Uniform Equicontinuity for a family of Zero Order operators approaching the fractional Laplacian

Patricio Felmer, Erwin Topp

UNIFORM EQUICONTINUITY FOR A FAMILY OF ZERO ORDER OPERATORS APPROACHING THE FRACTIONAL LAPLACIAN.

PATRICIO FELMER AND ERWIN TOPP

ABSTRACT. In this paper we consider a smooth bounded domain $\Omega \subset \mathbb{R}^N$ and a parametric family of radially symmetric kernels $K_\epsilon : \mathbb{R}^N \to \mathbb{R}_+$ such that, for each $\epsilon \in (0,1)$, its L^1-norm is finite but it blows up as $\epsilon \to 0$. Our aim is to establish an ϵ independent modulus of continuity in Ω, for the solution u_ϵ of the homogeneous Dirichlet problem

$$\begin{cases} -\mathcal{I}_\epsilon[u] &= f \quad \text{in } \Omega. \\ \qquad u &= 0 \quad \text{in } \Omega^c, \end{cases}$$

where $f \in C(\bar{\Omega})$ and the operator $\mathcal{I}_\epsilon$ has the form

$$\mathcal{I}_\epsilon[u](x) = \frac{1}{2} \int\limits_{\mathbb{R}^N} [u(x+z) + u(x-z) - 2u(x)] K_\epsilon(z) dz$$

and it approaches the fractional Laplacian as $\epsilon \to 0$. The modulus of continuity is obtained combining the comparison principle with the translation invariance of $\mathcal{I}_\epsilon$, constructing suitable barriers that allow to manage the discontinuities that the solution u_ϵ may have on $\partial\Omega$. Extensions of this result to fully non-linear elliptic and parabolic operators are also discussed.

1. INTRODUCTION.

Let $\Omega \subset \mathbb{R}^N$ be a bounded open domain with C^2 boundary, $f \in C(\bar{\Omega})$ and $\epsilon \in (0,1)$. In this paper we are concerned on study of the Dirichlet problem

$$-\mathcal{I}_\epsilon[u] = f \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{in } \Omega^c, \tag{1.2}$$

where $\mathcal{I}_\epsilon$ is a nonlocal operator approaching the fractional Laplacian as ϵ approaches 0. We focus our attention on $\mathcal{I}_\epsilon$ with the form

$$\mathcal{I}_\epsilon[u](x) := \int_{\mathbb{R}^N} [u(x+z) - u(x)] K_\epsilon(z) dz, \tag{1.3}$$

where, for $\sigma \in (0,1)$ fixed, K_ϵ is defined as

$$K_\epsilon(z) := \frac{1}{\epsilon^{N+2\sigma} + |z|^{N+2\sigma}} = \epsilon^{-(N+2\sigma)} K_1(z/\epsilon).$$

Notice that for each $\epsilon \in (0,1)$, K_ϵ is integrable in $\mathbb{R}^N$ with L^1 norm equal to $C\epsilon^{-2\sigma}$, where $C > 0$ is a constant depending only on N and σ. We point

Date: May 14, 2014.

out that operators with kernel in L^1, like $\mathcal{I}_\epsilon$, are known in the literature as *zero order nonlocal operators*.

Operator $\mathcal{I}_\epsilon$ is a particular case of a broad class of nonlocal elliptic operators. In fact, given a positive measure μ satisfying the *Lévy condition*

$$\int_{\mathbb{R}^N} \min\{1, |z|^2\}\mu(dz) < \infty,$$

and, for each $x \in \mathbb{R}^N$ and $u : \mathbb{R}^N \to \mathbb{R}$ bounded and sufficiently smooth at x, the operator $\mathcal{I}_\mu[u](x)$ defined as

$$\mathcal{I}_\mu[u](x) = \int_{\mathbb{R}^N} [u(x+z) - u(x) - \mathbf{1}_{B_1(0)}(z)\langle Du(x), z\rangle]\mu(dz), \tag{1.4}$$

has been a subject of study in a huge variety of contexts such as potential theory ([28]), probability ([13, 31]) and analysis ([32, 33, 3, 14, 15]). An interesting point of view of our problem comes from probability, since (1.4) represents the infinitesimal generator of a *jump Lévy process*, see Sato [31]. In our setting, the finiteness of the measure is associated with the so-called *Compound Poisson Process*. Dirichlet problems with the form of (1.1)-(1.2) arise in the context of exit time problems with trajectories driven by the jump Lévy process defined by $K_\epsilon(z)dz$, and the solution u_ϵ represents the expected value of the associated cost functional, see [29].

We may start our discussion with a natural notion of solution to our problem (1.1)-(1.2): we say that a bounded function $u : \mathbb{R}^N \to \mathbb{R}$, continuous in Ω, is a solution of (1.1)-(1.2) if it satisfies (1.1) pointwise in Ω and $u = 0$ on Ω^c. As we see in Section §2, this problem has a unique solution, more interestingly, through an example we will see that such a solution may not be continuous in $\mathbb{R}^N$, since a discontinuity may appear on the boundary of Ω. See Remark 2.4.

This situation is in great contrast with the limit case $\epsilon = 0$, where the kernel becomes $K(z)dz = |z|^{-(N+2\sigma)}dz$ and the associated nonlocal operator is the *fractional Laplacian of order* 2σ, denoted by $-(-\Delta)^\sigma$, see [23]. In this case, the corresponding Dirichlet problem becomes

$$\begin{cases} C_{N,\sigma}(-\Delta)^\sigma v &= f \quad \text{in } \Omega, \\ \qquad\qquad\quad v &= 0 \quad \text{in } \Omega^c, \end{cases} \tag{1.5}$$

where $C_{N,\sigma} > 0$ is a normalizing constant. In the context of the viscosity theory for nonlocal equations (see [3, 32, 33]), Barles, Chasseigne and Imbert [4] addressed a large variety of nonlocal elliptic problems including (1.5). In that paper, the authors proved the existence and uniqueness of a viscosity solution $v \in C(\bar{\Omega})$ of (1.5) satisfying $v = 0$ on $\partial\Omega$ that is, consequently, continuous when we regard it as a function on $\mathbb{R}^N$. This result is accomplished by the use of a nonlocal version of the notion of viscosity solution with *generalized boundary conditions*, see [21, 2, 8] for an introduction of this notion in the context of second-order equations.

Additionally, fractional problems like (1.5) enjoy a *regularizing effect* as in the classical second-order case. Roughly speaking, for a right-hand side which is merely bounded, the solution v of (1.5) is locally Hölder continuous in Ω, see [34]. In fact, we should mention here that interior Hölder regularity for more general fractional problems (for which (1.5) is a particular case) has been addressed by many authors, see for instance [4, 5, 9, 14, 15, 16, 34] and the classical book of Landkof [28], for a non-exhaustive list of references. The interior Hölder regularity is accomplished by well established elliptic techniques as the Harnack's inequality ([14, 10]) and the Ishii-Lions method ([4, 27]). In both cases, the nonintegrability of the kernel plays a key role. Hölder regularity for problems like (1.5) can be extended up to the boundary, as it is proved by Ros-Oton and Serra in [30], where a boundary Harnack's inequality is the key ingredient (see also [12]). Naturally, as a byproduct of these regularity results, compactness properties are available for certain families of solutions of fractional equations. For instance, the family $\{v_\eta\}$ of functions solving

$$\begin{cases} C_{N,\sigma}(-\Delta)^\sigma v_\eta &= f_\eta \qquad \text{in } \Omega \\ \qquad\qquad u_\eta &= 0, \qquad \text{in } \Omega^c, \end{cases}$$

satisfies compactness properties when $\{f_\eta\}$ is uniformly bounded in $L^\infty(\bar{\Omega})$.

For zero order problems, regularizing effects as arising in fractional problems are no longer available (see [18]). In fact, the finiteness of the kernel of zero order operators turns into degenerate ellipticity for which Ishii-Lions method cannot be applied. Thus, "regularity results" for zero order problems like (1.1)-(1.2) are circumscribed to the heritage of the modulus of continuity of the right-hand side f to the solution u_ϵ as it can be seen in [17]. However, the modulus of continuity found in [17] depends strongly on the size of the L^1 norm of K_ϵ, which explodes as $\epsilon \to 0$. A similar lack of stability as $\epsilon \to 0$ can be observed in the Harnack-type inequality results for nonlocal problems found by Coville in [20]. Hence, none of the mentioned tools are adequate for getting compactness for the family of solutions $\{u_\epsilon\}$ of problem (1.1)-(1.2), which is a paradoxical situation since, in the limit case, the solutions actually get higher regularity and stronger compactness control on its behavior.

In view of the discussion given above, a natural mathematical question is if there exists a uniform modulus of continuity in Ω, for the family of solutions $\{u_\epsilon\}$ to (1.1)-(1.2), and consequently compactness properties for it. In this direction, the main result of this paper is the following

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ a bounded domain with C^2 boundary and $f \in C(\bar{\Omega})$. For $\epsilon \in (0,1)$, let u_ϵ be a solution to problem (1.1)-(1.2). Then, there is a modulus of continuity m depending only on f, such that*

$$|u_\epsilon(x) - u_\epsilon(y)| \leq m(|x-y|), \quad \text{for } x, y \in \Omega.$$

The proof of this theorem is obtained combining the translation invariance of $\mathcal{I}_\epsilon$ and comparison principle, constructing suitable barriers to manage the discontinuities that u_ϵ may have on $\partial\Omega$ and to understand how they evolve as ϵ approaches zero, see Proposition 3.2.

As a consequence of Theorem 1.1 we have the following corollary, that actually was our original motivation to study the problem.

Corollary 1.2. *Let u_ϵ be the solution to equation (1.1), with f and Ω as in Theorem 4.1, and let u be the solution of the equation (1.5), then $u_\epsilon \to u$ in $L^\infty(\bar{\Omega})$ as $\epsilon \to 0$.*

We mention here that the application of the *half-relaxed limits method* introduced by Barles and Perthame in [7] (see also [6, 11, 3]) allows to obtain in a very direct way locally uniform convergence in Ω in the above corollary. At this point we emphasize on the main contribution of this paper, which is the analysis of the boundary behavior of the family $\{u_\epsilon\}$ of solutions to (1.1)-(1.2) coming from Theorem 1.1 and the subsequent *global* uniform convergence to the solution of (1.5).

There are many possible extensions of Theorem 1.1, for example, it can be readily extended to problems with the form

$$\begin{cases} -\mathcal{I}_\epsilon[u] &= f_\epsilon \qquad \text{in } \Omega \\ \qquad u &= 0 \qquad \text{in } \Omega^c, \end{cases}$$

with $\{f_\epsilon\} \subset C(\bar{\Omega})$ having a common modulus of continuity independent of $\epsilon \in (0,1)$. It can also be extended to fully nonlinear operators and to parabolic equations, as we discuss in Section §6. We could also consider different families of approximating zero order operators, but we do not pursue this direction. There are many other interesting lines of research that arises from this work. From the discussion given before Theorem 1.1, questions arises with respect to Harnack type inequalities and its relation with regularity and compactness properties of solutions, when $\epsilon \to 0$. Regarding operators $\mathcal{I}_\mu$, where μ might be singular with respect to the Lebesgue measure, an interesting question that arises is if the main results of this article can be extended to this case.

The paper is organized as follows: In Section §2 we establish the notion of pointwise solution and the comparison principle. Important estimates for the discontinuity of the solution at the boundary are given in Section §3, and the boundary equicontinuity result is presented in Section §4. The interior modulus of continuity is easily derived from the boundary equicontinuity, and therefore the proof of Theorem 1.1 is given in Section §5. Further related results are discussed in Section §6.

1.1. Notation.

For $x \in \mathbb{R}^N$ and $r > 0$, we denote $B_r(x)$ the ball centered at x with radius r and simply B_r if $x = 0$. For a set $U \subset \mathbb{R}^N$, we denote by $d_U(x)$ the signed distance to the boundary, this is $d_U(x) = \mathrm{dist}(x, \partial U)$, with $d_U(x) \leq 0$ if $x \in U^c$. Since many arguments in this paper concerns the set Ω, we write $d_\Omega = d$. We also define

$$\Omega_r = \{x \in \Omega \ : \ d(x) < r\}$$

Concerning the regularity of the boundary of Ω, we assume it is at least C^2, so the distance function d is a C^2 function in a neighborhood of $\partial\Omega$.

More precisely, there exists $\delta_0 > 0$ such that $x \mapsto d(x)$ is of class C^2 for $-\delta_0 < d(x) < \delta_0$.

In our estimates we will denote by c_i with $i = 1, 2, ...$ positive constants appearing in our proofs, depending only on N, σ and Ω. When necessary we will make explicit the dependence on the parameters. The index will be reinitiated in each proof.

2. Notion of Solution and Comparison Principle.

In the introduction we defined a notion of solution to problem (1.1)-(1.2), which is very natural for zero order operators and allows us to understand the main features of the mathematical problem that we have at hand. However, this notion is not suitable for a neat statement of the comparison principle and it is not adequate to understand the limit as $\epsilon \to 0$. For this reason, from now on, we adopt another notion of solution which is more adequate, that is the notion of viscosity solution with generalized boundary condition defined by Barles, Chasseigne and Imbert in [4].

We remark that results provided in this section are adequate for problems slightly more general than our problem (1.1)-(1.2). We will consider $J \in L^1(\mathbb{R}^N)$ a nonnegative function, and we define the nonlocal operator associated to J as

$$\mathcal{I}_J[u](x) = \int_{\mathbb{R}^N} [u(x+z) - u(x)] J(z) dz, \tag{2.1}$$

for $u \in L^\infty(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$, and a Dirichlet problem of the form

$$-\mathcal{I}_J[u] = f \qquad \text{in } \Omega, \tag{2.2}$$

$$u = 0 \qquad \text{on } \Omega^c, \tag{2.3}$$

with $f \in C(\bar{\Omega})$. Since we are interested in a Dirichlet problem for which the exterior data plays a role, we assume J and Ω satisfy the condition

$$\inf_{x \in \bar{\Omega}} \int_{\Omega^c - x} J(z) dz \geq \nu_0 > 0. \tag{2.4}$$

Notice that problem (1.1)-(1.2) is a particular case of (2.2)-(2.3).

In this situation, a bounded function $u : \mathbb{R}^N \to \mathbb{R}$, continuous in $\bar{\Omega}$ is a viscosity solution with generalized boundary condition to problem (2.2)-(2.3) if and only if it satisfies

$$-\mathcal{I}_J[u] \;= f \qquad \text{on } \bar{\Omega}, \tag{2.5}$$

$$u \;= 0 \qquad \text{in } \bar{\Omega}^c. \tag{2.6}$$

The sufficient condition is direct from the definition and the necessary condition follows from the lemma:

Lemma 2.1. *Let $f \in C(\bar{\Omega})$ and let $u : \mathbb{R}^N \to \mathbb{R}$ be a function satisfying*

$$-\mathcal{I}_J[u](x) \leq f(x) \quad \textit{for all } x \in \Omega, \tag{2.7}$$

where the above inequality is understood pointwise. Let $x_0 \in \partial\Omega$ and assume there exists a sequence $\{x_k\} \subset \Omega$ such that

$$x_k \to x_0, \quad u(x_k) \to u(x_0) \tag{2.8}$$

and

$$\limsup_{k \to +\infty} u(x_k + z) \le u(x_0 + z), \quad a.e. \tag{2.9}$$

Then, u satisfies (2.7) at x_0.

Here and in what follows the considered measure is the Lebesgue measure.

Proof. Consider $\{x_k\} \subset \Omega$ as in (2.8). Then, we can write

$$\int_{\mathbb{R}^N} u(x_k + z)J(z)dz - u(x_k) \int_{\mathbb{R}^N} J(z)dz \ge -f(x_k).$$

Hence, taking limsup in both sides of the last inequality, by (2.8) and the continuity of f, we arrive to

$$\int_{\mathbb{R}^N} \limsup_{k \to \infty} u(x_k + z)J(z)dz - u(x_0) \int_{\mathbb{R}^N} J(z)dz \ge -f(x_0),$$

where the exchange of the integral and the limit is justified by Fatou's Lemma. Then, using (2.9), we conclude the result. $\square$

We continue with our analysis with an existence result for (2.2)-(2.3).

Proposition 2.2. *Let $f \in C(\bar{\Omega})$. Then, there exists a unique bounded function $u : \mathbb{R}^N \to \mathbb{R}$, continuous in $\bar{\Omega}$, which is a viscosity solution with generalized boundary condition to problem (2.2)-(2.3).*

Proof. According with our discussion above, we need to find a solution to (2.5)-(2.6). Consider the map $T_a : C(\bar{\Omega}) \to C(\bar{\Omega})$ defined as

$$T_a(u)(x) = u(x) - a\Big(||J||_{L^1(\mathbb{R}^N)}u(x) - \int_{\Omega - x} u(x+z)J(z)dz - f(x)\Big).$$

We observe that $u \in C(\bar{\Omega})$ is a fixed points of T_a if and only if u is a solution to problem (2.5)-(2.6). Therefore, the aim is to prove that for certain $a > 0$ small enough, the map T_a is a contraction in $C(\bar{\Omega})$. By (2.4), there exists $\varrho_0 >$ such that

$$||J||_{L^1(\mathbb{R}^N)} - ||J||_{L^1(\Omega - x)} \ge \varrho_0, \quad \text{for each } x \in \bar{\Omega}.$$

Let $0 < a < \min\{\varrho_0^{-1}, ||J||^{-1}_{L^1(\mathbb{R}^N)}\}$ and consider $u_1, u_2 \in C(\bar{\Omega})$. Then, for all $x \in \bar{\Omega}$ we have

$$\begin{aligned} T_a(u_1)(x) - T_a(u_2)(x) &\le \Big(1 - a||J||_{L^1(\mathbb{R}^N)} + a\int_{\Omega - x} J(z)dz\Big)||u_1 - u_2||_\infty \\ &\le \Big(1 - a\varrho_0\Big)||u_1 - u_2||_\infty, \end{aligned}$$

concluding that

$$||T_a(u_1) - T_a(u_2)||_\infty \le (1 - a\varrho_0)||u_1 - u_2||_\infty,$$

that is, T_a is a contraction in $C(\bar{\Omega})$. From here existence and uniqueness follow. $\square$

Remark 2.3. *We observe that $u : \mathbb{R}^N \to \mathbb{R}$, a viscosity solution with generalized boundary condition to problem* (2.2)-(2.3)*, may be redefined on the boundary $\partial\Omega$ as $u = 0$, to obtain a solution to* (2.2)-(2.3) *in the sense defined in the introduction.*

Remark 2.4. *Let u be a solution of* (2.2)-(2.3) *in the sense defined in the introduction, with $f \ge \varrho_0 > 0$. Our purpose is to show that u has a discontinuity on the boundary of Ω. Let us assume, for contradiction, that $u : \mathbb{R}^N \to \mathbb{R}$ is a continuous function.*

Then $u \ge 0$ in Ω, otherwise there exists $x_0 \in \Omega$ such that $u(x_0) = \min_{\bar{\Omega}}\{u\} < 0$ and evaluating the equation at x_0 we arrive to

$$u(x_0)\int_{\Omega^c - x_0} J(z)dz \ge -\mathcal{I}_J[u](x_0) = f(x_0),$$

which is a contradiction to (2.4)*. Then, from the equation, we have for each $x \in \Omega$ the inequality*

$$-\mathcal{I}_J[u](x) = f(x) > \varrho_0.$$

Since u and f are continuous and $u = 0$ on $\partial\Omega$ and using that $u \ge 0$ in Ω, we obtain that, for each $x \in \partial\Omega$

$$0 \ge -\int_{\Omega - x}(u(x+z) - u(x))J(z)dz = -\mathcal{I}_J[u](x) = f(x) > \varrho_0,$$

which is a contradiction. Thus, $u > 0$ on $\partial\Omega$ which implies that u is discontinuous on $\partial\Omega$.

In what follows we prove that a solution to (2.2)-(2.3), in the sense defined in the introduction, can be extended continuously to $\bar{\Omega}$.

Proposition 2.5. *Let $f \in C(\bar{\Omega})$. Let $v : \mathbb{R}^N \to \mathbb{R}$ in $L^\infty(\mathbb{R}^N) \cap C(\Omega)$ be a solution to* (2.2)-(2.3)*, and $u : \mathbb{R}^N \to \mathbb{R}$ in $C(\bar{\Omega})$ be the viscosity solution to* (2.2)-(2.3) *given by Proposition 2.2. Then, $u = v$ in Ω.*

Proof: By contradiction, assume the existence of a point in Ω where u is different from v. Defining $w = u - v$, we will assume that

$$M := \sup_{\Omega}\{w\} > 0, \tag{2.10}$$

since the case $\inf_\Omega\{w\} < 0$ follows the same lines. Moreover, we assume that the supremum defining M is not attained, since this is the most difficult scenario. Let $\eta > 0$ and let $x_\eta \in \Omega \setminus \Omega_\eta$ such that

$$w(x_\eta) = \max_{\Omega\setminus\Omega_\eta}\{w\},$$

where Ω_η was defined at the end of the introduction. We clearly have $w(x_\eta) \to M$ as $\eta \to 0$ and since we assume M is not attained, then $x_\eta \to \partial\Omega$ as $\eta \to 0$. Now, using the equations for u and v at $x_\eta \in \Omega$, we can write

$$-\int_{\Omega - x_\eta} [w(x_\eta + z) - w(x_\eta)]J(z)dz + w(x_\eta)\int_{\Omega^c - x_\eta} J(z)dz \leq 0,$$

and by (2.4) and the fact that $w(x_\eta) \to M$ as $\eta \to 0$, we have

$$-\int_{\Omega - x_\eta} [w(x_\eta + z) - w(x_\eta)]J(z)dz + \nu_0 M - o_\eta(1) \leq 0, \tag{2.11}$$

where $o_\eta(1) \to 0$ as $\eta \to 0$. But writing

$$\begin{aligned}\int_{\Omega - x_\eta} [w(x_\eta + z) - w(x_\eta)]J(z)dz &= \int_{\Omega \setminus \Omega_\eta - x_\eta} [w(x_\eta + z) - w(x_\eta)]J(z)dz \\ &\quad + \int_{\Omega_\eta - x_\eta} [w(x_\eta + z) - w(x_\eta)]J(z)dz,\end{aligned}$$

by the boundedness of w and the integrability of J, the second integral term in the right-hand side of the last equality is $o_\eta(1)$, meanwhile, using the definition of x_η we have the first integral is nonpositive. Thus, we conclude

$$\int_{\Omega - x_\eta} [w(x_\eta + z) - w(x_\eta)]J(z)dz \leq o_\eta(1),$$

and replacing this into (2.11), we arrive to

$$\nu_0 M - o_\eta(1) \leq 0.$$

By making $\eta \to 0$, we see that this contradicts (2.10), since $\nu_0 > 0$. □

As a consequence of the last proposition, we have the following

Corollary 2.6. *Let $f \in C(\bar{\Omega})$. Then, there exists a unique solution $v \in L^\infty(\mathbb{R}^N) \cap C(\Omega)$ to problem* (2.2)-(2.3) *in the sense defined in the introduction. Moreover, v is uniformly continuous in Ω and its unique continuous extension to $\bar{\Omega}$ coincides with the unique viscosity solution to* (2.2)-(2.3).

The main tool in this paper is the comparison principle, and here the so-called *strong* comparison principle is the appropriate version to deal with discontinuities at the boundary.

Proposition 2.7. (Comparison Principle) *Assume $f \in L^\infty(\bar{\Omega})$. Let $u, v \in \mathbb{R}^N \to \mathbb{R}$ be bounded, upper and lower semicontinuous functions on $\bar{\Omega}$, respectively. Assume u and v satisfy*

$$-\mathcal{I}_J[u] \leq f \quad \textit{and} \quad -\mathcal{I}_J[v] \geq f, \quad \textit{on } \bar{\Omega}. \tag{2.12}$$

If $u \leq v$ in $\bar{\Omega}^c$, then $u \leq v$ in $\bar{\Omega}$.

Proof: Assume by contradiction that there exists $x_0 \in \bar{\Omega}$ such that

$$(u - v)(x_0) = \max_{x \in \bar{\Omega}}\{u - v\} > 0.$$

Evaluating inequalities in (2.12) at x_0 and substracting them, denoting $w = u - v$, we arrive to

$$-\int_{\Omega - x_0} [w(x_0 + z) - w(x_0)]J(z)dz - \int_{\Omega^c - x_0} [w(x_0 + z) - w(x_0)]J(z)dz \leq 0,$$

and therefore, using that x_0 is a maximum point for w in Ω and that $w \leq 0$ in Ω^c, we can write

$$w(x_0) \int_{\Omega^c - x_0} J(z)dz \leq 0,$$

and using (2.4) we arrive to a contradiction with the fact that $w(x_0) > 0$. $\square$

As a first consequence of this comparison principle, we obtain an a priori $L^\infty(\bar{\Omega})$ estimate for the solutions u_ϵ of (1.1)-(1.2), independent of ϵ.

Proposition 2.8. *Let $\epsilon \in (0, 1)$, $f \in C(\bar{\Omega})$ and u_ϵ be the viscosity solution of (1.1)-(1.2). Then, there exists a constant $C > 0$ such that*

$$||u_\epsilon||_{L^\infty(\bar{\Omega})} \leq C||f||_\infty$$

and this constant depends only on Ω, N and σ, but not on ϵ, for $\epsilon \in (0, 1)$.

Proof. Consider the bounded function $\chi(x) = \mathbf{1}_{\bar{\Omega}}(x)$. We clearly have that $\chi \in C(\bar{\Omega})$ and $\chi = 0$ in $\bar{\Omega}^c$. Denote $R = diam(\Omega) > 0$ and use the definition of the operator $\mathcal{I}_\epsilon$ to see that for each $x \in \bar{\Omega}$ we have

$$-\mathcal{I}_\epsilon[\chi](x) = \int_{\Omega^c - x} K_\epsilon(z)dz \geq \int_{B^c_{R+1}} \frac{dz}{2|z|^{N+2\sigma}} = \frac{\mathrm{Vol}(B_1)(R+1)^{-2\sigma}}{2\sigma}.$$

Hence, denoting $C = (2\sigma)^{-1}\mathrm{Vol}(B_1)(R + 1)^{-2\sigma}$ and $\tilde{\chi} = C^{-1}||f||_\infty \chi$, we may use the comparison principle to conclude $u_\epsilon \leq C||f||_\infty$ in $\bar{\Omega}$. A lower bound can be found in a similar way, concluding the result. $\square$

3. Estimates of the Boundary Discontinuity.

The aim of this section is to estimate the discontinuity jump on $\partial\Omega$ of the solution u_ϵ of (1.1)-(1.2). For this purpose, a flattening procedure on the boundary is required.

Recall that $\delta_0 > 0$ is such that the distance function to $\partial\Omega$ is smooth in Ω_{δ_0}, and for $x \in \Omega_{\delta_0}$ we denote $\hat{x}$ the unique point on $\partial\Omega$ such that $d_\Omega(x) = |x - \hat{x}|$. We can fix δ_0 small in order to have the existence of three constants $R_0, r_0, r_0' > 0$ depending only on the regularity of the boundary, satisfying the following properties:

(i) For each $x \in \Omega_{\delta_0}$, there exists $\mathcal{N}_x \subset \partial(\Omega - x)$, a $\partial(\Omega - x)$-neighborhood of $\hat{x} - x$, which is the graph of a C^2 function $\varphi_x : B_{R_0} \subset \mathbb{R}^{N-1} \to \mathbb{R}^N$, that is,

$$(\xi', \varphi_x(\xi')) \in \mathcal{N}_x, \quad \text{for all } \xi' \in B_{R_0}.$$

(ii) If we define the function Φ_x as

$$\Phi_x(\xi', s) = (\xi', \varphi(\xi')) + (d(x) + s)\nu_{\xi'}, \quad (\xi', s) \in B_{R_0} \times (-R_0, R_0), \tag{3.1}$$

where $\nu_{\xi'}$ is the unit inward normal to $\partial(\Omega - x)$ at $(\xi', \varphi_x(\xi'))$ and denoting $\mathcal{R}_x = \Phi_x(B_{R_0} \times (-R_0, R_0))$, then $\Phi_x : B_{R_0} \times (-R_0, R_0) \to \mathcal{R}_x$ is a C^1-diffeomorphism. Notice that $\Phi_x(0', 0)$ is the origin and therefore $\mathcal{R}_x$ is an $\mathbb{R}^N$-neighborhood of the origin.

(iii) The constant $r_0 > 0$ is such that $B_{r_0} \subset \mathcal{R}_x$ for all $x \in \Omega_{\delta_0}$.

(iv) The constant $r_0' > 0$ is such that $\Phi_x(B_{r_0'} \times (-r_0', r_0')) \subset B_{r_0}$.

We may assume $0 < r_0' \leq r_0 \leq \delta_0$. In addition, by the smoothness of the boundary there exists a constant $C_\Omega > 1$ such that

$$C_\Omega^{-1} K_\epsilon(\xi) \leq \tilde{K}_\epsilon(\xi) \leq C_\Omega K_\epsilon(\xi), \quad \xi \in \mathbb{R}^N, \tag{3.2}$$

where $\tilde{K}_\epsilon(\xi) = |\mathrm{Det}(D\Phi_x(\xi))| K_\epsilon(\Phi_x(\xi))|$.

The following Lemma is the key technical result of this paper

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain and $\epsilon \in (0,1)$. For $\beta \in (0,1)$, consider the function*

$$\psi(x) = \psi_\beta(x) := (\epsilon + d(x))^\beta \mathbf{1}_{\bar{\Omega}}(x), \tag{3.3}$$

where $d = d_\Omega$ is the distance function to $\partial\Omega$. Then, there exists $\bar{\delta} \in (0, \delta_0)$, $\beta_0 \in (0, \min\{1, 2\sigma\})$ and a constant $c^* > 0$, depending only on Ω, N and σ, such that, for all $\beta \leq \beta_0$ we have

$$-\mathcal{I}_\epsilon[\psi](x) \geq c^*(\epsilon + d(x))^{\beta - 2\sigma}, \quad \textit{for all } x \in \bar{\Omega}_{\bar{\delta}}, \ \epsilon \in (0, \bar{\delta}). \tag{3.4}$$

Proof: We start considering $\bar{\delta} < r_0'$ and $x \in \Omega_{\bar{\delta}}$. We split the integral

$$\mathcal{I}_\epsilon[\psi](x) = I_0(x) + I_1(x) + I_2(x) + I_3(x),$$

where

$$\begin{aligned}
I_0(x) &:= \int_{B_{r_0}^c} [\psi(x+z) - \psi(x)] K_\epsilon(z) dz, \\
I_1(x) &:= \int_{B_{d(x)/2}} [\psi(x+z) - \psi(x)] K_\epsilon(z) dz, \\
I_2(x) &:= -(\epsilon + d(x))^{\beta - 2\sigma} \int_{(\Omega^c - x) \cap B_{r_0}} K_\epsilon(z) dz \quad \text{and} \\
I_3(x) &:= \int_{(\Omega - x) \cap B_{r_0} \setminus B_{d(x)/2}} [(\epsilon + d(x+z))^\beta - (\epsilon + d(x))^\beta] K_\epsilon(z) dz.
\end{aligned}$$

In what follows we estimate each $I_i(x), i = 0,1,2,3$. Since ψ is bounded in $\mathbb{R}^N$ independent of ϵ, β when $\epsilon, \beta \in (0,1)$, we have

$$I_0(x) \leq c_1 r_0^{-2\sigma}, \tag{3.5}$$

where $c_1 > 0$ depends only on Ω and N. For $I_1(x)$, by the symmetry of K_ϵ we have

$$I_1(x) = \frac{1}{2} \int_{B_{d(x)/2}} [\psi(x+z) + \psi(x-z) - 2\psi(x)] K_\epsilon(z) dz.$$

Then we consider the function $\theta(z) = \psi(x+z) + \psi(x-z) - 2\psi(x)$, which is smooth in $\bar{B}_{d(x)/2}$ and therefore, we can write by Taylor expansion

$$\begin{aligned}\theta(z) =& \frac{\beta}{2}\Big[(\epsilon + d(x+\tilde{z}))^{\beta-1}\langle D^2 d(x+\tilde{z})z, z\rangle \\ &+ (\epsilon + d(x-\bar{z}))^{\beta-1}\langle D^2 d(x-\bar{z})z, z\rangle \\ &+ (\beta-1)(\epsilon + d(x+\tilde{z}))^{\beta-2}|\langle Dd(x+\tilde{z}), z\rangle|^2 \\ &+ (\beta-1)(\epsilon + d(x-\bar{z}))^{\beta-2}|\langle Dd(x-\bar{z}), z\rangle|^2\Big],\end{aligned}$$

where $\tilde{z}, \bar{z} \in B_{d(x)/2}$. With this, since we assume $\beta < 1$, by the smoothness of the distance function d inherited by the smoothness of $\partial\Omega$ we have

$$\theta(z) \leq c_2(\epsilon + d(x))^{\beta-1}|z|^2, \quad \text{for all } z \in B_{d(x)/2},$$

where $c_2 = C_\Omega \beta > 0$ depends on the domain, but not on ϵ or $d(x)$. From this, we get

$$I_1(x) \leq c_2(\epsilon + d(x))^{\beta-1} \int_{B_{d(x)/2}} |z|^2 K_\epsilon(z) dz,$$

and since $K_\epsilon(z) \leq K_0(z)$, we conclude that, for a constant $c_3 > 0$, we have

$$I_1(x) \leq c_3 \beta (\epsilon + d(x))^{\beta - 2\sigma + 1}. \tag{3.6}$$

Now we address the estimates of $I_2(x)$ and $I_3(x)$. For $I_2(x)$, recalling the change of variables Φ_x, we have

$$\Phi_x(B_{r_0'} \times (-r_0', -d(x))) \subset (\Omega^c - x) \cap B_{r_0}.$$

With this, using the change of variables Φ_x and applying (3.2), we have

$$I_2(x) \leq -C_\Omega^{-1}(\epsilon + d(x))^\beta \int_{B_{r_0'} \times (-r_0', -d(x))} K_\epsilon(\xi', s) d\xi' ds.$$

But there exists a constant $c_4 > 0$, depending only on N and σ, such that

$$\epsilon^{N+2\sigma} + |s|^{N+2\sigma} \leq c_4(\epsilon^{1+2\sigma} + |s|^{1+2\sigma})^{(N+2\sigma)/(1+2\sigma)},$$

and with this, defining $\rho(\epsilon, s) = (\epsilon^{1+2\sigma} + |s|^{1+2\sigma})^{1/(1+2\sigma)}$, we can write

$$\begin{aligned} I_2(x) \leq & \; -c_5(\epsilon + d(x))^\beta \int_{-r_0'}^{-d(x)} \frac{ds}{\rho(\epsilon, s)^{N+2\sigma}} \int_{B_{r_0'}} \frac{d\xi'}{1 + |\xi'/\rho(\epsilon, s)|^{N+2\sigma}} \\ = & \; -c_5(\epsilon + d(x))^\beta \int_{-r_0'}^{-d(x)} \frac{ds}{\rho(\epsilon, s)^{1+2\sigma}} \int_{B_{r_0'}/\rho(\epsilon, s)} \frac{dy}{1 + |y|^{N+2\sigma}} \\ \leq & \; -c_6(\epsilon + d(x))^\beta \int_{-r_0'}^{-d(x)} \frac{ds}{\epsilon^{1+2\sigma} + |s|^{1+2\sigma}}. \end{aligned}$$

Finally, making the change $t = -s/(\epsilon + d(x))$, we conclude

$$I_2(x) \le -c_6(\epsilon + d(x))^{\beta-2\sigma} \int\limits_{1-\tau}^{r_0'/(\epsilon+d(x))} \frac{dt}{\tau^{1+2\sigma} + |t|^{1+2\sigma}}, \tag{3.7}$$

where $\tau = \epsilon/(\epsilon + d(x)) \in (0,1)$. At this point, taking $\bar{\delta}$ small in order to have $\epsilon + d(x) < 2/r_0'$, we find that the interval

$$(1 - \tau, r_0'/(\epsilon + d(x)))$$

has at least lenght 1. Hence, we conclude the existence of $c_7 > 0$, depending only on Ω, N and σ, such that

$$I_2(x) \le -c_7(\epsilon + d(x))^{\beta-2\sigma}. \tag{3.8}$$

It remains to estimate $I_3(x)$. Defining $D_+(x) = \{z : d(x+z) \ge d(x)\}$, we clearly have

$$I_3(x) \le \int\limits_{(\Omega-x)\cap D_+(x)\cap B_{r_0}\setminus B_{d(x)/2}} [(\epsilon + d(x+z))^\beta - (\epsilon + d(x))^\beta] K_\epsilon(z)dz,$$

and since

$$(\Omega - x) \cap B_{r_0} \cap D_+ \subset \Phi_x(B_{r_0} \times (0, r_0)),$$

we have

$$\begin{aligned} I_3(x) &\le \int\limits_{\Phi_x(B_{r_0}\times(0,r_0))\setminus B_{d(x)/2}} [(\epsilon + d(x+z))^\beta - (\epsilon + d(x))^\beta] K_\epsilon(z)dz \\ &= C_\Omega(\epsilon + d(x))^\beta \int\limits_{\Phi_x(B_{r_0}\times(0,r_0))\setminus B_{d(x)/2}} \Big[\Big(\frac{\epsilon + d(x+z)}{\epsilon + d(x)}\Big)^\beta - 1\Big] K_\epsilon(z)dz, \end{aligned}$$

Thus, making a change of variables we have

$$I_3(x) \le C_\Omega(\epsilon+d(x))^\beta \int\limits_{B_{r_0}\times(0,r_0)\setminus\Phi_x^{-1}(B_{d(x)/2})} \Big[\Big(\frac{\epsilon + d(x+\Phi_x(\xi))}{\epsilon + d(x)}\Big)^\beta - 1\Big] \tilde{K}_\epsilon(\xi)d\xi.$$

Since Φ_x is a diffeomorphism, there exists a constant $c_8 > 0$ such that

$$d(x + \Phi_x(\xi', s)) \le d(x + \Phi_x(0, s)) + c_8|\xi'| = d(x) + s + c_8|\xi'|$$

and a constant $\lambda \in (0,1)$ small, depending only on the smoothness of $\partial\Omega$, such that $B_{\lambda d(x)} \subset \Phi_x^{-1}(B_{d(x)/2})$. Using this and (3.2) we arrive to

$$\begin{aligned} I_3(x) &\leq C_\Omega(\epsilon + d(x))^\beta \int\limits_{B_{r_0}\times(0,r_0)\setminus B_{\lambda d(x)}} [(1 + c_8|\xi/(\epsilon + d(x))|)^\beta - 1]K_\epsilon(\xi)d\xi \\ &= C_\Omega(\epsilon + d(x))^{\beta-2\sigma} \int\limits_{(\epsilon+d(x))^{-1}B_{r_0}\setminus B_{\lambda d(x)}} [(1 + c_8|y|)^\beta - 1]K_\tau(y)dy \\ &\leq C_\Omega(\epsilon + d(x))^{\beta-2\sigma} \int_{\lambda(1-\tau)}^{+\infty} \frac{[(1 + c_8t)^\beta - 1]t^{N-1}dt}{\tau^{1+2\sigma} + t^{N+2\sigma}}. \end{aligned}$$

At this point, we remark that for each $M > 2$, we have

$$\int_M^{+\infty} \frac{[(1 + c_8t)^\beta - 1]t^{N-1}dt}{\tau^{1+2\sigma} + t^{N+2\sigma}} \leq c_9 M^{\beta-2\sigma},$$

where $c_9 > 0$ depends only on N, σ and Ω. On the other hand, for each $M > 2$ there exists $\beta = \beta(M) > 0$ small such that

$$\int_{\lambda(1-\tau)}^{M} \frac{[(1 + c_8t)^\beta - 1]t^{N-1}dt}{\tau^{1+2\sigma} + t^{N+2\sigma}} \leq C_\Omega^{-1} c_7/2,$$

where $c_7 > 0$ is the constant arising in (3.8). From the last two estimates, we conclude that for each $M > 2$, there exists β small such that

$$I_3(x) \leq c_7(\epsilon + d(x))^{\beta-2\sigma}/2 + c_{10}M^{\beta-2\sigma}, \tag{3.9}$$

where $c_{10} > 0$ depends only on N, σ and Ω. Putting together (3.5), (3.6), (3.8) and (3.9), and fixing $M = \max\{2, r_0\}$, we have

$$\mathcal{I}_\epsilon[\psi](x) \leq (\epsilon + d(x))^{\beta-2\sigma}(-c_7/2 + c_2\beta(\epsilon + d(x))) + c_{11}r_0^{-2\sigma},$$

where $c_{11} > 0$ depends only on N, σ and Ω. Hence, fixing $\beta > 0$ smaller if it is necessary, we can write

$$\mathcal{I}_\epsilon[\psi](x) \leq -c_7(\epsilon + d(x))^{\beta-2\sigma}/4 + c_{11}r_0^{-2\sigma}.$$

Finally, taking $\epsilon + d(x)$ small in terms of c_7, c_{11}, r_0, β and σ (and therefore, depending only on N, σ and Ω), we conclude (3.4), where $c^* = c_7/8$. □

The last lemma allows us to provide the following control of the discontinuity at the boundary.

Proposition 3.2. *Let $\epsilon \in (0,1)$ and u_ϵ the solution of (1.1)-(1.2). Let $\bar\delta > 0$ and $\beta_0 \in (0, \min\{1, 2\sigma\})$ as in Lemma 3.1. Then, for each $d_0 \in (0, \bar\delta)$, there exists $C_0 > 0$ satisfying*

$$|u_\epsilon(x)| \leq C_0(\epsilon + d(x))^{\beta_0} \quad \text{for all } x \in \bar\Omega_{d_0}.$$

The constant C_0 depends on β_0, d_0, σ and Ω.

Proof: Let β_0 as in Lemma 3.1, $\psi = \psi_{\beta_0}$ as in (3.3) and consider the function
$$\zeta(x) = \min\{\psi(x), (\epsilon + d_0)^{\beta_0}\}, \quad x \in \mathbb{R}^N.$$
Observing that $\zeta = \psi$ in $\Omega_{d_0} \cup \Omega^c$ and $\psi \geq \zeta$ in $\mathbb{R}^N$, we easily conclude that
$$\mathcal{I}_\epsilon[\zeta](x) \leq \mathcal{I}_\epsilon[\tilde{\zeta}](x), \quad \text{for all } x \in \bar{\Omega}_{d_0},$$
and using Lemma 3.1 we get
$$-\mathcal{I}_\epsilon[\zeta](x) \geq c^*(\epsilon + d(x))^{\beta - 2\sigma} \quad \text{for all } x \in \bar{\Omega}_{d_0}.$$

Let $C > 0$ be the constant in Proposition 2.8 and define the function $\tilde{z}_+ = (Cd_0^{-\beta} + 2^\sigma c^{*-1})||f||_\infty \zeta$. By construction of $\tilde{z}_+$, we have
$$-\mathcal{I}_\epsilon[\tilde{z}_+] \geq ||f||_\infty \quad \text{in } \bar{\Omega}_{d_0}; \qquad \text{and} \qquad \tilde{z}_+ \geq u_\epsilon \quad \text{in } \bar{\Omega}_{d_0}^c,$$
and therefore, applying the comparison principle, we conclude $u_\epsilon \leq \tilde{z}_+$ in $\bar{\Omega}_{d_0}$. Similarly, we can conclude the function $\tilde{z}_- = -\tilde{z}_+$ satisfies $\tilde{z}_- \leq u_\epsilon$ in $\bar{\Omega}_{d_0}$, from which we get the result. □

4. Boundary Equicontinuity.

In this section we establish the boundary equicontinuity of the family of solutions $\{u_\epsilon\}_{\epsilon \in (0,1)}$ of problem (1.1)-(1.2). The main result of this section is the following

Theorem 4.1. *Let $\epsilon \in (0,1)$ and u_ϵ be the solution to (1.1)-(1.2). There exists a modulus of continuity m_0 depending only on N, σ, f and Ω, such that*
$$|u_\epsilon(x) - u_\epsilon(y)| \leq m_0(|x-y|) \quad \textit{for all } x, y \in \bar{\Omega}_{\bar{\delta}},$$
with $\bar{\delta} > 0$ given in Lemma 3.1.

The idea of the proof is based on the fact $w(x) = u(x+y) - u(x)$, where y is fixed, satisfies an equation (near the boundary) for which the comparison principle holds. Using this, we get the result constructing a barrier to this problem, independent of ϵ and associated to m in Theorem 4.1.

In what follows we discuss the precise elements on the proof. We consider $y \in \mathbb{R}^N$ with $0 < |y| < \bar{\delta}/2$, with $\bar{\delta}$ as in Lemma 3.1. Define the sets
$$\mathcal{O} = \mathcal{O}(y) := \Omega \setminus \bar{\Omega}_{|y|}, \quad \mathcal{U} = \mathcal{U}(y) := \{x \in \mathbb{R}^N : -|y| \leq d_\Omega(x) < |y|\}.$$
and the function
$$w(x) = w_{y,\epsilon}(x) := u_\epsilon(x+y) - u_\epsilon(x), \; x \in \mathbb{R}^N. \tag{4.1}$$

Notice that $w \equiv 0$ in $\mathbb{R}^N \setminus (\bar{\mathcal{O}} \cup \mathcal{U})$ and, by Proposition 3.2, there exists $C_0, \beta_0 > 0$ such that $|w(x)| \leq C_0(\epsilon + |y|)^{\beta_0}$ for all $x \in \mathcal{U}$. Since we have that w satisfies
$$-\mathcal{I}_\epsilon[w](x) = f(x+y) - f(x) \quad \text{for all } \; x \in \bar{\mathcal{O}},$$

denoting by m_f the modulus of continuity of f, we conclude that $w \in C(\bar{\mathcal{O}})$ satisfies the inequality

$$-\mathcal{I}_\epsilon[w](x) \leq m_f(|y|) \quad \text{in } \bar{\mathcal{O}}, \tag{4.2}$$

and the exterior inequality

$$w(x) \leq C_0(\epsilon + |y|)^{\beta_0} \mathbf{1}_{\mathcal{U}}(x) \quad \text{in } \bar{\mathcal{O}}^c. \tag{4.3}$$

Let ζ and η the functions defined as

$$\begin{aligned} \zeta(x) =& \min\{(\epsilon + \bar{\delta} - |y|)^\epsilon, (\epsilon + d_\Omega(x) - |y|)^\epsilon\} \mathbf{1}_{\bar{\mathcal{O}}}(x) \quad \text{and} \\ \eta(x) =& C_0(\epsilon + |y|)^{\beta_0} \mathbf{1}_{\mathcal{U}}(x), \end{aligned}$$

and consider the function

$$W(x) = \eta(x) + Am(|y|)\zeta(x), \tag{4.4}$$

where $A > 0$ and m is a modulus of continuity satisfying $m(|y|) \geq m_f(|y|)$. We have the following

Proposition 4.2. *There exists $A > 0$ large, depending on Ω, N and σ, such that*

$$-\mathcal{I}_\epsilon[W](x) \geq m_f(|y|), \quad \textit{for all } x \in \bar{\mathcal{O}},$$

for all $\epsilon \in (0, \bar{\delta})$, with $\bar{\delta}$ given in Lemma 3.1.

Proof: Without loss of generality we may assume the existence of a number $0 < \alpha < \min\{1, \beta_0\}$ and a constant c_1 such that

$$m(t) \geq c_1 t^\alpha, \quad \text{for all } t \geq 0. \tag{4.5}$$

By linearity of $\mathcal{I}_\epsilon$, we have

$$\mathcal{I}_\epsilon[W](x) = \mathcal{I}_\epsilon[\eta](x) + Am(|y|)\mathcal{I}_\epsilon[\zeta](x).$$

Thus, we may estimate each term in the right-hand side separately.

1.- Estimate for $\mathcal{I}_\epsilon[\zeta](x)$: We first notice that for $x \in \Omega$ with $|y| \leq d_\Omega(x) \leq \bar{\delta}$ we can write

$$\zeta(x) = (\epsilon + d_\Omega(x) - |y|)^\epsilon \mathbf{1}_{\bar{\mathcal{O}}} = (\epsilon + d_{\mathcal{O}}(x))^\epsilon \mathbf{1}_{\bar{\mathcal{O}}}(x).$$

Then, applying Lemma 3.1, for all ϵ small we have

$$-\mathcal{I}_\epsilon[\zeta](x) \geq c^*(\epsilon + d(x) - |y|)^{\epsilon - 2\sigma}, \quad \text{for all } x \in \bar{\Omega}_{\bar{\delta}} \cap \bar{\mathcal{O}},$$

for some $c^* > 0$ not depending on $d(x), |y|$ or ϵ. In fact, for all $\epsilon \in (0,1)$ the term $(\epsilon + d(x) - |y|)^{-\epsilon}$ is bounded below by a strictly positive constant, independent of ϵ, driving us to

$$-\mathcal{I}_\epsilon[\zeta](x) \geq c^*(\epsilon + d(x) - |y|)^{-2\sigma}, \quad \text{for all } x \in \bar{\Omega}_{\bar{\delta}} \cap \bar{\mathcal{O}}. \tag{4.6}$$

On the other hand, when $x \in \Omega \setminus \bar{\Omega}_{\bar{\delta}}$, for all $\epsilon \in (0,1)$ we have

$$\mathcal{I}_\epsilon[\zeta](x) \leq -(\epsilon + \bar{\delta} - |y|)^\epsilon \int\limits_{(\Omega \setminus \Omega_{|y|})^c - x} K_\epsilon(z)dz \leq -\epsilon^\epsilon \int\limits_{\Omega^c - x} K_1(z)dz,$$

and therefore, there exists $c_2 > 0$, not depending on $\epsilon, d(x)$ or $|y|$, such that

$$\mathcal{I}_\epsilon[\zeta](x) \leq -c_2, \quad \text{for all } x \in \Omega \setminus \bar{\Omega}_{\bar{\delta}}.$$

Since $|y| \leq \bar{\delta}/2$, making c^* smaller if necessary, the last inequality and (4.6) drives us to

$$-\mathcal{I}_\epsilon[\zeta](x) \geq c^*(\epsilon + d(x) - |y|)^{-2\sigma}, \quad \text{for all } x \in \bar{\mathcal{O}}, \tag{4.7}$$

2.- Estimate for $\mathcal{I}_\epsilon[\eta](x)$: By its very definition, for $x \in \bar{\mathcal{O}}$ we have

$$\mathcal{I}_\epsilon[\eta](x) = C_0(\epsilon + |y|)^{\beta_0} \int_{\mathcal{U}-x} K_\epsilon(z)dz. \tag{4.8}$$

We start considering the case $x \in \Omega \setminus \Omega_{\bar{\delta}}$, where we have $\mathrm{dist}(x, \mathcal{U}) \geq \bar{\delta}/2$ and then, there exists a constant $c_3 > 0$ depending only on $\bar{\delta}$ (which in turn depends only on the smoothness of the domain), such that $K_\epsilon(z)\mathbf{1}_{\mathcal{U}-x} \leq c_3$. Using this, we have

$$\mathcal{I}_\epsilon[\eta](x) \leq c_3(\epsilon + |y|)^{\beta_0} \int_{\mathcal{U}-x} dz.$$

By the boundedness of Ω, there exists $c_4 > 0$ depending only on N such that $\mathrm{Vol}(\mathcal{U} - x) \leq c_4|y|$. Using this and (4.5), we conclude that

$$\mathcal{I}_\epsilon[\eta](x) \leq c_5 m(|y|), \tag{4.9}$$

where $c_5 > 0$ depends only on N, σ and Ω.

Now we deal with the case $x \in \mathcal{O} \cap \Omega_{\bar{\delta}}$ (notice that in this case we are assuming $d_\Omega(x) > |y|$). Using (4.8) and recalling the change of variables Φ_x introduced in (3.1), we can write

$$\mathcal{I}_\epsilon[\eta](x) \leq C_0(\epsilon + |y|)^{\beta_0}\Big(\int_{(\mathcal{U}-x)\setminus B_{r_0}} K_\epsilon(z)dz + \int_{(\mathcal{U}-x)\cap\mathcal{R}_x} K_\epsilon(z)dz\Big),$$

where $\mathcal{R}_x$ was defined at the beginning of Section §3. Using a similar analysis as the one leading to (4.9), there exists a universal constant $c_6 > 0$ such that

$$\mathcal{I}_\epsilon[\eta](x) \leq c_6(\epsilon + |y|)^{\beta_0}\Big(|y| + \int_{(\mathcal{U}-x)\cap\mathcal{R}_x} K_\epsilon(z)dz\Big).$$

Now, we have that $(\mathcal{U} - x) \cap \mathcal{R}_x = \Phi_x(B_{R_0} \times (-d(x) - |y|, |y| - d(x)))$, and therefore, applying the change of variables Φ_x and the estimate (3.2), we arrive to

$$\mathcal{I}_\epsilon[\eta](x) \leq c_7(\epsilon + |y|)^{\beta_0}\Big(|y| + \int_{B_{R_0}\times(-d(x)-|y|,|y|-d(x))} \frac{d\xi' ds}{\epsilon^{N+2\sigma} + |(\xi', s)|^{N+2\sigma}}\Big),$$

and from this, using a similar argument as the one leading to (3.7) to treat the last integral term, and applying (4.5), we conclude that

$$\mathcal{I}_\epsilon[\eta](x) \leq c_8(\epsilon + |y|)^{\beta_0}\Big(m(|y|) + \int_{d(x)-|y|}^{d(x)+|y|} \frac{ds}{\epsilon^{1+2\sigma} + |s|^{1+2\sigma}}\Big). \tag{4.10}$$

Now, the core of this estimate is the computation of the last integral. Denoting

$$I(x) := (\epsilon + |y|)^{\beta_0} \int_{d(x)-|y|}^{d(x)+|y|} \frac{ds}{\epsilon^{1+2\sigma} + |s|^{1+2\sigma}},$$

we claim the existence of a constant $c_9 > 0$ not depending on $\epsilon, d(x)$ or $|y|$ such that

$$I(x) \leq c_9 m(|y|)(\epsilon + d(x) - |y|)^{-2\sigma}. \tag{4.11}$$

We get this estimate considering various cases. When $|y| \leq \epsilon$ and $d(x) - |y| \leq 2\epsilon$ we write

$$I(x) = (\epsilon + |y|)^{\beta_0}\epsilon^{-2\sigma} \int_{(d(x)-|y|)/\epsilon}^{(d(x)+|y|)/\epsilon} K_1(z)dz \leq 2^{\beta_0+1}\epsilon^{\beta_0-2\sigma-1}|y|,$$

and using that $m(|y|) \geq |y|^\alpha$ for some $\alpha \in (0, \beta_0)$, we have

$$\begin{aligned} I(x) &\leq 2^{\beta_0+1} m(|y|)\epsilon^{\beta_0-1}\epsilon^{-2\sigma}|y|^{1-\alpha} \\ &\leq 2^{\beta_0+1}3^{2\sigma} m(|y|)\epsilon^{\beta_0-\alpha}(\epsilon + d(x) - |y|)^{-2\sigma}, \end{aligned}$$

and from this, we conclude

$$I(x) \leq c_{10}\epsilon^{\beta_0-\alpha} m(|y|)(\epsilon + d(x) - |y|)^{-2\sigma}, \tag{4.12}$$

for some constant $c_{10} > 0$.

When $|y| \leq \epsilon$ and $d(x) - |y| > 2\epsilon$, we have

$$I(x) \leq 2^{\beta_0}\epsilon^{\beta_0} \int_{d(x)-|y|}^{d(x)+|y|} |z|^{-(1+2\sigma)}dz \leq 2^{\beta_0}\epsilon^{\beta_0}(d(x) - |y|)^{-(1+2\sigma)}|y|,$$

and using that $m(|y|) \geq |y|^\alpha$, we arrive to

$$I(x) \leq 2^{\beta_0-1} m(|y|)\epsilon^{\beta_0-\alpha}(d(x)-|y|)^{-2\sigma} \leq 2^{\beta_0-1+2\sigma} m(|y|)\epsilon^{\beta_0-\alpha}(\epsilon+d(x)-|y|)^{-2\sigma},$$

concluding the same estimate (4.12).

In the case $|y| > \epsilon$ and $d(x) - |y| \leq 2\epsilon$, performing the change $\xi = z/\epsilon$ in the integral defining $I(x)$, we have

$$I(x) \leq (\epsilon+|y|)^{\beta_0}\epsilon^{-2\sigma}||K_1||_{L^1} \leq ||K_1||_{L^1}3^{2\sigma}2^{\beta_0}|y|^{\beta_0-\alpha} m(|y|)(\epsilon+d(x)-|y|)^{-2\sigma},$$

and therefore we conclude

$$I(x) \leq C|y|^{\beta_0-\alpha} m(|y|)(\epsilon + d(x) - |y|)^{-2\sigma}. \tag{4.13}$$

Finally, in the case $|y| > \epsilon$ and $d(x) - |y| > 2\epsilon$ we have

$$I(x) \leq (\epsilon + |y|)^{\beta_0} \epsilon^{-2\sigma} \int_{(d(x)-|y|)/\epsilon}^{(d(x)+|y|)/\epsilon} K_1(z)dz \leq 2^{\beta_0 - 1} \sigma^{-1} |y|^{\beta_0} (d(x) - |y|)^{-2\sigma},$$

from which we arrive to (4.13). From (4.12) and (4.13) we arrive to (4.11). Hence, there exists $c_{11} > 0$ depending only on N, Ω and σ such that

$$-\mathcal{I}_\epsilon[\eta](x) \geq -c_{11} m(|y|) \Big((\epsilon + |y|)^{\beta_0} + (\epsilon + d(x) - |y|)^{-2\sigma} \Big),$$

for $x \in \mathcal{O} \cap \Omega_{\bar{\delta}}$. Taking this inequality and (4.9), since $|y| \leq \bar{\delta}/2$ there exists a constant $c_{12} > 0$ such that

$$-\mathcal{I}_\epsilon[\eta](x) \geq -c_{12} m(|y|) \Big((\epsilon + |y|)^{\beta_0} + (\epsilon + d(x) - |y|)^{-2\sigma} \Big), \tag{4.14}$$

for all $x \in \bar{\mathcal{O}}$, where the estimate for $x \in \partial\mathcal{O}$ is valid by Lemma 2.1.

3.- Conclusion: For each $x \in \bar{\mathcal{O}}$, by (4.7) and (4.14) we have

$$-\mathcal{I}_\epsilon[W](x) \geq \Big[(Ac^* - c_{12})(\epsilon + d(x) - |y|)^{-2\sigma} - c_{12}(\epsilon + |y|)^{\beta_0} \Big] m(|y|),$$

and therefore, by taking A large in terms of N, σ, c_{12}, c^* and $\mathrm{diam}(\Omega)$, we conclude by the choice of m that

$$-\mathcal{I}_\epsilon[W](x) \geq m(|y|) \geq m_f(|y|), \quad \text{for all } x \in \bar{\mathcal{O}},$$

and the proof follows. $\square$

This proposition allows us to give the
Proof of Theorem 4.1: Since w defined in (4.1) satisfies problem (4.2)-(4.3) and recalling W defined in (4.4), by Proposition 4.2 and the form of W in $\bar{\mathcal{O}}^c$, we can use the comparison principle to conclude that $w \leq W$ in $\bar{\mathcal{O}}$. This means that

$$u_\epsilon(x+y) - u_\epsilon(x) = w(x) \leq W(x) \leq c_1 A m(y), \quad x \in \bar{\Omega}_y,$$

for some constant $c_1 > 0$. Since a similar lower bound can be stated, by the arbitrariness of y we conclude the result with $m_0 = c_1 A m$. $\square$

5. Proof of Theorem 1.1.

Consider $\bar{\delta}$ as in Lemma 3.1, let $y \in \mathbb{R}^N$ such that $|y| \leq \bar{\delta}/8$ and consider the sets

$$\Sigma_1 = \overline{(\Omega - y) \cup \Omega}, \qquad \Sigma_2 = \Omega \cap (\Omega - y),$$
$$\Sigma_3 = \Sigma_1 \setminus \Sigma_2 \quad \text{and} \qquad \Sigma_4 = (\Omega \setminus \bar{\Omega}_{\bar{\delta}/2}) \cup ((\Omega \setminus \bar{\Omega}_{\bar{\delta}/2}) - y).$$

Notice that $\Sigma_4 \subset \Sigma_2 \subset \Sigma_1$. In addition, notice that if $z \in \Sigma_3$, then $z + y$ and z cannot be simultaneously in Ω. We also have

$$|\mathrm{dist}(z, \partial\Omega)|, |\mathrm{dist}(z + y, \partial\Omega)| \leq |y|$$

for each $z \in \Sigma_3$. Finally, observe that if $x \in \Sigma_2 \backslash \Sigma_4$, then $x, x+y \in \Omega_{\bar{\delta}}$. Thus, considering w as in (4.1), by Proposition 3.2 we can assure the existence of $C_0, \beta_0 > 0$ such that

$$w \leq C_0(\epsilon + |y|)^{\beta_0} \quad \text{in } \Sigma_3,$$

and by Theorem 4.1 we have

$$w \leq m_0(|y|) \quad \text{in } \Sigma_2 \setminus \bar{\Sigma}_4.$$

Now, consider the funtion

$$Z(x) = Am_0(|y|)\mathbf{1}_{\Sigma_2}(x) + C_0(\epsilon + |y|)^{\beta_0}\mathbf{1}_{\Sigma_3}(x),$$

where $A > 0$ is a constant to be fixed later. Notice that for each $x \in \bar{\Sigma}_3$, we have

$$\mathcal{I}_\epsilon[Z](x) = C_0(\epsilon + |y|)^{\beta_0} \int\limits_{\Sigma_3 - x} K_\epsilon(z)dz - Am_0(|y|) \int\limits_{\Sigma_2^c - z} K_\epsilon(z)dz. \tag{5.1}$$

At this point, we remark that there exists a constant $c_1 > 0$, independent of ϵ, y or x, such that

$$\int\limits_{\Sigma_2^c - z} K_\epsilon(z)dz \geq c_1.$$

On the other hand, since $\mathrm{dist}(x, \Sigma_3) \geq \bar{\delta}/2$ we have $K_\epsilon(z)\mathbf{1}_{\Sigma_3 - x} \leq c_2$ for some constant $c_2 > 0$, and by the boundedness of Ω, $\mathrm{Vol}(\Sigma_3 - x) \leq c_3|y|$ for some $c_3 > 0$. Using these facts on (5.1) and applying (4.5), we arrive to

$$\mathcal{I}_\epsilon[Z](x) \leq (c_4(\epsilon + |y|)^{\beta_0} - c_1 A)m_0(|y|).$$

Thus, taking A large in terms of c_1, c_4, we conclude that $-\mathcal{I}_\epsilon[Z] \geq m_f(|y|)$ in $\bar{\Sigma}_4$.

By the very definition of w, we have

$$-\mathcal{I}_\epsilon[w] = f(x+y) - f(x), \quad \text{for } x \in \bar{\Sigma}_4.$$

Then we have that $-\mathcal{I}_\epsilon[Z] \geq -\mathcal{I}_\epsilon[w]$ in $\bar{\Sigma}_4$ and by definition of W and the bounds of w in $\bar{\Sigma}_4^c$ stated above, we conclude that $w \leq W$ in $\bar{\Sigma}_4^c$. Using the comparison principle, we conclude $w \leq W$ in $\bar{\Sigma}_4$. A similar argument states the inequality $-W \leq w$ and the result follows. □

6. Further Results.

6.1. **Fully Nonlinear Equations.** The result obtained in Theorem 4.1 can be readily extended to a certain class of fully nonlinear equations. For example, consider two sets of indices $\mathcal{A}, \mathcal{B}$ and a two parameter family of radial continuous functions $a_{\alpha\beta} : \mathbb{R}^N \to \mathbb{R}$ satisfying the *uniform ellipticity condition*

$$\lambda_1 \leq a_{\alpha\beta}(z) \leq \lambda_2, \quad \forall \alpha \in \mathcal{A}, \beta \in \mathcal{B}, z \in \mathbb{R}^N \tag{6.1}$$

for certain constants λ_1, λ_2 such that $0 < \lambda_1 < \lambda_2 < +\infty$. Let us denote

$$K_{\alpha\beta,\epsilon}(z) := \frac{a_{\alpha\beta}(z)}{\epsilon^{n+2\sigma} + |z|^{n+2\sigma}}$$

and with this, for a suitable function u and $x \in \mathbb{R}^N$, define the linear operators

$$L_{\alpha\beta,\epsilon}[u](x) := \int_{\mathbb{R}^N} \delta(u,x,z) K_{\alpha\beta,\epsilon}(z) dz$$

and the corresponding *Isaacs Operator*

$$I_\epsilon[u](x) = \inf_{\alpha\in\mathcal{A}} \sup_{\beta\in\mathcal{B}} L_{\alpha\beta,\epsilon}[u](x).$$

Under these definitions, we may consider the corresponding nonlinear equation

$$\left\{ \begin{array}{rll} -I_\epsilon[u] = & f & \text{in } \Omega \\ u = & 0 & \text{in } \Omega^c. \end{array} \right. \tag{6.2}$$

Existence and uniqueness of a pointwise solution u_ϵ to (6.2), which is continuous in $\bar{\Omega}$ can be obtained in a very similar way as in the linear case, and Proposition 2.1 can be adapted to this nonlinear setting. This allows us to use the comparison principle stated in Proposition 2.7 as well.

The lack of linearity can be handled with the positive homogeneity of these operators and the so called *extremal operators*

$$\mathcal{M}_\epsilon^+[u](x) = \sup_{\alpha\in\mathcal{A},\beta\in\mathcal{B}} L_{\alpha\beta,\epsilon}[u](x), \quad \mathcal{M}_\epsilon^-[u](x) = \inf_{\alpha\in\mathcal{A},\beta\in\mathcal{B}} L_{\alpha\beta,\epsilon}[u](x),$$

since, for two functions u_1, u_2 and $x \in \mathbb{R}^N$, these operators satisfy the fundamental inequality

$$\mathcal{M}_\epsilon^-[u_1 - u_2](x) \leq \mathcal{I}_\epsilon[u_1](x) - \mathcal{I}_\epsilon[u_2](x) \leq \mathcal{M}_\epsilon^+[u_1 - u_2](x).$$

A priori estimates for the solution as it is stated in Proposition 3.2 can be found using the same barriers given in the proof of that proposition, as the following useful estimates hold: For each $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$, $D \subset \mathbb{R}^N$

$$\int_D hK_{\alpha\beta,\epsilon}(z)dz \leq \lambda_1 \int_D hK_\epsilon(z)dz,$$

for all $h : D \to \mathbb{R}$ bounded nonnegative function, and

$$\int_D hK_{\alpha\beta,\epsilon}(z)dz \leq \lambda_2 \int_D hK_\epsilon(z)dz$$

for all $h : D \to \mathbb{R}$ bounded nonpositive function.

Using these inequalities and (6.1), we can use the same barriers appearing in the proof of Theorem 1.1 (Theorem 4.1 included) and get similar result. Moreover, the same modulus of continuity for the linear case can be obtained in this nonlinear framework, up to a factor depending on λ_1 and λ_2.

6.2. **Parabolic Equations.** Let $T > 0$, $f : \bar{\Omega} \times [0, T] \to \mathbb{R}$ be a continuous function. A result similar to Theorem 4.1 can be readily obtained for the parabolic nonlocal equation

$$\left\{ \begin{array}{rll} u_t - \mathcal{I}_\epsilon[u] = & f & \text{in } \Omega \times [0, T), \\ u(x,t) = & 0 & \text{in } \Omega^c \times [0, T), \\ u(x, 0) = & 0 & \text{in } \bar{\Omega}. \end{array} \right. \tag{6.3}$$

Similar problem is adressed by the authors in [24] for the Cauchy problem in all $\mathbb{R}^N$. Inspired by techniques used by Ishii in [26], a modulus of continuity in time can be derived once a modulus of continuity in space is found. So, the key fact is the modulus in x and this can be obtained in the parabolic setting noting that Theorem 4.1 readily applies considering equations with the form

$$\lambda u - \mathcal{I}_\epsilon[u] = f \quad \text{in } \Omega$$

for $\lambda > 0$ and that each time $Z(x)$ is a suitable barrier for this problem, then the function $(x, t) \mapsto e^t Z(x)$ plays the role of a barrier for the evolution problem (6.3).

6.3. **Convergence Issues.** The proof of Corollary 1.2 is standard in the viscosity sense, once the uniform convergence is stated. However, following the ideas of Cortázar, Elgueta and Rossi in [19], and also in [24], under stronger assumptions over the regularity of u in Corollary 1.2, we can find a rate of convergence.

Theorem 6.1. *Let f, u_ϵ and u as in Corollary 1.2 and assume $u \in C^{2\sigma+\gamma}(\bar{\Omega})$ for some $\gamma > 0$. Then,*

$$||u_\epsilon - u||_{L^\infty(\bar{\Omega})} \leq C\epsilon^{\gamma_0}$$

for some $0 < \gamma_0 \leq \min\{2\sigma, \gamma\}$ and with C depending only on n and σ.

Proof. For simplicity, we will see the case $2\sigma < 1$ and $2\sigma + \gamma < 1$. Defining $w = u_\epsilon - u$, for $x \in \Omega$ we have

$$\begin{aligned} -\mathcal{I}_\epsilon[w](x) &= \mathcal{I}_\epsilon[u](x) + (-\Delta)^\sigma[u](x) \\ &= -\epsilon^{n+2\sigma} \int_{\Omega - x} \frac{u(x+z) - u(x)}{|z|^{n+2\sigma}(\epsilon^{n+2\sigma} + |z|^{n+2\sigma})} dz \\ &\quad -\epsilon^{n+2\sigma} u(x) \int_{(\Omega - x)^c} \frac{dz}{|z|^{n+2\sigma}(\epsilon^{n+2\sigma} + |z|^{n+2\sigma})} \\ &= I_1 + I_2. \end{aligned}$$

By the regularity of u we have

$$|I_1| \leq C\|u\|_{C^{2\sigma+\gamma}(\bar{\Omega})}\epsilon^\gamma,$$

where C does not depend on ϵ. On the other hand, for I_2 we split the analysis. First, if $\epsilon \leq d(x)$, then

$$|I_2| \leq C\|u\|_{C^{2\sigma+\gamma}(\bar{\Omega})}\epsilon^{n+2\sigma} d(x)^{-(n+2\sigma)+\gamma},$$

where we have used that there is no loss of boundary condition for u, hence $u = 0$ on $\partial\Omega$ and then $|u(x)| \leq [u]_{C^{2\sigma+\gamma}(\bar{\Omega})} d(x)^{2\sigma+\gamma}$. Hence, we conclude

$$|I_2| \leq C\epsilon^{\gamma}.$$

Second, when $d(x) < \epsilon$ we have

$$|I_2| \leq Cd(x)^{2\sigma+\gamma}\epsilon^{-2\sigma}(d(x)/\epsilon)^{-2\sigma} \leq C\epsilon^{\gamma}.$$

Since we know that $|w| \leq C\epsilon^{\beta_0}$ on $\partial\Omega$, by Proposition 3.2, we can get the result proceeding exactly as in the proof of Proposition 2.8. □

6.4. **An example of a scheme without boundary equicontinuity.** In this subsection we consider the reverse scheme, that is approximating zero order equations by fractional ones and we prove the absence of uniform modulus of continuity in $\bar{\Omega}$. For this, we recall some facts of Section §2. Let $f \in C(\bar{\Omega})$ with $f \geq \varrho_0 > 0$, $J : \mathbb{R}^N \to \mathbb{R}_+$ integrable and $\mathcal{I}_J$ as in (2.1). Consider the associated problem (2.2)-(1.2), that is

$$\left\{ \begin{array}{rll} -\mathcal{I}_J[u] & = f & \text{in } \Omega \\ u & = 0. & \text{in } \Omega^c \end{array} \right. \tag{6.4}$$

As we saw in Remark 2.4, the unique solution $u \in C(\bar{\Omega})$ for this problem is such that $u > 0$ in $\partial\Omega$.

Consider J, f as above, with J such that $J \geq m$ in B_r, for some $r, m > 0$. For $\epsilon \in (0, 1)$ and $\alpha > 1$ consider the family of kernels

$$J_\epsilon(z) = \min\{1, |z/\epsilon|^{\alpha}\}^{-1} J(z),$$

which are not integrable at the origin. If we define

$$\mathcal{J}_\epsilon[u](x) = \int_{\mathbb{R}^N} [u(x+z) - u(x)] J_\epsilon(z) dz$$

and consider the problems

$$\left\{ \begin{array}{rll} -\mathcal{J}_\epsilon[u] & = f & \text{in } \Omega \\ u & = 0, & \text{in } \Omega^c \end{array} \right. \tag{6.5}$$

it is known that the unique viscosity solution u_ϵ of (6.5) agrees the prescribed value of the equation on the boundary, and then $u_\epsilon = 0$ on $\partial\Omega$ for all $\epsilon \in (0, 1)$, see for example [4]. We have $\{u_\epsilon\}$ is uniformly bounded in $L^\infty(\bar{\Omega})$ and therefore, the application of half-relaxed limits together with viscosity stability results in [3], imply $u_\epsilon \to u$ locally uniform in Ω as $\epsilon \to 0$, where u is the unique solution to (6.4). Since u is strictly positive in $\partial\Omega$, the convergence of u_ϵ to u cannot be uniform in $\bar{\Omega}$, and therefore the family $\{u_\epsilon\}$ is not equicontinuous in this case.

This example resembles the behavior of the viscosity solutions u_ϵ of the equation

$$-\epsilon u'' + u' = 1 \quad \text{in } (0, 1), \quad \text{with } u(0) = u(1) = 0,$$

which approximate the solution of the equation

$$u' = 1 \quad \text{in } (0, 1), \quad \text{with } u(0) = u(1) = 0.$$

In this case, the family (u_ϵ) is not equicontinuous too, see [1].

Acknowledgements: The authors want to thank the referees for carefully reading the paper and making suggestions that resulted in a great improvement in the clarity and simplicity of the proofs. P.F. was partially supported by Fondecyt Grant # 1110291, BASAL-CMM projects and CAPDE, Anillo ACT-125. E.T. was partially supported by CONICYT, Grants Capital Humano Avanzado and Ayuda Realización Tesis Doctoral.

References

[1] Barles, G. *Solutions de Viscosite des Equations de Hamilton-Jacobi* Collection "Mathematiques et Applications" de la SIAM, no 17, Springer-Verlag (1994).

[2] Barles, G. and Burdeau, J. *The Dirichlet Problem for Semilinear Second-Order Degenerate Elliptic Equations and Applications to Stochastic Exit Time Control Problems* Comm. in PDE., 20 (1-2), 129-178 (1995).

[3] Barles, G. and Imbert, C. *Second-order Elliptic Integro-Differential Equations: Viscosity Solutions' Theory Revisited.* AIHP Anal. Non Lin., Vol. 25 (2008) no. 3, 567-585.

[4] Barles, G., Chasseigne, E. and Imbert, C. *On the Dirichlet Problem for Second Order Elliptic Integro-Differential Equations* Indiana U. Math. Journal, 2008.

[5] Barles, G., Chasseigne, E., Ciomaga, A. and Imbert, C. *Lipschitz Regularity of Solutions for Mixed Integro-Differential Equations.* J. Diff. Eq., 252 (2012), 6012-6060.

[6] Barles, G. and Perthame, B. *Exit Time Problems in Optimal Control And Vanishing Viscosity Method.* SIAM J. Control Opt., 26 (1988) 1133-1148.

[7] Barles, G. and Perthame, B., *Discontinuous solutions of deterministic optimal stopping-time problems*, Mod'l. Math. Anal. Num. 21 (1987), 557-579.

[8] Barles, G. and Rouy, E. *A Strong Comparison Result for the Bellaman Equation Arising in Stochastic Exit Time Control Problems ans its Applications.* Comm. in PDE, 23 (11 & 12)(1998), 1995-2033.

[9] Bass, R. and Kassmann M. *Hölder Continuity of Harmonic Functions with Respect to Operators of Variable Order.* Comm. Partial Differential Equations, 30(7-9), 1249-1259, (2005).

[10] Bass R. and and Levin D. *Harnack Inequalities for Jump Processes.* Potential Anal., 17(4), 375-388, (2002)

[11] Bensaoud, I, Sayah, A. *Stability results for Hamilton-Jacobi equations with integro-differential terms and discontinuous Hamiltonians.* Arch. Mat. 79, 392-395 (2002)

[12] Bogdan, K. *The Boundary Harnack Principle for the Fractional Laplacian.* Studia Math. 123, 43-80 (1997)

[13] Blumenthal, R. and Getoor, R. *Some Theorems on Stable Processes.* Transactions AMS, 95 (1960), 263-273.

[14] Caffarelli, L. and Silvestre, L. *Regularity Theory For Nonlocal Integro-Differential Equations.* Comm. Pure Appl. Math, Vol. 62 (2009), no. 5, 597-638.

[15] Caffarelli, L. and Silvestre, L. *Regularity Results for Nonlocal Equation by Approximation.* Archive for Rational Mechanics and Analysis, Vol. 200, no. 1, 59-88

[16] Cardaliaguet, P. and Cannarsa, P. *Hölder Estimates in Space-Time for Viscosity Solutions of Hamilton-Jacobi Equations* CPAM 63(5): 590-629 (2010).

[17] Chasseigne, E. *The Dirichlet problem for some nonlocal diffusion equations.* Differential Integral Equations 20 (2007), no. 12, 13891404.

[18] Chasseigne, E., Chavez, M. and Rossi, J.D. *Asymptotic behavior for Nonlocal Diffusion Equations.* J.Math. Pures Appl. 86 (2006) 271-291.

[19] Cortázar, C., Elgueta, M. and Rossi, J. D. *Nonlocal diffusion problems that approximate the heat equation with Dirichlet boundary conditions.* Israel J. Math. Vol. 170 (2009), 5360

[20] Coville, J. *Harnack type Inequality for Positive Solutions of Some Integral Equation.* Ann. Mat. Pura Appl. (4) 191 (2012), no. 3, 503528.

[21] M.G. Crandall, H. Ishii and P.-L. Lions. *User's Guide to Viscosity Solutions of Second Order Partial Differential Equations.* Bull. Amer. Math. Soc. (N.S.), Vol. 27 (1992), no. 1, 1-67.

[22] Da Lio, F. *Remarks on the Strong Maximum Principle for Viscosity Solutions to Fully Nonlinear Parabolic Equations.* Comm. on Pure and Appl. Aanal., Vol 3, No 3 (2004) pp. 395-415.

[23] Di Neza, E., Palatucci, G. and Valdinoci, E. *Hitchhiker's Guide to the Fractional Sobolev Spaces* Bull. Sci. Math., 136, (2012), no. 5, 521–573.

[24] Felmer, P. and Topp, E. *Convergence Results for a Class of Nonlinear Fractional Heat Equations* Israel J.

[25] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin 2001.

[26] Ishii, H. *Existence and Uniqueness of Solutions of Hamilton-Jacobi Equations.* Funkcialaj Ekvacioj, Vol. 29 (1986) 167-188.

[27] Ishii, H. and Lions, P.L. *Viscosity Solutions of Fully Nonlinear Second-Order Elliptic Partial Differential Equations* J. Differential Equations, 83(1) 26-78, 1990.

[28] Landkof, N. S. *Foundations of Modern Potential Theory.* Springer, New York, 1972.

[29] Oksendal, B. and Agnès, S. *Applied Stochastic Control of Jump Diffusions.* Springer-Verlag, 2007.

[30] Ros-Oton, X. and Serra, J. *The Dirichlet Problem for the Fractional Laplacian: Regularity up to the Boundary.* Preprint.

[31] Sato, K.-I. *Lévy Processes and Infinitely Divisible Distributions.* Cambridge Univ. Press., 1999.

[32] Sayah, A. *Équations d'Hamilton-Jacobi du premier Ordre Avec Termes Intégro-Différentiels. I. Unicité des solutions de viscosité.* Comm. Partial Differential Equations 16 (1991), 1057-1074.

[33] Sayah, A. *Équations d'Hamilton-Jacobi du premier Ordre Avec Termes Intégro-Différentiels. II. Existence de solutions de viscosité.* Comm. Partial Differential Equations 16 (1991), 1075-1093.

[34] Silvestre, L. *Hölder Estimates for Solutions of Integro-Differential Equations like the Fractional Laplace.* Indiana U. Math. J. 55 (3), 2006, 1155-1174.

Patricio Felmer - Departamento de Ingeniería Matemática and CMM (UMI 2807 CNRS), Universidad de Chile, Casilla 170 Correo 3, Santiago, CHILE. (pfelmer@dim.uchile.cl)

Erwin Topp - Departamento de Ingeniería Matemática (UMI 2807 CNRS), Universidad de Chile, Casilla 170, Correo 3, Santiago, CHILE; and Laboratoire de Mathématiques et Physique Théorique (CNRS UMR 6083), Féderation Denis Poisson, Université François Rabelais, Parc de Grandmont, 37200, Tours, FRANCE (etopp@dim.uchile.cl).