariant :

$$H^{euc}(N) = \inf_{\phi\in\cup_{m\in\mathbb{N}}\mathcal{I}(N,\mathbb{R}^m)} \|H(\phi)\|_\infty$$

where $H(\phi)$ stands for the mean curvature vector field of ϕ.

Theorem 3.7. *Let (M, θ) be a strictly pseudoconvex CR manifold of real dimension $2n + 1$ and let $f : (M, \theta) \to N$ be a Riemannian submersion over a Riemannian manifold of dimension $2n$ such that $df(T) = 0$. The eigenvalues of the operator $-\Delta_b + V$ in a bounded domain $\Omega \subset M$, with Dirichlet boundary conditions if $\Omega \neq M$, satisfy for every $k \geq 1$ and $p \in \mathbb{R}$,*

$$\sum_{i=1}^k (\lambda_{k+1}(\theta) - \lambda_i(\theta))^p \leq \frac{\max\{2, p\}}{n} \sum_{i=1}^k (\lambda_{k+1}(\theta) - \lambda_i(\theta))^{p-1}(\lambda_i(\theta) + \frac{1}{4}H^{euc}(N)^2 - T_i) \tag{3.19}$$

with $T_i = \int_\Omega V u_i^2 \Psi_\theta$. Moreover, if V is bounded below on Ω, then, for every $k \geq 1$,

$$\lambda_{k+1}(\theta) \leq (1 + \frac{2}{n})\frac{1}{k}\sum_{i=1}^k \lambda_i(\theta) + \frac{1}{2n}H^{euc}(N)^2 - \frac{2}{n}\inf_\Omega V \tag{3.20}$$

and

$$\lambda_{k+1}(\theta) \leq (1 + \frac{2}{n})k^{\frac{1}{n}}\lambda_1(\theta) + C \tag{3.21}$$

with $C = \Big((1 + \frac{2}{n})k^{\frac{1}{n}} - 1\Big)\Big(\frac{1}{4}H^{euc}(N)^2 - \inf_\Omega V\Big)$.

Proof. Let $\phi : N \to \mathbb{R}^m$ be any isometric immersion. It is straightforward to check that the map $\hat{f} = \phi \circ f : (M, \theta) \to \mathbb{R}^m$ is semi-isometric and that, $\forall X, Y \in H(M)$,

$$\beta_{\hat{f}}(X, Y) = d\phi(\beta_f(X, Y)) + B_\phi(df(X), df(Y)) = B_\phi(df(X), df(Y)),$$

where B_ϕ stands for the second fundamental form of ϕ and where the last equality follows from Corollary 3.2. Now, from the assumptions on f, the differential of f induces, for each $x \in M$, an isometry between $H_x(M)$ and $T_{f(x)}N$. Thus, if $X_1, \cdots, X_{2n}$ is a local orthonormal frame of $H(M)$, then $df(X_1), \cdots, df(X_{2n})$ is also an orthonormal frame of TN. This leads to the equality

$$H_b(\hat{f}) = H(\phi).$$

Therefore, it suffices to apply Theorem 3.3 to $\hat{f}$ and then take the infimum with respect to ϕ to finish the proof.

□

For example, when N is an open set of $\mathbb{R}^{2n}$ or, more generally, a minimal submanifold in $\mathbb{R}^m$, then $H^{euc}(N) = 0$ and the Theorem above gives a class of pseudoconvex CR manifolds including domains of the Heisenberg group, for which the following holds :

Corollary 3.8. *Let (M, θ) be a strictly pseudoconvex CR manifold of real dimension $2n + 1$ which admits a Riemannian submersion $f : (M, \theta) \to N$ over a minimal submanifold N of dimension $2n$ of $\mathbb{R}^m$ such that $df(T) = 0$. The eigenvalues of the operator $-\Delta_b + V$ in a bounded domain $\Omega \subset M$, with Dirichlet boundary conditions if $\Omega \neq M$, satisfy for every $k \geq 1$ and $p \in \mathbb{R}$,*

$$\sum_{i=1}^{k} (\lambda_{k+1}(\theta) - \lambda_i(\theta))^p \leq \frac{\max\{2, p\}}{n} \sum_{i=1}^{k} (\lambda_{k+1}(\theta) - \lambda_i(\theta))^{p-1} (\lambda_i(\theta) - T_i) \tag{3.22}$$

with $T_i = \int_\Omega V u_i^2 \Psi_\theta$. Moreover, if V is bounded below on Ω, then for every $k \geq 1$,

$$\lambda_{k+1}(\theta) \leq (1 + \frac{2}{n})\frac{1}{k} \sum_{i=1}^{k} \lambda_i(\theta) - \frac{2}{n} \inf_\Omega V \tag{3.23}$$

and

$$\lambda_{k+1}(\theta) \leq (1 + \frac{2}{n}) k^{\frac{1}{n}} \lambda_1(\theta) - \left((1 + \frac{2}{n}) k^{\frac{1}{n}} - 1\right) \inf_\Omega V. \tag{3.24}$$

The natural embedding $j : \mathbb{S}^{2n} \to \mathbb{R}^{2n+1}$ of the sphere into the Euclidean space satisfies $|H(j)|^2_{\mathbb{R}^{2n+1}} = 4n^2$. Thus, Theorem 3.7 leads to the following

Corollary 3.9. *Let (M, θ) be a strictly pseudoconvex CR manifold of real dimension $2n+1$. Assume that (M, θ) admits a Riemannian submersion $f : (M, \theta) \to D \subset \mathbb{S}^{2n}$ over a domain D of the standard sphere with $df(T) = 0$. The eigenvalues of the operator $-\Delta_b + V$ in a bounded domain $\Omega \subset M$, with Dirichlet boundary conditions if $\Omega \neq M$, satisfy for every $k \geq 1$ and $p \in \mathbb{R}$,*

$$\sum_{i=1}^{k} (\lambda_{k+1}(\theta) - \lambda_i(\theta))^p \leq \frac{\max\{2, p\}}{n} \sum_{i=1}^{k} (\lambda_{k+1}(\theta) - \lambda_i(\theta))^{p-1} (\lambda_i(\theta) + n^2 - T_i)$$

with $T_i = \int_\Omega V u_i^2 \Psi_\theta$. Moreover, if V is bounded below on Ω, then for every $k \geq 1$,

$$\lambda_{k+1}(\theta) \leq (1 + \frac{2}{n})\frac{1}{k}\sum_{i=1}^{k} \lambda_i(\theta) + 2n - \frac{2}{n}\inf_\Omega V$$

and

$$\lambda_{k+1}(\theta) \leq (1 + \frac{2}{n})k^{\frac{1}{n}}\lambda_1(\theta) + C$$

with $C(n, k, V) = \left((1 + \frac{2}{n})k^{\frac{1}{n}} - 1\right)\left(n^2 - \inf_\Omega V\right)$.

In the particular case of a manifold M without boundary that satisfies the assumptions of Corollary 3.9, one has, with $V = 0$, $\lambda_2(\theta) = 0$,

$$\lambda_2(\theta) \leq 2n$$

and, for every $k \geq 1$,

$$\lambda_{k+1}(\theta) \leq n(n + 2)k^{\frac{1}{n}} - n^2.$$

We denote by $\mathbb{F}P^m$ the m-dimensional real projective space if $\mathbb{F} = \mathbb{R}$, the complex projective space of real dimension $2m$ if $\mathbb{F} = \mathbb{C}$, and the quaternionic projective space of real dimension $4m$ if $\mathbb{F} = \mathbb{Q}$. The manifold $\mathbb{F}P^m$ carries a natural metric so that the Hopf fibration $\pi : \mathbb{S}^{d_\mathbb{F}(m+1)-1} \subset \mathbb{F}^{m+1} \to \mathbb{F}P^m$ is a Riemannian fibration, where $d_\mathbb{F} = \dim_\mathbb{R} \mathbb{F}$.

Let $\mathcal{H}_{m+1}(\mathbb{F}) = \{A \in \mathcal{M}_{m+1}(\mathbb{F}) \mid A^* := {}^t\overline{A} = A\}$ be the vector space of $(m + 1) \times (m + 1)$ Hermitian matrices with coefficients in $\mathbb{F}$, that we endow with the inner product

$$\langle A, B\rangle = \frac{1}{2}\mathrm{trace}(A\, B).$$

The map $\psi : \mathbb{S}^{d_\mathbb{F}(m+1)-1} \subset \mathbb{F}^{m+1} \longrightarrow \mathcal{H}_{m+1}(\mathbb{F})$ given by

$$\psi(z) = \begin{pmatrix} |z_0|^2 & z_0\bar{z}_1 & \cdots & z_0\bar{z}_m \\ z_1\bar{z}_0 & |z_1|^2 & \cdots & z_1\bar{z}_m \\ \cdots & \cdots & \cdots & \cdots \\ z_m\bar{z}_0 & z_m\bar{z}_1 & \cdots & |z_m|^2 \end{pmatrix}$$

induces through the Hopf fibration an isometric embedding ϕ from $\mathbb{F}P^m$ into $\mathcal{H}_{m+1}(\mathbb{F})$. Moreover, $\phi(\mathbb{F}P^m)$ is a minimal submanifold of the hypersphere $\mathbb{S}\left(\frac{I}{m+1}, \sqrt{\frac{m}{2(m+1)}}\right)$ of $\mathcal{H}_{m+1}(\mathbb{F})$ of radius $\sqrt{\frac{m}{2(m+1)}}$ centered at $\frac{I}{m+1}$. One deduces that the mean curvature $H(\phi)$ satisfies

$$|H(\phi)|^2 = 2m(m + 1)d_\mathbb{F}^2.$$

Therefore, $H^{euc}(\mathbb{F}P^m)^2 \leq 2m(m + 1)d_\mathbb{F}^2$ and Theorem 3.7 leads to the following

Corollary 3.10. *Let (M, θ) be a strictly pseudoconvex CR manifold of real dimension $2n + 1$ which admits a Riemannian submersion $f : (M, \theta) \to D \subset \mathbb{F}P^m$ over a domain of the projective space $\mathbb{F}P^m$ of real dimension $2n$ (i.e. $m = 2n/d_\mathbb{F}$) with $df(T) = 0$. The eigenvalues of the operator*

$-\Delta_b + V$ in a bounded domain $\Omega \subset M$, with Dirichlet boundary conditions if $\Omega \neq M$, satisfy for every $k \geq 1$ and $p \in \mathbb{R}$,

$$\sum_{i=1}^{k} (\lambda_{k+1}(\theta) - \lambda_i(\theta))^p \leq \frac{\max\{2, p\}}{n} \sum_{i=1}^{k} (\lambda_{k+1}(\theta) - \lambda_i(\theta))^{p-1} (\lambda_i(\theta) + n(2n + d_{\mathbb{F}}) - T_i)$$

with $T_i = \int_\Omega V u_i^2 \Psi_\theta$. Moreover, if V is bounded below on Ω, then for every $k \geq 1$,

$$\lambda_{k+1}(\theta) \leq (1 + \frac{2}{n}) \frac{1}{k} \sum_{i=1}^{k} \lambda_i(\theta) + 2(2n + d_{\mathbb{F}}) - \frac{2}{n} \inf_\Omega V$$

and

$$\lambda_{k+1}(\theta) \leq (1 + \frac{2}{n}) k^{\frac{1}{n}} \lambda_1(\theta) + C$$

with $C(n, k, V) = \left((1 + \frac{2}{n}) k^{\frac{1}{n}} - 1\right) (n(2n + d_{\mathbb{F}}) - \inf_\Omega V)$.

3.4 Semi-isometric maps into Heisenberg groups

Theorem 3.11. *Let (M, θ) be a strictly pseudoconvex CR manifold of dimension $2n + 1$ and let $f : M \longrightarrow \mathbb{H}^m$ be a C^2 semi-isometric map satisfying $df(H(M)) \subseteq H(\mathbb{H}^m)$. Then the eigenvalues of the operator $-\Delta_b + V$ in any bounded domain $\Omega \subset M$, with Dirichlet boundary conditions if $\Omega \neq M$, satisfy for every $k \geq 1$ and $p \in \mathbb{R}$,*

$$\sum_{i=1}^{k} (\lambda_{k+1}(\theta) - \lambda_i(\theta))^p \leq \frac{\max\{2, p\}}{n} \sum_{i=1}^{k} (\lambda_{k+1}(\theta) - \lambda_i(\theta))^{p-1} (\lambda_i(\theta) + \frac{1}{4} D_i) \tag{3.25}$$

with

$$D_i = \int_\Omega \left(|H_b(f)|^2_{\mathbb{H}^m} - 4V\right) u_i^2 \, \Psi_\theta.$$

Moreover, if V is bounded below on M, then for every $k \geq 1$,

$$\lambda_{k+1}(\theta) \leq (1 + \frac{2}{n}) \frac{1}{k} \sum_{i=1}^{k} \lambda_i(\theta) + \frac{1}{2n} D_\infty \tag{3.26}$$

and

$$\lambda_{k+1}(\theta) \leq (1 + \frac{2}{n}) k^{\frac{1}{n}} \lambda_1(\theta) + \frac{1}{4} \left((1 + \frac{2}{n}) k^{\frac{1}{n}} - 1\right) D_\infty \tag{3.27}$$

with $D_\infty = \sup_\Omega \left(|H_b(f)|^2_{\mathbb{H}^m} - 4V\right)$.

In the particular case where (M, θ) is the Heisenberg group $\mathbb{H}^n$ endowed with the standard contact form, this theorem provides an alternative way to derive Corollary 3.5

The following observation will be crucial for the proof of Theorem 3.11.

Proposition 3.12. *Let (M,θ) be a strictly pseudoconvex CR manifold and let*

$$\begin{array}{rcl} f & : & (M,\theta) \longrightarrow \mathbb{H}^m \simeq \mathbb{C}^m \times \mathbb{R} \\ & & x \quad \longrightarrow \quad f(x) = (F_1(x), ..., F_m(x), \alpha(x)) \end{array}$$

be a C^2 map such that $df(H(M)) \subset H(\mathbb{H}^m)$. Then

$$H_b(f) = \sum_{j=1}^{m} (\Delta_b \varphi_j X_j + \Delta_b \psi_j Y_j)$$

where $\varphi_j(x) = ReF_j(x)$ and $\psi_j(x) = ImF_j(x)$.
In particular, $H_b(f)$ is a horizontal vector field and

$$|H_b(f)|^2_{\mathbb{H}^m} = 4\sum_{j=1}^{m} [(\Delta_b \varphi_j)^2 + (\Delta_b \psi_j)^2].$$

Proof. One has, for any vector $W \in TM$,

$$df(W) = \sum_{j=1}^{m} \left(d\varphi_j(W) \frac{\partial}{\partial x_j} + d\psi_j(W) \frac{\partial}{\partial y_j} \right) + \theta(df(W))T.$$

For $W \in H(M)$, $df(W) \in H(\mathbb{H}^m)$ and, then,

$$df(W) = \sum_{j=1}^{m} \left(d\varphi_j(W) X_j + d\psi_j(W) Y_j \right). \tag{3.28}$$

Let $\{e_i\}$ be a local orthonormal frame of $H(M)$, then

$$\beta_f(e_i, e_i) = \nabla^f_{e_i} df(e_i) - df(\nabla_{e_i} e_i).$$

Since e_i and $\nabla_{e_i} e_i$ are horizontal and that $df(H(M)) \subset H(\mathbb{H}^m)$, we have

$$\beta_f(e_i, e_i) = \sum_{j=1}^{m} \nabla^f_{e_i} (d\varphi_j(e_i) X_j + d\psi_j(e_i) Y_j) - \sum_{j=1}^{m} [d\varphi_j(\nabla_{e_i} e_i) X_j + d\psi_j(\nabla_{e_i} e_i) Y_j]$$

with

$$\nabla^f_{e_i} (d\varphi_j(e_i) X_j) = e_i \cdot d\varphi_j(e_i) X_j + d\varphi_j(e_i) \nabla^{\mathbb{H}^m}_{df(e_i)} X_j$$

and

$$\nabla^f_{e_i} (d\psi_j(e_i) Y_j) = e_i \cdot d\psi_j(e_i) Y_j + d\psi_j(e_i) \nabla^{\mathbb{H}^m}_{df(e_i)} Y_j.$$

Therefore,

$$\begin{aligned} \beta_f(e_i, e_i) \quad = \quad & \sum_{j=1}^{m} \left[e_i \cdot d\varphi_j(e_i) - d\varphi_j(\nabla_{e_i} e_i) \right] X_j + \sum_{j=1}^{m} \left[e_i \cdot d\psi_j(e_i) - d\psi_j(\nabla_{e_i} e_i) \right] Y_j \\ + \quad & \sum_{j=1}^{m} \left[d\varphi_j(e_i) \nabla^{\mathbb{H}^m}_{df(e_i)} X_j + d\psi_j(e_i) \nabla^{\mathbb{H}^m}_{df(e_i)} Y_j \right]. \end{aligned} \tag{3.29}$$

Recall that the Levi-Civita connection of $\mathbb{H}^m$ is such that

$$\nabla^{\mathbb{H}^m}_{X_k} X_j = \nabla^{\mathbb{H}^m}_{Y_k} Y_j = \nabla^{\mathbb{H}^m}_{T} T = 0,$$

$$\nabla^{\mathbb{H}^m}_{X_k} Y_j = -2\delta_{kj} T, \qquad \nabla^{\mathbb{H}^m}_{X_k} T = 2Y_k, \qquad \nabla^{\mathbb{H}^m}_{Y_k} T = -2X_k,$$

$$\nabla^{\mathbb{H}^m}_{Y_k} X_j = 2\delta_{kj} T, \qquad \nabla^{\mathbb{H}^m}_{T} X_k = 2Y_k, \qquad \nabla^{\mathbb{H}^m}_{T} Y_k = -2X_k.$$

Thus,

$$\begin{aligned}\nabla^{\mathbb{H}^m}_{df(e_i)} X_j &= \sum_k (d\varphi_k(e_i)\nabla_{X_k} X_j + d\psi_k(e_i)\nabla_{Y_k} X_j) \\ &= d\psi_j(e_i)\nabla_{Y_j} X_j = 2d\psi_j(e_i)T.\end{aligned}$$

and

$$\nabla^{\mathbb{H}^m}_{df(e_i)} Y_j = -2d\varphi_j(e_i)T.$$

Replacing into (3.29) and summing up with respect to i, we get

$$\begin{aligned}H_b(f) &= \sum_{i=1}^{2n}\sum_{j=1}^{m}\Big([e_i \cdot d\varphi_j(e_i) - d\varphi_j(\nabla_{e_i}e_i)]X_j + [e_i \cdot d\psi_j(e_i) - d\psi_j(\nabla_{e_i}e_i)]Y_j\Big) \\ &= \sum_{j=1}^{m}\Big(\Delta_b\varphi_j X_j + \Delta_b\psi_j Y_j\Big).\end{aligned}$$

□

Proof of Theorem 3.11. As in the proof of Theorem 3.3, we will use the components of the map f as multiplication operators. Let us write $f(x) = (F_1(x), ..., F_m(x), \alpha(x)) \in \mathbb{C}^m \times \mathbb{R}$ and $F_j(x) = \varphi_j(x) + i\psi_j(x)$. The main difference with respect to the Euclidean case is that here, only the $\mathbb{C}^m$ components of f come in. All along this proof we will use the fact that, $\forall\, W \in H_x(M)$, the vector $df(W)$ is horizontal and (see (3.28))

$$|df(W)|^2_{\mathbb{H}^m} = 4\sum_{j=1}^{m}\Big(|d\varphi_j(W)|^2 + |d\psi_j(W)|^2\Big). \tag{3.30}$$

Repeating the same calculations as in the proof of the Theorem 3.3, we get

$$\begin{aligned}\sum_{j=1}^{m}\langle[-\Delta_b + V, \varphi_j]u_i, \varphi_j u_i\rangle_{L^2} \quad &+ \quad \langle[-\Delta_b + V, \psi_j]u_i, \psi_j u_i\rangle_{L^2} \\ &= \sum_{j=1}^{m}\int_\Omega \{|\nabla^H\varphi_j|^2_{G_\theta} + |\nabla^H\psi_j|^2_{G_\theta}\}u_i^2.\end{aligned}$$

Let $\{e_i\}$ be a G_θ-orthonormal basis of $H_x(M)$, then

$$\begin{aligned}
\sum_{j=1}^{m} |\nabla^H \varphi_j|^2_{G_\theta} + |\nabla^H \psi_j|^2_{G_\theta} &= \sum_{j=1}^{m} \sum_{i=1}^{2n} \langle \nabla^H \varphi_j, e_i \rangle^2_{G_\theta} + \langle \nabla^H \psi_j, e_i \rangle^2_{G_\theta} \\
&= \sum_{i=1}^{2n} \sum_{j=1}^{m} \langle \nabla \varphi_j, e_i \rangle^2_{G_\theta} + \langle \nabla \psi_j, e_i \rangle^2_{G_\theta} \\
&= \sum_{i=1}^{2n} \sum_{j=1}^{2m} (d\varphi_j(e_i)^2 + d\psi_j(e_i)^2) \\
&= \frac{1}{4} \sum_{i=1}^{2n} |df(e_i)|^2_{\mathbb{H}^m} = \frac{n}{2}.
\end{aligned}$$

Thus,

$$\sum_{j=1}^{m} \langle [-\Delta_b + V, \varphi_j] u_i, \varphi_j u_i \rangle_{L^2} + \langle [-\Delta_b + V, \psi_j] u_i, \psi_j u_i \rangle_{L^2} = \frac{n}{2}. \tag{3.31}$$

On the other hand,

$$\begin{aligned}
\|[-\Delta_b + V, \varphi_j] u_i\|^2_{L^2} &= \int_\Omega \Big((\Delta_b \varphi_j) u_i + 2 \langle \nabla^H \varphi_j, \nabla^H u_i \rangle_{G_\theta} \Big)^2 \\
&= \int_\Omega (\Delta_b \varphi_j)^2 u_i^2 + 4 \int_\Omega \langle \nabla^H \varphi_j, \nabla^H u_i \rangle^2_{G_\theta} \\
&+ 2 \int_\Omega (\Delta_b \varphi_j) \langle \nabla^H \varphi_j, \nabla^H u_i^2 \rangle_{G_\theta}.
\end{aligned}$$

We have a similar formula for $\|[-\Delta_b + V, \psi_j] u_i\|^2_{L^2}$. Since $\nabla^H u_i \in H(M)$, one has

$$\begin{aligned}
\sum_{j=1}^{m} \langle \nabla^H \varphi_j, \nabla^H u_i \rangle^2_{G_\theta} &+ \langle \nabla^H \psi_j, \nabla^H u_i \rangle^2_{G_\theta} \\
&= \sum_{j=1}^{m} \{ d\varphi_j(\nabla^H u_i)^2 + d\psi_j(\nabla^H u_i)^2 \} \\
&= \frac{1}{4} |df(\nabla^H u_i)^2|_{\mathbb{H}^m} = \frac{1}{4} |\nabla^H u_i|^2_{G_\theta}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{j=1}^{m} \int_\Omega \Big(\langle \nabla^H \varphi_j, \nabla^H u_i \rangle^2_{G_\theta} + \langle \nabla^H \psi_j, \nabla^H u_i \rangle^2_{G_\theta} \Big) &= \frac{1}{4} \int_\Omega |\nabla^H u_i|^2_{G_\theta} \\
&= \frac{1}{4} \left(\lambda_i(\theta) - \int_\Omega V u_i^2 \right).
\end{aligned}$$

For the two remaining terms, we have thanks to Proposition 3.12,

$$\sum_{j=1}^{m} \int_\Omega \Big((\Delta_b \varphi_j)^2 + (\Delta_b \psi_j)^2 \Big) u_i^2 = \frac{1}{4} \int_\Omega |H_b(f)|^2_{\mathbb{H}^m} u_i^2$$

and

$$\begin{aligned}\sum_{j=1}^{m}\int_{\Omega} & \Big(\Delta_b\varphi_j\langle\nabla^H\varphi_j,\nabla^H u_i^2\rangle_{G_\theta}+\Delta_b\psi_j\langle\nabla^H\psi_j,\nabla^H u_i^2\rangle_{G_\theta}\Big)\\ = & \ \frac{1}{4}\int_{\Omega}\langle H_b(f),\sum_{j=1}^{m}d\varphi_j(\nabla^H u_i^2)X_j+\sum_{j=1}^{m}d\psi_j(\nabla^H u_i^2)Y_j\rangle_{\mathbb{H}^m}\\ = & \ \frac{1}{4}\int_{\Omega}\langle H_b(f),df(\nabla^H u_i^2)\rangle_{\mathbb{H}^m}=0,\end{aligned}$$

where the last equality follows from the fact that $H_b(f)$ is orthogonal to $df(H(M))$ (Lemma 3.1). Finally,

$$\|[-\Delta_b+V,\varphi_j]u_i\|_{L^2}^2+\|[-\Delta_b+V,\psi_j]u_i\|_{L^2}^2=\lambda_i(\theta)+\frac{1}{4}\int_{\Omega}\Big(|H_b(f)|^2_{\mathbb{H}^m}-V\Big)u_i^2. \tag{3.32}$$

Applying Lemma 3.6 with $A=-\Delta_b+V$ and $B=\varphi_j$ then $B=\psi_j$, summing up with respect to j and using (3.31) and (3.32), we obtain the inequality (3.25).

As in the proof of Theorem 3.3, we derive the inequalities (3.26) and (3.27) from (3.25) with $p=2$. □

3.5 Reilly type inequalities on CR manifolds

Let (M,θ) be a compact strictly pseudo-convex CR manifold. If $f:(M,\theta)\longrightarrow\mathbb{R}^m$ is a semi-isometric C^2 map, then Theorem 3.3 (i.e. inequality (3.12) with $k=1$ and $p=1$) gives,

$$\lambda_2(\theta)\le(1+\frac{2}{n})\lambda_1(\theta)+\frac{1}{2n}\int_M\Big(|H_b(f)|^2_{\mathbb{R}^m}-4V\Big)u_1^2.$$

When M is a compact manifold without boundary and $V=0$, one has $\lambda_1(\theta)=0$ and $u_1^2=\frac{1}{V(M,\theta)}$. Therefore, the following Reilly type result holds (see[4] for details about Reilly inequalities)

$$\lambda_2(\theta)\le\frac{1}{2nV(M,\theta)}\int_M|H_b(f)|^2_{\mathbb{R}^m}.$$

This result can be obtained in an independent and simpler way, in the spirit of Reilly's proof, under weaker assumptions on f. Moreover, the equality case can be characterized. Indeed, we first have the following

Theorem 3.13. *Let (M,θ) be a compact strictly pseudoconvex CR manifold of dimension $2n+1$ without boundary. For every C^2 map $f:(M,\theta)\longrightarrow\mathbb{R}^m$ one has*

$$\lambda_2(\theta)E_b(f)\le\frac{1}{2}\int_M|H_b(f)|^2_{\mathbb{R}^m} \tag{3.33}$$

where the equality holds if and only if the Euclidean components $f_1,\dots,f_m$ of f satisfy $-\Delta_b f_\alpha=\lambda_2(\theta)\Big(f_\alpha-⨍ f_\alpha\Big)$ for every $\alpha\le m$.

Proof. Replacing if necessary f_α by $f_\alpha - \fint f_\alpha$ we can assume without loss of generality that the Euclidean components $f_1, \ldots, f_m$ of f satisfy $\int_M f_\alpha \, \Psi_\theta = 0$ so that, we have

$$\lambda_2(\theta) \int_M f_\alpha^2 \leq \int_M |\nabla^H f_\alpha|^2_{G_\theta}. \tag{3.34}$$

Summing up with respect to α, we get

$$\lambda_2(\theta) \int_M |f|^2_{\mathbb{R}^m} \leq \int_M \sum_{\alpha=1}^m |\nabla^H f_\alpha|^2_{G_\theta}.$$

Denoting by $\{\epsilon_\alpha\}$ the standard basis of $\mathbb{R}^m$ and by $\{X_i\}$ a local orthonormal frame of $H(M)$, we observe that

$$\begin{aligned} 2e_b(f) &= \sum_{i=1}^{2n} |df(X_i)|^2_{\mathbb{R}^m} = \sum_{i=1}^{2n} \sum_{\alpha=1}^m \langle df(X_i), \epsilon_\alpha \rangle^2_{\mathbb{R}^m} \\ &= \sum_{\alpha=1}^m \sum_{i=1}^{2n} |df_\alpha(X_i)|^2_{\mathbb{R}^m} = \sum_{\alpha=1}^m |\nabla^H f_\alpha|^2_{G_\theta}. \end{aligned}$$

Therefore,

$$\lambda_2(\theta) \int_M |f|^2_{\mathbb{R}^m} \leq \int_M \sum_{\alpha=1}^m |\nabla^H f_\alpha|^2_{G_\theta} = 2E_b(f). \tag{3.35}$$

On the other hand, we have

$$\begin{aligned} 4E_b(f)^2 &= \left(\sum_{\alpha=1}^m \int_M |\nabla^H f_\alpha|^2_{G_\theta} \right)^2 = \left(\sum_{\alpha=1}^m \int_M f_\alpha \Delta_b f_\alpha \right)^2 \\ &= \left(\int_M \langle f(x), \sum_\alpha^m (\Delta_b f_\alpha) \epsilon_\alpha \rangle_{\mathbb{R}^m} \right)^2 \\ &= \left(\int_M \langle f(x), H_b(f) \rangle_{\mathbb{R}^m} \right)^2 \leq \int_M |f|^2_{\mathbb{R}^m} \int_M |H_b(f)|^2_{\mathbb{R}^m}. \end{aligned}$$

Combining with (3.35), we get

$$4E_b(f)^2 \leq \frac{2E_b(f)}{\lambda_2(\theta)} \int_M |H_b(f)|^2_{\mathbb{R}^m}$$

which gives the desired inequality.

Now, if we have, for every $\alpha \leq m$, $-\Delta_b f_\alpha = \lambda_2(\theta) f_\alpha$, then $H_b(f) = (\Delta_b f_1, \ldots, \Delta_b f_m) = -\lambda_2(\theta) f$ and $\int_M |H_b(f)|^2_{\mathbb{R}^m} = \lambda_2(\theta)^2 \int_M |f|^2_{\mathbb{R}^m}$. On the other hand, $E_b(f) = \int_M \sum_{\alpha=1}^m |\nabla^H f_\alpha|^2_{G_\theta} = \lambda_2(\theta) \int_M |f|^2_{\mathbb{R}^m}$ which implies that the equality holds in (3.33). Reciprocally, if the equality holds in (3.33) for a nonconstant map f, then it also holds in (3.34) for each α. Thus, the functions $f_1, \ldots, f_m$ belong to the $\lambda_2(\theta)$-eigenspace of $-\Delta_b$. $\square$

If a map $f : (M, \theta) \longrightarrow \mathbb{R}^m$ preserves the metric with respect to horizontal directions (i.e., $|df(X)|_{\mathbb{R}^m} = |X|_{G_\theta}$ for any $X \in H(M)$), then its energy density $e_b(f)$ is constant equal to n and

$$E_b(f) = nV(M, \theta).$$

Inequality (3.33) becomes in this case

$$\lambda_2(\theta) \le \frac{1}{2nV(M,\theta)} \int_M |H_b(f)|^2_{\mathbb{R}^m}. \tag{3.36}$$

The characterization of the equality case is the last inequality requires the following Takahashi's type result.

Lemma 3.14. *Let (M,θ) be a strictly pseudoconvex CR manifold of dimension $2n+1$ and let $f : (M,\theta) \longrightarrow \mathbb{R}^m$ be C^2 map.*
i) Assume that $f(M)$ is contained in a sphere $\mathbb{S}^{m-1}(r)$ of radius r centered at the origin. Then f is pseudo-harmonic from (M,θ) to $S^{m-1}(r)$ if and only if its Euclidean components $f_1,\dots,f_m$ satisfy, $\forall\alpha \le m$,

$$-\Delta_b f_\alpha = \mu f_\alpha$$

with $\mu = \frac{2}{r^2} e_b(f) \in C^\infty(M)$.
ii) Assume that f is semi-isometric. If the Euclidean components $f_1,\dots,f_m$ of f satisfy, $\forall\alpha \le m$, $-\Delta_b f_\alpha = \lambda(\theta) f_\alpha$ for some $\lambda(\theta) \in \mathbb{R}$, then $f(M)$ is contained in the sphere $\mathbb{S}^{m-1}(r)$ of radius $r = \sqrt{\frac{2n}{\lambda(\theta)}}$ and f is a pseudo-harmonic map from (M,θ) to $S^{m-1}(r)$. Conversely, if $f(M)$ is contained in a sphere $S^{m-1}(r)$ and if f is a pseudo-harmonic map from (M,θ) to $S^{m-1}(r)$, then $\forall\alpha \le m$, $-\Delta_b f_\alpha = \frac{2n}{r^2} f_\alpha$.

Proof of Lemma 3.14. i) For convenience, let us write $f = j \circ \bar f$ where $j : \mathbb{S}^{m-1}(r) \to \mathbb{R}^m$ is the standard embedding and $\bar f : M \to \mathbb{S}^{m-1}(r)$ is defined by $\bar f(x) = f(x)$. It is straightforward to observe that, $\forall X, Y \in H(M)$,

$$\beta_f(X,Y) = B_j(d\bar f(X), d\bar f(Y)) + dj(\beta_{\bar f}(X,Y))$$

where $B_j(W,W) = -\frac{1}{r^2}|W|^2_{\mathbb{R}^m}\vec{x}$ is the second fundamental form of the sphere $\mathbb{S}^{m-1}(r)$. Taking the trace, we obtain

$$H_b(f) = -\frac{2e_b(\bar f)}{r^2}\bar f + dj(H_b(\bar f)) = -\frac{2e_b(f)}{r^2} f + dj(H_b(\bar f)).$$

Hence, if f is pseudo-harmonic from (M,θ) to $S^{m-1}(r)$, then $H_b(\bar f) = 0$ and, consequently, $H_b(f) = -\frac{2e_b(f)}{r^2} f$ with $H_b(f) = (\Delta_b f_1, \dots, \Delta_b f_m)$. Thus, $\forall\alpha \le m$, $-\Delta_b f_\alpha = \frac{2}{r^2} e_b(f) f_\alpha$.
Reciprocally, if there exists a function $\mu \in C^\infty$ such that $-\Delta_b f_\alpha = \mu f_\alpha$ for every $\alpha \le m$, then

$$0 = \Delta_b\left(\sum_{\alpha=1}^m f_\alpha^2\right) = 2\mu \sum_{\alpha=1}^m f_\alpha^2 + 2\sum_{\alpha=1}^m |\nabla^H f_\alpha|^2_{G_\theta} = 2\mu r^2 + 4e_b(f).$$

Hence, $\mu = \frac{2e_b(f)}{r^2}$, $H_b(f) = -\frac{2e_b(f)}{r^2} f$ and, then, $H_b(\bar f) = 0$, which means that f is pseudo-harmonic from (M,θ) to $\mathbb{S}^{m-1}(r)$.

ii) From the assumptions, one has $H_b(f) = -\lambda(\theta) f$. Since f is semi-isometric, we know that $H_b(f)$ is orthogonal to $df(H(M))$ (Lemma 3.1). Therefore, $\forall x \in M$ and $\forall X \in H_x(M)$, one has $\langle f(x), df_x(X)\rangle_{\mathbb{R}^m} = 0$ which implies that the function $x \mapsto |f(x)|^2_{\mathbb{R}^m}$ has zero derivative with respect to all horizontal directions. Since the distribution $H(M)$ is not integrable, this implies that $|f(x)|^2_{\mathbb{R}^m}$

is constant on M, that is $f(M)$ is contained in a sphere $\mathbb{S}^{m-1}(r)$ of radius r centered at the origin. The pseudo-harmonicity of f from M into $\mathbb{S}^{m-1}(r)$ then follows from (i). Moreover, one necessarily has $\lambda(\theta) = \frac{2e_b(f)}{r^2}$ with $e_b(f) = n$ since f is semi-isometric. Thus, the radius of the sphere is such that $r^2 = \frac{2n}{\lambda(\theta)}$

□

Theorem 3.13 and Lemma 3.14 lead to the following

Corollary 3.15. *Let (M, θ) be a compact strictly pseudoconvex CR manifold of dimension $2n + 1$ without boundary and let $f : (M, \theta) \longrightarrow \mathbb{R}^m$ be C^2 semi-isometric map. Then*

$$\lambda_2(\theta) \le \frac{1}{2nV(M, \theta)} \int_M |H_b(f)|^2_{\mathbb{R}^m}. \tag{3.37}$$

Moreover, the equality holds in this inequality if and only if $f(M)$ is contained in a sphere $\mathbb{S}^{m-1}(r)$ of radius $r = \sqrt{\frac{2n}{\lambda_2(\theta)}}$ and f is a pseudo-harmonic map from (M, θ) to the sphere $S^{m-1}(r)$.

Similarly, for CR manifolds mapped into the Heisenberg group, one has the following

Theorem 3.16. *Let (M, θ) be a compact strictly pseudoconvex CR manifold of dimension $2n + 1$ without boundary.*
i) Let $f : M \longrightarrow \mathbb{H}^m = \mathbb{R}^{2m} \times \mathbb{R}$ be any C^2 map satisfying $df(H(M)) \subseteq H(\mathbb{H}^m)$. Then

$$\lambda_2(\theta)E_b(f) \le \frac{1}{2} \int_M |H_b(f)|^2_{\mathbb{H}^m}$$

where the equality holds if and only if the first $2m$ components $f_1, \dots, f_{2m}$ of f satisfy $-\Delta_b f_\alpha = \lambda_2(\theta) \left(f_\alpha - \fint f_\alpha \right)$ for every $\alpha \le 2m$.
ii) Let $f : M \longrightarrow \mathbb{H}^m$ be any C^2 semi-isometric map satisfying $df(H(M)) \subseteq H(\mathbb{H}^m)$. Then

$$\lambda_2(\theta) \le \frac{1}{2nV(M, \theta)} \int_M |H_b(f)|^2_{\mathbb{H}^m}.$$

Moreover, the equality holds in this last inequality if and only if $f(M)$ is contained in the product $\mathbb{S}^{2m-1}(r) \times \mathbb{R} \subset \mathbb{H}^m$ with $r = \sqrt{\frac{2n}{\lambda_2(\theta)}}$, and $\pi \circ f$ is a pseudo-harmonic map from (M, θ) to the sphere $S^{2m-1}(r)$, where $\pi : \mathbb{H}^m \to \mathbb{R}^{2m}$ is the standard projection.

Proof. i) Let $f : M \longrightarrow \mathbb{H}^m = \mathbb{R}^{2m} \times \mathbb{R}$ be a C^2 map satisfying $df(H(M)) \subseteq H(\mathbb{H}^m)$ and set $\tilde{f} := \pi \circ f : M \longrightarrow \mathbb{R}^{2m}$ where $\pi : \mathbb{H}^m \to \mathbb{R}^{2m}$ is the standard projection. One has, for every pair (X, Y) of horizontal vectors,

$$\beta_{\tilde{f}}(X, Y) = B_\pi(df(X), df(Y)) + d\pi(\beta_f(X, Y))$$

Since for any $X \in H(\mathbb{H}^m)$, $|d\pi(X)|^2_{\mathbb{R}^{2m}} = \frac{1}{4}|X|^2_{\mathbb{H}^m}$ and $d\pi(T) = 0$, one can easily check that $\beta_\pi = 0$ (Corollary 3.2) and $\beta_{\tilde{f}}(X, Y) = d\pi(\beta_f(X, Y))$. Thus, $H_b(\tilde{f}) = d\pi(H_b(f))$ and, since $H_b(f)$ is horizontal (Proposition 3.12), $|H(\tilde{f})|^2_{\mathbb{R}^{2m}} = \frac{1}{4}|H(f)|^2_{\mathbb{H}^m}$. On the other hand, it is clear that $e_b(\tilde{f}) = \frac{1}{4}e_b(f)$ and, then, $E_b(\tilde{f}) = \frac{1}{4}E_b(f)$. Therefore, it suffices to apply Theorem 3.33 to complete the proof of the first part of the theorem.

ii) Assume now that the map f is semi-isometric. Using the assumption that f preserves horizontality, i.e., $df(H(M)) \subseteq H(\mathbb{H}^m)$, one checks that the map $2\pi \circ f$ is also semi-isometric. Applying Corollary 3.37 to the latter we easily deduce what is stated in part (ii) of the theorem.

□

3.6 Horizontal Laplacians on Carnot groups

A Carnot group of step r is a connected, simply connected, nilpotent Lie group G whose Lie algebra $\mathfrak{g}$ admits a stratification

$$\mathfrak{g} = V_1 \oplus ... \oplus V_r$$

so that $[V_1, V_j] = V_{j+1},\ \ j = 1, ..., r-1$ and $[V_i, V_j] \subset V_{i+j},\ \ j = 1, ..., r$, with $V_k = \{0\}$ for $k > r$. We also assume that $\mathfrak{g}$ carries a scalar product $\langle , \rangle_{\mathfrak{g}}$ for which the subspaces V_j are mutually orthogonal. The layer V_1 generates the whole $\mathfrak{g}$ and induces a sub-bundle HG of TG of rank $d_1 = \dim V_1$ that we call the horizontal bundle of the Carnot group. The Heisenberg group $\mathbb{H}^d$ is the simplest example of a Carnot group of step 2.

For each $i \leq r$, let $\{e^i_1, \cdots, e^i_{d_i}\}$ be an orthonormal basis of V_i and denote by $\{X^i_1, \cdots, X^i_{d_i}\}$ the system of left invariant vector fields that coincides with $\{e^i_1, \cdots, e^i_{d_i}\}$ at the identity element of G. We consider the Riemannian metric g_G on G with respect to which the family $\{X^1_1, \cdots, X^1_{d_1}, \cdots, X^r_1, \cdots, X^r_{d_r}\}$ constitute an orthonormal frame for TG. The corresponding Levi-Civita connection ∇ induces a connection on ∇^H on HG that we call "horizontal connection" : If X and Y are a smooth sections of HG, then $\nabla^H_X Y = \pi_H \nabla_X Y$, where $\pi_H : TG \to HG$ is the orthogonal projection. The horizontal Laplacian Δ_H is then defined for every C^2 function on G by

$$\Delta_H u := \mathrm{trace}_H \nabla^H du = \sum_{i \leq d_1} X^1_i \cdot \left(X^1_i \cdot u\right),$$

where the last equality follows from the fact that $\nabla^H_{X^i_1} X^j_1 = 0$ for any $i, j = 1 \ldots d_1$. The operator Δ_H is a hypoelliptic operator of Hörmander type.

Theorem 3.17. *Let G be a Carnot group and let Ω be a bounded domain in G. Let V be a function on Ω so that the operator $-\Delta_H + V$, with Dirichlet boundary conditions if $\Omega \neq G$, admits a purely discrete spectrum $\{\lambda_j\}_{j \geq 1}$ which is bounded from below. Then, for every $k \geq 1$ and $p \in \mathbb{R}$,*

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^p \leq \frac{\max\{4, 2p\}}{d} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{p-1} (\lambda_i - T_i),$$

where d is the rank of the horizontal distribution HG, $T_i = \int_\Omega V u_i^2 v_G$ and v_G is the Riemannian volume element associated with g_G. Moreover, if V is bounded below on Ω, then for every $k \geq 1$,

$$\lambda_{k+1} \leq \left(1 + \frac{4}{d}\right) \frac{1}{k} \sum_{i=1}^{k} \lambda_i - \frac{4}{d} \inf_\Omega V$$

and

$$\lambda_{k+1} \leq \left(1 + \frac{4}{d}\right) k^{\frac{2}{d}} \lambda_1 - C(d, k) \inf_\Omega V$$

with $C(d, k) = (1 + \frac{4}{d}) k^{\frac{2}{d}} - 1$.

Proof. Let $\{e_1, \ldots, e_d\}$ be an orthonormal basis of the subspace V_1 and denote by $\{X_1, \cdots, X_d\}$ the system of left invariant vector fields that coincides with $\{e_1, \ldots, e_d\}$ at the identity element of G. Since the group G is nilpotent, the exponential map $\exp : \mathfrak{g} \longrightarrow G$ is a global diffeomorphism. We can define, for each $i \leq d$, a smooth map $x_i : G \to \mathbb{R}$ by

$$x_i(g) := \left\langle \exp^{-1}(g), e_i \right\rangle_{\mathfrak{g}}.$$

These functions satisfy (see [21, Proposition 5.7]), $\forall i, j = 1, ..., m$,

$$X_j \cdot x_i = \delta_{ij} \text{ and } \Delta_H x_i = 0.$$

Again, we apply Lemma 3.6 with $A = -\Delta_H + V$ and $B = x_\alpha$, $1 \leq \alpha \leq m$. We need to deal with the calculation of $\langle [-\Delta_H + V, x_\alpha] u_i, x_\alpha u_i \rangle_{L^2}$ and $\|[-\Delta_H + V, x_\alpha] u_i\|^2_{L^2}$, where $\{u_i\}_{i \geq 1}$ a complete orthonormal family of eigenfunctions with $(-\Delta_b + V) u_i = \lambda_i u_i$. We have after a straightforward calculation :

$$[-\Delta_H + V, x_\alpha] u_i = -2 X_\alpha \cdot u_i.$$

Integrating by parts we get

$$\int_\Omega (X_\alpha \cdot u_i)\, x_\alpha u_i = \frac{1}{2} \int_\Omega \left(X_\alpha \cdot u_i^2 \right) x_\alpha = -\frac{1}{2} \int_\Omega u_i^2 \,(X_\alpha \cdot x_\alpha) = -\frac{1}{2} \int_\Omega u_i^2 = -\frac{1}{2}.$$

Thus,

$$\sum_{\alpha=1}^{d} \langle [-\Delta_H + V, x_\alpha] u_i, x_\alpha u_i \rangle_{L^2} = -2 \sum_{\alpha=1}^{d} \int_\Omega (X_\alpha \cdot u_i)\, x_\alpha u_i = d.$$

On the other hand, we have

$$\sum_{\alpha=1}^{d} \|[-\Delta_H + V, x_\alpha] u_i\|^2_{L^2} = 4 \sum_{\alpha=1}^{d} \int_\Omega |X_\alpha \cdot u_i|^2 = 4\,(\lambda_i - T_i)$$

Putting these identities in Lemma 3.1, we obtain the first inequality of the theorem.

The rest of the proof is identical to that of Theorem 3.3.

□

Chapter 4

Pseudohermitian Bochner-Lichnerowicz formula

4.1 CR Paneitz operator and Chang-Chiu's formula

Let $(M, T_{1,0}(M))$ be a strictly pseudoconvex CR manifold, of CR dimension n. For all local calculations in this chapter we consider a local frame $\{T_\alpha : 1 \le \alpha \le n\}$ of $T_{1,0}(M)$, defined on the open set U, and set

$$g_{\alpha\overline{\beta}} = G_\theta(T_\alpha \, , \, T_{\overline{\beta}}), \quad T_{\overline{\alpha}} = \overline{T}_\alpha \, ,$$

$$\nabla T_B = \omega_B{}^A T_A \, , \quad \omega_B{}^A = \Gamma^A_{CB}\theta^C \, ,$$

$$\tau(T_\alpha) = A^{\overline{\beta}}_\alpha T_{\overline{\beta}} \, , \quad A_{\alpha\beta} = g_{\alpha\overline{\gamma}} A^{\overline{\gamma}}_\beta \, ,$$

$$\alpha, \beta, \gamma, \cdots \in \{1, \cdots, n\}, \quad A, B, C, \cdots \in \{0, 1, \cdots, n, \overline{1}, \cdots, \overline{n}\}.$$

Here $\{\theta^\alpha : 1 \le \alpha \le n\}$ is the *adpated* coframe determined by

$$\theta^\alpha(T_\beta) = \delta^\alpha_\beta \, , \quad \theta^\alpha(T_{\overline{\beta}}) = 0, \quad \theta^\alpha(T) = 0.$$

Then (cf. e.g. (1.62) and (1.64) in [94], p. 39-40)

$$d\theta = 2i g_{\alpha\overline{\beta}} \theta^\alpha \wedge \theta^{\overline{\beta}} \, , \tag{4.1}$$

$$d\theta^\alpha = \theta^\beta \wedge \omega_\beta{}^\alpha + \theta \wedge \tau^\alpha \, , \quad \tau^\alpha \equiv A^\alpha_{\overline{\beta}} \theta^{\overline{\beta}} \, , \quad A^\alpha_{\overline{\beta}} = \overline{A^{\overline{\alpha}}_\beta} \, , \tag{4.2}$$

$$A_{\alpha\beta} = A_{\beta\alpha} \, . \tag{4.3}$$

Therefore, if we set $A(X, Y) = g_\theta(\tau X, Y)$ for any $X, Y \in \mathfrak{X}(M)$ then A is symmetric. Let R^∇ be the curvature tensor field of the Tanaka-Webster connection ∇. As to the local components of R^∇ we adopt the convention $R^\nabla(T_B, T_C)T_A = R_A{}^D{}_{BC} T_D$ (cf. [94], p. 50). The Ricci tensor of ∇ is

$$\mathrm{Ric}_\nabla(Y, Z) = \mathrm{trace}\left\{X \in T(M) \longmapsto R^\nabla(X, Z)Y\right\}, \quad Y, Z \in T(M).$$

Locally we set $R_{AB} = \mathrm{Ric}_\nabla(T_A, T_B)$. The *pseudohermitian Ricci tensor* is then $R_{\lambda\overline{\mu}}$. By a result of S. Webster, [100] (to whom the notion is due) $R_{\lambda\overline{\mu}} = R_\lambda{}^\alpha{}_{\alpha\overline{\mu}}$. The *pseudohermitian scalar curvature* is $\rho = g^{\lambda\overline{\mu}} R_{\lambda\overline{\mu}}$ where $\left[g^{\alpha\overline{\beta}}\right] = \left[g_{\alpha\overline{\beta}}\right]^{-1}$. Let us set

$$\Pi_\alpha{}^\beta = d\omega_\alpha{}^\beta - \omega_\alpha{}^\gamma \wedge \omega_\gamma{}^\beta \,,$$

$$\Omega_\alpha{}^\beta = \Pi_\alpha{}^\beta - 2i\theta_\alpha \wedge \tau^\beta + 2i\tau_\alpha \wedge \theta^\beta \,,$$

where

$$\theta_\alpha = g_{\alpha\overline{\beta}}\theta^{\overline{\beta}} \,, \quad \theta^{\overline{\alpha}} = \overline{\theta^\alpha} \,, \quad \tau_\alpha = g_{\alpha\overline{\beta}}\tau^{\overline{\beta}} \,, \quad \tau^{\overline{\beta}} = A^{\overline{\beta}}_\alpha \theta^\alpha \,.$$

By a result of S.M. Webster, [100] (cf. also Theorem 1.7 in [94], p. 55)

$$\Omega_\alpha{}^\beta = R_\alpha{}^\beta{}_{\lambda\overline{\mu}}\, \theta^\lambda \wedge \theta^{\overline{\mu}} + W^\beta_{\alpha\lambda}\theta^\lambda \wedge \theta - W^\beta_{\alpha\overline{\lambda}}\theta^{\overline{\lambda}} \wedge \theta \tag{4.4}$$

where

$$W^\beta_{\alpha\overline{\mu}} = g^{\beta\overline{\sigma}}\, \nabla_\alpha A_{\overline{\mu}\overline{\sigma}} \,, \quad W^\beta_{\alpha\lambda} = g^{\beta\overline{\sigma}} \nabla_{\overline{\sigma}} A_{\alpha\lambda} \,. \tag{4.5}$$

Given $u \in C^\infty(M, \mathbb{R})$ the *pseudohermitian Hessian* is

$$(\nabla^2 u)(X, Y) = (\nabla_X du)Y, \quad X, Y \in \mathfrak{X}(M).$$

Locally we set $\nabla_A u_B = (\nabla^2 u)(T_A, T_B)$. The pseudohermitian Hessian is not symmetric. Rather one has the commutation formulae

$$\nabla_\alpha u_\beta = \nabla_\beta u_\alpha \,, \tag{4.6}$$

$$\nabla_\alpha u_{\overline{\beta}} = \nabla_{\overline{\beta}} u_\alpha - 2i g_{\alpha\overline{\beta}} u_0 \,, \quad u_0 \equiv T(u), \tag{4.7}$$

$$\nabla_0 u_\beta = \nabla_\beta u_0 - u_{\overline{\alpha}} A^{\overline{\alpha}}_\beta \,. \tag{4.8}$$

The third order covariant derivative of u is given by

$$(\nabla^3 u)(X, Y, Z) = (\nabla_X H_u)(Y, Z) =$$

$$= X(H_u(Y, Z)) - H_u(\nabla_X Y, Z) - H_u(Y, \nabla_X Z), \quad H_u \equiv \nabla^2 u \,,$$

for any $X, Y, Z \in \mathfrak{X}(M)$. Locally we set $u_{ABC} = (\nabla^3 u)(T_A, T_B, T_C)$. Commutation formulae for u_{ABC} have been established by J.M. Lee, [60] (cf. also [94], p. 426) and are not needed through this chapter.

Let $\Delta_b u = -\mathrm{div}\left(\nabla^H u\right)$ be the sublaplacian of (M, θ). Another useful expression of Δ_b is

$$\Delta_b u = -\mathrm{trace}_{G_\theta} \Pi_H \, \nabla^2 u \tag{4.9}$$

or (locally)

$$\Delta_b u = -\sum_{a=1}^{2n} \{E_a(E_a(u)) - (\nabla_{E_a} E_a)(u)\}$$

for any local G_θ-orthonormal frame $\{E_a : 1 \le a \le 2n\}$ of $H(M)$ on $U \subset M$. If $\{T_\alpha : 1 \le \alpha \le n\}$ is a local frame of $T_{1,0}(M)$ on $U \subset M$ and $E_a = E^\lambda_a T_\lambda + E^{\overline{\lambda}}_a T_{\overline{\lambda}}$ for some $E^\lambda_a \in C^\infty(U, \mathbb{C})$ with $E^{\overline{\lambda}}_a = \overline{E^\lambda_a}$ then $G_\theta(E_a\,, E_b) = \delta_{ab}$ yields

$$\sum_{a=1}^{2n} E^\alpha_a E^\beta_a = 0, \quad \sum_{a=1}^{2n} E^\alpha_a E^{\overline{\beta}}_a = g^{\alpha\overline{\beta}} \,,$$

so that (4.9) may be written locally as

$$\Delta_b u = -\nabla_\alpha u^\alpha - \nabla_{\overline{\alpha}} u^{\overline{\alpha}} . \tag{4.10}$$

A complex valued differential p-form $\omega \in \Omega^p(M) \otimes \mathbb{C}$ is a $(p,0)$-*form* (respectively a $(0,p)$-*form*) if $T_{0,1}(M) \rfloor \omega = 0$ (respectively $T_{0,1}(M) \rfloor \omega = 0$ and $T \rfloor \omega = 0$). Let $\Lambda^{p,0}(M) \to M$ and $\Lambda^{0,p}(M) \to M$ be the relevant bundles and $\Omega^{p,0}(M)$ and $\Omega^{0,p}(M)$ the corresponding spaces of sections. Let $\mathcal{F}$ be the flow on M tangent to the Reeb vector T (i.e. $T(\mathcal{F}) = \mathbb{R}T$). Let $\Omega^{1,0}_B(\mathcal{F}) = \{\omega \in \Omega^{1,0}(M) : T \rfloor \omega = 0\}$ be the space of all basic $(1,0)$-forms (on the foliated manifold $(M, \mathcal{F})$, cf. also [29]). If $\omega \in \Omega^{1,0}_B(\mathcal{F})$ one may use the Levi form to define a unique complex vector field $\omega^\sharp \in C^\infty(T_{0,1}(M))$. Here $\omega^\sharp$ is determined by

$$\omega(Z) = G_\theta(Z, \omega^\sharp), \quad Z \in T_{1,0}(M),$$

hence locally $\omega^\sharp = \omega^{\overline{\beta}} T_{\overline{\beta}}$ where $\omega^{\overline{\beta}} = g^{\alpha\overline{\beta}} \omega_\alpha$ and $\omega = \omega_\alpha \theta^\alpha$. Let $\delta_b : \Omega^{1,0}_B(\mathcal{F}) \to C^\infty(M, \mathbb{C})$ be the differential operator (due to [60]) defined by

$$\delta_b \omega = \operatorname{div}\left(\omega^\sharp\right), \quad \delta_b \theta = 0, \quad \omega \in \Omega^{0,1}_B(\mathcal{F}).$$

Similarly, again by following [60], if $\eta \in \Omega^{0,1}(M)$ then let $\eta^\sharp \in C^\infty(T_{1,0}(M))$ be determined by

$$\eta(\overline{Z}) = G_\theta(\eta^\sharp, \overline{Z}), \quad Z \in T_{1,0}(M),$$

and let us consider

$$\overline{\delta}_b : \Omega^{0,1}(M) \to C^\infty(M, \mathbb{C}), \quad \overline{\delta}_b \eta = \operatorname{div}\left(\eta^\sharp\right), \quad \eta \in \Omega^{0,1}(M),$$

so that (locally) $\eta^\sharp = \eta^\alpha T_\alpha$ where $\eta = \eta_{\overline{\beta}} \theta^{\overline{\beta}}$ and $\eta^\alpha = g^{\alpha\overline{\beta}} \eta_{\overline{\beta}}$. Also (again locally) $\delta_b \omega = \nabla_{\overline{\beta}} \omega^{\overline{\beta}}$ and $\overline{\delta}_b \eta = \nabla_\alpha \eta^\alpha$. For each $f \in C^\infty(M, \mathbb{C})$ we set

$$(Pf)Z = g^{\alpha\overline{\beta}} \left(\nabla^3 f\right)(Z, T_\alpha, T_{\overline{\beta}}) + 2\, n\, i\, A\left(Z, (\nabla^H f)^{1,0}\right), \tag{4.11}$$

$$(Pf)\overline{Z} = 0, \quad (Pf)T = 0, \quad Z \in T_{1,0}(M).$$

Here $X^{1,0} = \Pi_{1,0} X$ for any $X \in H(M)$ and $\Pi_{1,0} : H(M) \otimes \mathbb{C} \to T_{1,0}(M)$ is the natural projection associated to $H(M) \otimes \mathbb{C} = T_{1,0}(M) \oplus T_{0,1}(M)$. Note that $g^{\alpha\overline{\beta}} \left(\nabla_{T_{\overline{\beta}}}(\nabla^2 f)\right)(T_\alpha, Z)$ is invariant under a transformation

$$T'_\alpha = U^\beta_\alpha T_\beta , \quad \det\left[U^\beta_\alpha\right] \neq 0 \;\text{ on }\; U \cap U',$$

hence $(Pf)Z$ is globally defined. Locally one has

$$Pf = \left(P_\beta f\right)\theta^\beta , \quad P_\beta f = f_\beta{}^{\overline{\alpha}}{}_{\overline{\alpha}} + 2ni\, A_{\beta\gamma} f^\gamma ,$$

(compare to Definition 1.1 and (1.2) in [92], p. 263). Similar to $P : C^\infty(M, \mathbb{C}) \to \Omega^{1,0}_B(\mathcal{F})$ we build $\overline{P} : C^\infty(M, \mathbb{C}) \to \Omega^{0,1}(M)$ given by

$$\left(\overline{P} f\right)\overline{Z} = g^{\alpha\overline{\beta}} \left(\nabla^3 f\right)(Z, T_{\overline{\beta}}, T_\alpha) - 2\, n\, i\, A\left(\overline{Z}, (\nabla^H f)^{0,1}\right), \tag{4.12}$$

$$(\overline{P} f)Z = 0, \quad (\overline{P} f)T = 0, \quad Z \in T_{1,0}(M),$$

where $X^{0,1} = \overline{X^{1,0}}$ for any $X \in H(M)$. Also let[1]

$$P_0 f = \delta_b(Pf) + \overline{\delta}_b(\overline{P}f), \quad f \in C^\infty(M,\mathbb{C}). \tag{4.13}$$

From now on we assume that M is a compact strictly pseudoconvex CR manifold and $\theta \in \mathcal{P}_+$. Then g_θ is a Riemannian metric on M. It should be observed that the operators above are complexifications of real operators familiar in Riemannian geometry, as follows. For instance let $\sharp$ be "raising of indices" with respect to g_θ i.e. $g_\theta\left(\alpha^\sharp, X\right) = \alpha(X)$ for any (real) 1-form $\eta \in \Omega^1(M)$ and any (real) vector field $X \in \mathfrak{X}(M)$. Then the musical isomorphisms $\sharp : \Omega_B^{1,0}(\mathcal{F}) \to C^\infty(T_{0,1}(M))$ and $\sharp : \Omega^{0,1}(M) \to C^\infty(T_{1,0}(M))$ (as built above) are restrictions of the $\mathbb{C}$-linear extension (to $\Omega^1(M) \otimes \mathbb{C} = C^\infty\left(T^*(M) \otimes \mathbb{C}\right)$) of $\sharp : \Omega^1(M) \to \mathfrak{X}(M)$ to $\Omega_B^{1,0}(\mathcal{F})$ and $\Omega^{0,1}(M)$ respectively.

Also let $\Omega_B^1(\mathcal{F})$ be the space of all basic 1-forms on $(M,\mathcal{F})$ and $d_b : C^\infty(M) \to \Omega_B^1(\mathcal{F})$ the first order differential operator given by

$$d_b u = du - u_0\,\theta, \quad u \in C^\infty(M,\mathbb{R}), \quad u_0 \equiv T(u).$$

Let $d_b^* : \Omega_B^1(\mathcal{F}) \to C^\infty(M,\mathbb{R})$ be the formal adjoint of d_b i.e.

$$\left(d_b^*\omega\,,\,u\right)_{L^2} = (\omega\,,\,d_b u)_{L^2}\;, \quad \omega \in \Omega_B^1(\mathcal{F}), \;\; u \in C^\infty(M),$$

with respect to the L^2 inner products

$$(u,v)_{L^2} = \int_M uv\,\Psi_\theta\,, \quad (\alpha,\beta)_{L^2} = \int_M g_\theta^*(\alpha\,,\beta)\,\Psi_\theta\,,$$

$$u,v \in C^\infty(M,\mathbb{R}), \quad \alpha,\beta \in \Omega^1(M).$$

Let $d_b : C^\infty(M,\mathbb{C}) \to \Omega_B^1(\mathcal{F}) \otimes \mathbb{C}$ and $d_b^* : \Omega_B^1(\mathcal{F}) \otimes \mathbb{C} \to C^\infty(M,\mathbb{C})$ be the $\mathbb{C}$-linear extensions of d_b and d_b^*. Then

Lemma 4.1. i) $\Omega_B^1(\mathcal{F}) \otimes \mathbb{C} = \Omega_B^{1,0}(\mathcal{F}) \oplus \Omega^{0,1}(M)$,

ii) $d_b f = \partial_b f + \overline{\partial}_b f$ *for any* $f \in C^\infty(M,\mathbb{C})$,

iii) $d_b^*\big|_{\Omega_B^{1,0}(\mathcal{F})} = \partial_b^* = -\delta_b$,

iv) $d_b^*\big|_{\Omega^{0,1}(M)} = \overline{\partial}_b^* = -\overline{\delta}_b$.

Here the tangential C-R operator $\overline{\partial}_b$ is thought of as $\Omega^{0,1}(M)$-valued (i.e. one requests that $Z \rfloor \overline{\partial}_b f =$ and $T \rfloor \overline{\partial}_b f = 0$ to start with). Also $\partial_b f$ is the unique element of $\Omega_B^{1,0}(\mathcal{F})$ coinciding with df on $T_{1,0}(M)$. Locally

$$\partial_b f = f_\alpha \theta^\alpha\,, \quad \overline{\partial}_b f = f_{\overline{\alpha}} \theta^{\overline{\alpha}}\,, \quad f_\alpha \equiv T_\alpha(f), \quad f_{\overline{\alpha}} \equiv T_{\overline{\alpha}}(f).$$

Also

$$\partial_b^* : \Omega_B^{0,1}(\mathcal{F}) \to C^\infty(M,\mathbb{C}), \quad \overline{\partial}_b^* : \Omega^{0,1}(M) \to C^\infty(M,\mathbb{C}),$$

are the formal adjoints of

$$\partial_b : C^\infty(M,\mathbb{C}) \to \Omega_B^{1,0}(\mathcal{F}), \quad \overline{\partial}_b : C^\infty(M,\mathbb{C}) \to \Omega^{0,1}(M),$$

[1] The operator P_0 in this thesis and [92] differ by a multiplicative factor $\frac{1}{4}$.

with respect to the L^2 inner products

$$(f, g)_{L^2} = \int_M f\overline{g}\,\Psi_\theta\,, \quad (\omega_1\,,\,\omega_2)_{L^2} = \int_M G^*_\theta(\omega_1\,,\,\overline{\omega}_2)\,\Psi_\theta\,,$$

for any $f, g \in C^\infty(M, \mathbb{C})$ and any complex 1-forms ω_1 , ω_2 either in $\Omega^{1,0}_B(\mathcal{F})$ or in $\Omega^{0,1}(M)$. Statements (i)-(ii) in Lemma 4.1 are immediate. The last equality in (iii) (respectively in (iv)) is due to [60] (cf. also [94], p. 280). To prove (iii) let $\omega \in \Omega^{1,0}_B(\mathcal{F})$ and $f \in C^\infty(M, \mathbb{C})$. Then

$$G^*_\theta(\omega\,,\,\overline{d_b f}) = \operatorname{div}(f\omega^\sharp) - \overline{f}\operatorname{div}(\omega^\sharp) \tag{4.14}$$

hence (by Green's lemma)

$$\left(d^*_b\omega\,,\,f\right)_{L^2} = \int_M G^*_\theta(\omega\,,\,\overline{d_b f})\,\Psi_\theta = -\int_M \overline{f}\operatorname{div}(\omega^\sharp)\,\Psi_\theta = -\,(\delta_b\omega\;f)_{L^2}$$

so that $d^*_b\omega = \delta_b\omega$. As to the proof of (4.14) one may locally compute

$$G^*_\theta(\omega\,,\,\overline{d_b f}) = \omega_\alpha T_{\overline{\beta}}(\overline{f})g^{\alpha\overline{\beta}} = T_{\overline{\beta}}(\overline{f}\omega^{\overline{\beta}}) - \overline{f}\,T_{\overline{\beta}}(\omega^{\overline{\beta}}) =$$

$$= \operatorname{div}\left(\overline{f}\,\omega^{\overline{\beta}}T_{\overline{\beta}}\right) - \overline{f}\left\{\omega^{\overline{\beta}}\operatorname{div}(T_{\overline{\beta}}) + T_{\overline{\beta}}(\omega^{\overline{\beta}})\right\} =$$

$$= \operatorname{div}\left(\overline{f}\omega^\sharp\right) - \overline{f}\left\{T_{\overline{\beta}}(\omega^{\overline{\beta}}) + \Gamma^{\overline{\alpha}}_{\overline{\alpha}\overline{\beta}}\omega^{\overline{\beta}}\right\} = \operatorname{div}\left(\overline{f}\omega^\sharp\right) - \overline{f}\,\nabla_{\overline{\beta}}\omega^{\overline{\beta}}$$

and

$$\nabla_{\overline{\beta}}\omega^{\overline{\beta}} = \operatorname{trace}\left\{T_{\overline{\alpha}} \longmapsto \nabla_{T_{\overline{\alpha}}}\omega^\sharp\right\} = \operatorname{div}(\omega^\sharp).$$

Finally one may complete the proof of (iii) by observing that $G^*_\theta(\omega\,,\,\overline{\partial_b f}) = G^*_\theta(\omega\,,\,\overline{d_b f})$ so that $d^*_b\omega = \partial^*_b\omega$. The proof of (iv) is similar (hence omitted). Lemma 4.1 is proved.

For every $f \in C^\infty(M, \mathbb{R})$

$$\int_M g^*_\theta\left((P + \overline{P})f\,,\,\overline{d_b f}\right)\,\Psi_\theta = \left(Pf + \overline{P}f\,,\,d_b f\right)_{L^2} =$$

$$= \left(d^*_b(Pf + \overline{P}f)\,,\,f\right)_{L^2} = -\,(P_0 f\,,\,f)_{L^2}$$

(compare to (1.3) in [92], p. 263). By a result of S-C. Chang & H-L. Chiu, [92], the operator P_0 is nonnegative i.e. $\int_M (P_0 u)u\,\Psi_\theta \geq 0$ for any $u \in C^\infty(M, \mathbb{R})$. We end the preparation of CR and pseudohermitian geometry by establishing

$$u^\alpha\;u_\alpha{}^\beta{}_\beta + u^{\overline{\alpha}}\;u_{\overline{\alpha}}{}^{\overline{\beta}}{}_{\overline{\beta}} = -u^\alpha\,P_\alpha u - u^{\overline{\alpha}}\,P_{\overline{\alpha}}u + \tag{4.15}$$

$$+2ni\left(A_{\alpha\beta}u^\alpha u^\beta - A_{\overline{\alpha}\overline{\beta}}u^{\overline{\alpha}}u^{\overline{\beta}}\right) - \left(\nabla^H u\right)(\Delta_b u)$$

(compare to (2.3) in [92], p. 267). Indeed (by (4.10))

$$g_\theta\left(\nabla^H u\,,\,\nabla^H\Delta_b u\right) = u^\alpha\,(\Delta_b u)^{\overline{\beta}}\,g_{\alpha\overline{\beta}} + u^{\overline{\alpha}}\,(\Delta_b u)^\beta\,g_{\overline{\alpha}\beta} =$$

$$= -u^\alpha\left(\nabla_\gamma u^\gamma + \nabla_{\overline{\gamma}}u^{\overline{\gamma}}\right)_\alpha - u^{\overline{\alpha}}\left(\nabla_\gamma u^\gamma + \nabla_{\overline{\gamma}}u^{\overline{\gamma}}\right)_{\overline{\alpha}} =$$

$$= -u^\alpha \left(\nabla_\alpha \nabla_\gamma u^\gamma + \nabla_\alpha \nabla_{\overline{\gamma}} u^{\overline{\gamma}}\right) - u^{\overline{\alpha}} \left(\nabla_{\overline{\alpha}} \nabla_\gamma u^\gamma + \nabla_{\overline{\alpha}} \nabla_{\overline{\gamma}} u^{\overline{\gamma}}\right) =$$

$$= -u^\alpha \left(g^{\gamma\overline{\beta}}\, u_{\alpha\gamma\overline{\beta}} + g^{\beta\overline{\gamma}}\, u_{\alpha\overline{\gamma}\beta}\right) - u^{\overline{\alpha}} \left(g^{\gamma\overline{\beta}}\, u_{\overline{\alpha}\gamma\overline{\beta}} + g^{\beta\overline{\gamma}}\, u_{\overline{\alpha}\overline{\gamma}\beta}\right) =$$

$$= -u^\alpha \left(u_\alpha{}^{\overline{\beta}}{}_{\overline{\beta}} + u_\alpha{}^\beta{}_\beta\right) - u^{\overline{\alpha}} \left(u_{\overline{\alpha}}{}^{\overline{\beta}}{}_{\overline{\beta}} + u_{\overline{\alpha}}{}^\beta{}_\beta\right) =$$

$$= -u^\alpha \left(P_\alpha u - 2ni\, A_{\alpha\beta} u^\beta + u_\alpha{}^\beta{}_\beta\right) - u^{\overline{\alpha}} \left(u_{\overline{\alpha}}{}^{\overline{\beta}}{}_{\overline{\beta}} + P_{\overline{\alpha}} u + 2ni\, A_{\overline{\alpha}\overline{\beta}} u^{\overline{\beta}}\right).$$

Q.e.d.

4.2 Bochner-Lichnerowicz formulae on Fefferman spaces

Let $S^1 \to C(M) \xrightarrow{\pi} M$ be the canonical circle bundle over a strictly pseudoconvex CR manifold M, of CR dimension n (cf. e.g. Definition 2.9 in [94], p. 119). We set $\mathfrak{M} = C(M)$ for simplicity. Let $\theta \in \mathcal{P}_+$ be a positively oriented contact form on M and let F_θ be the corresponding Fefferman metric on $\mathfrak{M}$ i.e.

$$F_\theta = \pi^* \tilde{G}_\theta + 2\,(\pi^*\theta) \odot \sigma, \tag{4.16}$$

$$\sigma = \frac{1}{n+2} \left\{ d\gamma + \pi^* \left(i\,\omega_\alpha{}^\alpha - \frac{i}{2}\, g^{\mu\overline{\nu}}\, dg_{\mu\overline{\nu}} - \frac{\rho}{4(n+1)}\,\theta \right) \right\}. \tag{4.17}$$

Cf. Definition 2.15 and Theorem 2.4 in [94], p. 128-129. As to the notations in (4.16)-(4.17) we set $\tilde{G}_\theta = G_\theta$ on $H(M) \otimes H(M)$ and $\tilde{G}_\theta(T, W) = 0$ for every $W \in \mathfrak{X}(M)$. Moreover γ is a local fibre coordinate on $\mathfrak{M}$. We recall that $F_\theta \in \mathrm{Lor}(\mathfrak{M})$ i.e. F_θ is a Lorentzian metric on $\mathfrak{M}$ (a semi-Riemannian metric of signature $(- + \cdots +)$) and its *restricted* conformal class $\{e^{2\,u\circ\pi} F_\theta : u \in C^\infty(M, \mathbb{R})\}$ is a CR invariant (cf. [59]).

Let D be the Levi-Civita connection of $(\mathfrak{M}, F_\theta)$. Given a point $z_0 \in \mathfrak{M}$ let $\{E_p : 1 \le p \le 2n+2\}$ be a local orthonormal (i.e. $F_\theta(E_p\,,\, E_q) = \epsilon_p\, \delta_{pq}$ with $\epsilon_p \in \{\pm 1\}$) frame of $T(\mathfrak{M})$, defined on an open neighborhood $\pi^{-1}(U) \subset \mathfrak{M}$ of z_0, such that

$$(D_{E_p} E_q)(z_0) = 0, \quad 1 \le p, q \le 2n+2.$$

Such a local frame may always be built by parallel translating a given orthonormal basis $\{e_p : 1 \le p \le 2n+2\} \subset T_{z_0}(\mathfrak{M})$ along the geodesics of $(\mathfrak{M}, F_\theta)$ issuing at z_0.

Let $\Box$ be the wave operator (the Laplace-Beltrami operator of $(\mathfrak{M}, F_\theta)$). If $f \in C^\infty(\mathfrak{M}, \mathbb{R})$ and $g = F_\theta(Df, Df)$ then

$$(\Box g)(z_0) = -\sum_{p=1}^{2n+2} \epsilon_p \left\{ E_p\left(E_p(g)\right) - \left(D_{E_p} E_p\right)(g) \right\}_{z_0} =$$

$$= -2 \sum_p \epsilon_p\, E_p \left(F_\theta(D_{E_p} Df\,,\, Df)\right)_{z_0} =$$

$$= -2 \sum_p \epsilon_p \left\{ F_\theta(D_{E_p} D_{E_p} Df\,,\, Df) + F_\theta(D_{E_p} Df\,,\, D_{E_p} Df) \right\}_{z_0}.$$

As $\{E_p : 1 \leq p \leq 2n+2\}$ is orthonormal, the first term may be written

$$F_\theta(D_{E_p} D_{E_p} Df \, , \, Df)_{z_0} = \sum_q \epsilon_q \, F_\theta(D_{E_p} D_{E_p} Df \, , \, E_q)_{z_0} \, E_q(f)_{z_0} \, .$$

On the other hand

$$F_\theta(D_{E_p} D_{E_p} Df \, , \, E_q)_{z_0} =$$

$$= E_p\left(F_\theta(D_{E_p} Df \, , \, E_q)\right)_{z_0} - F_\theta\left(D_{E_p} Df \, , \, D_{E_p} E_q\right)_{z_0} =$$

(by $(D_{E_p} E_q)_{z_0} = 0$ and $DF_\theta = 0$)

$$= E_{p,z_0}\left\{E_p\left(F_\theta(Df \, , \, E_q)\right) - F_\theta\left(Df \, , \, D_{E_p} E_q\right)\right\} =$$

$$= E_{p,z_0}\left\{E_p(E_q(f)) - (D_{E_p} E_q)(f)\right\} = E_{p,z_0}\left\{(D^2 f)(E_p \, , \, E_q)\right\}$$

where

$$(D^2 f)(X, Y) = X(Y(f)) - (D_X Y)(f), \quad X, Y \in \mathfrak{X}(\mathfrak{M}),$$

(the Hessian of f). As F_θ is a Lorentzian metric, $D^2 f$ is symmetric. Thus

$$E_{p,z_0}\left\{(D^2 f)(E_p \, , \, E_q)\right\} = E_{p,z_0}\left\{(D^2 f)(E_q \, , \, E_p)\right\} =$$

(by reversing the calculation above)

$$= F_\theta\left(D_{E_p} D_{E_q} Df \, , \, E_p\right)_{z_0} \, .$$

So far we obtained

$$-(1/2) \, \square \, (F_\theta(Df, Df))_{z_0} = \sum_p \epsilon_p \, F_\theta\left(D_{E_p} Df \, , \, D_{E_p} Df\right)_{z_0} + \tag{4.18}$$

$$+ \sum_{p,q} \epsilon_p \epsilon_q \, F_\theta\left(D_{E_p} D_{E_q} Df \, , \, E_p\right)_{z_0} \, E_q(f)_{z_0} \, .$$

Let (U, x^j) be a local coordinate system on M and let $(\pi^{-1}(U), Z^p)$ be the induced local coordinates on $\mathfrak{M}$ i.e. $Z^j = x^j \circ \pi$ and $Z^{2n+2} = \gamma$. If $\mathfrak{B}$ is a $C^\infty(\mathfrak{M})$-bilinear form on $\mathfrak{X}(\mathfrak{M})$ then

$$F_\theta^*(\mathfrak{B}, \mathfrak{B}) = F^{pr} \, F^{qs} \, \mathfrak{B}_{pq} \, \mathfrak{B}_{rs}$$

on $\pi^{-1}(U)$ where

$$F_{pq} = F_\theta\left(\partial_p \, , \, \partial_q\right), \quad \mathfrak{B}_{pq} = \mathfrak{B}\left(\partial_p \, , \, \partial_q\right), \quad \partial_p = \frac{\partial}{\partial Z^p} \, , \quad F_{pq} F^{qr} = \delta_p^r \, .$$

If $E_p = E_p^q \partial_q$ then $\sum_p \epsilon_p E_p^q E_p^r = F^{qr}$. Hence

$$F_\theta^*(D^2 f \, , \, D^2 f)_{z_0} = \sum_{p,q} \epsilon_p \epsilon_q (D^2 f)(E_p \, , \, E_q)_{z_0}^2 =$$

$$= \sum_{p,q} \epsilon_p \epsilon_q \left\{E_p(E_q(f)) - (D_{E_p} E_q)(f)\right\}_{z_0}^2 =$$

$$= \sum_{p,q} \epsilon_p \epsilon_q F_\theta \left(D_{E_p} Df \, , \, E_q \right)^2_{z_0}$$

that is

$$F^*_\theta (D^2 f \, , \, D^2 f)_{z_0} = \sum_p \epsilon_p \, F_\theta \left(D_{E_p} Df \, , \, D_{E_p} Df \right). \tag{4.19}$$

Let R^D be the curvature tensor field of R^D. Then

$$D_{E_p} D_{E_q} = D_{E_q} D_{E_p} + R^D(E_p \, , \, E_q) + \left[E_p \, , \, E_q \right],$$

$$\left[E_p \, , \, E_q \right]_{z_0} = 0,$$

$$\sum_p \epsilon_p F_\theta \left(D_{E_q} D_{E_p} Df \, , \, E_p \right)_{z_0} = \sum_p \epsilon_p E_q \left(F_\theta (D_{E_p} Df \, , \, E_p) \right)_{z_0} =$$

$$= E_q \left(\operatorname{div}(Df) \right) = -E_q \left(\Box f \right),$$

so that

$$\sum_{p,q} \epsilon_p \epsilon_q F_\theta \left(D_{E_p} D_{E_q} Df \, , \, E_p \right)_{z_0} \; E_q(f)_{z_0} = \tag{4.20}$$

$$= \sum_q \epsilon_q \left\{ -E_q(\Box f)_{z_0} + \sum_p \epsilon_p F_\theta \left(R^D(E_p \, , \, E_q) Df \, , \, E_p \right)_{z_0} \right\} E_q(f)_{z_0} \, .$$

Let Ric_D and K^D be respectively the Ricci curvature and the Christoffel 4-tensor of $(\mathfrak{M}, F_\theta)$i.e.

$$\mathrm{Ric}_D(X, Y) = \operatorname{trace} \left\{ Z \in T(\mathfrak{M}) \mapsto R^D(Z, Y) X \right\},$$

$$K^D(X, Y, Z, W) = F_\theta \left(R^D(Z, W) Y \, , \, X \right),$$

for any $X, Y, Z, W \in T(\mathfrak{M})$. Then (by taking into account the symmetries of the Christoffel tensor)

$$\sum_p \epsilon_p F_\theta \left(R^D(E_p \, , \, E_q) Df \, , \, E_p \right)_{z_0} =$$

$$= \sum_p \epsilon_p K^D \left(E_p \, , \, Df \, , \, E_p \, , \, E_q \right)_{z_0} = \sum_p \epsilon_p K^D \left(E_p \, , \, E_q \, , \, E_p \, , \, Df \right)_{z_0} =$$

$$= \sum_p \epsilon_p F_\theta \left(R^D(E_p \, , \, Df) E_q \, , \, E_p \right)_{z_0} =$$

$$= \operatorname{trace} \left\{ Z \in T(\mathfrak{M}) \mapsto R^D(Z \, , \, Df) E_q \right\}_{z_0} = \mathrm{Ric}_D(E_q \, , \, Df)$$

so that (by (4.20))

$$\sum_{p,q} \epsilon_p \epsilon_q F_\theta \left(D_{E_p} D_{E_q} Df \, , \, E_p \right)_{z_0} \; E_q(f)_{z_0} =$$

$$= \sum_q \epsilon_q \left\{ -E_q(\Box f)_{z_0} + \mathrm{Ric}_D(E_q \, , \, Df) \right\} E_q(f)_{z_0} =$$

$$= -(Df)(\Box f)_{z_0} + \mathrm{Ric}_D(Df, Df)_{z_0}$$

and (by taking into account (4.19)) one may write (4.18) as

$$-(1/2)\,\Box\,(F_\theta(Df, Df)) = F_\theta^*\left(D^2 f\,,\,D^2 f\right) - \tag{4.21}$$

$$-(Df)(\Box f) + \mathrm{Ric}_D(Df\,,\,Df).$$

Let us assume that M is a closed manifold (i.e. M is compact and $\partial M = \emptyset$). Then $\mathfrak{M}$ is a closed manifold, as well (as the total space of a locally trivial bundle over a compact manifold, with compact fibres). Integration of (4.21) over $\mathfrak{M}$ leads (by Green's lemma) to the (Lorentzian analog to the) L^2 Bochner-Lichnerowicz formula

$$\int_{\mathfrak{M}} \left\{F_\theta^*\left(D^2 f\,,\,D^2 f\right) + \mathrm{Ric}_D(Df\,,\,Df)\right\}\,d\mathrm{vol}(F_\theta) = \tag{4.22}$$

$$= \int_{\mathfrak{M}} (Df)(\Box f)\,d\,\mathrm{vol}(F_\theta).$$

Compare to (G.IV.5) in [71], p. 131.

4.3 Curvature theory

By a result in [18] the 1-form $\sigma \in \Omega^1(M)$ is a connection form in the canonical circle bundle $S^1 \to \mathfrak{M} \to M$. Let $X^\uparrow \in \mathfrak{X}(\mathfrak{M})$ denote the horizontal lift of $X \in \mathfrak{X}(M)$ i.e. $X_z^\uparrow \in \mathrm{Ker}(d_z\pi)$ and $(d_z\pi)X_z^\uparrow = X_{\pi(z)}$ for any $z \in \mathfrak{M}$. Let $S \in \mathfrak{X}(\mathfrak{M})$ be the tangent to the S^1-action i.e. locally

$$S = \frac{n+2}{2}\,\frac{\partial}{\partial\gamma}\,.$$

The Levi-Civita connection D of $(\mathfrak{M}, F_\theta)$ is given by (cf. Lemma 2 in [31], p. 03504-26)

$$D_{X^\uparrow} Y^\uparrow = (\nabla_X Y)^\uparrow + \tag{4.23}$$

$$+\{\Omega(X,Y)\circ\pi\}\;T^\uparrow + \left\{\sigma\left(\left[X^\uparrow, Y^\uparrow\right]\right) - 2\,A(X,Y)\circ\pi\right\}S,$$

$$D_{X^\uparrow} T^\uparrow = \{\tau(X) + \phi(X)\}^\uparrow\,, \tag{4.24}$$

$$D_{T^\uparrow} X^\uparrow = (\nabla_T X + \phi X)^\uparrow + 4\,(d\sigma)(X^\uparrow, T^\uparrow)\,S, \tag{4.25}$$

$$D_{X^\uparrow} S = D_S X^\uparrow = \frac{1}{2}\,(JX)^\uparrow\,, \tag{4.26}$$

$$D_{T^\uparrow} T^\uparrow = 2\,V^\uparrow\,, \quad D_S S = D_S T^\uparrow = D_{T^\uparrow} S = 0, \tag{4.27}$$

where $\Omega = -d\theta$ while $\phi : H(M) \to H(M)$ and $V \in H(M)$ are the bundle endomorphism and vector field determined by

$$G_\theta(\phi X, Y)\circ\pi = (d\sigma)(X^\uparrow\,,\,Y^\uparrow), \tag{4.28}$$

$$G_\theta(V, X) = (d\sigma)(T^\uparrow\,,\,X^\uparrow), \tag{4.29}$$

for any $X, Y \in H(M)$. The differential form $\Omega \in \Omega^2(M)$ bears a certain similarity to the canonical 2-form associated to a Kählerian metric, in that it may be written as $\Omega(X,Y) = g_\theta(X, JY)$ for any $X, Y \in \mathfrak{X}(M)$, yet similarity doesn't go any further e.g. the de Rham cohomology class $[\Omega] = 0$

(while statements on the Betti numbers of a Kählerian manifold may be got by a mere inspection of the powers $[\Omega]^k$, cf. [96]). Locally ϕ and V are given by

$$\phi_\alpha{}^\beta = \frac{i}{2(n+2)}\left\{R_\alpha{}^\beta - \frac{\rho}{2(n+1)}\,\delta_\alpha^\beta\right\}, \quad \phi_\alpha{}^{\overline{\beta}} = 0, \quad \phi_\alpha{}^0 = 0, \tag{4.30}$$

$$V^\alpha = g^{\alpha\overline{\beta}} V_{\overline{\beta}}, \quad V_{\overline{\beta}} = \frac{1}{2(n+2)}\left\{\frac{1}{4(n+1)}\,\rho_{\overline{\beta}} + i\,W^\alpha_{\alpha\overline{\beta}}\right\}. \tag{4.31}$$

In particular $[J,\phi] = 0$ (as a consequence of (4.30)). We recall (cf. (1.100) in [94], p. 58)

$$\mathrm{Ric}_{g_\theta}(T_\mu\,,\,T_{\overline{\nu}}) = -\frac{1}{2}\,R_{\mu\overline{\nu}} + g_{\mu\overline{\nu}}\,, \tag{4.32}$$

$$R_{\mu\nu} = i(n-1)\,A_{\mu\nu}\,, \tag{4.33}$$

$$R_{0\nu} = S^{\overline{\mu}}_{\overline{\mu}\nu}\,, \quad R_{\mu 0} = 0, \quad R_{00} = 0. \tag{4.34}$$

Here Ric_{g_θ} is the Ricci curvature of the Riemannian manifold (M, g_θ). Also

$$S(X,Y) = (\nabla_X \tau)Y - (\nabla_Y \tau)X, \quad X, Y \in \mathfrak{X}(M),$$

so that $S^{\overline{\mu}}_{\overline{\mu}\nu}$ are among $S^j_{k\ell}T_j = S(T_k\,,\,T_\ell)$. As a consequence of (4.32) one has $R_{\mu\overline{\nu}} = R_{\overline{\nu}\mu}$. Let us take the exterior derivative of (4.17)

$$(n+2)\,d\sigma = \pi^*\left\{i d\omega_\alpha{}^\alpha - \frac{i}{2}\,dg^{\mu\overline{\nu}} \wedge dg_{\mu\overline{\nu}} - \frac{1}{4(n+1)}\,d(\rho\theta)\right\}$$

and observe that $dg^{\mu\overline{\nu}} \wedge dg_{\mu\overline{\nu}} = 0$. Also (by Theorem 1.7 in [94], p. 55)

$$d\omega_\alpha{}^\alpha = R_{\mu\overline{\nu}}\,\theta^\mu \wedge \theta^{\overline{\nu}} + \left(W^\alpha_{\alpha\lambda}\,\theta^\lambda - W^\alpha_{\alpha\overline{\mu}}\,\theta^{\overline{\mu}}\right) \wedge \theta$$

where $\{\theta^\alpha : 1 \le \alpha \le n\}$ is an *admissible* local frame of $T_{1,0}(M)^*$ i.e.

$$\theta^\alpha(T_\beta) = \delta^\alpha_\beta\,, \quad \theta^\alpha(T_{\overline{\beta}}) = 0, \quad \theta^\alpha(T) = 0.$$

Throughout $\theta^{\overline{\alpha}} = \overline{\theta^\alpha}$. By taking into account (4.32)-(4.34)

$$\mathrm{Ric}_\nabla(X, JY) = -2i\,\left(R_{\mu\overline{\nu}}\theta^\mu \wedge \theta^{\overline{\nu}}\right)(X,Y) - (n-1)A(X,Y) \tag{4.35}$$

for any $X, Y \in H(M)$. Also $d(\rho\theta) = -\rho\,\Omega$ on $H(M) \otimes H(M)$. Consequently

$$2(d\sigma)(X^\uparrow, Y^\uparrow) = \frac{1}{n+2}\left\{\frac{\rho}{2(n+1)}\,\Omega(X,Y) - \right. \tag{4.36}$$

$$\left. - (n-1)\,A(X,Y) - \mathrm{Ric}_\nabla(X, JY)\right\}.$$

By a result in [98], Vol. I, p. 65, $[X,Y]^\uparrow$ is the horizontal component of $\left[X^\uparrow, Y^\uparrow\right]$ for any $X, Y \in \mathfrak{X}(M)$. When $X, Y \in H(M)$ the vertical component may be easily derived from (4.36). One obtains the decomposition

$$\left[X^\uparrow\,,\,Y^\uparrow\right] = [X,Y]^\uparrow + \frac{2}{n+2}\,\{\mathrm{Ric}_\nabla(X, JY) + \tag{4.37}$$

$$+(n-1)\,A(X,Y) - \frac{\rho}{2(n+1)}\,\Omega(X,Y)\Big\}\; S.$$

Similarly let us compute $f \in C^\infty(M)$ in $[X^\uparrow\,,\,T^\uparrow] = [X,T]^\uparrow + fS$. If $\varphi = i\left(W^\alpha_{\alpha\lambda}\theta^\lambda - W^\alpha_{\alpha\overline{\mu}}\theta^{\overline{\mu}}\right)$ then

$$i\,(d\omega_\alpha{}^\alpha)(X,T) = (\varphi \wedge \theta)(X,T) = \frac{1}{2}\,\varphi(X),$$

$$2(n+2)(d\sigma)(X^\uparrow\,,\,T^\uparrow) = \varphi(X) - \frac{1}{2(n+1)}\,d(\rho\theta)(X,T)$$

or

$$2(d\sigma)(X^\uparrow, T^\uparrow) = \frac{1}{n+2}\left\{\varphi(X) - \frac{1}{4(n+1)}\,X(\rho)\right\} \tag{4.38}$$

as $T \rfloor d\theta = 0$. We conclude (as $\sigma(S) = \frac{1}{2}$)

$$[X^\uparrow\,,\,T^\uparrow] = [X,T]^\uparrow + \frac{2}{n+2}\left\{\frac{1}{4(n+1)}\,X(\rho) - \varphi(X)\right\} S. \tag{4.39}$$

We need to establish

Lemma 4.2. *Let M be a strictly pseudoconvex CR manifold, of CR dimension n, and $\theta \in \mathcal{P}_+$ a positively oriented contact form. The curvature R^D of the Lorentzian manifold $(\mathfrak{M}, F_\theta)$ is given by*

$$R^D(X^\uparrow, Y^\uparrow)Z^\uparrow = \left(R^\nabla(X,Y)Z\right)^\uparrow - \tag{4.40}$$

$$-\frac{1}{2(n+1)(n+2)}\,\{X(\rho)\,\Omega(Y,Z) - Y(\rho)\,\Omega(X,Z)\}\,S -$$

$$-\frac{n+5}{n+2}\,\{(\nabla_X A)(Y,Z) - (\nabla_Y A)(X,Z)\}\,S +$$

$$+\frac{1}{n+2}\,\{(\nabla_X \mathrm{Ric}_\nabla)(Y,JZ) - (\nabla_Y \mathrm{Ric}_\nabla)(Y,JZ)\}\,S +$$

$$+\Omega(Y,Z)\left\{(\tau X)^\uparrow + (\phi X)^\uparrow - \frac{\rho}{4(n+1)(n+2)}\,(JX)^\uparrow\right\} -$$

$$-\Omega(X,Z)\left\{(\tau Y)^\uparrow + (\phi Y)^\uparrow - \frac{\rho}{4(n+1)(n+2)}\,(JY)^\uparrow\right\} +$$

$$+\frac{1}{2(n+2)}\,\{\mathrm{Ric}_\nabla(Y,JZ) - (n+5)\,A(Y,Z)\}\,(JX)^\uparrow -$$

$$-\frac{1}{2(n+2)}\,\{\mathrm{Ric}_\nabla(X,JZ) - (n+5)\,A(X,Z)\}\,(JY)^\uparrow -$$

$$-\frac{1}{n+2}\left\{\mathrm{Ric}_\nabla(X,JY)\,(JZ)^\uparrow - 2\Omega(X,Y)\,\mathrm{Ric}_\nabla(T,JZ)\,S\right\} -$$

$$-\frac{1}{n+2}\left\{(n-1)A(X,Y) - \frac{\rho}{2(n+1)}\,\Omega(X,Y)\right\}\,(JZ)^\uparrow -$$

$$-2\,\Omega(X,Y)\,\left\{(\phi Z)^\uparrow + \frac{2}{n+2}\left[\varphi(Z) - \frac{1}{4(n+1)}\,Z(\rho)\right] S\right\}.$$

$$\begin{aligned}
R^D(X^\uparrow, T^\uparrow)Z^\uparrow = &\left(R^\nabla(X,T)Z\right)^\uparrow + ((\nabla_X\phi)Z)^\uparrow - \\
&-\frac{1}{n+2}\{\mathrm{Ric}_\nabla(X, J\phi Z) + \mathrm{Ric}_\nabla(\tau X, JZ)\}S + \\
&+\frac{1}{n+2}\left\{\varphi(Z)(JX)^\uparrow + \varphi(X)(JZ)^\uparrow\right\} - \\
&-\frac{1}{4(n+1)(n+2)}\left\{Z(\rho)(JX)^\uparrow + X(\rho)(JZ)^\uparrow\right\} + \\
&+\frac{2}{n+2}\left\{(\nabla_X\varphi)Z - \frac{1}{4(n+1)}(\nabla_X d\rho)Z\right\}S - \\
&-\frac{1}{n+2}\{(\nabla_T\mathrm{Ric}_\nabla)(X, JZ) - (n+5)(\nabla_T A)(X,Z)\}S + \\
&+\{\Omega(X,\phi Z) - \Omega(\tau X, Z)\}\left\{T^\uparrow - \frac{\rho}{2(n+1)(n+2)}S\right\} - \\
&-2\Omega(X,Z)\left\{V^\uparrow - \frac{T(\rho)}{4(n+1)(n+2)}S\right\} - \\
&-\frac{3(n+3)}{n+2}\{A(X,\phi Z) - A(\tau X, Z)\}S,
\end{aligned} \tag{4.41}$$

$$\begin{aligned}
R^D(X^\uparrow, S)Z^\uparrow = & \\
= -\frac{1}{2(n+2)}&\{\mathrm{Ric}_\nabla(X,Z) + (n+5)A(X,JZ)\}S - \\
&-\frac{1}{2}G_\theta(X,Z)\left\{T^\uparrow - \frac{\rho}{2(n+1)(n+2)}S\right\},
\end{aligned} \tag{4.42}$$

$$\begin{aligned}
R^D(X^\uparrow, Y^\uparrow)T^\uparrow = &((\nabla_X\tau)Y + (\nabla_X\phi)Y)^\uparrow + 4\Omega(X,Y)V^\uparrow - \\
&-\frac{1}{n+2}\{\mathrm{Ric}_\nabla(J\tau X, Y) - \mathrm{Ric}_\nabla(X, J\tau Y) + \\
&+\mathrm{Ric}_\nabla(J\phi X, Y) - \mathrm{Ric}_\nabla(X, J\phi Y)\}S - \\
&-\frac{n+5}{2(n+2)^2}\{\mathrm{Ric}_\nabla(\tau X, JY) - \mathrm{Ric}_\nabla(JX, \tau Y) + 2(n-1)\Omega(\tau X, \tau Y)\},
\end{aligned} \tag{4.43}$$

$$R^D(X^\uparrow, Y^\uparrow)S = 0, \tag{4.44}$$

$$\begin{aligned}
R^D(T^\uparrow, S)Z^\uparrow = & \\
= \frac{1}{n+2}&\left\{\varphi(JZ) - \frac{1}{4(n+1)}(JZ)(\rho)\right\}S - \\
&-\frac{2}{n+2}\left\{\varphi(Z) - \frac{1}{4(n+1)}Z(\rho)\right\}S,
\end{aligned} \tag{4.45}$$

$$R^D(T^\uparrow, S)T^\uparrow = 0, \tag{4.46}$$

$$R^D(T^\uparrow, S)S = 0, \tag{4.47}$$

for any $X, Y, Z \in H(M)$.

Proof. As $H(M)$ is parallel with respect to ∇ one has $\nabla_Y Z \in H(M)$. Then (by (4.23) and (4.36))

$$D_{X^\uparrow}(\nabla_Y Z)^\uparrow = (\nabla_X \nabla_Y Z)^\uparrow + \tag{4.48}$$

$$+\Omega(X, \nabla_Y Z)\left\{T^\uparrow - \frac{\rho}{2(n+1)(n+2)}\, S\right\} +$$

$$+\frac{1}{n+2}\left\{\mathrm{Ric}_\nabla(X, J\nabla_Y Z) - (n+5)\, A(X, \nabla_Y Z)\right\}\, S.$$

Next (by (4.23)-(4.24), (4.26), (4.36) and (4.48))

$$D_{X^\uparrow} D_{Y^\uparrow} Z^\uparrow = (\nabla_X \nabla_Y Z)^\uparrow + \tag{4.49}$$

$$+\left\{X(\Omega(Y,Z)) + \Omega(X, \nabla_Y Z)\right\}\left\{T^\uparrow - \frac{\rho}{2(n+1)(n+2)}\, S\right\} +$$

$$-\frac{X(\rho)}{2(n+1)(n+2)}\, \Omega(Y,Z) S -$$

$$-\frac{n+5}{n+2}\left\{X(A(Y,Z)) + A(X, \nabla_Y Z)\right\} S +$$

$$+\frac{1}{n+2}\left\{X(\mathrm{Ric}_\nabla(Y, JZ)) + \mathrm{Ric}_\nabla(X, J\nabla_Y Z)\right\} S +$$

$$+\Omega(Y,Z)\left\{(\tau X)^\uparrow + (\phi X)^\uparrow - \frac{\rho}{4(n+1)(n+2)}\, (JX)^\uparrow\right\} +$$

$$+\frac{1}{2(n+2)}\left\{\mathrm{Ric}_\nabla(Y, JZ) - (n+5)\, A(Y,Z)\right\}\, (JX)^\uparrow .$$

The calculation of $D_{[X^\uparrow, Y^\uparrow]} Z^\uparrow$ is a bit trickier as $[X, Y] \notin H(M)$ in general. To start with one uses the decomposition (4.37) followed by $[X, Y] = \Pi_H[X, Y] + \theta([X, Y]) T$. This yields (by (4.26))

$$D_{[X^\uparrow, Y^\uparrow]} Z^\uparrow = D_{[X,Y]^\uparrow} Z^\uparrow + \frac{2}{n+2}\, B(X,Y)\, D_S Z^\uparrow =$$

$$= D_{(\Pi_H[X,Y])^\uparrow} Z^\uparrow + \theta([X,Y])\, D_{T^\uparrow} Z^\uparrow + \frac{1}{n+2}\, B(X,Y)\, (JZ)^\uparrow$$

where we have set

$$B(X,Y) = \mathrm{Ric}_\nabla(X, JY) + (n-1)\, A(X,Y) - \frac{\rho}{2(n+1)}\, \Omega(X,Y)$$

for simplicity. At this point we may use (4.23) (as $\Pi_H[X,Y] \in H(M)$) and (4.25) so that

$$D_{[X^\uparrow, Y^\uparrow]} Z^\uparrow = (\nabla_{\Pi_H[X,Y]} Z)^\uparrow + \Omega(\Pi_H[X,Y], Z)\, T^\uparrow -$$

$$-2\left\{(d\sigma)\left((\Pi_H[X,Y])^\uparrow,\, Z^\uparrow\right) + A\,(\Pi_H[X,Y],\, Z)\right\} S +$$

$$+\theta([X,Y])\left\{(\nabla_T Z)^\uparrow + (\phi Z)^\uparrow + 4(d\sigma)(Z^\uparrow,\, T^\uparrow) S\right\} +$$

$$+\frac{1}{n+2}\, B(X,Y)\, (JZ)^\uparrow .$$

Next (by $T\rfloor\Omega = T\rfloor A = 0$ and the identities (4.36) and (4.38))

$$D_{[X^\uparrow, Y^\uparrow]}Z^\uparrow = (\nabla_{[X,Y]}Z)^\uparrow + \tag{4.50}$$

$$+\Omega([X,Y],Z)\left\{T^\uparrow - \frac{\rho}{2(n+1)(n+2)}\,S\right\} - \frac{n+5}{n+2}\,A([X,Y],Z)S +$$

$$+\frac{1}{n+2}\left\{\mathrm{Ric}_\nabla(X,JY)\,(JZ)^\uparrow + \mathrm{Ric}_\nabla\,(\Pi_H[X,Y]\,,\;JZ)\,S\right\} +$$

$$+\frac{1}{n+2}\left\{(n-1)A(X,Y) - \frac{\rho}{2(n+1)}\,\Omega(X,Y)\right\}\,(JZ)^\uparrow +$$

$$+\theta([X,Y])\left\{(\phi Z)^\uparrow + \frac{2}{n+2}\left[\varphi(Z) - \frac{1}{4(n+1)}\,Z(\rho)\right]S\right\}.$$

Moreover (by (4.49)-(4.50))

$$R^D(X^\uparrow\,,\;Y^\uparrow)Z^\uparrow = \left([D_{X^\uparrow}\,,\;D_{Y^\uparrow}] - D_{[X^\uparrow\,,\,Y^\uparrow]}\right)Z^\uparrow = \tag{4.51}$$

$$= (\nabla_X\nabla_Y Z)^\uparrow +$$

$$+\{X(\Omega(Y,Z)) + \Omega(X,\nabla_Y Z)\}\left\{T^\uparrow - \frac{\rho}{2(n+1)(n+2)}\,S\right\} -$$

$$-\frac{X(\rho)}{2(n+1)(n+2)}\,\Omega(Y,Z)S -$$

$$-\frac{n+5}{n+2}\,\{X(A(Y,Z)) + A(X,\nabla_Y Z)\}\,S +$$

$$+\frac{1}{n+2}\,\{X(\mathrm{Ric}_\nabla(Y,JZ)) + \mathrm{Ric}_\nabla(X,J\nabla_Y Z)\}\,S +$$

$$+\Omega(Y,Z)\left\{(\tau X)^\uparrow + (\phi X)^\uparrow - \frac{\rho}{4(n+1)(n+2)}\,(JX)^\uparrow\right\} +$$

$$+\frac{1}{2(n+2)}\,\{\mathrm{Ric}_\nabla(Y,JZ) - (n+5)\,A(Y,Z)\}\,(JX)^\uparrow -$$

$$-(\nabla_Y\nabla_X Z)^\uparrow -$$

$$-\{Y(\Omega(X,Z)) + \Omega(Y,\nabla_X Z)\}\left\{T^\uparrow - \frac{\rho}{2(n+1)(n+2)}\,S\right\} +$$

$$+\frac{Y(\rho)}{2(n+1)(n+2)}\,\Omega(X,Z)S +$$

$$+\frac{n+5}{n+2}\,\{Y(A(X,Z)) + A(Y,\nabla_X Z)\}\,S -$$

$$-\frac{1}{n+2}\,\{Y(\mathrm{Ric}_\nabla(X,JZ)) + \mathrm{Ric}_\nabla(Y,J\nabla_X Z)\}\,S -$$

$$-\Omega(X,Z)\left\{(\tau Y)^\uparrow + (\phi Y)^\uparrow - \frac{\rho}{4(n+1)(n+2)}\,(JY)^\uparrow\right\} -$$

$$-\frac{1}{2(n+2)}\left\{\mathrm{Ric}_\nabla(X, JZ) - (n+5)\, A(X,Z)\right\} (JY)^\uparrow -$$

$$-\, (\nabla_{[X,Y]}Z)^\uparrow -$$

$$-\Omega([X,Y],Z)\left\{T^\uparrow - \frac{\rho}{2(n+1)(n+2)}\, S\right\} + \frac{n+5}{n+2}\, A([X,Y],Z)S -$$

$$-\frac{1}{n+2}\left\{\mathrm{Ric}_\nabla(X, JY)\,(JZ)^\uparrow + \mathrm{Ric}_\nabla\,(\Pi_H[X,Y]\,,\, JZ)\, S\right\} -$$

$$-\frac{1}{n+2}\left\{(n-1)A(X,Y) - \frac{\rho}{2(n+1)}\,\Omega(X,Y)\right\}\,(JZ)^\uparrow -$$

$$-\theta([X,Y])\left\{(\phi Z)^\uparrow + \frac{2}{n+2}\left[\varphi(Z) - \frac{1}{4(n+1)}\, Z(\rho)\right] S\right\}.$$

Using the identity

$$[X,Y] = \nabla_X Y - \nabla_Y X + 2\Omega(X,Y)T, \quad X, Y \in H(M), \tag{4.52}$$

one has

$$X(\Omega(Y,Z)) + \Omega(X, \nabla_Y X) -$$

$$-Y(\Omega(X,Z)) - \Omega(Y, \nabla_X Z) - \Omega([X,Y],Z) =$$

$$= (\nabla_X \Omega)(Y,Z) - (\nabla_Y \Omega)(X,Z) - 2\Omega(X,Y)\Omega(T,Z) = 0$$

as $\nabla\Omega = 0$ and $T \rfloor \Omega = 0$. Similarly (again by (4.51) and $T \rfloor A = 0$)

$$-X(A(Y,Z)) - A(X, \nabla_Y Z) +$$

$$+Y(A(X,Z)) + A(Y, \nabla_X Z) + A([X,Y],Z) =$$

$$= -(\nabla_X A)(Y,Z) + (\nabla_Y A)(X,Z).$$

Next (by $\nabla J = 0$)

$$X(\mathrm{Ric}_\nabla(Y, JZ)) + \mathrm{Ric}_\nabla(X, J\nabla_Y Z) -$$

$$-Y(\mathrm{Ric}_\nabla(X, JZ)) - \mathrm{Ric}_\nabla(Y, J\nabla_X Z) - \mathrm{Ric}_\nabla(\Pi_H[X,Y], JZ) =$$

$$= (\nabla_X \mathrm{Ric}_\nabla)(Y, JZ) - (\nabla_Y \mathrm{Ric}_\nabla)(Y, JZ) + 2\Omega(X,Y)\,\mathrm{Ric}_\nabla(T, JZ).$$

Consequently (4.51) yields (4.40). The proof of the remaining identities (4.41)-(4.47) is relegated to § 4.6.

Using Lemma 4.2 one may compute the Ricci curvature of $(\mathfrak{M}, F_\theta)$. Let $\{E_a : 1 \le a \le 2n\}$ be an orthonormal frame of $H(M)$ i.e. $G_\theta(E_a, E_b) = \delta_{ab}$. Then

$$\left\{\tilde{E}_p : 1 \le p \le 2n+2\right\} \equiv \{E_a^\uparrow\,,\, T^\uparrow \pm S \,:\, 1 \le a \le 2n\},$$

$$\tilde{E}_a = E_a^\uparrow, \quad \tilde{E}_{2n+1} = T^\uparrow - S, \quad \tilde{E}_{2n+2} = T^\uparrow + S,$$

is a local F_θ-orthonormal frame of $T(\mathfrak{M})$, so that for any $U, W \in \mathfrak{X}(\mathfrak{M})$

$$\mathrm{Ric}_D(U,W) = \sum_{p=1}^{2n+2} \epsilon_p F_\theta\left(R^D(\tilde{E}_p\,,\, W)U\,,\, \tilde{E}_p\right) =$$

$$= \sum_{a=1}^{2n} F_\theta\Big(R^D(E_a^\uparrow \, , \, W)U \, , \, E_a^\uparrow\Big) -$$

$$-F_\theta\Big(R^D(T^\uparrow - S \, , \, W)U \, , \, T^\uparrow - S\Big) + F_\theta\Big(R^D(T^\uparrow + S \, , \, W)U \, , \, T^\uparrow + S\Big)$$

i.e.

$$\mathrm{Ric}_D(U, W) = \sum_{a=1}^{2n} F_\theta\Big(R^D(E_a^\uparrow \, , \, W)U \, , \, E_a^\uparrow\Big) + \tag{4.53}$$

$$+2\Big\{F_\theta\Big(R^D(T^\uparrow, W)U \, , \, S\Big) + F_\theta\Big(R^D(S, W)U \, , \, T^\uparrow\Big)\Big\}.$$

We may state the following

Lemma 4.3. *For any $X, Y \in H(M)$*

$$\mathrm{Ric}_D(X^\uparrow, Y^\uparrow) = \frac{n+1}{n+2}\,\{\mathrm{Ric}_\nabla(X, Y) + 3\,A(X, JY)\} + \tag{4.54}$$

$$+\frac{\rho}{2(n+1)(n+2)}\,G_\theta(X, Y),$$

$$\mathrm{Ric}_D(X^\uparrow, T^\uparrow) = \mathrm{Ric}_\nabla(X, T) + \mathrm{trace}\,\{\Pi_H(\nabla\phi)X\} + \tag{4.55}$$

$$+\frac{1}{n+2}\,\varphi(JX) - 2\,\Omega(V, X) + \frac{1}{4(n+1)(n+2)}\,\Omega(X \, , \, \nabla^H\rho),$$

$$\mathrm{Ric}_D(X^\uparrow, S) = 0, \tag{4.56}$$

$$\mathrm{Ric}_D(T^\uparrow, T^\uparrow) = \frac{1}{n+2}\,\mathrm{trace}\left\{\frac{\rho}{4(n+1)}\,J\phi - 3(n+3)\,\tau^2\right\} + \tag{4.57}$$

$$+\frac{1}{n+2}\,\mathrm{trace}_{G_\theta}\Pi_H\,\{\mathrm{Ric}_\nabla(\cdot\, , \, J\phi\,\cdot) + \mathrm{Ric}_\nabla(\tau\,\cdot\, , \, J\,\cdot) -$$

$$-\nabla\varphi + \frac{1}{4(n+1)}\,\nabla d\rho +$$

$$+\frac{n+5}{2}\,\nabla_T A - \frac{1}{2}\,(\nabla_T \mathrm{Ric}_\nabla)(\cdot\, , \, J\,\cdot)\Big\},$$

$$\mathrm{Ric}_D(T^\uparrow, S) = \frac{\rho}{4(n+1)}\,, \tag{4.58}$$

$$\mathrm{Ric}_D(S, S) = \frac{n}{2}\,. \tag{4.59}$$

Proof. Let $X, Y, E \in H(M)$ and let us replace (X, Y, Z) in (4.40) by (E, Y, X) and take the inner product of the resulting identity with $E^\uparrow$. As

$$F_\theta(X^\uparrow \, , \, Y^\uparrow) = G_\theta(X, Y) \circ \pi,$$

$$F_\theta(X^\uparrow \, , \, T^\uparrow) = 0, \quad F_\theta(X^\uparrow \, , \, S) = 0,$$

$$G_\theta(JX, JY) = G_\theta(X, Y),$$

we obtain

$$F_\theta\left(R^D(E^\uparrow, Y^\uparrow)X^\uparrow, E^\uparrow\right) = G_\theta\left(R^\nabla(E, Y)X, E\right) +$$
$$+\Omega(Y, X)\{G_\theta(\tau E, E) + G_\theta(\phi E, E)\} -$$
$$-\Omega(E, X)\left\{G_\theta(\tau Y, E) + G_\theta(\phi Y, E) - \frac{\rho}{4(n+1)(n+2)}\, G_\theta(JY, E)\right\} -$$
$$-\frac{1}{2(n+2)}\{\mathrm{Ric}_\nabla(E, JX) - (n+5)\, A(E, X)\}\, G_\theta(JY, E) -$$
$$-\frac{1}{n+2}\,\mathrm{Ric}_\nabla(E, JY)\, G_\theta(JX, E) -$$
$$-\frac{1}{n+2}\left\{(n-1)A(E, Y) - \frac{\rho}{2(n+1)}\,\Omega(E, Y)\right\} G_\theta(JX, E).$$

Let us replace E by E_a and sum over $1 \leq a \leq 2n$. Since

$$\mathrm{trace}(\tau) = 0, \quad X = \sum_a G_\theta(X, E_a)E_a\,, \quad X \in H(M),$$

one obtains

$$\sum_a F_\theta\left(R^D(E_a^\uparrow\,,\ Y^\uparrow)X^\uparrow\,,\ E_a^\uparrow\right) = \mathrm{Ric}_\nabla(X, Y) + \tag{4.60}$$
$$+\Omega(Y, X)\,\mathrm{trace}(\phi) - \Omega(\tau Y, X) - \Omega(\phi Y, X) +$$
$$+\frac{\rho}{4(n+1)(n+2)}\,\Omega(JY, X) -$$
$$-\frac{1}{2(n+2)}\ \{\mathrm{Ric}_\nabla(JY, JX) - (n+5)\, A(JY, X)\} -$$
$$-\frac{1}{n+2}\,\mathrm{Ric}_\nabla(JX, JY) -$$
$$-\frac{1}{n+2}\ \left\{(n-1)A(JX, Y) - \frac{\rho}{2(n+1)}\,\Omega(JX, Y)\right\}.$$

Note that (by the symmetry of A together with $\tau \circ J + J \circ \tau = 0$)

$$A(JX, Y) = A(X, JY), \quad G_\theta(JX, JY) = G_\theta(X, Y),$$

$$\Omega(\tau Y, X) = A(X, JY).$$

To further simplify (4.60) we need some preparation. Let us replace X by JX in (4.35). One has

$$\mathrm{Ric}_\nabla(JX, JY) = -2i\left(R_{\mu\bar{\nu}}\theta^\mu \wedge \theta^{\bar{\nu}}\right)(JX, Y) - (n-1)\, A(JX, Y) =$$

$$= 2i\left(R_{\mu\bar{\nu}}\theta^\mu \wedge \theta^{\bar{\nu}}\right)(Y, JX) - (n-1)\, A(X, JY) =$$

(by applying (4.35) once again)

$$= -\mathrm{Ric}_\nabla(Y, J^2X) - (n-1)\, A(Y, JX) - (n-1)\, A(X, JY)$$

or (as $J^2 = -I$ on $H(M)$)

$$\mathrm{Ric}_\nabla(JX, JY) = \mathrm{Ric}_\nabla(X, Y) - 2(n-1)\, A(X, JY) \tag{4.61}$$

for any $X, Y \in H(M)$. Here we have also used the symmetry of Ric_∇ on $H(M) \otimes H(M)$

$$\mathrm{Ric}_\nabla(X, Y) = \mathrm{Ric}_\nabla(Y, X)$$

which is an immediate consequence of (4.32)-(4.33). Moreover

$$\mathrm{trace}(\phi) = 0 \tag{4.62}$$

as a corollary of (4.30) and the fact that the trace of the endomorphism $\phi : H(M) \to H(M)$ coincides with the trace of its extension by $\mathbb{C}$-linearity to $H(M) \otimes \mathbb{C}$ (and $\phi_\alpha{}^\beta$ is purely imaginary).

Next one needs to compute $\Omega(\phi Y, X)$. If $\{T_\alpha : 1 \le \alpha \le n\}$ is a local frame of $T_{1,0}(M)$ and $X = X^\alpha T_\alpha + X^{\overline{\alpha}} T_{\overline{\alpha}}$ for some $X^\alpha \in C^\infty(U, \mathbb{C})$ (with $\overline{X^\alpha} = X^{\overline{\alpha}}$) then

$$\Omega(\phi Y, X) = G_\theta(\phi Y, JX) =$$

$$= -i Y^\alpha X^{\overline{\sigma}} g_{\beta\overline{\sigma}} \phi_\alpha{}^\beta + i Y^{\overline{\alpha}} X^\sigma g_{\overline{\beta}\sigma} \phi_{\overline{\alpha}}{}^{\overline{\beta}} =$$

(by identity (4.30))

$$= \frac{1}{2(n+2)} \left\{ Y^\alpha X^{\overline{\sigma}} \left[R_{\alpha\overline{\sigma}} - \frac{\rho}{2(n+1)} g_{\alpha\overline{\sigma}} \right] + \right.$$

$$\left. + Y^{\overline{\alpha}} X^\sigma \left[R_{\overline{\alpha}\sigma} - \frac{\rho}{2(n+1)} g_{\overline{\alpha}\sigma} \right] \right\}$$

or

$$\Omega(\phi Y\,,\, X) = \tag{4.63}$$

$$= \frac{1}{2(n+2)} \left\{ \mathrm{Ric}_\nabla \left(Y^{1,0}\,,\, X^{0,1} \right) + \mathrm{Ric}_\nabla \left(Y^{0,1}\,,\, X^{1,0} \right) \right\} -$$

$$- \frac{\rho}{4(n+1)(n+2)} \left\{ G_\theta \left(Y^{1,0}\,,\, X^{0,1} \right) + G_\theta \left(Y^{0,1}\,,\, X^{1,0} \right) \right\}$$

where we have set $X^{1,0} = X^\alpha T_\alpha$ and $X^{0,1} = \overline{X^{1,0}}$ (so that $X = X^{1,0} + X^{0,1}$). To further compute (4.63) let us observe that (by (4.33))

$$\mathrm{Ric}_\nabla \left(Y^{1,0}\,,\, X^{0,1} \right) + \mathrm{Ric}_\nabla \left(Y^{0,1}\,,\, X^{1,0} \right) =$$

$$= \mathrm{Ric}_\nabla \left(Y^{1,0}\,,\, X \right) - \mathrm{Ric}_\nabla \left(Y^{1,0}\,,\, X^{1,0} \right) +$$

$$+ \mathrm{Ric}_\nabla \left(Y^{0,1}\,,\, X \right) - \mathrm{Ric}_\nabla \left(Y^{0,1}\,,\, X^{0,1} \right) =$$

$$= \mathrm{Ric}_\nabla(Y, X) - i(n-1) Y^\alpha X^\sigma A_{\alpha\sigma} + i(n-1) Y^{\overline{\alpha}} X^{\overline{\sigma}} A_{\overline{\alpha\sigma}} =$$

$$= \mathrm{Ric}_\nabla(X, Y) - i(n-1) \left\{ A\left(Y^{1,0}\,,\, X^{1,0} \right) - A\left(Y^{0,1}\,,\, X^{0,1} \right) \right\} =$$

(as A vanishes on $T_{1,0}(M) \otimes T_{0,1}(M)$, as a consequence of $\tau\, T_{1,0}(M) \subset T_{0,1}(M)$)

$$= \mathrm{Ric}_\nabla(X, Y) - i(n-1) \left\{ A\left(Y^{1,0}\,,\, X \right) - A\left(Y^{0,1}\,,\, X \right) \right\}$$

or (as $JY = i(Y^{1,0} - Y^{0,1})$)

$$\mathrm{Ric}_{\nabla}\left(Y^{1,0}\,,\,X^{0,1}\right) + \mathrm{Ric}_{\nabla}\left(Y^{0,1}\,,\,X^{1,0}\right) = \tag{4.64}$$

$$= \mathrm{Ric}_{\nabla}(X,Y) - (n-1)\,A(X,JY).$$

Substitution from (4.64) into (4.63) leads to

$$\Omega(\phi Y, X) = \frac{1}{2(n+2)}\,\{\mathrm{Ric}_{\nabla}(X,Y) - (n-1)A(X,JY)\} - \tag{4.65}$$

$$-\frac{\rho}{4(n+1)(n+2)}\,G_\theta(X,Y)$$

for any $X, Y \in H(M)$. Substitution from (4.61)-(4.62) and (4.65) into (4.60) leads to (after simplifications)

$$\sum_{a=1}^{2n} F_\theta\left(R^D(E_a^\uparrow\,,\,Y^\uparrow)X^\uparrow\,,\,E_a^\uparrow\right) = \tag{4.66}$$

$$= \frac{n}{n+2}\,\mathrm{Ric}_{\nabla}(X,Y) + \frac{2(n-1)}{n+2}\,A(X,JY) + \frac{\rho}{(n+1)(n+2)}\,G_\theta(X,Y).$$

Let us take the inner product of (4.41) with S and use

$$F_\theta(S,S) = 0, \quad F_\theta(T^\uparrow, S) = \frac{1}{2}\,, \quad F_\theta(X^\uparrow, S) = 0, \quad X \in H(M).$$

Since (by (4.41))

$$R^D\left(X^\uparrow, T^\uparrow\right)Z^\uparrow \equiv \{\Omega(X,\phi Z) - \Omega(\tau X, Z)\}T^\uparrow, \quad \mathrm{mod}\ H(M)^\perp,\ S,$$

we obtain

$$F_\theta(R^D(X^\uparrow, T^\uparrow)Z^\uparrow\,,\,S) = \frac{1}{2}\{\Omega(X,\phi Z) - \Omega(\tau X, Z)\}. \tag{4.67}$$

Therefore the last two terms in (4.53) (with $U = X^\uparrow$ and $W = Y^\uparrow$) may be computed (by (4.67) and (4.65)) as

$$F_\theta\left(R^D(T^\uparrow, Y^\uparrow)X^\uparrow, S\right) + F_\theta\left(R^D(S, Y^\uparrow)X^\uparrow, T^\uparrow\right) = \tag{4.68}$$

$$= \frac{1}{2(n+2)}\,\{\mathrm{Ric}_{\nabla}(X,Y) + (n+5)\,A(X,JY)\} -$$

$$-\frac{\rho}{4(n+1)(n+2)}\,G_\theta(X,Y).$$

Finally formulae (4.53) and (4.68) lead to (4.54). The proof of the remaining identities (4.55)-(4.59) is given in § 4.6.

4.4 Pseudohermitian Bochner-Lichnerowicz formula

Let $f \in C^\infty(\mathfrak{M})$. Then

$$Df = \sum_{j=1}^{2n+2} \epsilon_j \tilde{E}_j(f)\tilde{E}_j = \sum_a E_a^\uparrow(f)E_a^\uparrow + 2\left\{T^\uparrow(f)S + S(f)T^\uparrow\right\}$$

hence

$$D(u \circ \pi) = \sum_a E_a(u)E_a^\uparrow + 2T(u)S = \left(\nabla^H u\right)^\uparrow + 2u_0\, S \tag{4.69}$$

for any $u \in C^\infty(M)$, where we have set $u_0 = T(u)$. Next (by (4.54), (4.56) and (4.59) in Lemma 4.3)

$$\mathrm{Ric}_D(D(u\circ\pi)\,,\, D(u\circ\pi)) =$$

$$= \sum_{a,b=1}^{2n} E_a(u)E_b(u)\mathrm{Ric}_D(E_a^\uparrow\,,\, E_b^\uparrow) + 4u_0^2\,\mathrm{Ric}_D(S,S) =$$

$$= 2nu_0^2 + \sum_{a,b} E_a(u)E_b(u)\left\{\frac{n+1}{n+2}\left[\mathrm{Ric}_\nabla(E_a,E_b) + 3A(E_a,E_b)\right] + \right.$$

$$\left. + \frac{\rho}{2(n+1)(n+2)}\, G_\theta(E_a\,,\, E_b)\right\}$$

or

$$\mathrm{Ric}_D(D(u\circ\pi)\,,\, D(u\circ\pi)) = 2nu_0^2 + \tag{4.70}$$

$$+\frac{n+1}{n+2}\left\{\mathrm{Ric}_\nabla\left(\nabla^H u, \nabla^H u\right) + 3A(\nabla^H u\,,\, J\nabla^H u)\right\} +$$

$$+\frac{\rho}{2(n+1)(n+2)}\,\left\|\nabla^H u\right\|^2\,.$$

Lemma 4.4. *Let $u \in C^\infty(M)$ and $f = u\circ\pi \in C^\infty(\mathfrak{M})$. Then*

$$(D^2 f)(X^\uparrow, Y^\uparrow) = (\nabla^2 u)(X,Y) - \Omega(X,Y)u_0\,, \tag{4.71}$$

$$(D^2 f)(X^\uparrow, T^\uparrow) = (\nabla^2 u)(T,X) - (\phi X)(u), \tag{4.72}$$

$$(D^2 f)(X^\uparrow, S) = -\frac{1}{2}\,(JX)(u), \tag{4.73}$$

$$(D^2 f)(T^\uparrow, T^\uparrow) = T(u_0) - 2V(u), \tag{4.74}$$

$$(D^2 f)(T^\uparrow, S) = 0, \tag{4.75}$$

$$(D^2 f)(S,S) = 0, \tag{4.76}$$

for every $X, Y \in H(M)$. Consequently

$$F_\theta^*(D^2 f\,,\, D^2 f) = \left\|\Pi_H \nabla^2 u\right\|^2 + 2nu_0^2 - 2\,\mathrm{div}\left(J\nabla^H u\right)(u_0) + \tag{4.77}$$

$$+4\left\{(J\nabla^H u)(u_0) - \left(\tau J\nabla^H u + \phi J\nabla^H u\right)(u)\right\}.$$

Proof. By (4.23) and $S(f) = 0$

$$(D^2 f)(X^\uparrow, Y^\uparrow) = X^\uparrow(Y^\uparrow(f)) - (D_{X^\uparrow} Y^\uparrow)(f) =$$

$$= X(Y(u)) - (\nabla_X Y)(u) - \Omega(X, Y) u_0$$

yielding (4.71). Similarly (4.72) follows from (4.24)

$$(D^2 f)(X^\uparrow, T^\uparrow) = X(u_0) - (\tau X)(u) - (\phi X)(u)$$

and $\tau(X) = \nabla_T X - [T, X]$. Next (4.73) is an immediate consequence of (4.26). Also the first identity in (4.27) yields (4.74). Finally the last identity in (4.27) implies (4.75)-(4.76). The proof of (4.77) is more involved. One has (by (4.75)-(4.76))

$$F^*_\theta\left(D^2 f, D^2 f\right) = \sum_{p,q=1}^{2n+2} \epsilon_p \epsilon_q (D^2 f)\left(\tilde{E}_p\,,\, \tilde{E}_q\right)^2 =$$

$$= \sum_{a,b} (D^2 f)(E_a^\uparrow, E_b^\uparrow)^2 - 2(D^2 f)(T^\uparrow - S, T^\uparrow + S)^2 +$$

$$+2 \sum_a \left\{(D^2 f)(E_a^\uparrow, T^\uparrow + S)^2 - (D^2 f)(E_a^\uparrow, T^\uparrow - S)^2\right\} +$$

$$+(D^2 f)(T^\uparrow - S, T^\uparrow - S)^2 + (D^2 f)(T^\uparrow + S, T^\uparrow + S)^2 =$$

$$= \sum_{a,b} (D^2 f)(E_a^\uparrow, E_b^\uparrow)^2 + 8 \sum_a (D^2 f)(E_a^\uparrow, T^\uparrow)\,(D^2 f)(E_a^\uparrow, S)$$

hence (by (4.71)-(4.73))

$$F^*_\theta(D^2 f, D^2 f) = \sum_{a,b} \left[(\nabla^2 u)(E_a, E_b) - \Omega(E_a, E_b) u_0\right]^2 - \tag{4.78}$$

$$-4 \sum_a \left\{(\nabla^2 u)(T, E_a) - (\phi E_a)(u)\right\}\, (J E_a)(u).$$

On the other hand $\sum_a (J E_a)(u)\, E_a = -J \nabla^H u$ so that (4.78) becomes

$$F^*_\theta(D^2 f, D^2 f) = \left\|\Pi_H \nabla^2 u\right\|^2 + 2n u_0^2 + 2\, u_0\, \mathrm{trace}_{G_\theta} \left\{\Pi_H (\nabla^2 u)_J\right\} + \tag{4.79}$$

$$+4 \left\{(\nabla^2 u)_J (T, \nabla^H u) - (\phi J \nabla^H u)(u)\right\}$$

where we have set

$$\left\|\Pi_H \nabla^2 u\right\|^2 = \sum_{a,b=1}^{2n} (\nabla^2 u)(E_a, E_b),$$

$$(\nabla^2 u)_J (X, Y) = (\nabla^2 u)(X, JY), \quad X, Y \in H(M).$$

Moreover (by $\nabla g_\theta = 0$)

$$\mathrm{trace}_{G_\theta} \left\{\Pi_H (\nabla^2 u)_J\right\} = \sum_a \{E_a ((J E_a)(u)) - (\nabla_{E_a} J E_a)(u)\} =$$

$$= -\sum_a \left\{ E_a \left(g_\theta(J\nabla^H u, E_a) \right) - g_\theta(J\nabla^H u\,,\, \nabla_{E_a} E_a) \right\} =$$

$$= -\sum_a g_\theta(\nabla_{E_a} J\nabla^H u\,,\, E_a)$$

i.e.

$$\mathrm{trace}_{G_\theta} \left\{ \Pi_H (\nabla^2 u)_J \right\} = -\mathrm{div}\left(J\nabla^H u \right). \tag{4.80}$$

Also

$$(\nabla^2 u)(T, X) = X(u_0) - (\tau X)(u), \quad X \in H(M),$$

and substitution from (4.80) into (4.79) leads to (4.77). By a result of J.M. Lee, [59], if $f = u \circ \pi$ then $\Box f = (\Delta_b u) \circ \pi$ hence (by (4.69))

$$(Df)(\Box f) = (\nabla^H u)(\Delta_b u), \tag{4.81}$$

$$F_\theta(Df, Df) = \|\nabla^H u\|^2 . \tag{4.82}$$

Finally (by taking into account the identities (4.70), (4.77) and (4.81)-(4.82)) the Bochner-Lichnerowicz formula (4.21) becomes

$$-\frac{1}{2}\,\Delta_b \left(\|\nabla^H u\|^2 \right) = \left\| \Pi_H \nabla^2 u \right\|^2 + 4n u_0^2 - 2\,\mathrm{div}\left(J\nabla^H u \right) u_0 + \tag{4.83}$$

$$+4 \left\{ (J\nabla^H u)(u_0) - \left(\tau J\nabla^H u + \phi J\nabla^H u \right)(u) \right\} - (\nabla^H u)(\Delta_b u) +$$

$$+\frac{n+1}{n+2} \left\{ \mathrm{Ric}_\nabla(\nabla^H u, \nabla^H u) + 3\,A(\nabla^H u, J\nabla^H u) \right\} +$$

$$+\frac{\rho}{2(n+1)(n+2)}\, \|\nabla^H u\|^2 .$$

The term $(\phi J\nabla^H u)(u)$ may be expressed in terms of pseudohermitian Ricci curvature and torsion. As

$$J\nabla^H u = i\left(u^\alpha T_\alpha - u^{\overline{\alpha}} T_{\overline{\alpha}} \right), \quad u^\alpha = g^{\alpha\overline{\beta}} u_{\overline{\beta}}, \quad u_{\overline{\beta}} = T_{\overline{\beta}}(u),$$

one has (by (4.30))

$$\phi J\nabla^H u = i\left(u^\alpha \phi_\alpha{}^\beta T_\beta - u^{\overline{\alpha}} \phi_{\overline{\alpha}}{}^{\overline{\beta}} T_{\overline{\beta}} \right) =$$

$$= -\frac{1}{2(n+2)}\, u^\alpha \left(R_\alpha{}^\beta - \frac{\rho}{2(n+1)}\, \delta_\alpha^\beta \right) T_\beta + \text{complex conjugate} =$$

$$= -\frac{1}{2(n+2)}\, g^{\beta\overline{\nu}} \mathrm{Ric}_\nabla \left(\left(\nabla^H u \right)^{1,0} \,,\, T_{\overline{\nu}} \right) T_\beta +$$

$$+\frac{\rho}{4(n+1)(n+2)} \left(\nabla^H u \right)^{1,0} + \text{complex conjugate} =$$

$$= -\frac{1}{2(n+2)} \left\{ g^{\beta\overline{\nu}} \mathrm{Ric}_\nabla \left(\left(\nabla^H u \right)^{1,0} \,,\, T_{\overline{\nu}} \right) T_\beta + \right.$$

$$\left. + g^{\overline{\beta}\nu} \mathrm{Ric}_\nabla \left(\left(\nabla^H u \right)^{0,1} \,,\, T_\nu \right) T_{\overline{\beta}} \right\} + \frac{\rho}{4(n+1)(n+2)}\, \nabla^H u$$

hence (as Ric_∇ is symmetric on $H(M) \otimes H(M)$)

$$(\phi J \nabla^H u)(u) = \frac{\rho}{4(n+1)(n+2)} \|\nabla^H u\|^2 - \tag{4.84}$$

$$-\frac{1}{n+2} \mathrm{Ric}_\nabla \left(\left(\nabla^H u\right)^{1,0}, \left(\nabla^H u\right)^{0,1} \right).$$

Formula (4.33) implies

$$\mathrm{Ric}_\nabla \left(X^{1,0}, X^{0,1} \right) = \frac{1}{2} \{ \mathrm{Ric}_\nabla (X, X) - (n-1) A(X, JX) \} \tag{4.85}$$

for any $X \in H(M)$. Hence (by (4.85) with $X = \nabla^H u$) formula (4.84) becomes

$$(\phi J \nabla^H u)(u) = \frac{\rho}{4(n+1)(n+2)} \|\nabla^H u\|^2 - \tag{4.86}$$

$$-\frac{1}{2(n+2)} \left\{ \mathrm{Ric}_\nabla \left(\nabla^H u, \nabla^H u \right) - (n-1) A \left(\nabla^H u, J \nabla^H u \right) \right\}.$$

Let us substitute from (4.86) and $(\tau J \nabla^H u)(u) = A(\nabla^H u, J \nabla^H u)$ into (4.83). We obtain

$$-\frac{1}{2} \Delta_b \left(\|\nabla^H u\|^2 \right) = \left\| \Pi_H \nabla^2 u \right\|^2 - (\nabla^H u)(\Delta_b u) + 4 n u_0^2 + \tag{4.87}$$

$$+4 (J \nabla^H u)(u_0) - 2 \,\mathrm{div}(J \nabla^H u)\, u_0 +$$

$$+\frac{n+3}{n+2} \mathrm{Ric}_\nabla \left(\nabla^H u, \nabla^H u \right) - \frac{\rho}{2(n+1)(n+2)} \|\nabla^H u\|^2 -$$

$$-\frac{3(n+1)}{n+2} A(\nabla^H u, J \nabla^H u).$$

Lemma 4.5. *For any $u \in C^\infty(M)$*

$$\mathrm{div} \left(J \nabla^H u \right) = 2 n u_0 . \tag{4.88}$$

Proof. One has

$$\nabla_{T_\beta} J \nabla^H u = i \left\{ \left(\nabla_\beta u^\alpha \right) T_\alpha - \left(\nabla_\beta u^{\overline{\alpha}} \right) T_{\overline{\alpha}} \right\},$$

$$\nabla_{T_{\overline{\beta}}} J \nabla^H u = i \left\{ \left(\nabla_{\overline{\beta}} u^\alpha \right) T_\alpha - \left(\nabla_{\overline{\beta}} u^{\overline{\alpha}} \right) T_{\overline{\alpha}} \right\},$$

hence (by $\mathrm{div}(X) = \mathrm{trace}\,\{Y \mapsto \nabla_Y X\}$)

$$\mathrm{div} \left(J \nabla^H u \right) = i \left\{ \nabla_\alpha u^\alpha - \nabla_{\overline{\alpha}} u^{\overline{\alpha}} \right\}. \tag{4.89}$$

On the other hand

$$(\nabla^2 u)(X, Y) = (\nabla^2 u)(Y, X) + 2\, \Omega(X, Y) u_0 , \quad X, Y \in H(M),$$

yields (for $X = T_\alpha$ and $Y = T_{\overline{\beta}}$)

$$\nabla_\alpha u_{\overline{\beta}} = \nabla_{\overline{\beta}} u_\alpha - 2 i g_{\alpha \overline{\beta}}\, u_0$$

or (by contraction with $g^{\alpha\overline{\beta}}$)

$$\nabla_\alpha u^\alpha = \nabla_{\overline{\alpha}} u^{\overline{\alpha}} - 2inu_0 . \tag{4.90}$$

Finally substitution from (4.90) into (4.89) leads to (4.89). Q.e.d.

As a consequence of Lemma 4.5 the identity (4.87) simplifies to

$$-\frac{1}{2}\Delta_b\left(\|\nabla^H u\|^2\right) = \left\|\Pi_H \nabla^2 u\right\|^2 - (\nabla^H u)(\Delta_b u) + \tag{4.91}$$
$$+4(J\nabla^H u)(u_0) +$$
$$+\frac{n+3}{n+2}\,\mathrm{Ric}_\nabla\left(\nabla^H u\,,\,\nabla^H u\right) - \frac{\rho}{2(n+1)(n+2)}\,\|\nabla^H u\|^2 -$$
$$-\frac{3(n+1)}{n+2}\,A(\nabla^H u\,,\,J\nabla^H u).$$

(the *pseudohermitian Bochner-Lichnerowicz* formula). Let us integrate over M and observe that (by Green's lemma and (4.88))

$$\int_M (J\nabla^H u)(u_0)\,\Psi_\theta = -\int_M u_0\,\mathrm{div}(J\nabla^H u)\,\Psi_\theta = -2n\,\|u_0\|^2_{L^2} .$$

We obtain

$$\left\|\Pi_H \nabla^2 u\right\|^2_{L^2} - 8n\,\|u_0\|^2_{L^2} + \tag{4.92}$$
$$+\int_M \left\{\frac{n+3}{n+2}\,\mathrm{Ric}_\nabla\left(\nabla^H u\,,\,\nabla^H u\right) - \frac{3(n+1)}{n+2}\,A\left(\nabla^H u\,,\,J\nabla^H u\right)\right\}\Psi_\theta =$$
$$= \int_M \left(\nabla^H u\right)(\Delta_b u)\;\Psi_\theta + \frac{1}{2(n+1)(n+2)}\int_M \rho\,\left\|\nabla^H u\right\|^2\;\Psi_\theta$$

(the *integral* pseudohermitian Bochner-Lichnerowicz formula).

4.5 A lower bound on $\lambda_1(\theta)$

Let $\lambda \in \sigma(\Delta_b)$ be an eigenvalue of Δ_b and $u \in \mathrm{Eigen}(\Delta_b\,,\,\lambda)$ an eigenfunction corresponding to λ. With these data

$$\int_M \left(\nabla^H u\right)(\Delta_b u)\;\Psi_\theta = \lambda\,\left\|\nabla^H u\right\|^2_{L^2} . \tag{4.93}$$

On the other hand (cf. (27) in [32], p. 88)

$$\left\|\Pi_H \nabla^2 u\right\|^2 \geq \frac{1}{2n}\,(\Delta_b u)^2 \tag{4.94}$$

everywhere on M. Moreover (by Green's lemma)

$$\|\Delta_b u\|^2_{L^2} = \lambda \int_M u\,\Delta_b u\;\Psi_\theta = \lambda\,\|\nabla^H u\|^2_{L^2} . \tag{4.95}$$

By our assumption (29)

$$\int_M \mathrm{Ric}_\nabla\left(\nabla^H u\,,\,\nabla^H u\right)\Psi_\theta \geq k\,\left\|\nabla^H u\right\|^2_{L^2} . \tag{4.96}$$

Moreover (by (29) with $X = E_a$ and (4.117))

$$\rho \geq nk. \tag{4.97}$$

In particular $\rho_0 \equiv \sup_{x \in M} \rho(x) > 0$ and

$$\int_M \rho \|\nabla^H u\|^2 \, \Psi_\theta \leq \rho_0 \, \left\|\nabla^H u\right\|^2_{L^2} . \tag{4.98}$$

For any $X, Y \in H(M)$ (by Cauchy-Schwartz inequality)

$$|A(X,Y)| = |G_\theta(X, \tau Y)| \leq \|X\| \, \|\tau Y\| \leq \|\tau\| \, \|X\| \, \|Y\|,$$

$$\|\tau\|_x = \sup\{G_{\theta,x}(\tau_x v \, , \, \tau_x v) : v \in H(M)_x \, , \ \ G_{\theta,x}(v,v) = 1\}, \quad x \in M.$$

Consequently (by $G_\theta(JX, JY) = G_\theta(X,Y)$)

$$\int_M A\left(\nabla^H u \, , \, J\nabla^H u\right) \leq \tau_0 \left\|\nabla^H u\right\|^2_{L^2} \tag{4.99}$$

where $\tau_0 = \sup_{x \in M} \|\tau\|_x$. The integral Bochner-Lichnerowicz formula (4.92) reads (by (4.93))

$$0 = \left\|\Pi_H \nabla^2 u\right\|^2_{L^2} - 8n \, \|u_0\|^2_{L^2} +$$

$$+ \int_M \left\{ \frac{n+3}{n+2} \, \mathrm{Ric}_\nabla \left(\nabla^H u \, , \, \nabla^H u\right) - \frac{3(n+1)}{n+2} \, A\left(\nabla^H u \, , \, J\nabla^H u\right) \right\} \Psi_\theta -$$

$$-\lambda \left\|\nabla^H u\right\|^2_{L^2} - \frac{1}{2(n+1)(n+2)} \int_M \rho \, \left\|\nabla^H u\right\|^2 \, \Psi_\theta \geq$$

(by (4.94) and (4.96)-(4.99))

$$\geq \frac{1}{2n} \, \|\Delta_b u\|^2_{L^2} - 8n \, \|u_0\|^2_{L^2} + \left[\frac{(n+3)k}{n+2} - \frac{3(n+1)\tau_0}{n+2} \right] \left\|\nabla^H u\right\|^2_{L^2} -$$

$$-\lambda \left\|\nabla^H u\right\|^2_{L^2} - \frac{\rho_0}{2(n+1)(n+2)} \, \left\|\nabla^H u\right\|^2_{L^2}$$

so that (by (4.95))

$$\left\{ \frac{1}{2n} - 1 + \frac{1}{\lambda} \left[\frac{(n+3)k}{n+2} - \frac{3(n+1)\tau_0}{n+2} - \right.\right.$$

$$\left.\left. - \frac{\rho_0}{2(n+1)(n+2)} \right] \right\} \|\Delta_b u\|^2_{L^2} \leq 8n \, \|u_0\|^2_{L^2} \, .$$

Finally (by (4.95) and Chang-Chiu inequality (4.118) in § 4.7)

$$-\frac{2n+3}{n+2} + \frac{1}{\lambda} \left\{ \frac{(n+3)k}{2(n+1)} - \frac{(11n+19)\tau_0}{n+2} - \frac{\rho_0}{2(n+1)(n+2)} \right\} \leq 0$$

or

$$\lambda \geq \frac{2n}{(n+2)(n+3)} \left\{ (n+3)k - (11n+19)\tau_0 - \frac{\rho_0}{2(n+1)} \right\} \tag{4.100}$$

which the announced lower bound on $\lambda_1(\theta)$ (cf. (30) above). Of course this is useful only when

$$k > \frac{(11n+19)\tau_0}{n+3} + \frac{\rho_0}{2(n+1)(n+3)} \, . \tag{4.101}$$

In particular (by (4.97)) it must be $k > 2(n+1)(11n+19)\tau_0 / [(n+2)(2n+3)]$.

4.6 Curvature of the Fefferman metric

The main purpose of § 4.6 is to complete the proof of Lemmas 4.2 and 4.3. We start with the calculation of

$$R^D\left(X^\uparrow, T^\uparrow\right)Z^\uparrow = [D_{X^\uparrow}, D_{T^\uparrow}]Z^\uparrow - D_{[X^\uparrow, T^\uparrow]}Z^\uparrow$$

for any $X, Z \in H(M)$. By (4.25) (followed by (4.23) and (4.26))

$$\begin{gathered} D_{X^\uparrow}D_{T^\uparrow}Z^\uparrow = (\nabla_X\nabla_T Z)^\uparrow + (\nabla_X \phi Z)^\uparrow + \\ + \{\Omega(X, \nabla_T Z) + \Omega(X, \phi Z)\}\, T^\uparrow - \\ -2\left\{(d\sigma)\left(X^\uparrow, (\nabla_T Z)^\uparrow\right) + (d\sigma)\left(X^\uparrow, (\phi Z)^\uparrow\right) + \right. \\ \left. + A(X, \nabla_T Z) + A(X, \phi Z)\right\} S + \\ +4X^\uparrow\left((d\sigma)(Z^\uparrow, T^\uparrow)\right) S + 2(d\sigma)(Z^\uparrow, T^\uparrow)(JX)^\uparrow. \end{gathered} \tag{4.102}$$

Similarly

$$\begin{gathered} D_{T^\uparrow}D_{X^\uparrow}Z^\uparrow = (\nabla_T\nabla_X Z)^\uparrow + (\phi\nabla_X Z)^\uparrow + \\ + T\left(\Omega(X, Z)\right) T^\uparrow + 2\,\Omega(X, Z) V^\uparrow + \\ +4(d\sigma)\left((\nabla_X Z)^\uparrow, T^\uparrow\right) S - \\ -2\,T^\uparrow\left((d\sigma)\left(X^\uparrow, Z^\uparrow\right)\right) S - 2T(A(X, Z))S, \end{gathered} \tag{4.103}$$

$$\begin{gathered} D_{[X^\uparrow, T^\uparrow]}Z^\uparrow = (\nabla_{[X,T]}Z)^\uparrow + \Omega([X, T], Z)T^\uparrow - \\ -2\left\{(d\sigma)\left([X, T]^\uparrow, Z^\uparrow\right) + A([X, T], Z)\right\} S + \\ + \frac{1}{n+2}\left\{\frac{1}{4(n+1)}X(\rho) - \varphi(X)\right\}(JZ)^\uparrow. \end{gathered} \tag{4.104}$$

The identities

$$[X, T] = -\nabla_T X + \tau(X), \quad \nabla\Omega = 0,$$

together with (4.102)-(4.104) lead to

$$\begin{gathered} R^D(X^\uparrow, T^\uparrow)Z^\uparrow = \left(R^\nabla(X, T)Z\right)^\uparrow + ((\nabla_X\phi)Z)^\uparrow + \\ + \{\Omega(X, \phi Z) - \Omega(\tau X, Z)\}\, T^\uparrow - 2\Omega(X, Z)\, V^\uparrow + \\ +4\left\{A(\tau X, Z) - A(X, \phi Z) + \frac{1}{2}(\nabla_T A)(X, Z)\right\} S - \\ -2\left\{(d\sigma)\left(X^\uparrow, (\nabla_T Z)^\uparrow\right) + (d\sigma)\left(X^\uparrow, (\phi Z)^\uparrow\right)\right\} S + \\ +4\,X^\uparrow\left((d\sigma)\left(Z^\uparrow, T^\uparrow\right)\right) S + \\ +2(d\sigma)\left(Z^\uparrow, T^\uparrow\right)(JX)^\uparrow - 4(d\sigma)\left((\nabla_X Z)^\uparrow, T^\uparrow\right) S + \\ +2T^\uparrow\left((d\sigma)\left(X^\uparrow, Z^\uparrow\right)\right) S - \\ -2\left\{(d\sigma)\left((\nabla_T X)^\uparrow, Z^\uparrow\right) - (d\sigma)\left((\tau X)^\uparrow, Z^\uparrow\right)\right\} S - \end{gathered}$$

$$-\frac{1}{n+2}\left\{\frac{1}{4(n+1)}\, X(\rho) - \varphi(X)\right\}\, (JZ)^{\uparrow},$$

and then (by (4.36) and (4.38)) to (4.41). Next one needs to compute $R^D(X^{\uparrow}, S)Z^{\uparrow}$. One has (by (4.23)-(4.27))

$$D_{X^{\uparrow}} D_S Z^{\uparrow} = \frac{1}{2}\left\{(\nabla_X JZ)^{\uparrow} - G_{\theta}(X, Z)T^{\uparrow}\right\} - \tag{4.105}$$

$$-\left\{(d\sigma)(X^{\uparrow}, (JZ)^{\uparrow}) + A(X, JZ)\right\} S,$$

$$D_S D_{X^{\uparrow}} Z^{\uparrow} = \frac{1}{2}\,(J\nabla_X Z)^{\uparrow} - 2\, S\left((d\sigma)(X^{\uparrow}\,,\, Z^{\uparrow})\right) S. \tag{4.106}$$

Finally

$$D_{[X^{\uparrow}, S]} Z^{\uparrow} = 0 \tag{4.107}$$

because of (by (4.26))

$$\left[X^{\uparrow}, S\right] = D_{X^{\uparrow}} S - D_S X^{\uparrow} = 0.$$

Then (4.105)-(4.107) lead to

$$R^D(X^{\uparrow}, S)Z^{\uparrow} = -\frac{1}{2}\, G_{\theta}(X, Z)T^{\uparrow} - A(X, JZ)S +$$

$$+\left\{2\, S\left((d\sigma)(X^{\uparrow}, Z^{\uparrow})\right) - (d\sigma)(X^{\uparrow}, (JZ)^{\uparrow})\right\} S$$

and then (by (4.36) i.e. $S\left((d\sigma)(X^{\uparrow}, Z^{\uparrow})\right) = 0$) to (4.42). Next one computes $R^D(X^{\uparrow}, Y^{\uparrow})T^{\uparrow}$. To this end (by (4.24))

$$D_{X^{\uparrow}} D_{Y^{\uparrow}} T^{\uparrow} = D_{X^{\uparrow}}\, (\tau Y + \phi Y)^{\uparrow} =$$

or (by (4.23))

$$D_{X^{\uparrow}} D_{Y^{\uparrow}} T^{\uparrow} = (\nabla_X \tau Y + \nabla_X \phi Y)^{\uparrow} \tag{4.108}$$

$$+\{\Omega(X, \tau Y) + \Omega(X, \phi Y)\} T^{\uparrow} -$$

$$-2\left\{(d\sigma)\left(X^{\uparrow}, (\tau Y)^{\uparrow}\right) + (d\sigma)\left(X^{\uparrow}, (\phi Y)^{\uparrow}\right)\right\} S -$$

$$-2\{A(X, \tau Y) + A(X, \phi Y)\} S.$$

The identities

$$\Omega(X, \tau Y) = -A(X, JY), \quad A(X, \tau Y) = G_{\theta}(\tau X, \tau Y),$$

and (4.65) show that $\Omega(X, \tau Y)$, $A(X, \tau Y)$ and $\Omega(X, \phi Y)$ are symmetric in (X, Y). Let us interchange X and Y and subtract the resulting identity from (4.108). We obtain

$$D_{X^{\uparrow}} D_{Y^{\uparrow}} T^{\uparrow} - D_{Y^{\uparrow}} D_{X^{\uparrow}} T^{\uparrow} = \tag{4.109}$$

$$= (\nabla_X \tau Y - \nabla_Y \tau X + \nabla_X \phi Y - \nabla_Y \phi X)^{\uparrow} -$$

$$-2\{A(X, \phi Y) - A(Y, \phi X)\} S -$$

$$-2\left\{(d\sigma)\left(X^{\uparrow}, (\tau Y)^{\uparrow}\right) - (d\sigma)\left(Y^{\uparrow}, (\tau X)^{\uparrow}\right)\right\} S -$$

$$-2\left\{(d\sigma)\left(X^{\uparrow}, (\phi Y)^{\uparrow}\right) - (d\sigma)\left(Y^{\uparrow}, (\phi X)^{\uparrow}\right)\right\} S.$$

On the other hand (by (4.65))

$$A(X, \phi Y) = G_\theta(\tau X, \phi Y) = G_\theta(J\tau X, J\phi Y) = \Omega(\phi JY, \tau X) =$$

$$= \frac{1}{2(n+2)} \{\mathrm{Ric}_\nabla(\tau X, JY) - (n-1)A(\tau X, JY)\} -$$

$$- \frac{\rho}{4(n+1)(n+2)} A(X, JY)$$

where

$$A(\tau X, JY) = G_\theta(\tau^2 X, JY) = -G_\theta(\tau X, J\tau Y) = -\Omega(\tau X, \tau Y)$$

is skew-symmetric in (X, Y). Thus

$$A(X, \phi Y) - A(Y, \phi X) = \tag{4.110}$$

$$= \frac{1}{2(n+2)} \{\mathrm{Ric}_\nabla(\tau X, JY) - \mathrm{Ric}_\nabla(\tau Y, JX)\} + \frac{n-1}{n+2} \Omega(\tau X, \tau Y).$$

Moreover (by (4.37), (4.24) and (4.27))

$$D_{[X^\uparrow, Y^\uparrow]} T^\uparrow = (\tau[X, Y])^\uparrow + (\phi[X, Y])^\uparrow - 4\Omega(X, Y) V^\uparrow\,. \tag{4.111}$$

Then (4.109)-(4.111) and (4.52) (together with $\tau T = \phi T = 0$) lead to

$$R^D(X^\uparrow, Y^\uparrow) T^\uparrow = ((\nabla_X \tau)Y + (\nabla_X \phi)Y)^\uparrow - \tag{4.112}$$

$$- \frac{1}{n+2} \{\mathrm{Ric}_\nabla(\tau X, \tau Y) - \mathrm{Ric}_\nabla(\tau Y, JX)\} S -$$

$$- \frac{2(n-1)}{n+2} \Omega(\tau X, \tau Y) S + 4\,\Omega(X, Y)\, V^\uparrow -$$

$$-2 \left\{ (d\sigma)(X^\uparrow, (\tau Y)^\uparrow) - (d\sigma)(Y^\uparrow, (\tau X)^\uparrow) \right\} S -$$

$$-2 \left\{ (d\sigma)(X^\uparrow, (\phi Y)^\uparrow) - (d\sigma)(Y^\uparrow, (\phi X)^\uparrow) \right\} S$$

hence (by (4.36))

$$R^D(X^\uparrow\,,\ Y^\uparrow) T^\uparrow = ((\nabla_X \tau)Y + (\nabla_X \phi)Y)^\uparrow - \tag{4.113}$$

$$- \frac{1}{n+2} \{\mathrm{Ric}_\nabla(\tau X, JY) - \mathrm{Ric}_\nabla(\tau Y, JX) +$$

$$+ \mathrm{Ric}_\nabla(Y, J\tau X) - \mathrm{Ric}_\nabla(X, J\tau Y) +$$

$$+ \mathrm{Ric}_\nabla(Y, J\phi X) - \mathrm{Ric}_\nabla(X, J\phi Y)\}\ S -$$

$$- \frac{2(n-1)}{n+2}\, \Omega(\tau X, \tau Y) S + 4\,\Omega(X, Y) V^\uparrow -$$

$$- \frac{\rho}{2(n+1)(n+2)} \{\Omega(X, \tau Y) - \Omega(Y, \tau X) + \Omega(X, \phi Y) - \Omega(Y, \phi X)\} S +$$

$$+ \frac{n-1}{n+2} \{A(X, \tau Y) - A(Y, \tau X) + A(X, \phi Y) - A(Y, \phi X)\} S\,.$$

Yet the quantities $\Omega(X, \tau Y) = -A(X, JY)$ and (by (4.65)) $\Omega(X, \phi Y)$ and $A(X, \tau Y) = G_\theta(\tau X, \tau Y)$ are symmetric in (X, Y) hence (4.113) simplifies (again by (4.110)) to (4.43). Next we compute

$$R^D(X^\uparrow, Y^\uparrow)S = D_{X^\uparrow} D_{Y^\uparrow} S - D_{Y^\uparrow} D_{X^\uparrow} S - D_{[X^\uparrow, Y^\uparrow]} S =$$

(by (4.26), (4.37) and (4.27))

$$= D_{X^\uparrow}\left(\frac{1}{2}(JY)^\uparrow\right) - D_{Y^\uparrow}\left(\frac{1}{2}(JX)^\uparrow\right) - D_{[X,Y]^\uparrow} S -$$

$$-\frac{2}{n+2}\left\{\mathrm{Ric}_\nabla(X, JY) - (n-1)A(X, Y) - \frac{\rho}{2(n+1)}\,\Omega(X, Y)\right\} D_S S =$$

(by (4.23) and (4.26)-(4.27))

$$= \frac{1}{2}\left\{(\nabla_X JY)^\uparrow + \Omega(X, JY)T^\uparrow - 2\left[(d\sigma)(X^\uparrow, (JY)^\uparrow) + A(X, JY)\right] S -\right.$$

$$\left. -(\nabla_Y JX)^\uparrow - \Omega(Y, JX)T^\uparrow + 2\left[(d\sigma)(Y^\uparrow, (JX)^\uparrow) + A(Y, JX)\right] S\right\} -$$

$$-\frac{1}{2}\,(J\Pi_H[X, Y])^\uparrow - \theta([X, Y])D_{T^\uparrow} S =$$

(by (4.52))

$$= \frac{1}{2}\,((\nabla_X J)Y - (\nabla_Y J)X)^\uparrow -$$

$$-\left\{(d\sigma)(X^\uparrow (JY)^\uparrow) + (d\sigma)((JX)^\uparrow, Y^\uparrow)\right\} S =$$

(by $\nabla J = 0$ and (4.36))

$$= -\frac{1}{2(n+2)}\left\{\frac{\rho}{2(n+1)}\Omega(X, JY) - (n-1)A(X, JY) - \right.$$

$$\left. -\mathrm{Ric}_\nabla(X, J^2 Y)\right\} -$$

$$-\frac{1}{2(n+2)}\left\{\frac{\rho}{2(n+1)}\Omega(JX, Y) - (n-1)A(JX, Y) - \right.$$

$$-\mathrm{Ric}_\nabla(JX, JY)\} =$$

$$= \frac{1}{2(n+2)}\,\{2(n-1)\,A(X, JY) +$$

$$+\mathrm{Ric}_\nabla(JX, JY) - \mathrm{Ric}_\nabla(X, Y)\} = 0$$

(by applying (4.61)) thus leading to (4.44). Next we compute

$$R^D(T^\uparrow, S)Z^\uparrow = D_{T^\uparrow} D_S Z^\uparrow - D_S D_{T^\uparrow} Z^\uparrow - D_{[T^\uparrow, S]} Z^\uparrow =$$

(by (4.25)-(4.26) and $[T^\uparrow, S] = 0$)

$$= D_{T^\uparrow}\left(\frac{1}{2}\,(JZ)^\uparrow\right) - D_S\left((\nabla_T Z + \phi Z)^\uparrow + 4(d\sigma)(Z^\uparrow, T^\uparrow)S\right) =$$

(by (4.25))

$$= \frac{1}{2}\left\{(\nabla_T JZ + \phi JZ)^{\uparrow} + 4(d\sigma)((JZ)^{\uparrow}, T^{\uparrow})S\right\} -$$

$$-\frac{1}{2}\,(J(\nabla_T Z + \phi Z))^{\uparrow} - 4\,S\,((d\sigma)(Z^{\uparrow}, T^{\uparrow}))S \; - 4(d\sigma)(Z^{\uparrow}, T^{\uparrow})D_S S \;=$$

(by (4.27))

$$= \frac{1}{2}\,((\nabla_T J)Z + [\phi, J]Z)^{\uparrow} +$$

$$+2(d\sigma)((JZ)^{\uparrow}, T^{\uparrow})S \; - 4(d\sigma)(Z^{\uparrow}, T^{\uparrow})S.$$

Finally $\nabla J = 0$, $[\phi, J] = 0$ and (4.38) yield (4.45). The proof of (4.46)-(4.47) follows (by (4.27)) from

$$R^D(T^{\uparrow}, S)T^{\uparrow} = D_{T^{\uparrow}} D_S T^{\uparrow} - D_S D_{T^{\uparrow}} T^{\uparrow} - D_{[T^{\uparrow}, S]} T^{\uparrow} =$$

$$= -D_S\left(2V^{\uparrow}\right) = -(JV)^{\uparrow},$$

$$R^D(T^{\uparrow}, S)S = D_{T^{\uparrow}} D_S S - D_S D_{T^{\uparrow}} S - D_{[T^{\uparrow}, S]} S = 0.$$

The proof of Lemma 4.2 is complete.

To prove (4.55) let $X \in H(M)$. Then (by (4.53))

$$\mathrm{Ric}_D(X^{\uparrow}, T^{\uparrow}) = \sum_a F_\theta(R^D(E_a^{\uparrow}, T^{\uparrow})X^{\uparrow}, E_a^{\uparrow}) +$$

$$+2\,F_\theta(R^D(S, T^{\uparrow})X^{\uparrow}, T^{\uparrow})$$

and (by (4.41))

$$R^D(E_a^{\uparrow}, T^{\uparrow})X^{\uparrow} \equiv \left(R^{\nabla}(E_a, T)X\right)^{\uparrow} + ((\nabla_{E_a}\phi)X)^{\uparrow} +$$

$$+\frac{1}{n+2}\left\{\varphi(X)(JE_a)^{\uparrow} + \varphi(E_a)(JX)^{\uparrow}\right\} -$$

$$-\frac{1}{4(n+1)(n+2)}\left\{X(\rho)(JE_a)^{\uparrow} + E_a(\rho)(JX)^{\uparrow}\right\} -$$

$$-2\Omega(E_a, X)V^{\uparrow}\,, \quad \mathrm{mod}\ T^{\uparrow}, S.$$

Let us take the inner product with $E_a^{\uparrow}$ and sum over $1 \le a \le 2n$. One obtains

$$\sum_a F_\theta(R^D(E_a^{\uparrow}, T^{\uparrow})X^{\uparrow}, E_a^{\uparrow}) = \mathrm{Ric}_{\nabla}(X, T) + \tag{4.114}$$

$$+\mathrm{trace}\ \{\Pi_H(\nabla\phi)X\} + \frac{1}{n+2}\,\varphi(JX) +$$

$$+\frac{1}{4(n+1)(n+2)}\,\Omega(X, \nabla^H\rho) - 2\Omega(V, X).$$

Also (by the symmetries of the Riemann-Christoffel tensor and (4.46))

$$F_\theta(R^D(S, T^{\uparrow})X^{\uparrow}, T^{\uparrow}) = F_\theta(T^{\uparrow}S)T^{\uparrow}, X^{\uparrow}) = 0$$

so that (4.114) yields (4.55). Next (again by (4.53))

$$R^D(X^\uparrow, S) = \sum_a F_\theta(R^D(E_a^\uparrow, S)X^\uparrow, E_a^\uparrow) +$$

$$+ 2\, F_\theta(R^D(T^\uparrow, S)X^\uparrow, S) = 0$$

by (4.42) and (4.47). Indeed (by (4.42)) $R^D(E_a^\uparrow, S)X^\uparrow \equiv 0$, mod $T^\uparrow$, S and $H(M)^\perp$ is orthogonal on $\mathbb{R}T^\uparrow \oplus \mathbb{R}S$. This yields (4.56). Moreover (by (4.46))

$$\mathrm{Ric}_D(T^\uparrow, T^\uparrow) = \sum_a F_\theta(R^D(E_a^\uparrow, T^\uparrow)T^\uparrow, E_a^\uparrow) =$$

$$= -\sum_a F_\theta(R^D(E_a^\uparrow, T^\uparrow)E_a^\uparrow, T^\uparrow)$$

and (by (4.41))

$$R^D(X^\uparrow, T^\uparrow)X^\uparrow \equiv -\frac{1}{n+2}\{\mathrm{Ric}_\nabla(X, J\phi X) + \mathrm{Ric}_\nabla(\tau X, JX) -$$

$$-2\left[(\nabla_X \varphi)X - \frac{1}{4(n+1)}(\nabla_X d\rho)X\right] +$$

$$+ (\nabla_T \mathrm{Ric}_\nabla)(X, JX) + (n+5)(\nabla_T A)(X, X) +$$

$$+ \frac{\rho}{2(n+1)}\left[\Omega(X, \phi X) - \Omega(\tau X, X)\right] +$$

$$+ 3(n+3)\left[A(X, \phi X) - A(\tau X, X)\right]\}, \quad \mathrm{mod}\ H(M)^\uparrow,\ T^\uparrow,$$

hence (for $X = E_a$)

$$\mathrm{Ric}_D(T^\uparrow, T^\uparrow) = \tag{4.115}$$

$$= \frac{1}{n+2}\,\mathrm{trace}_{G_\theta}\,\Pi_H\,\{\mathrm{Ric}_\nabla(\cdot\,, J\phi\,\cdot) + \mathrm{Ric}_\nabla(\tau\,\cdot\,, J\,\cdot) +$$

$$\left. + \frac{1}{4(n+1)}\,\nabla d\rho - \nabla\varphi - \frac{1}{2}\,(\nabla_T \mathrm{Ric}_\nabla)(\cdot\,, J\,\cdot) + \frac{n+5}{2}\,\nabla_T A\right\} +$$

$$+ \frac{\rho}{4(n+1)(n+2)}\,\mathrm{trace}(J\phi - \tau J) + \frac{3(n+3)}{n+2}\,\mathrm{trace}(\tau\phi - \tau^2).$$

Since $\mathrm{trace}(\tau J) = \mathrm{trace}(\tau\phi) = 0$ the identity (4.115) implies (4.57). Moreover (by (4.53))

$$\mathrm{Ric}_D(T^\uparrow, S) = \sum_a F_\theta(R^D(E_a^\uparrow, S)T^\uparrow, E_a^\uparrow) =$$

$$= -\sum_a F_\theta(R^D(E_a^\uparrow, S)E_a^\uparrow, T^\uparrow)$$

or (by (4.42))

$$\mathrm{Ric}_D(T^\uparrow, S) = -\frac{n\rho}{4(n+1)(n+2)} + \tag{4.116}$$

$$+ \frac{1}{4(n+2)}\,\mathrm{trace}_{G_\theta}\Pi_H\,\mathrm{Ric}_\nabla$$

and

$$\mathrm{trace}_{G_\theta}\Pi_H \mathrm{Ric}_\nabla = \sum_a \mathrm{Ric}_\nabla(E_a, E_a) =$$

$$= \sum_a \left\{ E_a^\lambda E_a^\mu R_{\lambda\mu} + E_a^\lambda E_a^{\overline{\mu}} R_{\lambda\overline{\mu}} + E_a^{\overline{\lambda}} E_a^\mu R_{\overline{\lambda}\mu} + E_a^{\overline{\lambda}} E_a^{\overline{\mu}} R_{\overline{\lambda}\overline{\mu}} \right\},$$

$$E_a = E_a^\lambda T_\lambda + E_a^{\overline{\lambda}} T_{\overline{\lambda}}, \quad E_a^{\overline{\lambda}} = \overline{E_a^\lambda}, \quad \sum_a E_a^\lambda E_a^{\overline{\mu}} = g^{\lambda\overline{\mu}},$$

so that (by (4.32)-(4.33))

$$\mathrm{trace}_{G_\theta}\Pi_H \mathrm{Ric}_\nabla = 2 g^{\lambda\overline{\mu}} R_{\lambda\overline{\mu}} +$$

$$+ \sum_a i(n-1) \left\{ E_a^\lambda E_a^\mu A_{\lambda\mu} - E_a^{\overline{\lambda}} E_a^{\overline{\mu}} A_{\overline{\lambda}\overline{\mu}} \right\} =$$

$$= 2\rho + i(n-1) \sum_a \left\{ A(E_a^{1,0}\,,\, E_a^{1,0}) - A(E_a^{0,1}\,,\, E_a^{0,1}) \right\} =$$

$$= 2\rho + (n-1) \sum_a A(JE_a\,,\, E_a) = 2\rho + (n-1)\,\mathrm{trace}(\tau J)$$

i.e.

$$\mathrm{trace}_{G_\theta}\,\Pi_H\,\mathrm{Ric}_\nabla = 2\rho. \tag{4.117}$$

Substitution from (4.117) into (4.116) leads to (4.58). Finally (by (4.42))

$$\mathrm{Ric}_D(S,S) = \sum_a F_\theta(R^D(E_a^\uparrow, S)S, E_a^\uparrow) =$$

$$= -\sum_a F_\theta(R^D(E_a^\uparrow, S)E_a^\uparrow, S) = \frac{1}{4}\sum_a G_\theta(E_a, E_a) = \frac{n}{2}$$

i.e. (4.59) holds. Lemma 4.3 is proved.

4.7 The Chang-Chiu inequality

The purpose of § 4.7 is to give a proof of

$$4n\ \|u_0\|_{L^2}^2 \le \frac{1}{n}\ \|\Delta_b u\|_{L^2}^2 + 4\,\tau_0\ \left\|\nabla^H u\right\|_{L^2}^2 \tag{4.118}$$

for any $u \in C^\infty(M,\mathbb{R})$ (compare[2] to (3.5) in [92], p. 270). This is referred to as the *Chang-Chiu inequality*. To prove (4.118) let us contract (4.8) by u^β so that to obtain

$$u^\beta \nabla_0 u_\beta = u^\beta \nabla_\beta u_0 - A_{\alpha\beta} u^\alpha u^\beta$$

or

$$u^\beta\,\nabla_0 u_\beta = \nabla_\beta\left(u_0 u^\beta\right) - u_0\,\nabla_\beta u^\beta - A_{\alpha\beta} u^\alpha u^\beta\,. \tag{4.119}$$

[2] Discrepancies among (4.118) and (3.5) in [92], p. 270, are due to the different convention as to wedge products of 1-forms producing the additional 2 factor in (4.7). Cf. also (1.62) in [94], p. 39, and (9.7) in [94], p. 424. Through this thesis conventions as to wedge products and exterior differentiation calculus are those in [98], p. 35-36.

On the other hand (by (4.7))

$$\nabla_\beta u^\beta = \nabla_{\overline{\beta}} u^{\overline{\beta}} - 2in\, u_0$$

so that (by substitution into (4.119))

$$u^\beta \nabla_0 u_\beta + u_0\, \nabla_{\overline{\beta}} u^{\overline{\beta}} = 2in\, u_0^2 - A_{\alpha\beta} u^\alpha u^\beta + \nabla_\beta \left(u_0 u^\beta \right). \tag{4.120}$$

Next (again by (4.8))

$$u_0\, \nabla_{\overline{\beta}} u^{\overline{\beta}} = \nabla_{\overline{\beta}} \left(u_0 u^{\overline{\beta}} \right) - u^{\overline{\beta}}\, \nabla_{\overline{\beta}} u_0 = \nabla_{\overline{\beta}} \left(u_0 u^{\overline{\beta}} \right) - u^{\overline{\beta}} \left(\nabla_0 u_{\overline{\beta}} + u_\gamma A^\gamma_{\overline{\beta}} \right)$$

hence (by substitution of $u_0\, \nabla_{\overline{\beta}} u^{\overline{\beta}}$ into (4.120))

$$i \left(u^{\overline{\beta}}\, \nabla_0 u_{\overline{\beta}} - u^\beta\, \nabla_0 u_\beta \right) = \tag{4.121}$$

$$= 2nu_0^2 + i \left(A_{\alpha\beta} u^\alpha u^\beta - A_{\overline{\alpha}\overline{\beta}} u^{\overline{\alpha}} u^{\overline{\beta}} \right) + i \left\{ \nabla_{\overline{\alpha}} \left(u_0 u^{\overline{\alpha}} \right) - \nabla_\alpha \left(u_0 u^\alpha \right) \right\}$$

(compare to (2.4) in Lemma 2.2, [92], p. 268). Calculations are performed with respect to an arbitrary local frame $\{T_\alpha : 1 \le \alpha \le n\}$ in $T_{1,0}(M)$ (rather than a G_θ-orthonormal frame, as in [92]). The next step is to evaluate the left hand side of (4.121) in terms of the operator $P + \overline{P}$. One has

$$u_0 = \frac{i}{2n} \left(\nabla_\beta u^\beta - \nabla_{\overline{\beta}} u^{\overline{\beta}} \right)$$

hence (by (4.8))

$$u^{\overline{\alpha}}\, \nabla_0 u_{\overline{\alpha}} = u^{\overline{\alpha}} \left(\nabla_{\overline{\alpha}} u_0 - u_\beta A^\beta_{\overline{\alpha}} \right) = \frac{i}{2n}\, u^{\overline{\alpha}} \nabla_{\overline{\alpha}} \left(\nabla_\beta u^\beta - \nabla_{\overline{\beta}} u^{\overline{\beta}} \right) - A_{\overline{\alpha}\overline{\beta}} u^{\overline{\alpha}} u^{\overline{\beta}} =$$

$$= \frac{i}{2n}\, u^{\overline{\alpha}} \left(\nabla_{\overline{\alpha}} \nabla_\beta u^\beta - \nabla_{\overline{\alpha}} \nabla_{\overline{\beta}} u^{\overline{\beta}} \right) - A_{\overline{\alpha}\overline{\beta}} u^{\overline{\alpha}} u^{\overline{\beta}} =$$

$$= \frac{i}{2n}\, u^{\overline{\alpha}} \left(g^{\beta\overline{\gamma}} u_{\overline{\alpha}\beta\overline{\gamma}} - g^{\overline{\beta}\gamma} u_{\overline{\alpha}\overline{\beta}\gamma} \right) - A_{\overline{\alpha}\overline{\beta}} u^{\overline{\alpha}} u^{\overline{\beta}}$$

or

$$u^{\overline{\alpha}} \nabla_0 u_{\overline{\alpha}} = \frac{i}{2n}\, u^{\overline{\alpha}} \left(u_{\overline{\alpha}}{}^{\overline{\gamma}}{}_{\overline{\gamma}} - u_{\overline{\alpha}}{}^{\gamma}{}_{\gamma} \right) - A_{\overline{\alpha}\overline{\beta}} u^{\overline{\alpha}} u^{\overline{\beta}}\,. \tag{4.122}$$

Using $P_{\overline{\alpha}} u \equiv u_{\overline{\alpha}}{}^{\gamma}{}_{\gamma} - 2ni\, A_{\overline{\alpha}\overline{\beta}} u^{\overline{\beta}}$ the identity (4.122) becomes

$$i\, u^{\overline{\alpha}}\, \nabla_0 u_{\overline{\alpha}} = \frac{1}{2n}\, u^{\overline{\alpha}} \left(P_{\overline{\alpha}} u - u_{\overline{\alpha}}{}^{\overline{\gamma}}{}_{\overline{\gamma}} \right). \tag{4.123}$$

Let us take the complex conjugate of (4.123) and add the resulting equation to (4.123). We obtain

$$2ni \left(u^{\overline{\alpha}}\, \nabla_0 u_{\overline{\alpha}} - u^\beta\, \nabla_0 u_\beta \right) = u^{\overline{\alpha}} P_{\overline{\alpha}} u + u^\alpha P_\alpha u - \left\{ u^{\overline{\alpha}}\, u_{\overline{\alpha}}{}^{\overline{\gamma}}{}_{\overline{\gamma}} + u^\alpha\, u_\alpha{}^\gamma{}_\gamma \right\} \tag{4.124}$$

where $P_\alpha u \equiv u_\alpha{}^{\overline{\gamma}}{}_{\overline{\gamma}} + 2niA_{\alpha\beta} u^\beta$. Let us replace $u^\alpha\ u_\alpha{}^\beta{}_\beta + u^{\overline{\alpha}}\ u_{\overline{\alpha}}{}^{\overline{\beta}}{}_{\overline{\beta}}$ from (4.15) into (4.124). We obtain

$$2ni \left(u^{\overline{\alpha}}\, \nabla_0 u_{\overline{\alpha}} - u^\alpha\, \nabla_0 u_\alpha \right) = 2 \left(u^\alpha\, P_\alpha u + u^{\overline{\alpha}} P_{\overline{\alpha}} u \right) - \tag{4.125}$$

$$-2ni\left(A_{\alpha\beta}u^{\alpha}u^{\beta} - A_{\overline{\alpha}\overline{\beta}}u^{\overline{\alpha}}u^{\overline{\beta}}\right) + \left(\nabla^{H}u\right)(\Delta_b u).$$

Finally substitution from (4.125) into (4.121) leads to

$$2\left(u^{\alpha}\,P_{\alpha} + u^{\overline{\alpha}}\,P_{\overline{\alpha}}u\right) + \left(\nabla^{H}u\right)(\Delta_b u) = \tag{4.126}$$

$$= 4n^2u_0^2 + 4ni\left(A_{\alpha\beta}u^{\alpha}u^{\beta} - A_{\overline{\alpha}\overline{\beta}}u^{\overline{\alpha}\overline{\beta}}\right) + 2ni\left\{\nabla_{\overline{\alpha}}\left(u_0u^{\overline{\alpha}}\right) - \nabla_{\alpha}\left(u_0u^{\alpha}\right)\right\}.$$

Let us observe that

$$i\left(A_{\alpha\beta}u^{\alpha}u^{\beta} - A_{\overline{\alpha}\overline{\beta}}u^{\overline{\alpha}\overline{\beta}}\right) = A\left(\nabla^{H}u\,,\ J\nabla^{H}u\right),$$

$$i\left\{\nabla_{\alpha}\left(u_0u^{\alpha}\right) - \nabla_{\overline{\alpha}}\left(u_0u^{\overline{\alpha}}\right)\right\} = \operatorname{div}\left(u_0\,J\nabla^{H}u\right),$$

$$u^{\alpha}\,P_{\alpha} + u^{\overline{\alpha}}\,P_{\overline{\alpha}}u = g_\theta^*(Lu\,,\ d_bu),$$

where $L = P + \overline{P}$. Then (4.126) becomes

$$2\,g_\theta^*\,(Lu\,,\ d_bu) + \left(\nabla^{H}u\right)(\Delta_b u) = 4n^2\,u_0^2 + \tag{4.127}$$

$$+4n\,A\left(\nabla^{H}u\,,\ J\nabla^{H}u\right) - 2n\operatorname{div}\left(u_0\,J\nabla^{H}u\right).$$

Let us integrate over M and use Green's lemma. Then (by Lemma 4.1)

$$-2\int_M (P_0u)u\,\Psi_\theta + \int_M \left(\nabla^{H}u\right)(\Delta_b u)\,\Psi_\theta = \tag{4.128}$$

$$= 4n^2\ \|u_0\|_{L^2}^2 + 4n\int_M A\left(\nabla^{H}u\,,\ J\nabla^{H}u\right)\,\Psi_\theta\,.$$

Also (again by Green's lemma)

$$\int_M \left(\nabla^{H}u\right)(\Delta_b u)\ \Psi_\theta = \int_M \left\{\operatorname{div}\left((\Delta_b u)\,\nabla^{H}u\right) - (\Delta_b u)\operatorname{div}\left(\nabla^{H}u\right)\right\}\ \Psi_\theta =$$

$$= \int_M (\Delta_b u)^2\ \Psi_\theta = \|\Delta_b u\|_{L^2}^2\ .$$

Finally as P_0 is nonnegative (4.99) and (4.128) lead to (4.118). Q.e.d.

Chapter 5

A New proof of the CR Pohožaev Identity and related Topics

5.1 Introduction and Main Results

We are concerned with non existence results for the following semilinear boundary value problems on a bounded domain Ω of the Heisenberg group $\mathbb{H}^n$

$$(P)\begin{cases} -\Delta_H u & = & g(u) & \text{in } \Omega \\ u & = & 0 & \text{in } \partial\Omega, \end{cases}$$

where Δ_H is the sublaplacian of $\mathbb{H}^n$, g is a C^1 function. Recall that the Heisenberg group $\mathbb{H}^n$ is the homogeneous Lie group whose underlying manifold is $\mathbb{R}^{2n+1}$ and group law given by

$$\tau_{\xi'}(\xi) = \xi' \cdot \xi = (x + x',\ y + y'\ t + t' + 2(< x,\ y' > - < x',\ y >))$$

where $< .,. >$ denotes the inner product in $\mathbb{R}^n$, $\xi = (x, y, t)$ and $\xi' = (x', y', t')$. The homogeneous norm of the space

$$\rho(\xi) = ((| x |^2 + | y |^2)^2 + t^2)^{\frac{1}{4}}$$

and the natural distance is accordingly defined by $d(\xi, \xi') = \rho(\xi^{-1} \cdot \xi')$. The Koranyi ball of center ξ_0 and radius r for this distance is given by $B_r(\xi) = \{\xi \in \mathbb{H}^n /\ d(\xi_0, \xi) < \ r\}$. There are a remarkable families of transformations groups on $\mathbb{H}^n$, the group of parabolic dilations and the groups of left translations. The parabolic $\mathbb{H}^n$-dilatations are the following transformations

$$\begin{array}{rcl} \delta_\lambda : \mathbb{H}^n & \longrightarrow & \mathbb{H}^n \\ (x, y, t) & \longrightarrow & (\lambda x, \lambda y, \lambda^2 t)\,,\ \ \lambda > 0. \end{array}$$

The Jacobian determinant of δ_λ is λ^{2n+2}, it yields that the homogeneous dimension of $\mathbb{H}^n$ is $Q = 2n + 2$. For a given $\xi' \in \mathbb{H}^n$, one can define a group of left translations by setting:

$$\tau_\alpha(\xi) = \tau_{\alpha\xi'}(\xi) = \alpha\xi' \cdot \xi, \quad \forall \xi \in \mathbb{H}^n$$

The generators of the group of dilations $\{\delta_\lambda, \lambda > 0\}$ and the group of left translations $\{\tau_{\alpha\xi'}, \alpha \in \mathbb{R}\}$ are given respectively by the following smooth vector fields

$$X = \sum_{i=1} (x_i \partial_{x_i} + y_i \partial_{y_i}) + 2t\partial_t \tag{5.1}$$

$$Y(\xi') = Y(x', y', t') = \sum_{i=1} (x'_i \partial_{x_i} + y'_i \partial_{y_i}) + (t' + 2(< x, y' > - < x', y >))\partial_t. \tag{5.2}$$

We say that a function u is homogeneous of degree k with respect to the parabolic dilations $\{\delta_\lambda, \lambda > 0\}$ if and only if $u \circ \delta_\lambda = \lambda^k u$ for $\lambda > 0$, which implies that its Lie derivative with respect to X satisfies $L_X u = X\ u = k\ u$. For example, the naturel distance function is homogenous of degree 1. In the other hand a function u is homogeneous of degree k with respect to the group of left translations $\{\tau_{\alpha\xi'}, \alpha \in \mathbb{R}\}$ if and only if its Lie derivative with respect to Y satisfies

$$L_{Y(\xi')}\ u = Y(\xi')\ u = k\ u.$$

The subelliptic gradient is given by $\nabla_{\mathbb{H}^n} = (X_1, ..., X_n, Y_1, ..., Y_n)$ where $X_i = \partial_{x_i} + 2y_i\partial_t,\ Y_i = \partial_{y_i} - 2x_i\partial_t,\ i \in \{1, 2...n\}$ span the horizontal subspace of the tangent space of $\mathbb{H}^n$ accordingly to the following decomposition $T\mathbb{H}^n = \mathcal{H} \oplus \mathbb{R}T$, where $\mathcal{H}$ is the horizontal subspace and T is the Reeb vector field given by $T = \partial_t$. The Lie Algebra of left invariant vector fields is generated by $\{(X_i, Y_i)_{1\le i\le n}, T\}$. Since $[X_i,\ Y_i] = -4T$, the Heisenberg laplacian $\Delta_H = \sum_{i=1}^n (X_i^2 + Y_i^2)$, is a second order degenerate elliptic operator of Hörmander type and hence it is hypoelliptic. If we denote by $A = (a_{ij})$ the $(2n+1)\times(2n+1)$ symmetric matrix given by $a_{ij} = \delta_{ij}$ if $i, j = 1, ...2n,\ a_{(2n+1)j} = -2x_j$ if $j = n+1, ..., 2n$, and $a_{(2n+1)(2n+1)} = 4|z|^2$. We remark that the matrix A is related to Δ_H by the formula $\Delta_H = div(A\ \nabla)$ where ∇ and div denote respectively the euclidian gradient and the euclidian divergence operator of $\mathbb{R}^{2n+1}$. The canonical contact and volume forms of $\mathbb{H}^n$ are given by $\theta_0 = dt + 2\sum_{1\le i\le n}(x_i\, dy_i - \ y_i dx_i)$ and $d\Psi_{\theta_0} = \theta_0 \wedge d\theta_0^n$. A fundamental solution of $-\Delta_H$ with pole at zero is given by (one can see [43])

$$\Gamma(\xi) = \frac{c_Q}{d(\xi)^{Q-2}}$$

where $c_Q = \frac{\Gamma^2(n/2)}{2^{4-2n}\pi^{n+1}}$ and $Q = 2n+2$. Moreover, a fundamental solution with pole at ξ is

$$\Gamma(\xi, \xi') = \frac{c_Q}{d(\xi, \xi')^{Q-2}}.$$

A basic role in the functional analysis on the Heisenberg group is played by the following Sobolev-type inequality

$$|\varphi|^2_{Q^*} \le c|\nabla_{\mathbb{H}^n}\varphi|_2^2,\ \ \forall\varphi \in C_0^\infty(\mathbb{H}^n)$$

where $Q^* = \frac{2Q}{Q-2}$. This inequality ensures in particular that for every domain Ω of $\mathbb{H}^n$, the function $|\varphi| = |\nabla_{\mathbb{H}^n}\varphi|_2$ is a norm on $C_0^\infty(\Omega)$. We denote by $S^{1,2}(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to this norm, $S^{1,2}(\Omega)$ becomes a Hilbert space with the inner product

$$< u, v >_{S^{1,2}} = \int_\Omega < \nabla_{\mathbb{H}^n} u, \nabla_{\mathbb{H}^n} v >\ d\Psi_{\theta_0}.$$

Define $S_0^{1,2}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ with respect to the norm above.

The Pohožaev Identity is the principle tool used here to investigate the relation between domain geometry and solvability of equation (P). We seek u a positive solution to equation (P), where g has critical or supercritical growth, meaning, $g(u) \geq ku^{1+\frac{2}{n}}$ for some positive constant k. We ask the question " for a prescribed domain and a nonlinearity g, can we find a positive solution u?". For Euclidean domains $\Omega \subset \mathbb{R}^N$, S.Pohožaev in [97] proved that there is no solution for starlike ones, on the other hand, A.Bahri and J.M.Coron, W.Y.Ding in [1] and [105], have shown that a solution exists when $g(u) = u^{p*}$, and the domain has nontrivial topology, here $p^* = (N+2)/(N-2)$ is the critical exponent for the compactness of the Sobolev inclusion $W_0^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$, for $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$, $1 < p < q < \infty$ where $W_0^{k,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm $||u||_{W^{k,p}(\Omega)} = Sup_{\ l(\alpha)\leq k}||D^\alpha u||_{L^p(\Omega)}$.

For the Heisenberg group and using arguments related to the topology of the domain, G.Citti and F.Uguzzoni [44] following the work of A. Bahri and Coron, gave the Kohn Laplacian counterpart of the celebrated theorem in [1], and proved an existence result for Yamabe type problem on domains which have a nontrivial homology group (with $\mathbb{Z}_2$-coefficients), I.Birendili, I.Capuzzo Dolcetta and A.Cutri in [53] used blow up techniques to prove existence results, while in [39] F.Uguzzoni gave a non-existence result for equation (P) involving the critical exponent on halfspaces of the Heisenberg group. We have also to mention the non existence results of E.Lanconelli and F.Uguzzoni on unbounded domains of the Heisenberg group in [33] and [34], and the existence of positives solutions on the Heisenberg group one can see [65] and[91].

For euclidian domains by strict-starlike, we mean that if $x \in R^n$ and ν is the boundary normal, then on the boundary of the domain $(x.\nu) > 0$ for all x. P.Pucci and J.Serrin noted that Pohožaev's result did not require strict starlikeness on the domain and what was needed was a domain with a vector function h that acted like the starlike vector field $h = x$. Several authors P.Pucci, J.Serrin, R.Schaaf, J.McGough, J.Mortesen, C.Rickett and G.Stubendieck in [82], [88], [61], [62] and [63] have examined this new class of h-starlike domains and the resulting extensions of the Pohožaev like results.

While for the Heisenberg group $\mathbb{H}^n$ using the geometry of the domain to give non existence and existence results for equation (P), N.Garofalo and E.Lanconelli in [78] have used the analogy with the hstarlike euclidean domains for a given vector field h. They defined for the Heisenberg group a notion of CR starlike domains for two special smooth vector fields, X and Y which are respectively the generator of the group of dilations and the generator of the group of left translations of $\mathbb{H}^n$ given by (5.1) and (5.2). Next we will introduce the definition given in [78] of domains starshapeness which will be used throughout the present work. Given a piecewise C^1 bounded domain $\Omega \subset \mathbb{H}^n$, we say that it is $\delta-$starshaped with respect to a point $\xi_0 \in \Omega$, if denoting by N the outer unit normal to the boundary of $\tau_{\xi_0^{-1}}(\Omega)$, we have

$$X.N \geq 0 \tag{5.3}$$

at every point of $\partial(\tau_{\xi_0^{-1}}(\Omega))$. For a bounded domain Ω of $\mathbb{H}^n$, we denote by $C(\overline{\Omega})$ the space of all continuous functions $f : \overline{\Omega} \to \mathbb{R}$ such that X_if, Y_if, X_i^2f and Y_i^2f for $i \in \{1, 2, ...n\}$ are continuous functions on Ω and continuous up to the boundary of Ω.

CR versions of the Pohoz̃aev identity

1. Let $u \in C(\overline{\Omega})$ be a solution of the equation (P), then we have

$$\int_{\Omega} \|\nabla_{\mathbb{H}^n} u\|^2 \, X.N d\sigma \quad = \quad -(Q-2)\int_{\Omega} ug(u)du + 2Q\int_{\Omega} G(u)du.$$

   where $G(u) = \displaystyle\int_0^u g(s)\, ds.$

2. We replace in equation (P) $g(u)$ by $g(\xi, u) = u^{1+\frac{2}{n}} + h(\xi)\, u$, with $\xi \in \mathbb{H}^n$ and $h \in C^{\infty}(\mathbb{H}^n)$, set (P') the equation thus obtained. If $u \in C(\overline{\Omega})$ is a solution of (P'), then we have

$$\int_{\partial\Omega} \|\nabla_{\mathbb{H}^n} u\|^2 \, X.N d\sigma \quad = \quad -2\int_{\Omega} \Big(h + \frac{1}{2}(Xh)\Big) u^2 \, d\Psi_{\theta_0}.$$

Pohožaev's non existence results

Let $\Omega \subset \mathbb{H}^n$ be a bounded and connected domain such that $0 = (0,0,0) \in \Omega$ and Ω is $\delta-$starshaped with respect to this point.

1. Then any positive solution u of equation (P) vanishes identically if

$$-(Q-2)ug(u) + 2QG(u) \leq 0. \tag{5.4}$$

2. If $g(u) = u^{1+\frac{2}{n}} + \lambda\, u$, $\lambda \leq 0$, then (P) has no positive solution u different of the trivial solution $u \equiv 0$.

3. Let the function h given in equation (P') satisfies

$$h + \frac{1}{2}(Xh) \leq \; 0. \tag{5.5}$$

   Then there is no positive solution $u \in S_0^{1,2}(\Omega)$ of equation (P') unless $u \equiv 0$.

The chapter is organized as follows. In section 5.2, we prove preliminary results and give the CR Pohoz̃aev Identity. The section 5.3 is devoted to establish some non existence result for equation (P) based on the theory of unique continuation property proved by N. Garofallo and E. Lanconelli for solutions of semi linear equations on Heisenberg group domains, one can see [77] and [78]. In section 5.4, we study a Yamabe like problem on a bounded domain of the Heisenberg group and deduce a non existence result using a related CR Pohoz̃aev Identity.

5.2 Description of the Problem

We will be interested on the existence of a positive solution to the following semilinear equation

$$(P)\begin{cases} -\Delta_H u & = & g(u) & \text{in } \Omega \\ u & = & 0 & \text{in } \partial\Omega, \end{cases}$$

where Δ_H is the sublaplacian of $\mathbb{H}^n$, g is a C^1 function on Ω a bounded domain of the Heisenberg group $\mathbb{H}^n$.

Lemma 5.1. *If u is a solution for problem (P), then we have*

$$-\int_\Omega \Delta_H u(Xu) = \int_\Omega g(u)(Xu) = \int_\Omega X(G(u)) = -(2n+2)\int_\Omega G(u)$$

where $G(u) = \int_0^u g(s)\, ds$.

Proof. We multiply equation (P) by Xu and integrate by parts, we obtain

$$-\int_\Omega \Delta_H u(Xu) = \int_\Omega g(u)(Xu).$$

Since $\frac{\partial}{\partial x_i}(x_i G(u)) = G(u) + x_i \frac{\partial}{\partial x_i} G(u)$ for $i \in \{1, ...n\}$, we have

$$\int_\Omega \frac{\partial}{\partial x_i}(x_i G(u)) = \int_\Omega G(u) + \int_\Omega x_i \frac{\partial}{\partial x_i} G(u)$$

thus it yields that $\int_\Omega G(u) + \int_\Omega x_i \frac{\partial}{\partial x_i} G(u) = 0$, since u is equal to zero on the boundary of Ω. In the same way we obtain

$$\int_\Omega G(u) + \int_\Omega y_i \frac{\partial}{\partial y_i} G(u) = 0,$$

for $i \in \{1, ...n\}$ and $\int_\Omega G(u) + \int_\Omega t \frac{\partial}{\partial t} G(u) = 0$, hence the proof of the lemma is complete.

□

In what follows, for a bounded domain Ω of $\mathbb{H}^n$, we denote by $C(\overline{\Omega})$ the space of all continuous functions $f : \overline{\Omega} \to \mathbb{R}$ such that $X_i f$, $Y_i f$, $X_i^2 f$ and $Y_i^2 f$ for $i \in \{1, 2, ...n\}$ are continuous functions up to the boundary of Ω. Next we will consider the following vector field on $\mathbb{H}^n$, $P = Xu(\nabla_{\mathbb{H}^n} u) = (P_1, P_2, ..., P_{2n})$, where u is in $C(\overline{\Omega})$. If we denote by $\widetilde{div}$ the horizontal divergence operator on $\mathbb{H}^n$, we remark that

$$\widetilde{div} P := div_{\mathbb{H}^n} P = \sum_{i=1}^n (X_i P + Y_i P) = div \widetilde{P}. \tag{5.6}$$

where $\widetilde{P} = (\widetilde{P}_1, \widetilde{P}_2, ..., \widetilde{P}_{2n}, \widetilde{P}_{2n+1})$ is the vector field on $\mathbb{R}^{2n+1}$ obtained from P as

$$\widetilde{P}_j = P_j, \text{ for } j = 1, ...2n \text{ and } \widetilde{P}_{2n+1} = 2\sum_{j=1}^n (y_j P_j - x_j P_{n+j})$$

Let Z be the vector field $\|\nabla_{\mathbb{H}^n} u\|^2\ X$, since $divX = 2n+2$, it yields

$$\int_\Omega divZ = (2n+2)\int_\Omega \|\nabla_{\mathbb{H}^n} u\|^2 + X < \nabla u, A\nabla u > . \tag{5.7}$$

Using (8) and (9), we obtain the following result:

Lemma 5.2. *Let Ω be a bounded domain of $\mathbb{H}^n$ and $u \in C(\overline{\Omega})$. Then*

$$\int_\Omega \widetilde{div}P = \int_\Omega Xu\ \Delta_H u + \int_\Omega divZ - 2n\int_\Omega \|\nabla_{\mathbb{H}^n} u\|^2 - \int_\Omega < A\nabla u, \nabla(Xu) > .$$

Proof. We have

$$\widetilde{div}P = (Xu)\widetilde{div}(\nabla_{\mathbb{H}^n} u) + \nabla_{\mathbb{H}^n} u \nabla_{\mathbb{H}^n}(Xu) = Xu\ \Delta_H u + \nabla_{\mathbb{H}^n} u \nabla_{\mathbb{H}^n}(Xu).$$

A simple computation gives

$$\widetilde{P}_{2n+1} = 2\sum_{j=1}^n (Xu)\ (y_j X_j - x_j Y_j)$$

therefore, since $\nabla_{\mathbb{H}^n} u \nabla_{\mathbb{H}^n}(Xu) =< \nabla u, A\nabla Xu >$ and

$$\begin{aligned} < \nabla u, A\nabla Xu > \quad &= \quad X < \nabla u, A\nabla u > - < A\nabla u, \sum_{j=1}^n \Big(X(\frac{\partial u}{\partial x_i})\partial_{x_i} + X(\frac{\partial u}{\partial y_i})\partial_{y_i}\Big) + X(\frac{\partial u}{\partial t})\partial_t) > \\ &+ \quad < \nabla u, A\nabla u > -2\frac{\partial u}{\partial t}\Big(\sum_{j=1}^n (y_j X_j(u) - x_j Y_j(u)\Big), \end{aligned}$$

we obtain

$$\begin{aligned} \int_\Omega \widetilde{div}P \quad &= \quad \int_\Omega Xu\ \Delta_H u + \int_\Omega divZ - (2n+2)\int_\Omega \|\nabla_{\mathbb{H}^n} u\|^2 \\ &+ \quad \int_\Omega < A\nabla u, \nabla u - \sum_{j=1}^n \Big(X(\frac{\partial u}{\partial x_i})\partial_{x_i} + X(\frac{\partial u}{\partial y_i})\partial_{y_i}\Big) + X(\frac{\partial u}{\partial t})\partial_t) > \\ &- \quad 2\int_\Omega \frac{\partial u}{\partial t}(\sum_{j=1}^n (y_j X_j(u) - x_j Y_j(u)) \\ &= \quad \int_\Omega Xu\ \Delta_H u + \int_\Omega divZ - 2n\int_\Omega \|\nabla_{\mathbb{H}^n} u\|^2 - \int_\Omega < A\nabla u, \nabla(Xu) > . \end{aligned}$$

□

Denoting by N the euclidian unit outer normal to $\partial\Omega$ and $d\sigma$ the $2n$-dimensional Hausdorff measure on $\mathbb{R}^{2n+1}$, if u is in $C(\overline{\Omega})$ the following holds

Theorem 5.3.

$$2\int_{\partial\Omega} X(u)(A\nabla u.N)d\sigma - \int_{\partial\Omega} \|\nabla_{\mathbb{H}^n} u\|^2\, X.Nd\sigma = 2\int_\Omega Xu\Delta_H u - 2n\int_\Omega \|\nabla_{\mathbb{H}^n} u\|^2 .$$

Proof. We have

$$\int_\Omega divZd\Psi_{\theta_0} = \int_{\partial\Omega} Z.Nd\sigma = \int_{\partial\Omega} < Z, N > d\sigma = \int_{\partial\Omega} \|\nabla_{\mathbb{H}^n} u\|^2 (X.N)d\sigma, \tag{5.8}$$

and

$$\int_\Omega \widetilde{div}Pd\Psi_{\theta_0} = \int_\Omega div\widetilde{P}dx = \int_{\partial\Omega} \widetilde{P}.Nd\sigma,$$

where

$$\begin{aligned} \widetilde{P} &= (P, 2\sum X(u)(y_jX_j(u) - x_jY_j(u))) = (Xu.\nabla_{\mathbb{H}^n} u, 2\sum_{i=1}^n (X(u)y_jX_j(u) - x_jY_j(u)) \\ &= X(u)(\nabla_{\mathbb{H}^n} u, 2\sum_{i=1}^n (y_jX_j(u) - x_jY_j(u)) = X(u)(A\nabla u). \end{aligned}$$

Therefore

$$\int_\Omega div\widetilde{P}dx = \int_{\partial\Omega} X(u)(A\nabla u.N)d\sigma. \tag{5.9}$$

On one hand, using Lemma 2.2 and (11), we obtain

$$\begin{aligned} \int_{\partial\Omega} X(u)(A\nabla u.N)d\sigma &= \int_\Omega Xu\Delta_H ud\Psi_{\theta_0} + \int_{\partial\Omega} \|\nabla_{\mathbb{H}^n} u\|^2 X.Nd\sigma \\ &- 2n\int_\Omega \|\nabla_{\mathbb{H}^n} u\|^2 d\Psi_{\theta_0} - \int_\Omega < A\nabla u, \nabla(Xu) > d\Psi_{\theta_0}. \end{aligned}$$

In the other hand, we have

$$\begin{aligned} \int_\Omega \widetilde{div}P = \int_\Omega div\widetilde{P} &= \int_\Omega div(X(u)A\nabla u) \\ &= \int_\Omega (X(u)div(A\nabla u) + DX(u)(A\nabla u) \\ &= \int_\Omega (X(u)div(A\nabla u) + \int_\Omega \nabla X(u).A\nabla u \\ &= \int_\Omega Xu.\Delta_H u + \int_\Omega < A\nabla u, \nabla(Xu) > . \end{aligned}$$

The result follows. □

We are now ready to state a CR version of the "Poho$\widetilde{z}$aev identity". Let $g : \mathbb{R} \to \mathbb{R}$ be a C^1 function with primitive $G(u) = \displaystyle\int_0^u g(s)ds$ and let $u \in C(\overline{\Omega})$ be a solution of the equation

$$(P)\begin{cases} -\Delta_H u &= g(u) \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \partial\Omega, \end{cases}$$

in a bounded domain $\Omega \subset \mathbb{H}^n$. Then there hold

$$\int_\Omega (-\Delta_H u)Xu = \int_\Omega g(u)X(u) = -(2n+2)\int_\Omega G(u),$$

and

$$\int_{\Omega} \|\nabla_{\mathbb{H}^n} u\|^2 = \int_{\Omega} ug(u)du. \tag{5.10}$$

In the other hand $X.u =< X, \nabla u >$, since the unit outer normal $N = -\dfrac{\nabla u}{\|\nabla u\|}$, we obtain

$$X(u) = - < X, N > \|\nabla u\|.$$

Therefore

$$\begin{aligned}\int_{\partial\Omega} \|\nabla_{\mathbb{H}^n} u\|^2 X.N d\sigma &= \int_{\partial\Omega} < A\nabla u, \nabla u > .X.N d\sigma \\ &= \int_{\partial\Omega} < A \|\nabla u\| N, \|\nabla u\| N > X.N d\sigma\end{aligned}$$

and computing this product, one obtain

$$\begin{aligned}< A\nabla u, \nabla u >< X, N > &= \|\nabla u\|^2 < AN, N > . < X, N > \\ &= \|\nabla u\|^2 < AN, N >< X, \frac{-\nabla u}{\|\nabla u\|} > \\ &= - \|\nabla u\| < AN, N >< X, \nabla u > \\ &= - \|\nabla u\| < AN, N > X.u \\ &= < A.\nabla u, N > X(u).\end{aligned}$$

It yields

$$\int_{\partial\Omega} \|\nabla_{\mathbb{H}^n} u\|^2 X.N d\sigma = \int_{\partial\Omega} X(u) A\nabla u.N d\sigma. \tag{5.11}$$

Therefore using (5.10) and (5.11), Theorem 2.3 reads as

Theorem 5.4. *Let $u \in C(\overline{\Omega})$ be a solution of the equation (P), then we have*

$$\int_{\partial\Omega} \|\nabla_{\mathbb{H}^n} u\|^2 X.N d\sigma = -(Q-2) \int_{\Omega} ug(u)du + 2Q \int_{\Omega} G(u)du.$$

Theorem 2.4 is a CR version of the "Poho$\widetilde{z}$aev identity".

5.3 Pohožaev's non existence results

We say that a family of functions has the unique continuation property, if no function besides possibly the zero function vanishes on a set of positive measure. In this section we proceed to establish some non existence result based on the theory of unique continuation property proved by N. Garofallo and E. Lanconelli for solutions of semi linear equations on Heisenberg group domains, one can see [77] and [78]. We begin this section by introducing the notion of starshapeness which will be used throughout this chapter.

Definition 5.5. *[78] Given a piecewise C^1 domain $\Omega \subset \mathbb{H}^n$, we say that is $\delta-$starshaped with respect to a point $\xi_0 \in \Omega$, if denoting by N the outer unit normal to the boundary of $\tau_{\xi_0^{-1}}(\Omega)$, we have*

$$X.N \geq 0 \tag{5.12}$$

at every point of $\partial(\tau_{\xi_0^{-1}}(\Omega))$.

We observe that if we left-translate ξ_0 to the origin then $v(\xi) = u(\tau_{\xi_0^{-1}}\xi)$ is in $C\overline{\tau_{\xi_0^{-1}}(\Omega)}$ and satisfies the same equation as u. Therefore we may assume without loss of generality that the origin belongs to the domain Ω. By using the definition 3.1, we obtain as a consequence of theorem 2.4 the following non existence result for equation (P).

Theorem 5.6. *Let $\Omega \subset \mathbb{H}^n$ be a connected and bounded domain containing $0 = (0, 0, 0)$, and assume that Ω is $\delta-$starshaped with respect to this point. Then any positive solution $u \in C(\overline{\Omega})$ of equation (P) vanishes identically if*

$$-(Q - 2)ug(u) + 2QG(u) \leq 0. \tag{5.13}$$

Proof. The proof is similar to the one given by N.Garofallo and E.Lanconelli for solution of such example of semi linear equations on Heisenberg group domains, one can see [78]. The proof is based on the theory of the unique continuation property developed in [77]. Since the domain is δ-starshaped i.e $X.N \geq 0$ on the boundary of Ω, hence from theorem 2.4, we deduce that $\|\nabla_{\mathbb{H}^n} u\|^2$ is identically equal to 0 in $\partial\Omega \cap B_r(\bar{\xi})$ for some $\bar{\xi} \in \partial\Omega$ and $r > 0$. Therefore if we set $u \equiv 0$ in $(\mathbb{H}^n \setminus \bar{\Omega}) \cap B_r(\bar{\xi})$, we obtain a positive solution of

$$-\Delta_H u = Vu \quad \text{in} \quad B_r(\bar{\xi}) \tag{5.14}$$

where Δ_H is the sublaplacian of $\mathbb{H}^n$, $V \in L^\infty(B_r(\bar{\xi}))$, $V = \dfrac{g(u)}{u}$ when $u \neq 0$ and $V = 0$ when $u = 0$ in $B_r(\bar{\xi})$. In the appendix of [78] Corollary A.1, by using the method of the unique continuation property for the solution u of (16) the authors prove that $u \equiv 0$ in $B_r(\bar{\xi})$. We can reformulate the result of Corollary A.1 as follows, if we denote by D the maximal open set of $B_r(\bar{\xi})$ on which u vanishes then there exist a sphere S such its interior is entirely contained in D and there exist $\xi \in \partial N \cap S$. As u vanishes in one side of S, it follows that $\xi \in D$, and hence the maximal open set D of $B_r(\bar{\xi})$ on which u vanishes is the hole ball i.e $D = B_r(\bar{\xi})$. To complete the proof i.e to show that $u \equiv 0$ on Ω, we use the fact that Ω is connected. □

Next we will focus on the special case where $g(u) = \lambda u + u^{p^*}$, $p^* = 1 + \dfrac{2}{n}$ is the critical exponent for the compactness of the Sobolev inclusion $S^{k,p}(\Omega) \hookrightarrow L^s(\Omega)$, for $\dfrac{1}{s} = \dfrac{1}{p} - \dfrac{k}{2n+2}$, $1 < p < s < \infty$; here $S^{k,p}(\Omega)$ is a Folland Stein space [24], the CR counterpart of The Sobolev space $W^{1,2}(\Omega)$ for euclidean domains. Define $S_0^{k,p}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{S^{k,p}(\Omega)} = Sup_{\, l(\alpha)\leq k}\|Z^\alpha u\|_{L^p(\Omega)}$, $Z^\alpha = (Z_{\alpha_1}, ...Z_{\alpha_k})$, where $\alpha = (\alpha_1, ..., \alpha_k)$, each α_j is an integer $1 \leq \alpha_j \leq 2n$, $l(\alpha) = \alpha_1 + ... + \alpha_k$ and

$$Z_{\alpha_j} = \begin{cases} X_{\alpha_j} & \text{for } 1 \leq \alpha_j \leq n \\ Y_{\alpha_j} & \text{for } n+1 \leq \alpha_j \leq 2n. \end{cases}$$

More precisely, given $\lambda \in \mathbb{R}$ we would like to solve the problem

$$E_{p^*}(\lambda)\begin{cases} -\Delta_H u &= u^{1+\frac{2}{n}} + \lambda u \quad \text{in } \Omega \\ u &> 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \partial\Omega \end{cases}$$

We obtain in this case the following non existence result

Corollary 5.7. *Suppose Ω is a bounded domain in $\mathbb{H}^n$, which is $\delta-$starshaped with respect to the origin $0 = (0, 0, 0)$ and let $\lambda \leq 0$. Then any solution $u \in S_0^{1,2}(\Omega)$ of the boundary value problem $E_{p^*}(\lambda)$ vanishes identically.*

Proof. we will proceed by contradiction and suppose that there exist a nontrivial solution of $E_{p^*}(\lambda)$. A simple computation shows that

$$-(Q-2)ug(u) + 2QG(u) = 2\lambda\, u^2. \tag{5.15}$$

Therefore using the result of Theorem 3.2, one deduce that $\lambda > 0$. The result follows. □

Let us remark that one can obtain the above result for a strict-$\delta-$starshaped domain by a direct method, in fact two cases occur
-If $\lambda < 0$, from equality (17) and theorem 2.4, we deduce that there is no positive solutions of $E_{p^*}(\lambda)$.
-If $\lambda = 0$, we use the Green formula for $u, v \in C(\overline{\Omega})$

$$\int_\Omega -\Delta_H u\; v\; d\Psi_{\theta_0} = \int_\Omega \nabla_{\mathbb{H}^n} u\; \nabla_{\mathbb{H}^n} v\; d\Psi_{\theta_0} - \int_{\partial\Omega} v\; A\nabla u.N d\sigma \tag{5.16}$$

and set $v \equiv 1$ in (18), since $N = \dfrac{-\nabla u}{||\nabla u||}$, we obtain for a solution u of (P)

$$\int_\Omega -\Delta_H\; u\; d\Psi_{\theta_0} \;= \int_{\partial\Omega} \frac{||\nabla_{\mathbb{H}^n} u||^2}{||\nabla u||} d\sigma \tag{5.17}$$

Since Ω is strict-$\delta-$starshaped with respect to $0 \in \mathbb{H}^n$, we have $X.N(\xi) > 0$ for all $\xi \in \partial\Omega$. Thus from theorem 2.4, we deduce that $||\nabla_{\mathbb{H}^n} u||^2$ is identically equal to 0 on the boundary of Ω, therefore

$$\int_\Omega -\Delta_H u = 0. \tag{5.18}$$

Hence $\displaystyle\int_\Omega u^{1+\frac{2}{n}} = 0$, which means $u = 0$, since $u \geq 0$.

Remarks

1. The result of corollary 3.3 still hold true for supercritical value of the exponent p, i.e $p > p^*$, for any value of $\lambda < \lambda^* = \dfrac{n(p-1)-2}{p+1}$.

2. If the domain Ω is not $\delta-$starshaped then equation (E_p) can have solutions even if (15) holds. In fact, if we choose a pseudo annulus $\Omega = \{\xi = (x, y, t) \in \mathbb{H}^n / R_1 < x^2 + y^2 < R_2,\ |t| < T\}$ for fixed $R_1, R_2, T > 0$, then for every fixed $p > 1$ and $\lambda \geq 0$ the problem (E_p) has a positive solution $u \in S_0^{1,2}(\Omega) \cap C^\infty(\Omega)$, which is Hölder continuous up to the boundary one can see [78].

However we can approch problem $E_{p^*}(\lambda)$ by a direct method and attempt to obtain non-trivial solutions as relative minima of the functional

$$J_\lambda(u) = \frac{1}{2}\int_\Omega (\|\nabla_{\mathbb{H}^n} u\|^2 - \lambda\, u^2)\theta_0 \wedge d\theta_0^n, \tag{5.19}$$

on the unit sphere of $L^{2+\frac{2}{n}}(\Omega)$

$$\sum = \{u \in S_0^{1,2}(\Omega),\ \|u\|^{2+\frac{2}{n}}_{L^{2+\frac{2}{n}}} = 1\}. \tag{5.20}$$

Equivalently, one may seek to minimize the Sobolev quotient

$$S_\lambda(u) = \frac{\int_\Omega (\|\nabla_{\mathbb{H}^n} u\|^2 - \lambda\, u^2)\theta_0 \wedge d\theta_0^n}{\|u\|^{2+\frac{2}{n}}_{L^{2+\frac{2}{n}}}}, \quad u \neq 0. \tag{5.21}$$

Let us note that for $\lambda = 0$

$$S_0(\Omega) = \inf_{u \in S_0^{1,2}(\Omega),\ u\neq 0} S_\lambda(u) = \inf_{u \in S_0^{1,2}(\Omega),\ u\neq 0} \frac{\int_\Omega \|\nabla_{\mathbb{H}^n} u\|^2\, \theta_0 \wedge d\theta_0^n}{\|u\|^{2+\frac{2}{n}}_{L^{2+\frac{2}{n}}}}, \quad u \neq 0 \tag{5.22}$$

is related to the best constant for the Sobolev embedding $S_0^{1,2}(\Omega) \hookrightarrow L^{2+\frac{2}{n}}(\Omega)$.

5.4 Yamabe like problems

In the sequel we will consider the case where λ is a function. More precisely let h be a smooth function on $\mathbb{H}^n$, we are looking for solutions of the semilinear equation on a bounded domain Ω

$$E_{p^*}(h)\begin{cases} -\Delta_H u &= u^{1+\frac{2}{n}} + h\,u & \text{in } \Omega \\ u &> 0 & \text{in } \Omega \\ u &= 0 & \text{in } \partial\Omega \end{cases}$$

This problem arises naturally in CR geometry, in fact let $(M;\theta)$ be a CR manifold of dimension $2n+1$, $n \geq 1$. We ask the question on whether there exist a contact form $\widetilde{\theta}$ on M conformal to θ i.e $\widetilde{\theta} = u^{\frac{2}{n}}\theta, u > 0$ which has a constant Webster scalar curvature. If we denote by R_θ (respectively $R_{\widetilde{\theta}}$) the Webster scalar curvature of the contact form θ (respectively $\widetilde{\theta}$), we have the following relation

$$(2 + \frac{2}{n})\Delta_b\, u + R_\theta u = R_{\widetilde{\theta}}\, u^{1+\frac{2}{n}} \tag{5.23}$$

where Δ_b is the sublaplacian (the real part of the Kohn Spencer laplacian) of the manifold M. The existence of such a conformal contact form of constant Webster scalar curvature is equivalent to the existence of a positive solution of (5.23). This problem is known to be the Yamabe problem, one can see [24], [25], [75] and [76].

We have the following result.

Lemma 5.8. *If u is a solution of problem $E_{p^*}(h)$, then*

$$\int_\Omega -\Delta_H u\ (Xu)\ d\Psi_{\theta_0} = -\int_\Omega \Big((n+1)\ h + \ \frac{1}{2}\ X\ h\Big)\ u^2\ d\Psi_{\theta_0} - n\int_\Omega u^{2+\frac{2}{n}}\ d\Psi_{\theta_0}.$$

Proof. We multiply equation $E_{p^*}(h)$ by Xu and integrate by parts, we obtain

$$\int_\Omega -\Delta_H u\ (Xu) = \int_\Omega h\ u(Xu) + \int_\Omega u^{1+\frac{2}{n}}\ (Xu).$$

on one hand, we have

$$2\ (h\ u)(X\ u) = X(h\ u^2) -\ (Xh)\ u^2, \tag{5.24}$$

and a simple computation as done in Lemma 2.1 gives

$$\int_\Omega X(h\ u^2) = -(2n+2)\int_\Omega h\ u^2. \tag{5.25}$$

On the other hand, we have

$$\int_\Omega u^{1+\frac{2}{n}}\ (Xu) = -n\ \int_\Omega u^{2+\frac{2}{n}}. \tag{5.26}$$

By using (26), (27) and (28), we obtain the desired result. □

Following the method used in section2, we obtain the CR version of the "Poho$\widetilde{z}$aev identity" for the present case

Lemma 5.9. *Let $u \in C(\overline{\Omega})$ be a solution of the equation $E_{p^*}(h)$, then we have*

$$\int_{\partial\Omega} \|\nabla_{\mathbb{H}^n} u\|^2\ X.N d\sigma = -2\int_\Omega \Big(h + \frac{1}{2}(Xh)\Big) u^2\ d\Psi_{\theta_0}.$$

Proof. Using theorem 2.3 and (13), we obtain

$$\int_\Omega -\Delta_H u\ (Xu) = -\frac{1}{2}\int_{\partial\Omega} \|\nabla_{\mathbb{H}^n} u\|^2\ X.N d\sigma - n\int_\Omega \|\nabla_{\mathbb{H}^n} u\|^2\ . \tag{5.27}$$

By comparing the result of lemma 4.1 and (29), the proof of lemma 4.2 is completed. □

We are now ready to state a non existence result for equation $E_{p^*}(h)$.

Corollary 5.10. *Suppose Ω is a connected and bounded domain in $\mathbb{H}^n$ containing* 0. *Suppose that Ω is $\delta-$starshaped with respect to this point and let $h \in C^\infty(\mathbb{H}^n)$ satisfying*

$$h + \frac{1}{2}(Xh) \leq \ 0. \tag{5.28}$$

Then there is no positive solution $u \in S_0^{1,2}(\Omega)$ of equation $E_{p^*}(h)$, $u \neq 0$.

Proof. The proof is similar to the one given for theorem 3.2 with $V = u^{\frac{2}{n}} + h$, when $u \neq 0$ and $V = 0$ when $u = 0$ in $B_r(\bar{\xi})$. □

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