Local and global properties of solutions of quasilinear Hamilton-Jacobi equations

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Abstract We study some properties of the solutions of (E) $-\Delta_p u + |\nabla u|^q = 0$ in a domain $\Omega \subset \mathbb{R}^N$, mostly when $p \geq q > p-1$. We give a universal priori estimate of the gradient of the solutions with respect to the distance to the boundary. We give a full classification of the isolated singularities of the nonnegative solutions of (E), a partial classification of isolated singularities of the negative solutions. We prove a general removability result expressed in terms of some Bessel capacity of the removable set. We extend our estimates to equations on complete non compact manifolds satisfying a lower bound estimate on the Ricci curvature, and derive some Liouville type theorems.

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1 Introduction

Let $N \geq p > 1$, $q > p-1$ and $\Omega \subset \mathbb{R}^N$ $(N > 1)$ be a domain. In this article we study some local and global properties of solutions of

$$-\Delta_p u + |\nabla u|^q = 0 \tag{1.1}$$

in Ω, where $\Delta_p u := \operatorname{div}\left(|\nabla \mathrm{u}|^{\mathrm{p}-2}|\nabla \mathrm{u}|\right)$. The main questions we consider are the following:

1- A priori estimates and Liouville type theorems.

2- Removability of singularities.

3- Description of isolated singularities of solutions.

Our technique allows us to handle both nonnegative and signed solutions. We will speak of a problem with *absorption* when we consider nonnegative solutions and a problem with *source* when we consider negative solutions (in which case we will often set $u = -\tilde{u}$). One of the main tools we use is a pointwise gradient estimate, valid for *any signed* solution of (1.1),

Theorem A. *Let $u \in C^1(\Omega)$ be a solution of* (1.1) *in Ω. Then*

$$|\nabla u(x)| \leq c_{N,p,q}(\operatorname{dist}(x, \partial\Omega))^{-\frac{1}{q+1-p}} \tag{1.2}$$

for any $x \in \Omega$. If $\Omega = \mathbb{R}^N$, u is a constant.

In the case $p = 2$ the existence of an upper bound of the gradient has first been obtained by Lasry and Lions [23] and then made explicit by Lions [24]; the idea there was based upon the Bernstein technique. In [32] Nguyen-Phuoc and Véron rediscovered this upper bound by a slightly different method. Our method of proof is a combination of the Bernstein approach and the Keller-Osserman construction of radial supersolutions of the elliptic inequality satisfied by $|\nabla u|^2$, a technique which will be fundamental for extension of this results in a geometric framework (see below).

Concerning solutions of (1.1) in a domain $\Omega \subset \mathbb{R}^N$ we obtain that if $p \neq q$, any solution satisfies

$$|u(x)| \leq c_{p,q}\left(\operatorname{dist}(x, \partial\Omega)^{\frac{p-q}{q+1-p}} - \delta_*^{\frac{p-q}{q+1-p}}\right) + \max\{|u(z)| : \operatorname{dist}(z, \partial\Omega) = \delta_*\} \tag{1.3}$$

if $\mathrm{dist}\,(x,\partial\Omega) \leq \delta_*$, where $\delta_* > 0$ depends on the curvature of $\partial\Omega$; when $p = q$ the formula holds provided the term $\mathrm{dist}\,(x,\partial\Omega)^{\frac{p-q}{q+1-p}} - \delta_*^{\frac{p-q}{q+1-p}}$ is replaced by $\ln(\mathrm{dist}\,(x,\partial\Omega)/\delta_*)$. In the case $p = 2$ this estimate was a key element for the study of boundary singularity developed in [32]. This aspect of equation (1.1) will be developed in a forthcoming article.

In the study of singularities, we first give a general removability result concerning interior singularities. The general removability result given in Theorem 3.5, is expressed in terms of the Bessel capacity $C_{1,\frac{q}{q+1-p}}$ relative to $\mathbb{R}^N$, and deals with locally renormalized solutions (see Definition 3.4).

Theorem B *Let $p-1 < q < p$ and $F \subset \Omega$ be a relatively closed set such that $C_{1,\frac{q}{q+1-p}}(F) = 0$. Then any locally renormalized solution u of* (1.1) *in $\Omega \setminus F$ can be extended as a locally renormalized solution in whole Ω. When u is nonnegative, u is therefore a C^1 solution in whole Ω. When u is a signed renormalized solution, it is a C^1 solution provided $q < \frac{N(p-1)}{N-1}$.*

Our result is actually stronger and deals with locally renormalized solutions of

$$-\Delta_p u + |\nabla u|^q = \mu \tag{1.4}$$

in $\Omega \setminus F$ where μ is a measure which is absolutely continuous with respect to the $C_{1,\frac{q}{q+1-p}}$ capacity.

A previous result in [27] shows the same result provided u is a p-subharmonic function in Ω, $\mu = 0$ and $q > p$. An interesting counter part in [28] asserts that if any weak solution of (1.1) in $\Omega \setminus F$ can be extended as a solution in Ω, then $C_{1,\frac{q}{q+1-p}}(F) = 0$. The construction therein is strongly associated with the solvability of problem (1.4) by the use the capacitary measure of F. When $p = 2$, necessary and sufficient conditions for the existence of a solution of (1.4) in $\mathbb{R}^N$ can be found in [14].

When (1.1) is replaced by

$$-\Delta_p u + \epsilon u^q = \mu \tag{1.5}$$

with $\epsilon = \pm 1$ and $p > q > p-1$, deep existence results of solutions (1.5) of have been obtained in [30], [31], with $\epsilon = -1$ and [3], [5] with $\epsilon = 1$. Their proofs (excepted for [3]) are strongly based upon fine study of Wolff potentials and their links with Bessel potentials. In the case $\epsilon = -1$, a necessary and sufficient condition for existence of a positive solution of (1.5) is obtained with an assumption of Lipschitz continuity of the measure $\mu \geq 0$ with respect to the $C_{p,\frac{q}{q+1-p}}$ capacity. In the case $\epsilon = 1$, a sufficient condition for existence of a signed solution of (1.5) for a signed measure μ is obtained with the assumption that μ is absolutely continuous with respect to the $C_{p,\frac{q}{q+1-p}}$ capacity.

If K is reduced to a single point $0 \in \Omega$, the threshold of removability of an isolated singularity corresponds to the exponent

$$q = q_c := \frac{N(p-1)}{N-1} \tag{1.6}$$

but the situation is different if we consider positive or negative solutions.

If $q > p-1$, we set

$$\beta_q = \frac{p-q}{q+1-p}. \tag{1.7}$$

When $p-1 < q < q_c$, there exists an explicit radial positive solution of (1.1) in $\mathbb{R}^N_* = \mathbb{R}^N \setminus \{0\}$

$$U(x) = U(|x|) = \lambda_{N,p,q}|x|^{-\beta_q} \tag{1.8}$$

where

$$\lambda_{N,p,q} = \beta_q^{-1}\left(\beta_q(p-1)+p-N\right)^{\frac{1}{q+1-p}}. \tag{1.9}$$

When $p=2$, Lions obtained in [24] the description of isolated singularities of nonnegative solutions of (1.1) in the subcritical case $1 < q < \frac{N}{N-1}$. We extend his result to the general case $1 < p \leq N$ and provide a full classification of isolated singularities of nonnegative solutions :

Theorem C *Assume $p-1 < q < q_c$ and $u \in C^1(\Omega \setminus \{0\})$ is a nonnegative solution of* (1.1) *in $\Omega \setminus \{0\}$. Then*
(i) either there exists $c \geq 0$ such that

$$\lim_{x\to 0} \frac{u(x)}{\mu_p(x)} = c, \tag{1.10}$$

where μ_p is the fundamental solution of the p-Laplacian defined by

$$\mu_p(x) = \frac{p-1}{N-p}|x|^{\frac{p-N}{p-1}} \qquad \text{if } 1 < p < N \qquad \text{and } \mu_N(x) = -\ln|x|.$$

Furthermore u satisfies

$$-\Delta_p u + |\nabla u|^q = c_{N,p} c\delta_0 \qquad \text{in } \mathcal{D}'(\Omega), \tag{1.11}$$

(ii) or

$$\lim_{x\to 0} |x|^{\beta_q} u(x) = \lambda_{N,p,q} \tag{1.12}$$

for some explicit positive constants $\lambda_{N,p,q}$ and $\beta_q = \frac{p-q}{q+1-p}$.

When $q \geq q_c$ the nonnegative solutions can be extended as C^1 functions. Concerning negative solutions there exists a radial negative singular solutions $V = -\tilde{U}$ of (1.1) in $\mathbb{R}^N_*$ with

$$\tilde{U}(x) = \tilde{U}(|x|) = \tilde{\lambda}_{N,p,q}|x|^{-\beta_q}, \tag{1.13}$$

where

$$\tilde{\lambda}_{N,p,q} = \beta_q^{-1}\left(N-p-\beta_q(p-1)\right)^{\frac{1}{q+1-p}}. \tag{1.14}$$

In this case we obtain a partial classification of isolated singularities of negative solutions of (1.1) in $\Omega \setminus \{0\}$.

Theorem D *Assume u is a negative C^1 solution of* (1.1) *in $\Omega \setminus \{0\}$. Then*
(i) When $p-1 < q < q_c$ there exists $c \leq 0$ such that (1.10) *and* (1.11) *hold.*

(ii) When $q > q_c$ (1.10) *and* (1.11) *hold with* $c = 0$.
Furthermore, when u *is radial, there holds:*
(iii) When $q = q_c$, *either*

$$\lim_{x\to 0} \frac{(-\ln|x|)^{\frac{N-1}{p-1}}\, u(x)}{\mu_p(x)} = c_{N,p} < 0 \tag{1.15}$$

or u *is regular at* 0.
(iv) When $q > q_c$,

$$\lim_{x\to 0} |x|^{\beta_q} u(x) = -\tilde{\lambda} c_{N,p,q}, \tag{1.16}$$

or u *is regular at* 0.

In the last section we obtain local and global estimates of solutions when $\mathbb{R}^N$ is replaced by a N-dimensional Riemannian manifold (M^N, g) and $-\Delta_p$ by the corresponding p-Laplacian $-\Delta_{g,p}$ in covariant derivatives. Our results emphasize the role of the Ricci curvature of the manifold if $p = 2$ and the sectional curvature if $p \neq 2$. In the case $1 < p < 2$ we need to introduce the notion of *convexity radius* of a point $x \in M$, denoted by $r_M(x)$, which is supremum of the $r > 0$ such that the geodesic ball $B_r(x)$ is strongly convex.

Theorem E. *Let* $q > p - 1 > 0$, (M^N, g) *be a Riemannian manifold with Ricci curvature* $Ricc_g$ *and sectional curvature* Sec_g *and* $\Omega \subset M$ *be a domain such that* $Ricc_g \geq (1 - N)B^2$ *in* Ω. *Assume also* $Sec_g \geq -\tilde{B}^2$ *in* Ω *if* $p > 2$ *or* $r_M(z) \geq \operatorname{dist}(z, \partial\Omega)$ *for any* $z \in \Omega$ *if* $1 < p < 2$. *Then any* C^1 *solution* u *of*

$$-\Delta_{g,p} u + |\nabla u|^q = 0 \tag{1.17}$$

in Ω *satisfies*

$$|\nabla u(x)|^2 \leq c_{N,p,q} \max\left\{ B^{\frac{2}{q+1-p}}, (1 + B_p d(x, \partial\Omega))^{\frac{1}{q+1-p}} (d(x, \partial\Omega))^{-\frac{2}{q+1-p}} \right\}, \tag{1.18}$$

for any $x \in \Omega$, *where* $B_p = B + (p-2)_+ \tilde{B}$ *and* $d(x, \partial\Omega)$ *is now the geodesic distance of* x *to* $\partial\Omega$.

Notice that $r_M(x)$ *is always infinite when* $Sec_g \leq 0$. Furthermore if for some $a \in M$ we have that $r_M(a) = \infty$, then $r_M(x) = \infty$ for any $x \in M$; in this case we say that the convexity radius r_M of M is infinite. As a consequence we obtain

Theorem F. *Let* $0 < p - 1 < q$ *and* (M^N, g) *be a complete noncompact Riemannian manifold such that* $Ricc_g \geq (1 - N)B^2$, $(B \geq 0)$. *Assume also* $r_M = \infty$ *if* $1 < p < 2$ *or*

$$\lim_{\operatorname{dist}(x,a)\to\infty} \frac{|Sec_g(x)|}{\operatorname{dist}(x, a)} = 0 \tag{1.19}$$

for some $a \in M$ *if* $p > 2$. *Then any solution* u *of* (1.17) *in* M *satisfies*

$$|\nabla u(x)| \leq c_{N,p,q} B^{\frac{1}{q+1-p}} \qquad \forall x \in M. \tag{1.20}$$

Since our estimate holds also in the case $p = q$, we obtain

Theorem G. *Assume M satisfies the assumptions of Theorem F. Then any positive p-harmonic function v on M satisfies*

$$v(a)e^{-\kappa B \mathrm{dist}\,(x,a)} \leq v(x) \leq v(a)e^{\kappa B \mathrm{dist}\,(x,a)} \tag{1.21}$$

for any points a, x in M, where $\kappa = \kappa(p, N) > 0$.

The case $p = 2$, $B = 0$ is due to Chen and Yau ([8]). Kortschwar and Li [19] obtain a similar estimate but a with a global estimate of the sectional curvature which implies our assumption on the Ricci curvature.

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2 A priori estimates in a domain of $\mathbb{R}^N$

2.1 The gradient estimates

The next result is the extension to the p-Laplacian of a result obtained by Lions [24] for the Laplacian. We denote by $d(x)$ the distance from $x \in \overline{\Omega}$ to $\partial\Omega$.

Proposition 2.1 *Assume $q > p - 1$ and u is a C^1 solution of* (1.1) *in a domain Ω. Then*

$$|\nabla u(x)| \leq c_{N,p,q}(d(x))^{-\frac{1}{q+1-p}} \qquad \forall x \in \Omega. \tag{2.1}$$

Proof. In any open subset G of Ω where $|\nabla u| > 0$ we write (1.1) under the form

$$-\Delta u - (p-2)\frac{D^2u(\nabla u, \nabla u)}{|\nabla u|^2} + |\nabla u|^{q+2-p} = 0 \tag{2.2}$$

and we recall the formula

$$\frac{1}{2}\Delta |\nabla u|^2 = \left|D^2u\right|^2 + \langle \nabla \Delta u, \nabla u \rangle. \tag{2.3}$$

By Schwarz inequality

$$\left|D^2u\right|^2 \geq \frac{1}{N}(\Delta u)^2,$$

hence we obtain

$$\frac{1}{2}\Delta |\nabla u|^2 \geq \frac{1}{N}(\Delta u)^2 + \langle \nabla \Delta u, \nabla u \rangle. \tag{2.4}$$

Next, we write $z = |\nabla u|^2$ and derive from (2.2)

$$\Delta u = -\frac{(p-2)}{2}\frac{\langle \nabla z, \nabla u \rangle}{z} + z^{\frac{q+2-p}{2}}, \tag{2.5}$$

thus

$$\langle \nabla \Delta u, \nabla u \rangle = -\frac{(p-2)}{2} \frac{\langle D^2 z(\nabla u), \nabla u \rangle}{z} - \frac{(p-2)}{4} \frac{|\nabla z|^2}{z} + \frac{(p-2)}{2} \frac{\langle \nabla z, \nabla u \rangle^2}{z^2} + \frac{q+2-p}{2} z^{\frac{q-p}{2}} \langle \nabla z, \nabla u \rangle. \tag{2.6}$$

From (2.5) and (2.6)

$$(\Delta u)^2 \geq \frac{1}{2} z^{q+2-p} - \frac{(p-2)^2}{4} \frac{\langle \nabla z, \nabla u \rangle^2}{z^2}, \tag{2.7}$$

and from (2.4),

$$\Delta z + (p-2) \frac{\langle D^2 z(\nabla u), \nabla u \rangle}{z} \geq \frac{1}{N} z^{q+2-p} - \frac{(p-2)^2}{2N} \frac{\langle \nabla z, \nabla u \rangle^2}{z^2} - \frac{(p-2)}{2} \frac{|\nabla z|^2}{z} + (p-2) \frac{\langle \nabla z, \nabla u \rangle^2}{z^2} + (q+2-p) z^{\frac{q-p}{2}} \langle \nabla z, \nabla u \rangle. \tag{2.8}$$

Noticing that $\frac{\langle \nabla z, \nabla u \rangle^2}{z^2} \leq \frac{|\nabla z|^2}{z}$, and that for any $\epsilon > 0$

$$z^{\frac{q-p}{2}} |\langle \nabla z, \nabla u \rangle| \leq z^{\frac{q+2-p}{2}} \frac{|\nabla z|}{\sqrt{z}} \leq \epsilon z^{q+2-p} + \frac{1}{4\epsilon} \frac{|\nabla z|^2}{z},$$

we obtain that the right-hand side of (2.8) is bounded from below by the quantity

$$\frac{1}{2N} z^{q+2-p} - D \frac{|\nabla z|^2}{z},$$

where $D = D(p, q, N) > 0$. We define the operator

$$v \mapsto \mathcal{A}(v) := -\Delta v - (p-2) \frac{\langle D^2 v(\nabla u), \nabla u \rangle}{|\nabla u|^2} = -\sum_{i,j=1}^{N} a_{ij} v_{x_i x_j} \tag{2.9}$$

where the a_{ij} depend on ∇u; since $|\nabla u|^2 = z$,

$$\theta |\xi|^2 := \min\{1, p-1\} |\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij} \xi_i \xi_j \leq \max\{1, p-1\} |\xi|^2 := \Theta |\xi|^2, \tag{2.10}$$

for all $\xi = (\xi_1, ..., \xi_N) \in \mathbb{R}^N$. Consequently, $\mathcal{A}$ is uniformly elliptic in G and z satisfies

$$\mathcal{L}(z) := \mathcal{A}(z) + \frac{1}{2N} z^{q+2-p} - D \frac{|\nabla z|^2}{z} \leq 0 \qquad \text{in } \Omega. \tag{2.11}$$

Consider a ball $B_a(R) \subset \Omega$ and set $w(x) = \tilde{w}(|x-a|) = \lambda (R^2 - |x-a|^2)^{-\frac{2}{q+1-p}}$. Put $r = |x-a|$, then

$$w_{x_i} = \frac{4\lambda}{q+1-p} (R^2 - |x-a|^2)^{-\frac{2}{q+1-p}-1} x_i,$$

$$w_{x_ix_j} = \frac{4\lambda}{q+1-p}(R^2-r^2)^{-\frac{2}{q+1-p}-1}\delta_{ij} + \frac{8(3+q-p))\lambda}{(q+1-p)^2}(R^2-r^2)^{-\frac{2}{q+1-p}-2}x_ix_j,$$

therefore

$$|\nabla w|^2 = \frac{16\lambda^2}{(q+1-p)^2}(R^2-r^2)^{-\frac{4}{q+1-p}-2}r^2,$$

$$\frac{|\nabla w|^2}{w} = \frac{16\lambda}{(q+1-p)^2}(R^2-r^2)^{-\frac{2}{q+1-p}-2}r^2,$$

$$w^{q+2-p} = \lambda^{q+2-p}(R^2-r^2)^{-2\frac{q+2-p}{q+1-p}} = \lambda^{q+2-p}(R^2-r^2)^{-\frac{2}{q+1-p}-2},$$

and finally

$$\mathcal{A}(w) \geq -\frac{4\Theta\lambda}{q+1-p}(R^2-r^2)^{-\frac{2}{q+1-p}-2}\left(NR^2 + \left(\frac{3+q-p}{q+1-p} - N\right)r^2\right).$$

At the end, using the fact that $r \leq R$, we obtain

$$\mathcal{L}(w) \geq \frac{\lambda}{2N}(R^2-r^2)^{-\frac{2}{q+1-p}-2}\left(\lambda^{q+1-p} - cR^2\right),$$

where $c = c_{N,p,q}$. We choose $\lambda = (cR^2)^{\frac{1}{q+1-p}}$ and derive $\mathcal{L}(w) \geq 0$. We take for G a connected component of $\{x \in B_R(a) : z(x) > w(x)\}$, thus $z(x) > 0$ in G and $\overline{G} \subset \overline{B}_R(a)$. If $x_0 \in G$ is such that $z(x_0) - w(x_0) = \max\{z(x) - w(x) : x \in G\}$, then $\nabla z(x_0) = \nabla w(x_0)$, $z(x_0) > w(x_0) > 0$ and

$$\mathcal{A}(z(x_0)-w(x_0)) + \frac{1}{2N}\left(z(x_0)^{q+2-p} - w(x_0)^{q+2-p}\right) - D\,|\nabla z|^2\left(\frac{1}{z(x_0)} - \frac{1}{w(x_0)}\right) \leq 0,$$

which contradicts the fact that all the terms are nonnegative with the exception of $z(x_0)^{q+2-p} - w(x_0)^{q+2-p}$ which is positive. Therefore $z \leq w$ in $B_R(a)$. In particular

$$z(a) \leq w(a) = c'_{N,p,q}R^{-\frac{2}{q+1-p}}. \tag{2.12}$$

Letting $R \to d(x)$ yields (2.1). □

2.2 Applications

The first estimate is a pointwise one for solutions with possible isolated singularities if $q \leq p$.

Corollary 2.2 *Assume $q > p-1 > 0$, Ω is a domain containing 0 and let $R^* > 0$ be such that $d(0) \geq 2R^*$. Then for any $x \in B_{R^*} \setminus \{0\}$, and $0 < R \leq R^*$, any $u \in C^2(\Omega \setminus \{0\})$ solution of (1.1) in $\Omega \setminus \{0\})$ satisfies*

$$|u(x)| \leq c_{N,p,q}\left||x|^{-\beta_q} - R^{-\beta_q}\right| + \max\{|u(z)| : |z| = R\}, \tag{2.13}$$

if $p \neq q$, and

$$|u(x)| \leq c_{N,p}\left(\ln|R| - \ln|x|\right) + \max\{|u(z)| : |z| = R\}, \tag{2.14}$$

if $p = q$.

Proof. Let $X = \frac{R}{|x|}x$, then $\operatorname{dist}(tx + (1-t)X, \partial B_R(X)) = t\,|x| + (1-t)R$ for any $0 < t < 1$, thus by (2.1) in $B_{R^*} \setminus \{0\}$

$$|u(x)| = \left| u(X) + \int_0^1 \frac{d}{dt} u(tx + (1-t)X)dt \right|$$
$$\leq |u(X)| + c_{N,p,q}\,|x - X| \int_0^1 (t\,|x| + (1-t)R)^{-\frac{1}{q+1-p}} dt.$$

By integration, we obtain (2.13) or (2.14). In the particular case where $p > q$ and $|x| \leq \frac{R}{2}$, we obtain

$$|u(x)| \leq c_{N,p,q}|x|^{-\beta_q} + \max\{|u(z)| : |z| = R\}. \tag{2.15}$$

□

The second estimate corresponds to solutions with an eventual boundary blow-up if $q \leq p$..

Corollary 2.3 *Assume $q > p - 1 > 0$, Ω is a bounded domain with a C^2 boundary. Then there exists $\delta_* > 0$ such that if we denote $\Omega_{\delta_*} := \{z \in \Omega : d(z) \leq \delta_*\}$, any $u \in C^2(\Omega)$ solution of (1.1) in Ω satisfies*

$$|u(x)| \leq c_{N,p,q} \left| (d(x))^{-\beta_q} - \delta^{*-\beta_q} \right| + \max\{|u(z)| : d(z) = \delta_*\} \quad \forall x \in \Omega_{\delta_*} \tag{2.16}$$

if $p \neq q$ and

$$|u(x)| \leq c_{N,p,q} (\ln \delta_* - \ln d(x)) + \max\{|u(z)| : d(z) = \delta_*\} \quad \forall x \in \Omega_{\delta_*} \tag{2.17}$$

if $p = q$.

Proof. We denote by δ_* the maximal $r > 0$ such that any boundary point a belongs to a ball $B_r(a_i)$ of radius r such that $B_r(a_i) \subset \overline{\Omega}$ and to a ball $B_r(a_s)$ with radius r too such that $B_r(a_s) \subset \overline{\Omega}^c$. If $x \in \Omega_{\delta_*}$, we denote by $\sigma(x)$ its projection onto $\partial\Omega$ and by $\mathbf{n}_{\sigma(x)}$ the outward normal unit vector to $\partial\Omega$ at $\sigma(x)$ and $z^* = \sigma(x) - 2\delta_*\mathbf{n}_{\sigma(x)}$. Then

$$u(x) = u(z^*) + \int_0^1 \frac{d}{dt} u(tx + (1-t)z^*)dt = \int_0^1 \langle \nabla u(tx + (1-t)z^*), x - z^* \rangle dt$$

thus

$$|u(x)| \leq |u(z^*)| + c_{N,p,q}(\delta_* - d(x)) \int_0^1 (td(x) + (1-t)d(z^*))^{-\frac{1}{q+1-p}} dt.$$

Integrating this relation we obtain (2.16) and (2.17). □

Remark. As a consequence of (2.16) there holds for $p > q > p - 1$

$$u(x) \leq (c_{N,p,q} + K \max\{|u(z)| : d(z) \geq \delta_*\})\,(d(x))^{-\beta_q} \quad \forall x \in \Omega \tag{2.18}$$

where $K = (\operatorname{diam}(\Omega))^{\frac{p-q}{q+1-p}}$, with the standard modification if $p = q$.

As a variant of Corollary 2.3 we have estimate of solutions in an exterior domain

Corollary 2.4 *Assume $q > p-1 > 0$, $R_0 > 0$ and let $u \in C^2(B^c_{R_0})$ be any solution of* (1.1) *in $B^c_{R_0}$. Then for any $R > R_0$ there holds*

$$|u(x)| \leq c_{N,p,q} \left|(|x| - R_0)^{-\beta_q} - (R - R_0)^{-\beta_q}\right| + \max\{|u(z)| : |z| = R\} \quad \forall x \in B^c_R \tag{2.19}$$

if $p \neq q$ and

$$|u(x)| \leq c_{N,p,q} \left(\ln(|x| - R_0) - \ln(R - R_0)\right) + \max\{|u(z)| : |z| = R\} \quad \forall x \in B^c_R \tag{2.20}$$

if $p = q$.

Proof. The proof is a consequence of the identity

$$u(x) = u(z) + \int_0^1 \frac{d}{dt} u(tx + (1-t)z)dt = \int_0^1 \langle \nabla u(tx + (1-t)z), x - z \rangle dt$$

where $z = \frac{R}{|x|}x$. Since

$$|\nabla u(tx + (1-t)z)| \leq C_{N,p,q}(t\,|x| + (1-t)R - R_0)^{-\frac{1}{q+1-p}}$$

by estimate (2.1), the result follows by integration. □

An important consequence of the gradient estimate is the Harnack inequality.

Proposition 2.5 *Assume $q > p - 1$ and let $u \in C^1(\Omega)$ be a positive solution of* (1.1) *in Ω. Then there exists a constant $C = C(n, p, q) > 0$ such that for any $a \in \Omega$ and $R > 0$ such that $\overline{B}_R(a) \subset \Omega$, there holds*

$$\max\{u(x) : x \in B_{R/2}(a)\} \leq C \min\{u(x) : x \in B_{R/2}(a)\}. \tag{2.21}$$

Proof. We can assume $a = 0$ in Ω and $R < d(0) = \operatorname{dist}(0, \partial\Omega)$. Then we write (1.1)

$$-\Delta_p u + C(x)\,|\nabla u|^{p-1} = 0$$

with $|C(x)| = |\nabla u|^{q+1-p} \leq c_{N,p,q} R^{-1}$ by (2.1). Set $u_R(y) = u(Ry)$, then u_R satisfies

$$-\Delta_p u_R + RC(Ry)\,|\nabla u_R|^{p-1} = 0 \qquad \text{in } B_1.$$

Since $RC(Ry)$ is bounded in B_1, we can apply Serrin's results (see [36]) and obtain

$$\max\{u_R(y) : y \in B_{1/2}(0)\} \leq C \min\{u(y) : y \in B_{1/2}(0)\}. \tag{2.22}$$

Then (2.21) follows. □

The following Liouville result which improves a previous one due to Farina and Serrin [11, Th 7], is a direct consequence of the gradient estimate.

Corollary 2.6 *Assume $q > p - 1 > 0$. Then any signed solution of* (1.1) *in $\mathbb{R}^N$ is a constant.*

Proof. We apply of (2.1) in $B_R(a)$ for any $R > 0$ and $a \in \mathbb{R}^N$ and let $R \to \infty$. □

3 Singularities in a domain

3.1 Radial solutions

If u is a radial function, we put $u(x) = u(|x|) = u(r)$, with $r = |x|$. If u is a radial solution of (1.1) in $B_1^* := B_1 \setminus \{0\}$, it satisfies

$$\left(|u'|^{p-2}u'\right)' + \frac{N-1}{r}|u'|^{p-2}u' - |u'|^q = 0 \tag{3.1}$$

in $(0,1)$. We suppose $q < p$, then $p-1 < q_c < p \leq N$. We set

$$b = \frac{N(p-1)-(N-1)q}{q+1-p} = \frac{(N-1)(q_c-q)}{q+1-p}. \tag{3.2}$$

The next result provides the classification of radial solutions according to their sign near 0.

Proposition 3.1 *Let u be a nontrivial solution of* (3.1), *then*

$$u'(r) = \begin{cases} -r^{\frac{1-N}{p-1}}\left(b^{-1}r^{\frac{q+1-p}{p-1}b} + K\right)^{\frac{1}{p-1-q}} & \text{if } q \neq q_c, \\ -r^{\frac{1-N}{p-1}}\left(\left|\ln r^{N-1}\right| + K\right)^{\frac{1-N}{p-1}} & \text{if } q = q_c. \end{cases} \tag{3.3}$$

As a consequence there holds

1- If u is positive near 0 and $p-1 < q < q_c$,

(i) either there exists $k > 0$ such that

$$u(r) = \begin{cases} k\frac{N-p}{p-1}r^{\frac{p-N}{p-1}} + O(r^{\frac{q+1-N}{p-1}b} \vee 1) & \text{if } p < N, \\ -k\ln r + O(1) & \text{if } p = N, \end{cases} \tag{3.4}$$

and u is a radial solution of

$$-\Delta_p u + |\nabla u|^q = c_{N,p}k^{p-1}\delta_0 \qquad \text{in } \mathcal{D}'(B_1), \tag{3.5}$$

(ii) or

$$u(r) = \lambda_{N,p,q}r^{-\beta_q} + M, \tag{3.6}$$

where

$$\lambda_{N,p,q} = \beta_q^{-1}b^{\frac{1}{q+1-p}}. \tag{3.7}$$

If u is positive near 0 and $q \geq q_c$, then u is constant.

2- If u is negative near 0: then for $p-1 < q < q_c$, there exists $k < 0$ such that

$$u(r) = \begin{cases} k\frac{N-p}{p-1}r^{\frac{p-N}{p-1}} + O(r^{\frac{q+1-N}{p-1}b} \vee 1) & \text{if } p < N, \\ -k\ln r + O(1) & \text{if } p = N, \end{cases} \tag{3.8}$$

and u is a radial solution of

$$\Delta_p u - |\nabla u|^q = -c_{N,p}(-k)^{p-1}\delta_0 \qquad \text{in } \mathcal{D}'(B_1). \tag{3.9}$$

If $q = q_c$, then

$$u(r) = \begin{cases} -\nu_{N,p} r^{\frac{p-N}{p-1}} (-\ln r)^{-\frac{p-1}{N-1}} (1+o(1)) & \text{if } p < N \\ -\nu_N \ln(-\ln r)(1+o(1)) & \text{if } p = N \end{cases} \tag{3.10}$$

for some for some constant $\nu_{N,p}$, $\nu_N > 0$.
If $q > q_c$,

$$u(r) = -\lambda_{N,p,q} r^{-\beta_q} + M. \tag{3.11}$$

Proof. We set

$$w(r) = r^{N-1} \left|u'\right|^{p-2} u', \tag{3.12}$$

then

$$w'(r) = r^{-\frac{(q+1-p)(N-1)}{p-1}} |w|^{\frac{q}{p-1}} \tag{3.13}$$

Thus

$$-|w|^{-\frac{q}{p-1}} w = \begin{cases} b^{-1} r^{\frac{q+1-p}{p-1}b} + K & \text{if } q \neq q_c \\ \ln\left(K r^{\frac{1}{N-1}}\right) & \text{if } q = q_c \end{cases} \tag{3.14}$$

for some K.
1-Case $p - 1 < q < q_c$, then $b > 0$. If $K > 0$ then w' and u' are negative and

$$u'(r) = -r^{\frac{1-N}{p-1}} \left[b^{-1} r^{\frac{q+1-p}{p-1}b} + K\right]^{-\frac{1}{q+1-p}} = -k' r^{\frac{p-N}{p-1}} + O(r^{\frac{q+2-p-N}{p-1}}). \tag{3.15}$$

Integrating again, we get (3.4) From the asymptotic of $u'(r)$ we derive that u is a radial solution of (3.5). If $K = 0$, then $u'(r) = -r^{-\frac{N-1+b}{p-1}} b^{\frac{1}{q+1-p}}$ and we get (3.6), (3.7). This is the explicit particular solution.
If $K = -\tilde{K} < 0$, then $w' > 0$ near 0. We set $\tilde{w} = -w$, $\tilde{u} = -u$ and $\tilde{w}(r) = r^{N-1} |\tilde{u}'|^{p-2} \tilde{u}'$. Thus,

$$\tilde{u}(r) = \tilde{k}'' r^{\frac{p-N}{p-1}} + O(r^{\frac{q+1-N}{p-1}b} \vee 1) \text{ or } u(r) = -\tilde{k}'' \ln r + O(1), \tag{3.16}$$

according $p < N$ or $p = N$, and $\tilde{u}$ satisfies

$$-\Delta_p \tilde{u} - |\nabla u|^q = c_{N,p} \tilde{k}' \delta_0 \qquad \text{in } \mathcal{D}'(B_1). \tag{3.17}$$

2-Case $q \geq q_c$. Then $b \leq 0$. If $q > q_c$ (equivalently $b < 0$), (3.6) implies

$$u'(r) = r^{\frac{1-N}{p-1}} \left[-b^{-1} r^{\frac{q+1-p}{p-1}b} + K\right]^{-\frac{1}{q+1-p}} = (-b)^{\frac{1}{q+1-p}} r^{-\frac{1}{q+1-p}} (1+o(1)), \tag{3.18}$$

then
$$u(r) = -\lambda_{N,p,q} r^{-\beta_q}(1+o(1)). \tag{3.19}$$
If $q = q_c$,
$$u'(r) = r^{\frac{1-N}{p-1}} \left[(1-N)^{-1}\ln r + K\right]^{-\frac{p-1}{N-1}} (1+o(1)), \tag{3.20}$$
and, either $p < N$ and
$$u(r) = -\nu_{N,p} r^{\frac{p-N}{p-1}} (-\ln r)^{-\frac{p-1}{N-1}} (1+o(1)), \tag{3.21}$$
or $p = N$ and
$$u(r) = -\nu_N \ln(-\ln r)(1+o(1)), \tag{3.22}$$
for some constant $\nu_{N,p}$, $\nu_N > 0$. □

Proposition 3.2 *Assume* $1 < p \leq N$ *and* $p - 1 < q < q_c$, *then for any* $k > 0$ *there exists a unique positive solution* $u = u_k$ *of* (3.1) *in* $(0, 1)$ *vanishing for* $r = 1$ *satisfying*
$$\lim_{r\to 0} \frac{u_k(r)}{\mu_p(r)} = k. \tag{3.23}$$
When $k \to \infty$, $u_k \uparrow u_\infty$ *which is a solution of* (3.1) *in* $(0, 1)$ *vanishing on* ∂B_1 *satisfying*
$$\lim_{r\to 0} r^{\beta_q} u_\infty(r) = \lambda_{N,p,q}. \tag{3.24}$$

Proof. Using (3.15) we see that K is completely determined by $K = k^{p-1-q}$ and u_k by
$$u_k(r) = \int_r^1 s^{\frac{1-N}{p-1}} \left[b^{-1} s^{\frac{q+1-p}{p-1}b} + k^{p-1-q}\right]^{-\frac{1}{q+1-p}} ds. \tag{3.25}$$
Conversely, asymptotic expansion in (3.25) yields to (3.14). The unique characterization of K yields to uniqueness although uniqueness is also a consequence of the maximum principle as we will see it in the non radial case. Clearly the function u_k defined by (3.25) is increasing and $u_\infty = \lim_{k\to\infty} u_k$ satisfies
$$u_\infty(r) = \int_r^1 s^{\frac{1-N}{p-1}} \left[b^{-1} s^{\frac{q+1-p}{p-1}b}\right]^{-\frac{1}{q+1-p}} ds = \lambda_{N,p,q}(r^{-\beta_q} - 1). \tag{3.26}$$

Proposition 3.3 *Assume* $1 < p \leq N$ *and* $p - 1 < q < q_c$. *If* u *is a nonnegative radial solution of* (3.1) *in* $(0, \infty)$. *Then*
(i) either $u(r) \equiv M$ *for some* $M \geq 0$, *or*
(ii) there exist $k > 0$ *and* $M \geq 0$ *such that*
$$u(r) = u_{k,M}(r) := \int_r^\infty s^{\frac{1-N}{p-1}} \left[b^{-1} s^{\frac{q+1-p}{p-1}b} + k^{p-1-q}\right]^{-\frac{1}{q+1-p}} ds + M, \tag{3.27}$$
(ii) or there exists $M \geq 0$ *such that*
$$u(r) = u_{\infty,M}(r) := \lambda_{N,p,q} r^{-\beta_q} + M \tag{3.28}$$

Proof. From identity (3.15), valid for any *nonconstant* solution u, we see that for a global positive solution we must have $K \geq 0$. If $K = 0$ then $u = u_\infty$ defined by (3.27). If $K > 0$, then $u' \in L^1(1,\infty)$, thus $u(\infty) = \lim_{s\to\infty} u(s)$ exists and

$$u(r) = u(\infty) + \int_r^\infty s^{\frac{1-N}{p-1}} \left[b^{-1} s^{\frac{q+1-p}{p-1}b} + K\right]^{-\frac{1}{q+1-p}} ds, \tag{3.29}$$

and $K = k^{p-1-q}$ in order to have (3.27). □

3.2 Removable singularities

3.2.1 Removable singularities of renormalized solutions with right-hand side measures

In this section Ω is any domain of $\mathbb{R}^N$. We denote by $\mathfrak{M}(\Omega)$ the set of Radon measures in Ω and we study a more general equation than (1.1)

$$-\Delta_p u + |\nabla u|^q = \mu, \tag{3.30}$$

where $\mu \in \mathfrak{M}(\Omega)$. For any $r > 1$, the $C_{1,r}$ capacity is defined by

$$C_{1,r}(K) = \inf\{\|\psi\|^r_{W^{1,r}} : \psi \in C_c^\infty(\mathbb{R}^N), \chi_K \leq \psi \leq 1\}$$

for any compact subset K of $\mathbb{R}^N$, and extended to capacitable sets by the classical method. We set

$$\mathfrak{M}^r(\Omega) = \{\mu \in \mathfrak{M}(\Omega) : \mu(K) = 0\ \forall K \subset \Omega,\ K \text{ compact s.t. } C_{1,r}(K) = 0\}.$$

We recall that any measure μ in Ω can be decomposed in a unique way as

$$\mu = \mu_0 + \mu_s^+ - \mu_s^- \tag{3.31}$$

where $\mu_0 \in \mathfrak{M}^r(\Omega)$ and $\mu_s^\pm$ are nonnegative measures concentrated on sets with zero $C_{1,r}$ capacity.

In order to study equation (3.30) it is natural to introduce other notions of solutions than the strong ones. We use the notion of locally renormalized solutions introduced in [3], which gives a local version of the notion of renormalized solutions very much used in [10], [26], [25].

For $k > 0$ and $s \in \mathbb{R}$, we define the truncation $T_k(s) = \max\{-k, \min\{k, s\}\}$. If u is measurable and finite a.e. and if $T_k(u) \in W^{1,p}_{loc}(\Omega)$, we define the gradient a.e. of u by $\nabla T_k(u) = \chi_{|u|\leq k}\nabla u$, for any $k > 0$. We denote by q_* the conjugate exponent of $\frac{q}{p-1}$

$$q_* = \frac{q}{q+1-p}, \tag{3.32}$$

i.e. the conjugate of q if $p = 2$.

Definition 3.4 *Let u be a measurable and finite a.e. function in Ω. Let $\mu = \mu_0 + \mu_s^+ - \mu_s^- \in \mathfrak{M}(\Omega)$ with $\mu_0 \in \mathfrak{M}^p(\Omega)$ and $\mu_s^\pm$ singular and nonnegative as in* (3.31)*.*

1- We say that u is weak solution of (3.30) *if $T_k(u) \in W^{1,p}_{loc}(\Omega)$ for any $k > 0$, $|\nabla u|^q \in L^1_{loc}(\Omega)$ and* (3.30) *holds in the sense of distributions in Ω.*

2- Assuming Ω is bounded, we say that u is a renormalized (abridged R-solution) solution of (1.1) *such that $u = 0$ on $\partial\Omega$ if $T_k(u) \in W^{1,p}_0(\Omega)$ for any $k > 0$, $|\nabla u|^q \in L^1(\Omega)$ and*

$$|u|^{p-1} \in L^\sigma(\Omega),\ \forall \sigma \in [1, \frac{N}{N-p}) \text{ and } |\nabla u|^{p-1} \in L^\tau(\Omega),\ \forall \tau \in [1, \frac{N}{N-1}), \tag{3.33}$$

and for any $h \in W^{1,\infty}(\mathbb{R})$ such that h' has compact support and any $\phi \in W^{1,m}(\Omega)$ for some $m > N$ such that $h(u)\phi \in W^{1,m}_0(\Omega)$ there holds

$$\begin{aligned}\int_\Omega \left(|\nabla u|^{p-1}\nabla u.\nabla(h(u)\phi) + |\nabla u|^q h(u)\phi\right) dx \\ = \int_\Omega h(u)\phi d\mu_0 + h(\infty)\int_\Omega \phi d\mu_s^+ - h(-\infty)\int_\Omega \phi d\mu_s^-.\end{aligned} \tag{3.34}$$

3- We say that u is a local renormalized (abridged LR-solution) of solution of (1.1) *if $T_k(u) \in W^{1,p}_{loc}(\Omega)$ for any $k > 0$, $|\nabla u|^q \in L^1_{loc}(\Omega)$ and*

$$|u|^{p-1} \in L^\sigma_{loc}(\Omega),\ \forall \sigma \in [1, \frac{N}{N-p}) \text{ and } |\nabla u|^{p-1} \in L^\tau_{loc}(\Omega),\ \forall \tau \in [1, \frac{N}{N-1}) \tag{3.35}$$

and for any $h \in W^{1,\infty}(\mathbb{R})$ such that h' has compact support and any $\phi \in W^{1,m}(\Omega)$ for some $m > N$ with compact support and such that $h(u)\phi \in W^{1,m}_0(\Omega)$ identity (3.34) *holds.*

Remark. If $q \geq 1$ and u is a weak solution, then it satisfies (3.34), see [10, Lemmas 2.2, 2.3].

Our main removability result is the following.

Theorem 3.5 *Assume $0 < p - 1 < q \leq p$. If $F \subset \Omega$ is a relatively closed set such that $C_{1,q_*}(F) = 0$, and $\mu \in \mathfrak{M}^{q^*}(\Omega)$.*

(i) Let $p - 1 < q \leq p$ and u be a LR-solution of (3.30) *in $\Omega \setminus F$. Then u is a LR-solution of* (3.30) *in Ω.*

(ii) Let $q > p$ and u be a weak solution of (3.30) *in $\Omega \setminus F$. Then u is a weak solution of* (3.30) *in Ω.*

Proof. Notice that a set F with $C_{1,q_*}(F) = 0$, has zero measure; since u is defined up to a set of zero measure, any extension of u to F is valid. Notice also that if $p - 1 < q < q_c$ then $W^{1,q_*}(\mathbb{R}^N)$ is imbedded into $C(\mathbb{R}^N)$, therefore only the empty set has zero C_{1,q_*} capacity.

(i) From our assumption $T_k(u)\in W^{1,p}_{loc}(\Omega\setminus\{F\})$ for any $k>0$, $|u|^{p-1}\in L^{\sigma}_{loc}(\Omega)$ for all $\sigma\in[1,\frac{N}{N-p})$, $|\nabla u|^{p-1}\in L^{\tau}_{loc}(\Omega)$ for all $\tau\in[1,\frac{N}{N-1})$, and $|\nabla u|^q\in L^1_{loc}(\Omega)$. Since $p\leq q_*$, for any compact $K\subset\Omega$, $C_{1,p}(F\cap K)=0$. Thus $T_k(u)\in W^{1,p}_{loc}(\Omega)$ by [15, Th 2.44]. Because u is measurable and finite a.e. on Ω, we can define ∇u a.e. in Ω by the formula $\nabla u=\nabla T_k(u)$ a.e. on the set $\{x\in\Omega:|u(x)|\leq k\}$.

Let $\zeta\in C^{\infty}_c(\Omega)$ with support in $\omega\subset\bar\omega\subset\Omega$, $\zeta\geq 0$. Set $K_\zeta=F\cap supp\,\zeta$. Then K_ζ is compact and $C_{1,q_*}(K_\zeta)=0$, thus there exists $\zeta_n\in C^\infty_0(\Omega)$ such that $0\leq\zeta_n\leq 1$, $\zeta_n=1$ in a neighborhood of K_ζ that we can assumed to be contained in ω, such that $\zeta_n\to 0$ in $W^{1,q_*}(\mathbb{R}^N)$. It can also be assumed that $\zeta_n(x)\to 0$ for all $x\in\mathbb{R}^N\setminus E$ where E is a Borel set such that $C_{1,q_*}(E)=0$ (see e.g. [1, Lemmas 2.1, 2.2]). Let $\xi_n=\zeta(1-\zeta_n)$. Since u is a weak solution of (3.30) in $\Omega\setminus F$, we can take $\xi_n^{q_*}$ as a test function and get

$$\int_\Omega\left(|\nabla u|^q\xi_n^{q_*}+q_*\xi_n^{q_*-1}|\nabla u|^{p-2}\nabla u.\nabla\xi_n\right)dx=\int_\Omega\xi_n^{q_*}\,d\mu.$$

By Hölder's inequality, for any $\eta>0$,

$$\begin{aligned}\int_\Omega|\nabla u|^q\xi_n^{q_*}\,dx&\leq q_*\int_\Omega\xi_n^{q_*-1}|\nabla u|^{p-1}|\nabla\xi_n|dx\\&\leq(q_*-1)\eta^{\frac{q}{p-1}}\int_\Omega|\nabla u|^q\xi_n^{q_*}\,dx+\eta^{-q_*}\int_\Omega|\nabla\xi_n|^{q_*}\,dx.\end{aligned}$$

Hence, taking η small enough,

$$\begin{aligned}\int_\Omega|\nabla u|^q\xi_n^{q_*}\,dx&\leq c\left(\int_\Omega|\nabla\xi_n|^{q_*}\,dx+\left|\int_\Omega\xi_n^{q_*}\,d\mu\right|\right)\\&\leq c\left(\int_\Omega\left(|\nabla\zeta|^{q_*}+|\nabla\zeta_n|^{q_*}\right)dx+\int_{\mathrm{supp}(\zeta)}d\,|\mu|\right).\end{aligned}$$

with $c=c(p,q)>0$. From Fatou's lemma, we get $|\nabla u|^q\zeta^{q_*}\in L^1(\Omega)$ and

$$\int_\Omega|\nabla u|^q\zeta^{q_*}\,dx\leq c_\zeta:=c\left(\int_\Omega|\nabla\zeta|^{q_*}\,dx+\int_{\mathrm{supp}(\zeta)}d\,|\mu|\right).\tag{3.36}$$

Taking now $T_k(u)\xi_n^{q_*}$ as test function, we obtain

$$\begin{aligned}\int_\Omega|\nabla(T_k(u))|^p\xi_n^{q_*}\,dx+\int_\Omega|\nabla u|^qT_k(u)\xi_n^{q_*}\,dx&=-\int_\Omega T_k(u)|\nabla u|^{p-2}\nabla u.\nabla\xi_n^{q_*}\,dx\\&\quad+\int_\Omega T_k(u)\xi_n^{q_*}\,d\mu_0+k\left(\int_\Omega\xi_n^{q_*}(d\mu_s^+-d\mu_s^-)\right).\end{aligned}$$

Then we deduce, from Hölder's inequality,

$$\begin{aligned}\frac{1}{k}\left|\int_\Omega T_k(u)|\nabla u|^{p-2}\nabla u.\nabla\xi_n^{q_*}dx\right| &\leq q_*\int_\Omega\left(\zeta^{q_*-1}|\nabla u|^{p-1}|\nabla\zeta|+\zeta^{q_*}|\nabla u|^{p-1}|\nabla\zeta_n|\right)dx\\ &\leq (2q_*-1)\int_\Omega\left(|\nabla u|^q\zeta^{q_*}+|\nabla\zeta|^{q_*}+q_*\zeta^{q_*}|\nabla\zeta_n|^{q_*}\right)dx\\ &\leq 2q_*c_\zeta+\int_\Omega|\nabla\zeta|^{q_*}dx+o(1).\end{aligned}$$

Therefore, up to changing c_ζ into another constant c_ζ depending on ζ,

$$\int_\Omega|\nabla(T_k(u))|^p\xi_n^{q_*}dx\leq(k+1)c_\zeta+o(1),$$

and by Fatou's lemma,

$$\int_\Omega|\nabla(T_k(u))|^p\zeta^{q_*}dx\leq(k+1)c_\zeta. \tag{3.37}$$

By a variant of the results in [6],[7] due to [33] it follows that the regularity statements (3.35) of Definition 3.4 hold.

Finally, we show that u is a LR-solution in Ω. Let $h\in W^{1,\infty}(\mathbb{R})$ such h' has compact support and let $\phi\in W^{1,m}(\Omega)$ with $m>N$ with compact support in Ω, such that $h(u)\phi\in W^{1,p}(\Omega)$. Consider again ζ, ζ_n and ξ_n as above. Then $(1-\zeta_n)\phi\in W^{1,m}(\Omega\setminus F)$ and $h(u)(1-\zeta_n)\phi\in W^{1,p}(\Omega\setminus F)$ and has compact support in $\Omega\setminus F$. We can write

$$I_{1,n}+I_{2,n}+I_{3,n}+I_{4,n}=I_{5,n}+I_{6,n}^{+}+I_{6,n}^{-},$$

where

$$I_{1,n}=\int_\Omega|\nabla u|^p h'(u)(1-\zeta_n)\phi dx\,,\ I_{2,n}=-\int_\Omega h(u)\phi|\nabla u|^{p-2}\nabla u.\nabla\zeta_n dx,$$

$$I_{3,n}=\int_\Omega h(u)(1-\zeta_n)|\nabla u|^{p-2}\nabla u.\nabla\phi dx\,,\ I_{4,n}=\int_\Omega|\nabla u|^q(1-\zeta_n)\phi dx.$$

$$I_{5,n}=\int_\Omega h(u)\phi(1-\zeta_n)d\mu_0\ \text{ and }\ I_{6,n}^{\pm}=h(\pm\infty)\int_\Omega\phi(1-\zeta_n)d\mu_s^{\pm}.$$

We get $\lim_{n\to\infty}I_{1,n}=\displaystyle\int_\Omega|\nabla u|^p h'(u)\phi dx$ since there exists some $a>0$, independent of n, such that

$$I_{1,n}=\int_\Omega|\nabla T_a(u)|^p\,h'(T_a(u))(1-\zeta_n)\phi dx.$$

Furthermore $\lim_{n\to\infty}I_{2,n}=0$ since

$$\left|\int_\Omega h(u)\phi|\nabla u|^{p-2}\nabla u.\nabla\zeta_n dx\right|\leq\|h\|_{L^\infty}\left(\int_\Omega|\nabla u|^q\right)^{\frac{p-1}{q}}\|\nabla\zeta_n\|_{L^{q_*}}.$$

Next $\lim_{n\to\infty} I_{3,n} = \displaystyle\int_\Omega h(u)|\nabla u|^{p-2}\nabla u.\nabla\phi dx$ because $\nabla\phi \in L^m(\Omega)$ and $|\nabla u|^{p-1} \in L^\tau_{loc}(\Omega)$ for all $\tau \in [1, \frac{N}{N-1})$. From (3.36) $\lim_{n\to\infty} I_{4,n} = \displaystyle\int_\Omega h(u)\phi\,|\nabla u|^q\,dx$. But $h(u)\phi \in L^1(\Omega, d\,|\mu_0|)$ by [10, Remark 2.26]. Since $\zeta_n \to 0$ everywhere in $\mathbb{R}^N \setminus E$ and $\mu(E) = 0$, it follows $\lim_{n\to\infty} I_{5,n} = \displaystyle\int_\Omega h(u)\phi d\mu_0$. Clearly $\lim_{n\to\infty} I^{\pm}_{6,n} = h(\pm\infty)\displaystyle\int_\Omega \phi d\mu^{\pm}_s$. Hence u is a LR solution in whole Ω.

(ii) Let u be a weak solution in $\Omega \setminus F$. SInce $q > p$, $1 < q_* < p$, hence $u \in W^{1,q_*}_{loc}(\Omega\setminus F) = W^{1,q_*}_{loc}(\Omega)$ and $|\nabla u| \in L^1_{loc}(\Omega)$. Let $\zeta \in C^\infty_c(\Omega)$ and ζ_n and ξ_n as in (i). We obtain again $|\nabla u|^q\zeta^{q_*} \in L^1(\Omega)$. Hence $\nabla u \in L^q_{loc}(\Omega)$. Next we take ξ_n as a test function in equation (3.30) in $\Omega \setminus F$. We obtain $J_{1,n} + J_{2,n} + J_{3,n} = J_{4,n}$ with

$$J_{1,n} = \int_\Omega (1-\zeta_n)\,|\nabla u|^{p-2}\,\nabla u.\nabla\zeta dx\,,\; J_{2,n} = -\int_\Omega \zeta\,|\nabla u|^{p-2}\,\nabla u.\nabla\zeta_n dx,$$

$$J_{3,n} = \int_\Omega |\nabla u|^q\,\zeta(1-\zeta_n)dx\,,\; J_{4,n} = \int_\Omega \zeta(1-\zeta_n)d\mu.$$

We can let $n \to \infty$ in $J_{1,n}$ and $J_{3,n}$ using the dominated convergence theorem and the fact that $\nabla u \in L^q_{loc}(\Omega)$ and $q > p-1$. Furthermore $\lim_{n\to\infty} J_{2,n} = 0$ because $|\nabla u|^{p-1} \in L^{\frac{q}{p-1}}_{loc}(\Omega)$ and $|\nabla\zeta_n| \to 0$ in $L^{q_*}(\Omega)$. Since $J_{4,n} \to \displaystyle\int_\Omega \zeta d\mu$ as above, it follows that u is a weak solution in Ω. □

3.2.2 Regularity results

The natural question concerning LR-solutions obtained in Theorem 3.5 is their regularity. It is noticeable that the results are very different according to whether we consider nonnegative or signed solutions. Here we give some regularity properties of solutions of (1.1). We first consider nonnegative solutions of (1.1).

Theorem 3.6 *Let $p-1 < q$, $N \geq 2$ and u is a nonnegative LR-solution of* (1.1). *Then $u \in L^\infty_{loc}(\Omega) \cap W^{1,p}_{loc}(\Omega)$. As a consequence, if $q \leq p$, $u \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$.*

Proof. Since $-\Delta_p \leq 0$, then $u \in L^\infty_{loc}(\Omega)$ by a recent argument due to Kilpelainen and Kuusi [18] and u satisfies the weak Harnack inequality

$$\sup_{B_\rho(x_0)} \leq \rho^{-N}\left(\int_{B_{2\rho}(x_0)} u^{q_*}dx\right)^{\frac{1}{q_*}}$$

with $C = C(N, p, q_*)$. Then u coincides with $T_k(u)$ in any ball $B_\rho(x_0)$ such that $\overline{B}_{2\rho}(x_0) \subset \Omega$, for k large enough. Thus $u \in W^{1,p}_{loc}(\Omega)$. If $q \leq p$, it follows by Tolksdorff's result [39] that $u \in C^{1,\alpha}(\Omega)$. □

When we deal with signed solutions of (1.1), there is another critical value involved when $q \leq p$,

$$\tilde{q} = p - 1 + \frac{p}{N}. \tag{3.38}$$

Observe that $q_c < \tilde{q} < p$ if $1 < p < N$ and $q_c = \tilde{q} = N$ if $p = N$. For simplicity we consider solutions of

$$\begin{aligned} -\Delta_p u + |\nabla u|^q = 0 \qquad & \text{in } \Omega \\ u = 0 \qquad & \text{on } \partial\Omega, \end{aligned} \tag{3.39}$$

and we first recall some local estimates of the gradient of renormalized solutions.

Lemma 3.7 *Assume Ω is a bounded C^2 domain. Let u be a renormalized solution of the problem*

$$\begin{aligned} -\Delta_p u = f \qquad & \text{in } \Omega \\ u = 0 \qquad & \text{on } \partial\Omega \end{aligned} \tag{3.40}$$

where $f \in L^m(\Omega)$ with $1 < m < N$ and set $\bar{m} = \frac{Np}{Np-N-p} = \frac{p}{\tilde{q}}$, where $\tilde{q}$ is defined in (3.32).
(i) If $m > \frac{N}{p}$, then $u \in L^\infty(\Omega)$. If $m = \frac{N}{p}$, then $u \in L^k(\Omega)$ for $1 \leq k < \infty$. If $m < \frac{N}{p}$, then $|u|^{p-1} \in L^k(\Omega)$ with $k = \frac{Nm}{N-mp}$.
(ii) $\nabla u^{p-1} \in L^{m^*}(\Omega)$ with $m^* = \frac{Nm}{N-m}$. Furthermore, if $\bar{m} \leq m$, then $u \in W_0^{1,p}(\Omega)$.

Proof. The estimates in the case $m < \bar{m}$ are obtained in [4] following [7] and [19], by using for test functions $\phi_{\beta,\epsilon}(T_k(u))$ where

$$\phi_{\beta,\epsilon}(w) = \int_0^w (\epsilon + |t|)^{-\beta} dt$$

for some $\beta < 1$. In the case $m \geq \bar{m}$ and $1 < p < N$ there holds $L^m(\Omega) \subset W^{-1,p'}(\Omega)$, thus u is a variational solution in $W_0^{1,p}(\Omega)$. In the case $m = \bar{m}$, then $m^* = p'$ and the conclusion follows. Next, if $m > \bar{m}$ or equivalently $m^* > p'$, then for any $\sigma > p$ and $F \in (L^\sigma(\Omega))^N$, there exists a unique $w \in W_0^{1,\sigma}(\Omega)$, weak solution of

$$\begin{aligned} -\Delta_p w = div(|F|^{p-2}F) \qquad & \text{in } \Omega \\ w = 0 \qquad & \text{on } \partial\Omega, \end{aligned} \tag{3.41}$$

see [16], [20], [21]. Let v be the unique solution in $W_0^{1,1}(\Omega)$

$$\begin{aligned} -\Delta v = f \qquad & \text{in } \Omega \\ v = 0 \qquad & \text{on } \partial\Omega. \end{aligned} \tag{3.42}$$

From the classical L^p-theory, $v \in W^{2,m}(\Omega)$, thus $\nabla v \in L^{m^*}(\Omega)$. Let F be defined by $|F|^{p-2}F = \nabla v$. Then $F \in (L^\sigma(\Omega))^N$ with $\sigma = (p-1)m^* > p$. Then

$$-\Delta_p w = -\Delta v = f.$$

Thus $w = u$. This implies $u \in W_0^{1,\sigma}(\Omega)$ and therefore $|\nabla u|^{p-1} \in L^{m^*}(\Omega)$. □

Our first result is valid without any sign assumption on the solution.

Theorem 3.8 *Assume Ω is a bounded C^2 domain. Let $p-1<q<\tilde{q}$, $N\geq 2$ and u be a renormalized solution of problem* (3.39)*, such that*

$$|\nabla u|^q \in L^{m_0}(\Omega) \quad \textit{for some } m_0 > \max\{1, \tfrac{N(q+1-p)}{q}\}. \tag{3.43}$$

Then $u \in C^{1,\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$. In particular (3.43) *is satisfied if $q < q_c$, or if $q_c \leq q < \tilde{q}$ and $u \in W_0^{1,p}(\Omega)$.*

Proof. Set $f = -|\nabla u|^q \in L^{m_0}(\Omega)$. If $m_0 \geq N$, $f \in L^{N-\delta}(\Omega)$ for any $\delta \in (0, N-1]$. Then $f \in L^{m_1}(\Omega)$ with $m_1 = \frac{(p-1)m_0^*}{q}$. Note that $\frac{m_1}{m_0} = \frac{N(p-1)}{qN-p} > 1$ since $q < \tilde{q}$. By induction, starting from m_1, we can defined m_n as long as it is smaller than N by $m_n = \frac{(p-1)m_{n-1}^*}{q}$, and we find $m_n < m_{n+1}$. If $m_n < N$ for any $n \in \mathbb{N}$, the sequence $\{m_n\}$ would converge to $L = \frac{N(q+1-p)}{q}$, which is impossible since we have assumed $m_0 > L$. Therefore there exists some n_0 such that $m_{n_0} \geq N$. If $m_{n_0} = N$, (or if $m_0 = N$ we can modify it so that $m_{n_0} < N$, but $m_{n_0+1} > N$. Then we conclude as above.

If $q < q_c$, then $|\nabla u|^{p-1} \in L^{\frac{N(1-\delta)}{N-1}}(\Omega)$ for $\delta > 0$ small enough. Then we can choose m_0 such that $\max\{1, \frac{N(q+1-p)}{q}\} < m_0 < \frac{N}{N-1}$. If $q_c \leq q < \tilde{q}$ and $u \in W_0^{1,p}(\Omega)$, we choose $m_0 = \frac{p}{q}$. □

Remark. The result which holds without sign assumption on u is sharp. Indeed, the function $v = V + \tilde{\lambda}_{N,p,q}$ defined above does not satisfy assumption (3.43), since $|\nabla u|^q \in L^m(\Omega)$ if and only if $m < \frac{N}{q(\beta_q+1)} = \frac{N(q+1-p)}{q}$.

3.3 Classification of isolated singularities

3.3.1 Positive solutions

The next result provides the complete classifications of isolated singularities of nonnegative solutions of (1.1). We suppose that Ω is an open subset of $\mathbb{R}^N$ containing 0 and set $\Omega^* = \Omega \setminus \{0\}$. Without loss of generality, we can suppose that $\Omega \supset \overline{B}_1$ and we also recall that $B_1^* = B_1 \setminus \{0\}$. We recall that the fundamental solution of the p-Laplacian is defined in $\mathbb{R}_*^N$ by

$$\mu_p(x) = \begin{cases} |x|^{\frac{p-N}{p-1}} & \text{if } 1 < p < N \\ -\ln|x| & \text{if } p = N, \end{cases} \tag{3.44}$$

and it satisfies

$$-\Delta_p \mu_p = c_{N,p}\delta_0 \quad \text{in } \mathcal{D}'(\Omega). \tag{3.45}$$

Theorem 3.9 *Let $p-1 < q < q_c$ and $1 < p \leq N$. If $u \in C^1(\Omega^*)$ is a nonnegative solution of* (1.1) *in Ω^*, then we have the following alternative.*
(i) Either there exists $k \geq 0$ such that

$$\lim_{x\to 0} \frac{u(x)}{\mu_p(x)} = k \tag{3.46}$$

and u satisfies

$$-\Delta_p v + |\nabla v|^q = c_{N,p} k^{p-1} \delta_0 \quad in\ \mathcal{D}'(\Omega). \tag{3.47}$$

(ii) Or

$$\lim_{x \to 0} |x|^{\beta_q} u(x) = \lambda_{N,p,q}, \tag{3.48}$$

where β_q *and* $\lambda_{N,p,q}$ *are defined in* (1.9).

Furthermore, if Ω *is bounded, a nonnegative solution* u *in* $C(\overline{\Omega} \setminus \{0\})$ *is uniquely determined by its data on* $\partial\Omega$ *and its behaviour* (3.46) *or* (3.48) *at* 0.

We need several lemmas for proving this theorem. The method developed below for obtaining point wise estimates of the derivatives is an adaptation of a technique introduced in [12].

Lemma 3.10 *Assume* p, q *are as in Theorem 3.9 and* $\phi : (0,1] \mapsto \mathbb{R}_+$ *is a continuous decreasing function such that* $\phi(2r) \leq a\phi(r)$ *and* $r^{\frac{p-q}{q+1-p}} \phi(r) \leq c$ *for some* $a, c > 0$ *and any* $r > 0$. *If* u *is a solution of* (1.1) *in* B_1^* *such that*

$$|u(x)| \leq \phi(|x|) \qquad \forall x \in B_1^*. \tag{3.49}$$

Then there exists $C > 0$ *and* $\alpha \in (0,1)$, *both depending on* N, p, q, *such that*

$$|\nabla u(x)| \leq C\phi(|x|)\, |x|^{-1} \qquad \forall x \in B_{\frac{1}{2}} \setminus \{0\}. \tag{3.50}$$

$$\left|\nabla u(x) - \nabla u(x')\right| \leq C\phi(|x|)\, |x|^{-1-\alpha} \left|x - x'\right|^{\alpha} \qquad \forall x, x' \ s.t.\ 0 < |x| \leq \left|x'\right| \leq \frac{1}{2}. \tag{3.51}$$

Proof. Define $\Gamma := \{y \in \mathbb{R}^N : 1 < |y| < 7\}$ and $\Gamma' = \{y \in \mathbb{R}^N : 2 \leq |y| \leq 6\}$. For $0 < |x| < \frac{1}{2}$ there exists $\ell \in (0, \frac{1}{4})$ such that $2\ell \leq |x| \leq 3\ell$. We set

$$u_\ell(y) = \frac{1}{\phi(\ell)} u(\ell y).$$

Then the equation

$$-\Delta_p u_\ell + \ell^{p-q} (\phi(\ell))^{q+1-p} |\nabla u_\ell|^q = 0$$

holds in Γ. Because of (3.49) and the fact that ϕ is decreasing, $u_\ell(y) \leq 1$ on Γ. Since $\ell^{p-q}(\phi(\ell))^{q+1-p}$ remains bounded for $\ell \in (0,1]$, we can apply Tolksdorff's a priori estimates [39] and derive that

$$|\nabla u_\ell(y)| \leq C \qquad \forall y \in \Gamma^*, \tag{3.52}$$

$$\left|\nabla u_\ell(y) - \nabla u_\ell(y')\right| \leq C \left|y - y'\right|^{\alpha} \qquad \forall y, y' \in \Gamma^*, \tag{3.53}$$

for some $C = C\left(N, p, q, \|u_\ell\|_{L^\infty(\Gamma)}\right)$ and $\alpha \in (0,1)$. Putting $x = \ell y$, $x' = \ell y'$ where x, x' are such that $0 < |x| \leq |x'| \leq 2|x| \leq 1$ we have $|y'| = \frac{|x'|}{\beta} \leq \frac{2|x|}{\beta} \leq 6$ and thus

$$\left|\nabla u(x) - \nabla u(x')\right| \leq C\phi(\ell)\ell^{-1-\alpha} \left|x - x'\right|^{\alpha} \leq C\phi(|x|)\, |x|^{-1-\alpha} \left|x - x'\right|^{\alpha}.$$

If $|x'| > 2|x|$ we have

$$\left|\nabla u(x) - \nabla u(x')\right| \le C\left(\frac{\phi(|x|)}{|x|} + \frac{\phi(|x'|)}{|x|}\right) \le \frac{2C\phi(|x|)}{|x|} \le \frac{2C\phi(|x|)}{|x|^{1+\alpha}}|x-x'|^{\alpha}.$$

$\square$

Lemma 3.11 *Assume p, q are as in Theorem 3.9. Let u be a nonnegative solution of* (1.1) *in* Ω^* *such that*

$$\liminf_{x\to 0} \frac{u(x)}{\mu_p(x)} < \infty. \tag{3.54}$$

Then there exists $k \ge 0$ such that (3.46) *and* (3.47) *hold.*

Proof. Let $y \in B_{\frac{1}{2}}$. By Proposition 2.5 there exists $C = C(N,p,q) > 0$ such that

$$\max_{|z-y|\le\frac{|y|}{4}} u(z) \le C \min_{|z-y|\le\frac{|y|}{4}} u(z)$$

By a simple 2-D geometric construction and the help of trigonometric estimates it is easy to check that if $|y'| = |y|$ there exist a chain of at most 7 balls $B_{\frac{|y|}{4}}(y_j)$ with center y_j on $\{z : |z| = |y|\}$ such that $y_1 = y$, $y_{13} = y'$ and $B_{\frac{|y|}{4}}(y_j) \cap B_{\frac{|y|}{4}}(y_{j+1}) \neq \emptyset$. This implies $u(y) \le C^7 u(y')$ and, since μ_p is radial (and we note $\mu_p(x) = \mu_p(|x|)$),

$$\limsup_{x\to 0} \frac{u(x)}{\mu_p(x)} = k < \infty. \tag{3.55}$$

If $k = 0$, then $u \le \epsilon\mu_p + M$ for any $\epsilon > 0$, where $M = \max\{u(x) : |x| = 1\}$, by the comparison principle. Thus u remains bounded near 0 and therefore the singularity is removable. Next we assume $k > 0$, thus $u \le k(\mu_p - \mu_p(1)) + M$ by applying the comparison principle in $\overline{B}_1 \setminus \{0\}$.

Up to changing B_1 into $B_{r_0}a$ for some $r_0 \in (0,1)$ it implies $u(x) \le m\mu_p(x)$ for some $m \ge k$. Since $q \le q_c$, $(\mu_p(r))^{q+1-p} r^{p-q} \le c$, it follows from Lemma 3.10 that

$$|\nabla u(x)| \le C\mu_p(|x|)\,|x|^{-1} \qquad \forall x \in B_{\frac{r_0}{2}} \setminus \{0\} \tag{3.56}$$

and

$$\left|\nabla u(x) - \nabla u(x')\right| \le C\mu_p(|x|)\,|x|^{-1-\alpha}\left|x-x'\right|^{\alpha} \quad \forall x, x' \text{ s.t. } 0 < |x| \le \left|x'\right| \le \frac{r_0}{2}. \tag{3.57}$$

If we define

$$u_r(y) = \frac{u(ry)}{\mu_p(r)} \qquad \forall y \in B_{\frac{r_0}{r}}, \tag{3.58}$$

it satisfies

$$-\Delta_p u_r + (\mu_p(r))^{q+1-p} r^{p-q}\,|\nabla u_r|^q = 0 \tag{3.59}$$

in $B_{\frac{r_0}{r}}$ and the following estimates:

$$0 \leq u_r(y) \leq m\frac{\mu_p(r\,|y|)}{\mu_p(r)} \qquad \forall y \in B_{\frac{r_0}{r}} \setminus \{0\}, \tag{3.60}$$

$$|\nabla u_r(y)| \leq C\frac{\mu_p(r\,|y|)}{\mu_p(r)}\,|y|^{-1} \qquad \forall y \in B_{\frac{r_0}{2r}} \setminus \{0\}, \tag{3.61}$$

and

$$\left|\nabla u_r(y) - \nabla u_r(y')\right| \leq C\frac{\mu_p(r\,|y|)}{\mu_p(r)}\,|y|^{-1-\alpha}\left|y-y'\right|^{\alpha} \quad \forall y, y' \text{ s.t. } 0 < |y| \leq \left|y'\right| \leq \frac{1}{2r_0}. \tag{3.62}$$

Let $0 < a < b$. If we assume that $0 < a \leq |y| \leq b$, then $\frac{\mu_p(r|y|)}{\mu_p(r)}$ remains bounded independently of $r \in (0,1]$ and the set of functions $\{u_r\}_{0<r<1}$ is relatively sequentially compact in the C^1 topology of $\overline{B}_b \setminus B_a$. There exist a sequences $\{r_n\}$ converging to 0 and a function $v \in C^1(\overline{B}_b \setminus B_a)$ such that $u_{r_n} \to v$ in $C^1(\overline{B}_b \setminus B_a))$. Since $(\mu_p(r_n))^{q+1-p} r_n^{p-q} \to 0$ as $q < q_c$, v is p-harmonic in $B_b \setminus \overline{B}_a$ and nonnegative. Notice that a and b are arbitrary, therefore, using Cantor diagonal process, we can assume that v is defined in $\mathbb{R}^N_*$ and $u_{r_n} \to v$ in the C^1-loc topology of $\mathbb{R}^N_*$. If $p = N$, the positivity of v implies that v is a constant [17, Corollary 2.2], say θ. If $1 < p < N$, there holds, by [17, Theorem 2.2] and (3.60)

$$v(y) = \theta\mu_p(y) + \sigma \leq m\mu_p(y) \qquad \forall y \in \mathbb{R}^N_*, \tag{3.63}$$

for some $\theta, \sigma \geq 0$, thus $\sigma = 0$. In order to make θ precise we set

$$\gamma(r) = \sup_{|x|=r} \frac{u(x)}{\mu_p(x)},$$

then $u(x) \leq \gamma(r)\mu_p(x)$ in $B_{r_0} \setminus B_r$. This implies in particular that, for $r < s < 1$, $u(x) \leq \gamma(r)\mu_p(x)$ for any x such that $|x| = s$ and finally

$$\gamma(s) \leq \gamma(r). \tag{3.64}$$

It follows from (3.55), (3.64) that $\lim_{r\to 0} \gamma(r) = k$. There exists y_{r_n} with $|y_{r_n}| = 1$ such that $u(r_n y_{r_n}) = \mu_p(r_n)\gamma(r_n)$. Therefore

$$\lim_{r_n \to 0} \frac{u(r_n y_{r_n})}{\mu_p(r_n)} = k = \theta. \tag{3.65}$$

Consequently

$$\lim_{r\to 0} \frac{u(ry)}{\mu_p(r)} = \lim_{r\to 0} u_r(y) = \begin{cases} k\mu(y) & \text{if } 1 < p < N \\ k & \text{if } p = N. \end{cases} \tag{3.66}$$

This implies in particular

$$\lim_{x\to 0} \frac{u(x)}{\mu_p(|x|)} = k. \tag{3.67}$$

Since the convergence of u_r holds in the C^1_{loc}-topology, we also deduce that

$$\lim_{x\to 0}|x|^{\frac{N-1}{p-1}}\nabla u(x)=\begin{cases}\frac{p-N}{p-1}k\frac{x}{|x|} & \text{if } 1<p<N\\ -k\frac{x}{|x|} & \text{if } p=N.\end{cases}\tag{3.68}$$

If we plug these two estimates into the weak formulation of (1.1) we obtain (3.47). □

Lemma 3.12 *Assume p, q are as in Theorem 3.9. Let u be a positive solution of* (1.1) *in Ω^* such that*

$$\liminf_{x\to 0}\frac{u(x)}{\mu_p(x)}=\infty.\tag{3.69}$$

Then (3.48) *holds.*

Proof. If (3.54) holds, then for any $k>0$ the function u is larger than the radial solution u_k of (1.1) in B_1^* which vanishes on ∂B_1 and satisfies (3.46). When $k\to\infty$ we derive from Proposition 3.2 that

$$u(x)\geq u_\infty(|x|)=\lambda_{N,p,q}(|x|^{-\beta_q}-1).\tag{3.70}$$

Next, for any $\epsilon>0$ we denote by $\tilde u_\epsilon$ the solution of (3.1) on $(\epsilon,1)$ which satisfies $\tilde u_\epsilon(\epsilon)=\infty$. This solution is expressed from (3.15) with a negative K, namely

$$\tilde u_\epsilon(r)=b^{\frac{1}{q+1-p}}\int_r^1 s^{\frac{1-N}{p-1}}\left[s^{\frac{q+1-p}{p-1}b}-\epsilon^{\frac{q+1-p}{p-1}b}\right]^{-\frac{1}{q+1-p}}ds.\tag{3.71}$$

and existence of the blow-up at $r=\epsilon$ follows from $p>q$. By comparison principle $u\leq\tilde u_\epsilon+M$ in $B_1\setminus B_\epsilon$ where $M=\sup\{u(z):|z|=1\}$. When $\epsilon\to 0$, formula (3.71) implies that

$$\lim_{\epsilon\to 0}\tilde u_\epsilon(r)=b^{\frac{1}{q+1-p}}\int_r^1 s^{\frac{1-N}{p-1}}\left[s^{\frac{q+1-p}{p-1}b}\right]^{-\frac{1}{q+1-p}}ds=u_\infty(r).\tag{3.72}$$

Therefore $u_\infty(|x|)\leq u(x)\leq u_\infty(|x|)+M$. □

Proof of Theorem 3.9. By combining Lemma 3.11 and Lemma 3.12 we have the alternative between (i) and (ii). Assuming now that Ω is bounded and u and u' are two solutions of (1.1) in Ω^* continuous in $\overline{\Omega}\setminus\{0\}$ coinciding on $\partial\Omega$ and satisfying either (i) with the same k or (ii), then, for any $\epsilon>0$, $(1+\epsilon)u+\epsilon$ is a supersolution which dominates u' in a neighborhood of 0 and a neighborhood of $\partial\Omega$. Therefore $(1+\epsilon)u+\epsilon\geq u'$, which implies $u\leq u'$, and vice versa. □

We end this section with a result dealing with global singular solutions.

Theorem 3.13 *Let $p-1<q<q_c$ and $1<p\leq N$. If u is a nonnegative solution of* (1.1) *in $\mathbb{R}^N_*$, then u is radial and we have the following dichotomy:*

(i) either there exists $M\geq 0$ such that $u(x)\equiv M$,

(ii) either there exist $k>0$, $M\geq 0$ such that $u(x)=u_{k,M}(|x|)$ defined by (3.27)*,*

(ii) or there exists some $M\geq 0$ such that $u(x)=u_{\infty,M}(|x|)$ defined by (3.28)*.*

Proof. Step 1: Asymptotic behaviour. If u is a solution of (1.1) in an exterior domain $G \supset B_R^c$, it is bounded by Corollary 2.4. By Proposition 2.1, it satisfies

$$|\nabla u(x)| = (u_r^2 + r^{-2} \left|\nabla' u\right|^2)^{\frac{1}{2}}(r,\sigma) \leq C_{N,p,q}(r-R)^{-\frac{1}{q+1-p}} \tag{3.73}$$

for all $x = (r,\sigma) \in [R,\infty) \times S^{N-1}$. Since $q < p$,

$$\int_{R+1}^{\infty} \int_{S^{N-1}} |u_r| \, d\sigma dt < \infty,$$

therefore there exists $\phi \in L^1(S^{N-1})$ such that $u(r,.) \to \phi(.)$ in $L^1(S^{N-1})$ as $r \to \infty$. The gradient estimate implies that the set of functions $\{u(r,.)\}_{r \geq R+1}$ is relatively compact in $C(S^{N-1})$, therefore $u(r,.) \to \phi(.)$ uniformly on S^{N-1} when $r \to \infty$. If σ and σ' belong to S^{N-1}, there exists a smooth path $\gamma := \{\gamma(t) : t \in [0,1]\}$ such that $\gamma(t) \in S^{N-1}$, $\gamma(0) = \sigma$, $\gamma(1) = \sigma'$. Then

$$u(r,\sigma) - u(r,\sigma') = \int_0^1 \frac{d}{dt} u(r,\gamma(t)) dt = \int_0^1 \langle \nabla' u(r,\gamma(t)), \gamma'(t) \rangle dt,$$

and finally, using (3.73),

$$\left|u(r,\sigma) - u(r,\sigma')\right| \leq \left\|\gamma'\right\|_{L^\infty} \left|\nabla' u(r,\gamma(t))\right| \leq C_{N,p,q} \left\|\gamma'\right\|_{L^\infty} r(r-R)^{-\frac{1}{q+1-p}} \tag{3.74}$$

Letting $r \to \infty$, it implies that ϕ is a constant, say M. As a consequence we have proved that

$$\lim_{|x| \to \infty} u(x) = M. \tag{3.75}$$

Notice that we did not use the fact that u is a nonnegative solution in order to derive (3.74). Next we assume the positivity.

Step 2: End of the proof. If u satisfies (3.46) for some $k > 0$, then for any $\epsilon > 0$, there holds with the notations of Proposition 3.3

$$(1-\epsilon) u_{k,M}(|x|) \leq u(x) \leq (1-\epsilon) u_{k,M}(|x|) \qquad \forall x \in \mathbb{R}_*^N$$

This implies $u = u_{k,M}$. Similarly, if satisfies (3.48), we derive $u = u_{\infty,M}$. □

3.3.2 Negative solutions

The next result make explicit the behaviour of negative solutions near an isolated singularity.

Theorem 3.14 *Let $p - 1 < q < q_c$ and $1 < p \leq N$. If u is a negative solution of* (1.1) *in $\Omega \setminus \{0\}$, then there exists $k \leq 0$ such that (3.46) and (3.47) hold. Furthermore, if $k = 0$, u can be extended as a $C^{1,\alpha}$ solution of* (1.1) *in Ω.*

Proof. We can assume $\overline{B}_1 \subset \Omega$. Since $\tilde{u} := -u$ satisfies

$$-\Delta_p \tilde{u} = |\nabla u|^q \quad \text{in } \Omega \setminus \{0\}.$$

It follows from [2, Th 1.1] that $|\nabla u|^q \in L^1_{loc}(\Omega)$ and there exists $k \geq 0$ such that

$$-\Delta_p \tilde{u} = |\nabla u|^q + k\delta_0 \quad \text{in } \mathcal{D}'(\Omega). \tag{3.76}$$

Furthermore $|\nabla u|^{p-1} \in M_{loc}^{\frac{N}{N-1}}(\Omega)$, where M^p denotes the Marcinkiewicz space (or weak L^p space). This implies

$$B := |\nabla u|^{q+1-p} \in M_{loc}^{\frac{N(p-1)}{(q+1-p)(N-1)}}(\Omega) \subset L_{loc}^{\frac{N(p-1)}{(q+1-p)(N-1)}-\sigma}(\Omega)$$

for any $\sigma > 0$. Since $q < q_c$, it follows $B \in L^{N+\epsilon}_{loc}(\Omega)$ for some $\epsilon > 0$. We write the equation under the form

$$-\Delta_p \tilde{u} = B\,|\nabla \tilde{u}|^{p-1}. \tag{3.77}$$

As a consquence of [37, Th 1] that either there exists $k' > 0$ such as

$$\frac{1}{c'} \leq \frac{\tilde{u}}{\mu_p} \leq c' \quad \text{near } 0, \tag{3.78}$$

or u has a removable singularity at 0. If the singularity is removable, then (3.76) holds with $k = 0$. If the singularity is not removable, we set

$$\gamma = \limsup_{x \to 0} \frac{\tilde{u}(x)}{\mu_p(x)}. \tag{3.79}$$

Then there exists a sequence $\{x_n\}$ converging to 0 such that

$$\gamma = \lim_{n \to \infty} \tilde{u}(x_n)/\mu_p(x_n) \tag{3.80}$$

We set $\delta_n = |x_n|$, $\xi_n = x_n/\delta_n$ and

$$\tilde{u}_{\delta_n}(\xi) = \frac{\tilde{u}(\delta_n \xi)}{\mu_p(\delta_n)}.$$

Then

$$-\Delta_p \tilde{u}_{\delta_n} - C(\delta_n)\,|\nabla \tilde{u}_{\delta_n}|^q = 0$$

in $B_{\delta_n^{-1}} \setminus \{0\}$ where

$$C(\delta_n) = \delta_n^{p-q}(\mu(\delta_n))^{q+1-p}.$$

Since $u_{\delta_n}(\xi) \leq c\mu_p(\xi)$, we derive from Lemma 3.10

$$\begin{aligned}
&|\nabla u_{\delta_n}(\xi)| \leq c\,|\xi|^{-1}\,\mu_p(\xi) && \text{for } |\xi| \leq \tfrac{1}{2\delta_n}\\
&|\nabla u_{\delta_n}(\xi) - \nabla u_{\delta_n}(\xi')| \leq c\,|\xi - \xi'|^{\alpha}\,|\xi|^{-1-\alpha}\,\mu_p(\xi) && \text{for } |\xi| \leq |\xi'| \leq \tfrac{1}{2\delta_n}.
\end{aligned}$$

Thus, by Ascoli's theorem, the set of functions $\{u_{\delta_n}\}$ is relatively compact in the C^1_{loc}-topology of $\mathbb{R}^N_*$. Since $C(\delta_n) \to 0$, there exists a subsequence $\{\tilde{u}_{\delta_{n_k}}\}$ and a nonnegative p-harmonic function $\tilde{w}$ such that $\tilde{u}_{\delta_{n_k}} \to \tilde{v}$ as well as its gradient, uniformly on any compact subset of $\mathbb{R}^N_*$. All the positive p-harmonic functions in $\mathbb{R}^N_*$ are known (see[17]: either they are a positive constant, if $N = p$ or have the form $\lambda\mu_p + \tau$ for some λ, $\tau \geq 0$ if $1 < p < N$.
If $p = N$, we obtain from (3.80)

$$\lim_{n_k\to\infty} \frac{\tilde{u}(x_{n_k})}{\mu_p(x_{n_k})} = \gamma = \lim_{n\to\infty} \frac{\tilde{u}(x_n)}{\mu_p(x_n)} \tag{3.81}$$

Thus $\tilde{w} = \gamma$ and the limit is locally uniform with respect to ξ. Therefore for any $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that for $n \geq n_0$, there holds

$$\tilde{u}(x) \geq (\gamma - \epsilon)\mu_N(x) \quad \forall x \text{ s.t. } |x| = \delta_n.$$

By comparison it implies

$$\tilde{u}(x) \geq (\gamma - \epsilon)\mu_N(x) \quad \forall x \text{ s.t. } \delta_n \leq |x| \leq \delta_{n_0}.$$

This holds for any $n \geq n_0$ and any $\epsilon > 0$, therefore,

$$\liminf_{x\to 0} \frac{\tilde{u}(x)}{\mu_N(x)} \geq \gamma. \tag{3.82}$$

Combining with (3.79), it implies

$$\lim_{x\to 0} \frac{\tilde{u}(x)}{\mu_N(x)} = \gamma. \tag{3.83}$$

If $1 < p < N$, estimate $u_{\delta_n}(\xi) \leq C\mu_p(\xi)$ implies $\tau = 0$, thus $\tilde{w} = \lambda\mu_p$. Clearly $\lambda = \gamma$ because of (3.79). Similarly as in the case $p = N$, (3.82) and (3.83) hold. Since the convergence is in C^1, we also get

$$\lim_{x\to 0} \frac{\tilde{u}_{x_j}(x)}{\mu_{N x_j}(x)} = \gamma. \tag{3.84}$$

From (3.76) it implies that there holds

$$-\Delta_p \tilde{u} = |\nabla \tilde{u}|^q + c_{N,p}\gamma\delta_0 \quad \text{in } \mathcal{D}'(\Omega). \tag{3.85}$$

□

Remark. In the case $q > q_c$ the description of the isolated singularities is much more difficult, as it is the case if one considers the positive solutions of

$$-\Delta_p \tilde{u} = \tilde{u}^m \qquad \text{in } \Omega \setminus \{0\} \tag{3.86}$$

for $m > m_c := \frac{N(p-1)}{N-p}$ (see [38] for partial but very deep results). In the case of equation

$$-\Delta_p \tilde{u} = |\nabla \tilde{u}|^q \quad \text{in } \mathbb{R}^N_* \tag{3.87}$$

the main difficulty is to prove that there exists only one positive solution under the form $\tilde{u}(x) = \tilde{u}(r, \sigma)$, which is the function $\tilde{U}$. Equivalently it is to prove that the only positive solution of

$$\begin{aligned} -div\left(\left(\beta_q^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p-2}{2}}\nabla\omega\right) - \left(\beta_q^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{q}{2}} \\ - \beta_q(\beta_q(p-1+p-N)(\left(\beta_q^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p-2}{2}}\omega = 0 \quad \text{in } S^{N-1} \end{aligned} \tag{3.88}$$

is the constant $\tilde{\lambda}_{N,p,q}$.

4 Quasilinear equations on Riemannian manifolds

4.1 Gradient geometric estimates

In this section we assume that (M^N, g) is a N-dimensional Riemannian manifold, TM its tangent bundle, ∇u is the covariant gradient, $\langle ., . \rangle$ the scalar product expressed in the metric $g := (g_{ij})$, $Ricc_g$ the Ricci tensor and Sec_g the sectional curvature. Formula (2.3) is a particular case of the Böchner-Weitzentböck formula which is the following: if $u \in C^3(M)$ there holds

$$\frac{1}{2}\Delta_g |\nabla u|^2 = \left|D^2 u\right|^2 + \langle \nabla \Delta_g u, \nabla u \rangle + Ricc_g(\nabla u, \nabla u), \tag{4.1}$$

where $D^2 u$ is the Hessian, $\Delta_g = \text{div}_g(\nabla u)$ is the Laplace-Beltrami operator on (M^N, g) and div_g is the divergence operator acting on $C^1(M, TM)$. For $p > 1$, we also denote by $\Delta_{g,p}$ the p-Laplacian operator on M defined by

$$\Delta_{g,p} u = \text{div}_g(|\nabla u|^{p-2}\nabla u), \tag{4.2}$$

with the convention $\Delta_{2,g} = \Delta_g$. A natural geometric assumption is that the Ricci curvature is bounded from below and more precisely

$$Ricc_g(x)(\xi, \xi) \geq -(N-1)B^2 |\xi|^2 \qquad \forall \xi \in T_x M \tag{4.3}$$

for some $B \geq 0$. If $u \in C^3(M)$ is a solution of

$$-\Delta_{p,g} u + |\nabla u|^q = 0 \qquad \text{in } M, \tag{4.4}$$

then (2.8) is replaced by

$$\begin{aligned} \Delta_g z + (p-2)\frac{\langle D^2 z(\nabla u), \nabla u \rangle}{z} \geq \frac{2a^2}{N} z^{q+2-p} - \frac{1}{Na^2}\frac{\langle \nabla z, \nabla u \rangle^2}{z^2} - \frac{(p-2)}{2}\frac{|\nabla z|^2}{z} \\ + (p-2)\frac{\langle \nabla z, \nabla u \rangle^2}{z^2} + (q+2-p)z^{\frac{q-p}{2}}\langle \nabla z, \nabla u \rangle - (N-1)B^2 z. \end{aligned} \tag{4.5}$$

and $\mathcal{L}$ in (2.11) by

$$\mathcal{L}^*(z) := \mathcal{A}(z) + Cz^{q+2-p} - D\frac{|\nabla z|^2}{z} - (N-1)B^2 z \leq 0 \qquad \text{in } \Omega. \tag{4.6}$$

We recall that the convexity radius $r_M(a)$ of some $a \in M$ is the supremum of all the $r > 0$ such that the ball $B_r(a)$ is convex. Note that, in order to obtain estimates on the gradient of solution, when $p \neq 2$ an extra assumption besides (4.3) is needed; it concerns the sectional curvature.

Lemma 4.1 *Assume $q > p-1 \geq 0$ and let $a \in M$, $R > 0$ and $B \geq 0$ such that $Ricc_g \geq -(N-1)B^2$ in $B_R(a)$. Assume also $Sec_g \geq -\tilde{B}^2$ in $B_R(a)$ for some $\tilde{B} \geq 0$ if $p > 2$, or $r_M(a) \geq R$ if $1 < p < 2$. Then there exists $c = c(N,p,q) > 0$ such that the function*

$$w(x) = \lambda \left(R^2 - r^2(x)\right)^{-\frac{2}{q+1-p}} + \mu, \tag{4.7}$$

where $r = r(x) = d(x,a)$, satisfies

$$\mathcal{L}^*(w) \geq 0 \qquad \text{in } B_R(a), \tag{4.8}$$

provided that

$$\lambda = c \max\left\{(R^4B^2)^{\frac{1}{q+1-p}}, ((1 + (B + (p-2)_+\tilde{B})R)R^2)^{\frac{1}{q+1-p}}\right\} \tag{4.9}$$

and

$$\mu \geq ((N-1)B^2)^{\frac{1}{q+1-p}}. \tag{4.10}$$

Proof. Let w as in (4.7)). We will show that by choosing λ and μ as in (4.9)) and (4.10) respectively, then (4.8)) holds. We recall that

$$\Delta_g w = w'' + w'\Delta_g r, \tag{4.11}$$

and (see [34, Lemma 1])

$$\Delta_g r \leq (N-1)B\coth(Br) \leq \frac{N-1}{r}\left(1 + Br\right).$$

Then

$$\begin{aligned}\Delta_g w \leq \frac{4}{q+1-p}&(R^2-r^2)^{-\frac{2(q+2-p)}{q+1-p}} \\ &\times \left(\frac{2r^2(q+3-p)}{q+1-p} + (R^2-r^2)(1+(N-1)\left(1+Br\right)\right).\end{aligned} \tag{4.12}$$

Moverover [13, Chapt 2, Theorem A]

$$D^2 w = w'' dr \otimes dr + w' D^2 r. \tag{4.13}$$

If $r_M(a) \geq r(x)$, then the ball $B_{r(x)}(a)$ is convex. This implies that r is convex and therefore $D^2 r \geq 0$ (see [35, IV-5]). Furthermore, if $Sec_g(x) \geq -\tilde{B}^2$, then from [35, IV-Lemma 2.9],

$$D^2 r \leq \tilde{B}\coth(\tilde{B}r)g_{ij} \leq \frac{\tilde{B}}{r}\left(1+\tilde{B}r\right)g_{ij}, \tag{4.14}$$

therefore

$$\begin{aligned} 0 &\leq \frac{\langle D^2 w(\nabla u), \nabla u\rangle}{|\nabla u|^2} \\ &\leq \frac{4}{q+1-p}(R^2-r^2)^{-\frac{2(q+2-p)}{q+1-p}}\left(\frac{2r^2(q+3-p)}{q+1-p} + (R^2-r^2)(2+\tilde{B}r)\right). \end{aligned} \tag{4.15}$$

We obtain

$$\begin{aligned} \mathcal{A}(w) &= -\Delta w - (p-2)\frac{\langle D^2 w(\nabla u), \nabla u\rangle}{|\nabla u|^2} \\ &\geq -k\lambda(R^2-r^2)^{-\frac{2(q+2-p)}{q+1-p}}(R^2+(p-2)_+(R^2-r^2)\tilde{B}r\coth\tilde{B}r) \\ &\geq -k\lambda(R^2-r^2)^{-\frac{2(q+2-p)}{q+1-p}}(R^2+(R^2-r^2)B_p r) \end{aligned} \tag{4.16}$$

for some $k = k(N,p,q)$, where $B_p = B + (p-2)_+\tilde{B}$. Since

$$w^{q+2-p} \geq \lambda^{q+2-p}\left(R^2-r^2\right)^{-\frac{2(q+1-p)}{q+1-p}} + \mu^{q+2-p},$$

we have

$$\begin{aligned} &\mathcal{L}^*(w) \\ &\quad\geq \lambda(R^2-r^2)^{-\frac{2(q+2-p)}{q+1-p}}\left(-k(R^2+(R^2-r^2)B_p r) - \frac{16D}{(q+1-p)^2}r^2 + C\lambda^{q+1-p}\right) \\ &\qquad + \mu^{q+2-p} - (N-1)B^2\lambda\left(R^2-r^2\right)^{-\frac{2}{q+1-p}} - (N-1)B^2\mu. \end{aligned} \tag{4.17}$$

Take $\mu \geq ((N-1)B^2)^{\frac{1}{q+1-p}}$ as in (4.10). Next we choose λ in order to have, uniformly for $0 \leq r < R$,

$$2^{-1}C\lambda^{q+1-p} \geq k(R^2+(R^2-r^2)B_p r) + D\frac{16}{(q+1-p)^2}r^2,$$

so that

$$\begin{aligned} \mathcal{L}^*(w) &\geq 2^{-1}C\lambda(R^2-r^2)^{-\frac{2(q+2-p)}{q+1-p}}\lambda^{q+1-p} - (N-1)B^2\lambda(R^2-r^2)^{-\frac{2}{q+1-p}} \\ &= \lambda(R^2-r^2)^{-\frac{2(q+2-p)}{q+1-p}}\left(2^{-1}C\lambda^{q+1-p} - (N-1)B^2(R^2-r^2)^2\right) \end{aligned}$$

uniformly for $0 \leq r < R$. Then we enlarge λ if necessary to have

$$2^{-1}C\lambda^{q+2-p}(R^2-r^2)^{-\frac{2(q+2-p)}{q+1-p}} \geq (N-1)B^2\lambda\left(R^2-r^2\right)^{-\frac{2}{q+1-p}},$$

also uniformly for $0 \le r < R$. Hence we see that there exists $c = c(N, p, q)$ such that, if we choose

$$\lambda = c \max \left\{ (R^4 B^2)^{\frac{1}{q+1-p}}, ((1 + B_p R) R^2)^{\frac{1}{q+1-p}} \right\} \tag{4.18}$$

then (4.8) holds. □

Proposition 4.2 *Assume $q > p - 1 > 0$. Let Ω be an open subset of M such that $Ricc_g \ge (1 - N)B^2$ in Ω. Assume also $Sec_g \ge -\tilde{B}^2$ in Ω if $p > 2$, or $r_M(x) \ge$* dist $(x, \partial\Omega)$ *for any $x \in M$ if $1 < p < 2$. Then any solution u of* (4.4) *in Ω satisfies*

$$|\nabla u(x)|^2 \le c_{N,p,q} \max \left\{ B^{\frac{2}{q+1-p}}, (1 + B_p d(x, \partial\Omega)))^{\frac{1}{q+1-p}} (d(x, \partial\Omega))^{-\frac{2}{q+1-p}} \right\} \quad \forall x \in \Omega, \tag{4.19}$$

where $B_p = B + (p - 2)_+ \tilde{B}$.

Proof. Assume $a \in \Omega$ and $R < d(a, \partial\Omega)$. Let w be as in Lemma 4.1, then

$$\mathcal{A}(z - w) + C \left(z^{q+2-p} - w^{q+2-p} \right) - (N - 1)B^2 (z - w) - D \left(\frac{|\nabla z|^2}{z} - \frac{|\nabla w|^2}{w} \right) \le 0 \tag{4.20}$$

in $B_R(a)$. Let G be a connected component of the set $\{x \in B_R(a) : z(x) - w(x) > 0\}$. Then, if $C(q + 2 - p)(w(a))^{q+1-p} > (N - 1)B^2$, by the mean value theorem and the fact that $w(a)$ is the minimum of w, there holds that

$$C \left(z^{q+2-p} - w^{q+2-p} \right) - (N - 1)B^2 (z - w) > 0 \qquad \text{in } G. \tag{4.21}$$

Since $w(a) \ge \mu \ge ((N - 1)B^2)^{\frac{1}{q+1-p}}$ and $q + 2 - p > 1$, this condition is fulfilled by choosing the right μ as in (4.10). We conclude as in the proof of Proposition 2.1 that $G = \emptyset$. Therefore $z \le w$ in $B_R(a)$. In particular,

$$z(a) \le c_{N,p,q} \max \{ B^{\frac{2}{q+1-p}}, (1 + B_p R)^{\frac{1}{q+1-p}} R^{-\frac{2}{q+1-p}} \} \tag{4.22}$$

where $c_{N,p,q} > 0$. Then (4.19) follows. □

Remark. Since $Ricc_g(x)(\xi, \xi) = (N - 1) \sum_V Sec_g(x)(V)$, where V denotes the set of two planes in $T_x M$ which contain ξ, there holds

$$Sec_g \ge -\tilde{B}^2 \Longrightarrow Ricc_g \ge (1 - N)\tilde{B}^2.$$

However, in the previous estimate, the long range estimate on ∇u depends only on the Ricci curvature.

4.2 Growth of solutions and Liouville type results

Corollary 4.3 *Assume (M^N, g) is a complete noncompact N-dimensional Riemannian manifold such that $Ricc_g \geq (1-N)B^2$ and let $q > p-1 > 0$. Assume also if $r_M(x) = \infty$ if $1 < p < 2$ or that the sectional curvature Sec_g satisfies for some $a \in M$*

$$\lim_{\text{dist}\,(a,x)\to\infty} \frac{|Sec_g(x)|}{\text{dist}\,(a,x)} = 0, \tag{4.23}$$

if $p > 2$. Then any solution u of (4.4) satisfies

$$|\nabla u(x)|^2 \leq c_{N,p,q} B^{\frac{2}{q+1-p}} \qquad \forall x \in M. \tag{4.24}$$

In particular, u is constant if $Ricc_g \geq 0$, while in the general case u has at most a linear growth with respect to the distance function.

Application An example of a complete manifold with constant negative Ricci curvature is the standard hyperbolic space $(\mathbb{H}^N, g_0)$ for which $Ricc_{g_0} = -(N-1)g_0$. Another application deals with positive p-harmonic functions (for related results with $p = 2$ see [41], [9]).

Corollary 4.4 *Assume (M^N, g) is as in Corollary 4.3. Let $p > 1$ and assume that (4.23) holds if $p > 2$ or $Sec_g \leq 0$ if $1 < p < 2$. If v is a positive p-harmonic function, then*

(i) if $Ricc_g \geq 0$, v is constant.

(ii) if $\inf\{Ricc_g(x) : x \in M\} = (1-N)B^2 < 0$, v satisfies

$$v(a)e^{-c_{N,p}B\text{dist}\,(x,a)} \leq v(x) \leq v(a)e^{c_{N,p}B\text{dist}\,(x,a)} \qquad \forall x \in M. \tag{4.25}$$

Proof. We take $q = p$ and assume that v is p-harmonic and positive. If we write $v = e^{-\frac{u}{p-1}}$, then u satisfies

$$-\Delta_{g,p} u + |\nabla u|^p = 0.$$

If $Ricc_g(x) \geq 0$, u, and therefore v is constant by Corollary 4.3. If $\inf\{Ricc_g(x) : x \in M\} = (1-N)B^2 < 0$ we apply (4.24) to ∇u. If γ is a minimizing geodesic from a to x, then $|\gamma'(t)| = 1$ and

$$u(x) - u(a) = \int_0^{d(x,a)} \frac{d}{dt} u \circ \gamma(t) dt = \int_0^{d(x,a)} \langle \nabla u \circ \gamma(t), \gamma'(t) \rangle dt.$$

Since

$$\left| \langle \nabla u \circ \gamma(t), \gamma'(t) \right| \leq |\nabla u \circ \gamma(t)| \leq c_{N,p} \kappa,$$

we obtain

$$u(a) - c_{N,p}B\text{dist}\,(x,a) \leq u(x) \leq u(a) + c_{N,p}B\text{dist}\,(x,a) \qquad \forall x \in M, \tag{4.26}$$

Then (4.25) follows since $u = (1-p)\ln v$. Notice that (i) follows from (ii) and that in the case $1 < p < 2$ the assumption (i) implies that $Ricc_g = 0$. $\square$

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