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Abstract

Let Ω be a bounded domain of $\mathbb{R}^N$, and $Q = \Omega \times (0,T)$. We consider problems of the type

$$\begin{cases} u_t - \Delta_p u \pm \mathcal{G}(u) = \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

where Δ_p is the p-Laplacian, μ is a bounded Radon measure, $u_0 \in L^1(\Omega)$, and $\pm\mathcal{G}(u)$ is an absorption or a source term. In the model case $\mathcal{G}(u) = \pm |u|^{q-1} u$ $(q > p-1)$, or $\mathcal{G}$ has an exponential type. We prove the existence of renormalized solutions for any measure μ in the subcritical case, and give sufficient conditions for existence in the general case, when μ is good in time and satisfies suitable capacitary conditions.

1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$, and $Q = \Omega \times (0,T)$, $T > 0$. We consider the quasilinear parabolic problem

$$\begin{cases} u_t - \mathcal{A}(u) \pm \mathcal{G}(u) = \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where μ is a bounded Radon measure on Q, $u_0 \in L^1(\Omega)$. We assume that $\mathcal{A}(u) = \mathrm{div}(A(x, \nabla u))$ and A is a Carathéodory function on $\Omega \times \mathbb{R}^N$, such that, for *a.e.* $x \in \Omega$, and any $\xi, \zeta \in \mathbb{R}^N$,

$$A(x,\xi).\xi \geq \Lambda_1 |\xi|^p, \qquad |A(x,\xi)| \leq \Lambda_2 |\xi|^{p-1}, \qquad \Lambda_1, \Lambda_2 > 0, \tag{1.2}$$

$$(A(x,\xi) - A(x,\zeta)).(\xi - \zeta) > 0 \text{ if } \xi \neq \zeta, \tag{1.3}$$

for $p > 1$; and $\mathcal{G}(u) = \mathcal{G}(x,t,u)$, where $(x,t,r) \mapsto \mathcal{G}(x,t,r)$ is a Caratheodory function on $Q \times \mathbb{R}$ with

$$\mathcal{G}(x,t,r)r \geq 0, \quad \text{for } a.e.(x,t) \in Q \quad \text{and any } r \in \mathbb{R}. \tag{1.4}$$

The model problem is relative to the p-Laplace operator: $\mathcal{A}(u) = \Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, and $\mathcal{G}$ has a power-type $\mathcal{G}(u) = \pm |u|^{q-1} u$ $(q > p-1)$, or an exponential type. Our aim is to give sufficient conditions on

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the measure μ in terms of capacity to obtain existence results. We denote by $\mathcal{M}_b(\Omega)$ and $\mathcal{M}_b(Q)$ the sets of bounded Radon measures on Ω and Q respectively.

Next we make a brief survey of the main works on problem (1.1). First we consider the case of an *absorption term*:

$$\begin{cases} u_t - \mathcal{A}(u) + \mathcal{G}(u) = \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \tag{1.5}$$

For $p = 2$, $\mathcal{A}(u) = \Delta u$ and $\mathcal{G}(u) = |u|^{q-1}u$ $(q > 1)$, the pionnier results concern the case $\mu = 0$ and u_0 is a Dirac mass in Ω, see [12]: existence holds if and only if $q < (N+2)/N$. Then optimal results are given in [3], for any $\mu \in \mathcal{M}_b(Q)$ and $u_0 \in \mathcal{M}_b(\Omega)$. Here two capacities are involved: the elliptic Bessel capacity $\mathrm{Cap}_{\mathbf{G}_\alpha,s}$ defined, for $\alpha > 0, s > 1$ and any Borel set $E \subset \mathbb{R}^N$, by

$$\mathrm{Cap}_{\mathbf{G}_\alpha,s}(E) = \inf\{||\varphi||^s_{L^s(\mathbb{R}^N)} : \varphi \in L^s(\mathbb{R}^N), \varphi \geq 0 \quad G_\alpha * \varphi \geq 1 \text{ on } E\},$$

where $\mathbf{G}_\alpha$ is the Bessel kernel of order α; and the capacity $\mathrm{Cap}_{2,1,s}$ defined, for any compact set $K \subset \mathbb{R}^{N+1}$ by

$$\mathrm{Cap}_{2,1,s}(K) = \inf\left\{||\varphi||^s_{W^{2,1}_s(\mathbb{R}^{N+1})} : \varphi \in \mathcal{S}(\mathbb{R}^{N+1}), \quad \varphi \geq 1 \text{ on a neighborhood of } K\right\},$$

and extended classically to Borel sets, where

$$||\varphi||_{W^{2,1}_s(\mathbb{R}^{N+1})} = ||\varphi||_{L^s(\mathbb{R}^{N+1})} + ||\varphi_t||_{L^s(\mathbb{R}^{N+1})} + ||\,|\nabla\varphi|\,||_{L^s(\mathbb{R}^{N+1})} + \sum_{i,j=1,2,\ldots,N} ||\varphi_{x_i x_j}||_{L^s(\mathbb{R}^{N+1})}.$$

In [3], Baras and Pierre proved that there exists a solution if and only if μ does not charge the sets of $\mathrm{Cap}_{2,1,\frac{q}{q-1}}$-capacity zero and u_0 does not charge the sets of $\mathrm{Cap}_{\mathbf{G}_{\frac{2}{q}},\frac{q}{q-1}}$-capacity zero.

The case where $\mathcal{G}$ has an exponential type was initiated by [17], and studied in the framework of Orlicz spaces in [29, 19], and very recently by [24] in the context of Wolff parabolic potentials.

For $p \neq 2$, most of the contributions are relative to the case $\mathcal{G}(u) = |u|^{q-1}u$, $\mu = 0$, with Ω bounded, or $\Omega = \mathbb{R}^N$. The case where u_0 is a Dirac mass in Ω was studied in [18, 20] when $p > 2$, and [13] when $p < 2$. Existence and uniqueness hold in the subcritical case

$$q < p_c := p - 1 + \frac{p}{N}. \tag{1.6}$$

If $q \geq p_c$ and $q > 1$, there is no solution with an isolated singularity at $t = 0$. For $q < p_c$, and $u_0 \in \mathcal{M}^+_b(\Omega)$, the existence was obtained in the sense of distributions in [30], and for any $u_0 \in \mathcal{M}_b(\Omega)$ in [8]. The case $\mu \in L^1(Q)$, $u_0 = 0$ was treated in [14], and with $\mu \in L^1(Q)$, $u_0 \in L^1(\Omega)$ in [1], where $\mathcal{G}$ can be multivalued. A larger set of measures, introduced in [16], was studied in [26]. Let $\mathcal{M}_0(Q)$ be the set of Radon measures μ on Q that do not charge the sets of zero c^Q_p-capacity, where for any Borel set $E \subset Q$,

$$c^Q_p(E) = \inf\Big(\inf_{E \subset U \text{ open} \subset Q} \{||u||_W : u \in W, u \geq \chi_U \quad a.e. \text{ in } Q\}\Big),$$

and W is the space of functions $z \in L^p((0,T); W^{1,p}_0(\Omega) \cap L^2(\Omega))$ such that $z_t \in L^{p'}((0,T); W^{-1,p'}(\Omega) + L^2(\Omega))$ imbedded with the norm

$$\|z\|_W = \|z\|_{L^p((0,T);W^{1,p}_0(\Omega)\cap L^2(\Omega))} + \|z_t\|_{t \in L^{p'}((0,T);W^{-1,p'}(\Omega)+L^2(\Omega))}\,.$$

It was shown that existence and uniqueness hold for any measure $\mu \in \mathcal{M}_b(Q) \cap \mathcal{M}_0(Q)$, called regular, or diffuse, and $p > 1$, and for any function $\mathcal{G} \in C(\mathbb{R})$ such that $\mathcal{G}(u)u \geq 0$. Up to our knowledge, up to now no existence results have been obtained for a measure $\mu \notin \mathcal{M}_0(Q)$.

The case of a source term

$$\begin{cases} u_t - \mathcal{A}(u) = \mathcal{G}(u) + \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.7}$$

with $\mathcal{G}(u) = u^q$ with nonnegative u and μ, u_0 was treated in [2] for $p = 2$, giving optimal conditions for existence. As in the absorption case the arguments of proofs cannot be extended to general p.

2 Main results

In Section 3, we introduce the notion of renormalized solutions, called R-solutions, of problem (1.1), and we recall at Theorem 3.4 the stability result that we proved in [7] for the problem without perturbation

$$\begin{cases} u_t - \mathcal{A}(u) = \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \tag{2.1}$$

under the assumption

$$p > p_1 := (2N+1)/(N+1),$$

that *we make in all the sequel.* This condition ensures that the functions u and $|\nabla u|$ are well defined in $L^1(Q)$. Combined with some approximation properties of the measures, Theorem 3.4 is the key point of our results.

In Section 4, we first give existence results of subcritical type, valid for any measure $\mu \in \mathcal{M}_b(Q)$. Let $G \in C(\mathbb{R}^+)$ be a nondecreasing function with values in $\mathbb{R}^+$, such that

$$|\mathcal{G}(x,t,r)| \leq G(|r|) \quad \text{for } a.e.\ x \in \Omega \text{ and any } \ r \in \mathbb{R}, \tag{2.2}$$

$$\int_1^\infty G(s) s^{-1-p_c} ds < \infty, \tag{2.3}$$

where p_c is defined at (1.6).

Theorem 2.1 *Assume (1.4), (2.2), (2.3). Then, for any $\mu \in \mathcal{M}_b(Q)$ and $u_0 \in L^1(\Omega)$, there exists a R-solution u of problem*

$$\begin{cases} u_t - \mathcal{A}(u) + \mathcal{G}(u) = \mu & in\ Q, \\ u = 0 & in\ \partial\Omega \times (0,T), \\ u(0) = u_0 & in\ \Omega. \end{cases} \tag{2.4}$$

Theorem 2.2 *Assume (1.4), (2.2), (2.3). There exists $\varepsilon > 0$ such that, for any $\lambda > 0$, any $\mu \in \mathcal{M}_b^+(Q)$ and any nonneagtive $u_0 \in L^1(\Omega)$, if $\lambda + \mu(Q) + ||u_0||_{L^1(\Omega)} \leq \varepsilon$, then there exists a nonnegative R-solution u of problem*

$$\begin{cases} u_t - \mathcal{A}(u) = \lambda\mathcal{G}(u) + \mu & in\ Q, \\ u = 0 & in\ \partial\Omega \times (0,T), \\ u(0) = u_0 & in\ \Omega, \end{cases} \tag{2.5}$$

In particular for any if $\mathcal{G}(u) = |u|^{q-1} u$, condition (2.3) is equivalent to the fact that q is subcritical: $0 < q < p_c$, where p_c is defined at (1.6).

Next we consider the general case, with no subcriticality assumptions, when $\mathcal{G}$ *is nondecreasing in* u, and $\mathcal{G}$ has a power type, or an exponential type. For $\mathcal{G}(u) = |u|^{q-1} u$ for $q \geq p_c$, and $p \neq 2$, up to now *the good capacities for solving the problem are not known.* In the following, we search sufficient conditions on the measures μ and u_0 ensuring that there exists a solution.To our knowledge, the question of finding necessary conditions for existence is still an open problem.

In the sequel we give sufficient conditions for existence for *measures that have a good behaviour in* t, based on recent results of [9] relative to the elliptic case. We recall the notion of (truncated) Wolff potential: for any nonnegative measure $\omega \in \mathcal{M}^+(\mathbb{R}^N)$ any $R > 0$, $x_0 \in \mathbb{R}^N$,

$$\mathbf{W}_{1,p}^R[\omega](x_0) = \int_0^R \left(r^{p-N}\omega(B(x_0,r))\right)^{\frac{1}{p-1}} \frac{dr}{r}. \tag{2.6}$$

Any measure $\omega \in \mathcal{M}_b(\Omega)$ is identified with its extension by 0 to $\mathbb{R}^N$. In case of absorption, we obtain the following:

Theorem 2.3 *Let $p < N$, $q > p-1$, $\mu \in \mathcal{M}_b(Q)$, $f \in L^1(Q)$ and $u_0 \in L^1(\Omega)$. Assume that*

$$|\mu| \leq \omega \otimes F, \qquad \text{with } \omega \in \mathcal{M}_b^+(\Omega), F \in L^1((0,T)), F \geq 0. \tag{2.7}$$

If ω does not charge the sets of $\mathrm{Cap}_{\mathbf{G}_p,\frac{q}{q+1-p}}$-capacity zero, then there exists a R-solution u of problem

$$\begin{cases} u_t - \mathcal{A}(u) + |u|^{q-1}u = f + \mu & \text{in } Q, \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 \quad \text{in } \Omega. \end{cases} \tag{2.8}$$

From [3, Proposition 2.3], a measure $\omega \in \mathcal{M}_b(\Omega)$ does not charge the sets of $\mathrm{Cap}_{\mathbf{G}_2,\frac{q}{q-1}}$-capacity zero if and only if $\omega \otimes \chi_{(0,T)}$ does not charge the sets of $\mathrm{Cap}_{2,1,\frac{q}{q-1}}$-capacity zero. Therefore, when $\mathcal{A}(u) = \Delta u$ and $\mu = \omega \otimes \chi_{(0,T)}$, $u_0 \in L^1(\Omega)$, we find again the existence result of [3]. Besides, in view of [16, Theorem 2.16], there exists data $\mu \in \mathcal{M}_b(Q)$ in Theorem 2.3 such that $\mu \notin \mathcal{M}_0(Q)$, see Remark 5.7, thus our result is the first one of existence for non diffuse measure. Otherwise our result can be extended to a more general function $\mathcal{G}$, see Remark 5.9.

We also consider a source term. Denoting by $D = \sup_{x,y\in\Omega}|x-y|$ the diameter of Ω, we obtain the following:

Theorem 2.4 *Let $p < N$, $q > p-1$. Let $\mu \in \mathcal{M}_b^+(Q)$, and nonnegative $u_0 \in L^\infty(\Omega)$. Assume that*

$$\mu \leq \omega \otimes \chi_{(0,T)}, \qquad \text{with } \omega \in \mathcal{M}_b^+(\Omega).$$

Then there exist λ_0 and b_0, depending of $N, p, q, \Lambda_1, \Lambda_2, D$, such that, if

$$\omega(E) \leq \lambda_0 \mathrm{Cap}_{\mathbf{G}_p,\frac{q}{q+1-p}}(E), \quad \forall E \text{ compact set} \subset \mathbb{R}^N, \qquad \text{and} \quad ||u_0||_{L^\infty(\Omega)} \leq b_0, \tag{2.9}$$

there exists a nonnegative R-solution u of problem

$$\begin{cases} u_t - \mathcal{A}(u) = u^q + \mu & \text{in } Q, \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 \quad \text{in } \Omega, \end{cases} \tag{2.10}$$

which satisfies, a.e. in Q,

$$u(x,t) \le C\mathbf{W}^{2D}_{1,p}[\omega](x) + 2||u_0||_{L^\infty(\Omega)}, \tag{2.11}$$

where $C = C(N,p,\Lambda_1,\Lambda_2)$.

In case where $\mathcal{G}$ is an exponential, we introduce the notion of maximal fractional operator, defined for any $\eta \ge 0$, $R > 0$, $x_0 \in \mathbb{R}^N$ by

$$\mathbf{M}^{\eta}_{p,R}[\omega](x_0) = \sup_{r\in(0,R)} \frac{\omega(B(x_0,r))}{r^{rN-p}h_\eta(r)}, \qquad \text{where } h_\eta(r) = \inf((-\ln r)^{-\eta}, (\ln 2)^{-\eta})).$$

In the case of absorption, we obtain the following:

Theorem 2.5 *Let $p < N$ and $\tau > 0, \beta > 1, \mu \in \mathcal{M}_b(Q)$, $f \in L^1(Q)$ and $u_0 \in L^1(\Omega)$. Assume that*

$$|\mu| \le \omega \otimes F, \qquad \textit{with } \omega \in \mathcal{M}^+_b(\Omega),\ F \in L^1((0,T)), F \ge 0,$$

and that one of the following assumptions is satisfied:
(i) $||F||_{L^\infty((0,T))} \le 1$, and for some $M_0 = M_0(N,p,\beta,\tau,\Lambda_1,\Lambda_2,D)$,

$$||\mathbf{M}^{\frac{p-1}{\beta'}}_{p,2D}[\omega]||_{L^\infty(\mathbb{R}^N)} < M_0; \tag{2.12}$$

(ii) there exists $\beta_0 > \beta$ such that $\mathbf{M}^{\frac{p-1}{\beta_0'}}_{p,2D}[\omega] \in L^\infty(\mathbb{R}^N)$.

Then there exists a R-solution to the problem

$$\begin{cases} u_t - \mathcal{A}(u) + (e^{\tau|u|^\beta} - 1)\mathrm{sign}u = f + \mu & \textit{in } Q, \\ u = 0 & \textit{on } \partial\Omega\times(0,T), \\ u(0) = u_0 & \textit{in } \Omega. \end{cases}$$

In the case of a source term, we obtain:

Theorem 2.6 *Let $\tau > 0, l \in \mathbb{N}$ and $\beta \ge 1$ such that $l\beta > p-1$. We set*

$$\mathcal{E}(s) = e^s - \sum_{j=0}^{l-1} \frac{s^j}{j!}, \qquad \forall s \in \mathbb{R}. \tag{2.13}$$

Let $\mu \in \mathcal{M}^+_b(Q)$, such that

$$\mu \le \omega \otimes \chi_{(0,T)}, \quad \textit{with } \omega \in \mathcal{M}^+_b(\Omega).$$

Then, there exist b_0 and M_0 depending on $N,p,\beta,\tau,l,\Lambda_1,\Lambda_2,D$, such that if

$$||\mathbf{M}^{\frac{(p-1)(\beta-1)}{\beta}}_{p,2D}[\omega]||_{L^\infty(\mathbb{R}^N)} \le M_0, \quad \textit{and} \quad ||u_0||_{L^\infty(\Omega)} \le b_0,$$

the problem

$$\begin{cases} u_t - \mathcal{A}(u) = \mathcal{E}(\tau u^\beta) + \mu & \textit{in } Q, \\ u = 0 & \textit{on } \partial\Omega\times(0,T), \\ u(0) = u_0 & \textit{in } \Omega, \end{cases} \tag{2.14}$$

admits a nonnegative R-solution u, which satisfies, a.e. in Q, for some $C = C(N,p,\Lambda_1,\Lambda_2)$,

$$u(x,t) \le C\mathbf{W}^{2D}_{1,p}[\omega](x) + 2b_0. \tag{2.15}$$

3 Renormalized solutions and stability theorem

Here we recall the definition of renormalized solutions of the problem without perturbation (2.1), given in [25] for $p > p_1$.

Let $\mathcal{M}_s(Q)$ be the set of measures $\mu \in \mathcal{M}_b(Q)$ with support on a set of zero c_p^Q-capacity, also called *singular*. Let $\mathcal{M}_b^+(Q), \mathcal{M}_0^+(Q), \mathcal{M}_s^+(Q)$ be the positive cones of $\mathcal{M}_b(Q), \mathcal{M}_0(Q), \mathcal{M}_s(Q)$.
Recall that any measure $\mu \in \mathcal{M}_b(Q)$ can be written (in a unique way) under the form

$$\mu = \mu_0 + \mu_s, \text{ where } \mu_0 \in \mathcal{M}_0(Q), \quad \mu_s = \mu_s^+ - \mu_s^-, \quad \text{with } \mu_s^+, \mu_s^- \in \mathcal{M}_s^+(Q).$$

In turn $\mu_0 \in \mathcal{M}_0(Q)$ admits (at least) a decomposition under the form

$$\mu_0 = f - \operatorname{div} g + h_t, \qquad f \in L^1(Q), \quad g \in (L^{p'}(Q))^N, \quad h \in L^p((0,T); W_0^{1,p}(\Omega)),$$

see [16]; and we write $\mu_0 = (f, g, h)$.

We set $T_k(r) = \max\{\min\{r,k\}, -k\}$, for any $k > 0$ and $r \in \mathbb{R}$. If u is a measurable function defined and finite *a.e.* in Q, such that $T_k(u) \in L^p((0,T); W_0^{1,p}(\Omega))$ for any $k > 0$, there exists a measurable function w from Q into $\mathbb{R}^N$ such that $\nabla T_k(u) = \chi_{|u| \leq k} w$, *a.e.* in Q, and for any $k > 0$. We define the gradient ∇u of u by $w = \nabla u$.

Definition 3.1 *Let $u_0 \in L^1(\Omega)$, $\mu = \mu_0 + \mu_s \in \mathcal{M}_b(Q)$. A measurable function u is a renormalized solution, called* ***R-solution*** *of (2.1) if there exists a decompostion (f, g, h) of μ_0 such that*

$$U = u - h \in L^\sigma(0,T; W_0^{1,\sigma}(\Omega) \cap L^\infty(0,T; L^1(\Omega)), \quad \forall \sigma \in [1, m_c)\,; \qquad T_k(U) \in L^p((0,T); W_0^{1,p}(\Omega)), \quad \forall k > 0;$$

and:

(i) for any $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support on $\mathbb{R}$, and $S(0) = 0$,

$$\begin{aligned} &-\textstyle\int_\Omega S(u_0)\varphi(0)dx - \int_Q \varphi_t S(U)dxdt + \int_Q S'(U)A(x,t,\nabla u).\nabla\varphi dxdt \\ &+ \textstyle\int_Q S''(U)\varphi A(x,t,\nabla u).\nabla U dxdt = \int_Q fS'(U)\varphi dxdt + \int_Q g.\nabla(S'(U)\varphi)dxdt, \end{aligned}$$

for any $\varphi \in L^p((0,T); W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ such that $\varphi_t \in L^{p'}((0,T); W^{-1,p'}(\Omega)) + L^1(Q)$ and $\varphi(.,T) = 0$;

(ii) for any $\phi \in C(\overline{Q})$,

$$\lim_{m\to\infty} \frac{1}{m} \int\limits_{\{m \leq U < 2m\}} \phi A(x,t,\nabla u).\nabla U dxdt = \int_Q \phi d\mu_s^+,$$

$$\lim_{m\to\infty} \frac{1}{m} \int\limits_{\{-m \geq U > -2m\}} \phi A(x,t,\nabla u).\nabla U dxdt = \int_Q \phi d\mu_s^-.$$

In the sequel we consider the problem (1.1) where $\mu \in \mathcal{M}_b(Q)$, $u_0 \in L^1(\Omega)$. We say that u is a R-solution of problem (1.1) if $\mathcal{G}(u) \in L^1(Q)$ and u is a R-solution of (2.1) with data $(\mu \mp \mathcal{G}(u), u_0)$.

We recall some properties of R-solutions which we proved in [7, Propositions 2.8,2.10 and Remark 2.9]:

Proposition 3.2 *Let $\mu \in L^1(Q)$ and $u_0 \in L^1(\Omega)$, and u be the (unique) R-solution of problem (1.1) with data μ and u_0. Then*

$$\text{meas}\{|u| > k\} \leq C(||u_0||_{L^1(\Omega)} + |\mu|(Q))^{\frac{p+N}{N}} k^{-p_c}, \qquad \forall k > 0, \tag{3.1}$$

for some $C = C(N, p, \Lambda_1, \Lambda_2)$.

Proposition 3.3 *Let $\{\mu_n\} \subset \mathcal{M}_b(Q)$, and $\{u_{0,n}\} \subset L^1(\Omega)$, with*

$$\sup_n |\mu_n|(Q) < \infty, \quad \text{and} \quad \sup_n ||u_{0,n}||_{L^1(\Omega)} < \infty.$$

Let $\{u_n\}$ be a sequence of R-solutions of (1.1) with data $\mu_n = \mu_{n,0} + \mu_{n,s}$ and $u_{0,n}$, relative to a decomposition (f_n, g_n, h_n) of $\mu_{n,0}$. Assume that $\{f_n\}$ is bounded in $L^1(Q)$, $\{g_n\}$ bounded in $(L^{p'}(Q))^N$ and $\{h_n\}$ converges in $L^p(0,T; W_0^{1,p}(\Omega))$.

Then, up to a subsequence, $\{u_n\}$ converges to a function u a.e in Q and in $L^s(Q)$ for any $s \in [1, m_c)$. Moreover, if $\{\mu_n\}$ is bounded in $L^1(Q)$, then $\{u_n\}$ converges to u in $L^s(0,T; W_0^{1,s}(\Omega))$ in $s \in [1, p - \frac{N}{N+1})$.

Our results are based on the *stability theorem* that we obtained for problem (2.1) in [7], extending the elliptic result of [15, Theorem 3.4] to the parabolic case. Note that it is valid under more general assumptions on the operator $\mathcal{A}$, see [7]. Recall that a sequence $\{\mu_n\} \subset \mathcal{M}_b(Q)$ converges to $\mu \in \mathcal{M}_b(Q)$ in the *narrow topology* of measures if

$$\lim_{n\to\infty} \int_Q \varphi d\mu_n = \int_Q \varphi d\mu \qquad \forall \varphi \in C(Q) \cap L^\infty(Q).$$

Theorem 3.4 *Let $p > p_1$, $u_0 \in L^1(\Omega)$, and*

$$\mu = f - \operatorname{div} g + h_t + \mu_s^+ - \mu_s^- \in \mathcal{M}_b(Q),$$

with $f \in L^1(Q), g \in (L^{p'}(Q))^N$, $h \in L^p((0,T); W_0^{1,p}(\Omega))$ and $\mu_s^+, \mu_s^- \in \mathcal{M}_s^+(Q)$. Let $u_{0,n} \in L^1(\Omega)$,

$$\mu_n = f_n - \operatorname{div} g_n + (h_n)_t + \rho_n - \eta_n \in \mathcal{M}_b(Q),$$

with $f_n \in L^1(Q), g_n \in (L^{p'}(Q))^N, h_n \in L^p((0,T); W_0^{1,p}(\Omega))$, and $\rho_n, \eta_n \in \mathcal{M}_b^+(Q)$, such that

$$\rho_n = \rho_n^1 - \operatorname{div} \rho_n^2 + \rho_{n,s}, \qquad \eta_n = \eta_n^1 - \operatorname{div}\eta_n^2 + \eta_{n,s},$$

with $\rho_n^1, \eta_n^1 \in L^1(Q), \rho_n^2, \eta_n^2 \in (L^{p'}(Q))^N$ and $\rho_{n,s}, \eta_{n,s} \in \mathcal{M}_s^+(Q)$. Assume that

$$\sup_n |\mu_n|(Q) < \infty,$$

and $\{u_{0,n}\}$ converges to u_0 strongly in $L^1(\Omega)$, $\{f_n\}$ converges to f weakly in $L^1(Q)$, $\{g_n\}$ converges to g strongly in $(L^{p'}(Q))^N$, $\{h_n\}$ converges to h strongly in $L^p((0,T); W_0^{1,p}(\Omega))$, $\{\rho_n\}$ converges to μ_s^+ and $\{\eta_n\}$ converges to μ_s^- in the narrow topology of measures; and $\{\rho_n^1\}, \{\eta_n^1\}$ are bounded in $L^1(Q)$, and $\{\rho_n^2\}, \{\eta_n^2\}$ bounded in $(L^{p'}(Q))^N$.

Let $\{u_n\}$ be a sequence of R-solutions of

$$\begin{cases} u_{n,t} - \mathcal{A}(u_n) = \mu_n & \text{in } Q, \\ u_n = 0 & \text{on } \partial\Omega \times (0,T), \\ u_n(0) = u_{0,n} & \text{in } \Omega. \end{cases}$$

relative to the decomposition $(f_n + \rho_n^1 - \eta_n^1, g_n + \rho_n^2 - \eta_n^2, h_n)$ *of* $\mu_{n,0}$*. Let* $U_n = u_n - h_n$*.*

Then up to a subsequence, $\{u_n\}$ *converges a.e. in* Q *to a R-solution* u *of (2.1), and* $\{U_n\}$ *converges a.e. in* Q *to* $U = u - h$*. Moreover,* $\{\nabla u_n\}, \{\nabla U_n\}$ *converge respectively to* $\nabla u, \nabla U$ *a.e. in* Q*, and* $\{T_k(U_n)\}$ *converge to* $T_k(U)$ *strongly in* $L^p((0,T); W_0^{1,p}(\Omega))$ *for any* $k > 0$*.*

For applying Theorem 3.4, we require some approximation properties of measures, see [7]:

Proposition 3.5 *Let* $\mu = \mu_0 + \mu_s \in \mathcal{M}_b^+(Q)$ *with* $\mu_0 \in \mathcal{M}_0^+(Q)$ *and* $\mu_s \in \mathcal{M}_s^+(Q)$*.*
(i) Then, we can find a decomposition $\mu_0 = (f, g, h)$ *with* $f \in L^1(Q), g \in (L^{p'}(Q))^N, h \in L^p(0,T; W_0^{1,p}(\Omega))$ *such that*

$$||f||_{L^1(Q)} + \|g\|_{(L^{p'}(Q))^N} + ||h||_{L^p(0,T;W_0^{1,p}(\Omega))} + \mu_s(\Omega) \le 2\mu(Q). \tag{3.2}$$

(ii) Furthermore, there exists sequences of measures $\mu_{0,n} = (f_n, g_n, h_n)$ *and* $\mu_{s,n}$ *such that* $f_n, g_n, h_n \in C_c^\infty(Q)$ *strongly converge to* f, g, h *in* $L^1(Q), (L^{p'}(Q))^N$ *and* $L^p(0,T; W_0^{1,p}(\Omega))$ *respectively, and* $\mu_{s,n} \in (C_c^\infty(Q))^+$ *converges to* μ_s *and* $\mu_n := \mu_{0,n} + \mu_{s,n}$ *converges to* μ *in the narrow topology of measures, and satisfying* $|\mu_n|(Q) \le \mu(Q)$*,*

$$||f_n||_{L^1(Q)} + \|g_n\|_{(L^{p'}(Q))^N} + ||h_n||_{L^p(0,T;W_0^{1,p}(\Omega))} + \mu_{s,n}(Q) \le 2\mu(Q). \tag{3.3}$$

In particular we use in the sequel a property of approximation by *nondecreasing sequences*:

Proposition 3.6 *Let* $\mu \in \mathcal{M}_b^+(Q)$*. Let* $\{\mu_n\}$ *be a nondecreasing sequence in* $\mathcal{M}_b^+(Q)$ *converging to* μ *in* $\mathcal{M}_b(Q)$*. Then, there exist* $f_n, f \in L^1(Q)$*,* $g_n, g \in (L^{p'}(Q))^N$ *and* $h_n, h \in L^p(0,T; W_0^{1,p}(\Omega))$*,* $\mu_{n,s}, \mu_s \in \mathcal{M}_s^+(Q)$ *such that*

$$\mu = f - \operatorname{div} g + h_t + \mu_s, \qquad \mu_n = f_n - \operatorname{div} g_n + (h_n)_t + \mu_{n,s},$$

and $\{f_n\}, \{g_n\}, \{h_n\}$ *strongly converge to* f, g, h *in* $L^1(Q), (L^{p'}(Q))^N$ *and* $L^p(0,T; W_0^{1,p}(\Omega))$ *respectively, and* $\{\mu_{n,s}\}$ *converges to* μ_s *(strongly) in* $\mathcal{M}_b(Q)$ *and*

$$||f_n||_{L^1(Q)} + ||g_n||_{(L^{p'}(Q))^N} + ||h_n||_{L^p(0,T;W_0^{1,p}(\Omega))} + \mu_{n,s}(\Omega) \le 2\mu(Q). \tag{3.4}$$

As a consequence of the above results, we get the following:

Corollary 3.7 *(i) Let* $u_0 \in L^1(\Omega)$ *and* $\mu \in \mathcal{M}_b(Q)$*. Then there exists a R-solution* u *to the problem 2.1 with data* (μ, u_0) *such that* u *satisfies (3.1).*

(ii) Furthermore, if $v_0 \in L^1(\Omega)$ *and* $\nu \in \mathcal{M}_b(Q)$ *such that* $u_0 \le v_0$ *and* $\mu \le \nu$*, then one can find R-solutions* u *and* v *to the problem 2.1 with respective data* (μ, u_0) *and* (ω, v_0) *such that* $u \le v$*,* u *satisfies (3.1) and*

$$\operatorname{meas}\{|v| > k\} \le C(||v_0||_{L^1(\Omega)} + |\nu|(Q))^{\frac{p+N}{N}} k^{-p_c}, \qquad \forall k > 0. \tag{3.5}$$

Proof. (i) We approximate μ by a smooth sequence $\{\mu_n\}$ defined at Proposition 3.5-(ii) and apply Proposition 3.2 and Theorem 3.4.

(ii) We set $w_0 = v_0 - u_0 \ge 0$ and $\lambda = \omega - \mu \ge 0$. In the same way, we consider a nonnegative, smooth sequence $(\lambda_n, w_{0,n})$ of approximations of (λ, w_0) defined at Proposition 3.5-(ii). Let v_n be the solution of the problem with data $(\lambda_n + \mu_n, w_{0,n} + u_{0,n})$. Clearly, $u_n \le v_n$ and $(\lambda_n + \mu_n, w_{0,n} + u_{0,n})$ is an approximation of data (ω, v_0) in the sense of Theorem 3.4, then we reach the conclusion. ∎

4 Subcritical case

We first consider the subcritical case with absorption. We obtain Theorem 2.1 as a direct consequence of Theorem 3.4 and Proposition 3.5. We follow the well-known technique introduced in [4] for the elliptic problem with absorption

$$-\mathcal{A}(u) + G(u) = \omega \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{4.1}$$

where $\omega \in \mathcal{M}_b(\Omega), p > 1$, and G is nondecreasing and odd, and $\int_1^\infty G(s)s^{-(N-1)p/(N-p)}ds < \infty$.

Proof of Theorem 2.1. Let $\mu = \mu_0 + \mu_s \in \mathcal{M}_b(Q)$, with $\mu_0 \in \mathcal{M}_0(Q), \mu_s \in \mathcal{M}_s(Q)$, and $u_0 \in L^1(\Omega)$. By Proposition 3.5, we can find $f_{n,i}, g_{n,i}, h_{n,i} \in C_c^\infty(Q)$ which strongly converge to f_i, g_i, h_i in $L^1(Q), (L^{p'}(Q))^N$ and $L^p((0,T); W_0^{1,p}(\Omega))$ respectively, for $i = 1, 2$, such that $\mu_0^+ = (f_1, g_1, h_1)$, $\mu_0^- = (f_2, g_2, h_2)$, and $\mu_{n,0,i} = (f_{n,i}, g_{n,i}, h_{n,i})$, converge respectively for $i = 1, 2$ to μ_0^+, μ_0^- in the narrow topology; and we can find nonnegative $\mu_{n,s,i} \in C_c^\infty(Q), i = 1, 2$, converging respectively to μ_s^+, μ_s^- in the narrow topology.
Furthermore, if we set

$$\mu_n = \mu_{n,0,1} - \mu_{n,0,2} + \mu_{n,s,1} - \mu_{n,s,2},$$

then $|\mu_n|(Q) \le |\mu|(Q)$. Consider a sequence $\{u_{0,n}\} \subset C_c^\infty(\Omega)$ which strongly converges to u_0 in $L^1(\Omega)$ and satisfies $||u_{0,n}||_{1,\Omega} \le ||u_0||_{L^1(\Omega)}$.

Let u_n be a solution of

$$\begin{cases} (u_n)_t - \mathcal{A}(u_n) + \mathcal{G}(u_n) = \mu_n & \text{in } Q, \\ u_n = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u_n(0) = u_{0,n} \quad \text{in } \Omega. \end{cases}$$

We can choose $\varphi = \varepsilon^{-1}T_\varepsilon(u_n)$ as test function of above problem. Since

$$\int_Q \left(\varepsilon^{-1}\overline{T_\varepsilon}(u_n)\right)_t dxdt = \int_\Omega \varepsilon^{-1}\overline{T_\varepsilon}(u_n(T))dx - \int_\Omega \varepsilon^{-1}\overline{T_\varepsilon}(u_{0,n})dx \ge -||u_{0,n}||_{L^1(\Omega)},$$

there holds from (1.2)

$$\int_Q \mathcal{G}(x,t,u_n)\varepsilon^{-1}T_\varepsilon(u_n)dxdt \le |\mu_n|(Q) + ||u_{0,n}||_{L^1(\Omega)} \le |\mu|(Q) + ||u_0||_{L^1(\Omega)}.$$

Letting $\varepsilon \to 0$, we obtain

$$\int_Q |\mathcal{G}(x,t,u_n)|\, dxdt \le |\mu|(Q) + ||u_0||_{L^1(\Omega)}.$$

Next we apply the estimate (3.1) of Proposition 3.2 to u_n, with initial data $u_{0,n}$ and measure data $\mu_n - \mathcal{G}(u_n) \in L^1(Q)$. We get for any $s > 0$ and any $n \in \mathbb{N}$,

$$\text{meas}\,\{|u_n| \ge s\} \le Ms^{-p_c}, \qquad M = C(|\mu|(Q) + ||u_0||_{L^1(\Omega)})^{\frac{p+N}{N}}, \quad C = C(N,p,\Lambda_1,\Lambda_2).$$

For any $L > 1$, we set $G_L(s) = \chi_{[L,\infty)}(s)G(s)$, and $|u_n|^*(s) = \inf\{a > 0 : \text{meas}\,\{|u_n| > a\} \le s\}$. For any $s \ge 0$, we obtain

$$\int_{\{|u_n| \ge L\}} G(|u_n|)dxdt = \int_Q G_L(|u_n|)dxdt \le \int_0^\infty G_L(|u_n|^*(s))ds \tag{4.2}$$

Since $|\mathcal{G}(x,t,u_n)| \le G(|u_n|)$, we deduce that $\{|\mathcal{G}(u_n)|\}$ is equi-integrable. Then, from Proposition 3.3, up to a subsequence, $\{u_n\}$ converges to some function u, *a.e.* in Q, and $\{\mathcal{G}(u_n)\}$ converges to $\mathcal{G}(u)$ in $L^1(Q)$. Therefore, applying Theorem 3.4, u is a R-solution of (2.4). ∎

Next we study the subcritical case with a source term. We proceed by induction by constructing an nondecreasing sequence of solutions. Here we meet a difficulty, due to the possible nonuniqueness of the solutions, that we solve by using Corollary 3.7.

Proof of Theorem 2.2. Let $\{u_n\}_{n\geq 1}$ be defined by induction as nonnegative R-solutions of

$$\left\{\begin{array}{l}(u_1)_t - \mathcal{A}(u_1) = \mu \qquad \text{in } Q,\\ u_1 = 0 \qquad \text{on } \partial\Omega\times(0,T),\\ u_1(0) = u_0 \qquad \text{in } \Omega,\end{array}\right. \qquad \left\{\begin{array}{l}(u_{n+1})_t - \mathcal{A}(u_{n+1}) = \mu + \lambda\mathcal{G}(u_n) \qquad \text{in } Q,\\ u_{n+1} = 0 \qquad \text{on } \partial\Omega\times(0,T),\\ u_{n+1}(0) = u_0 \qquad \text{in } \Omega,\end{array}\right.$$

From Corollary 3.7 we can assume that $\{u_n\}$ is nondecreasing and satisfies, for any $s>0$ and $n\in\mathbb{N}$

$$\text{meas}\{|u_n|\geq s\} \leq C_1 K_n s^{-p_c}, \tag{4.3}$$

where C_1 does not depend on s,n, and

$$K_1 = (||u_0||_{L^1(\Omega)} + |\mu|(Q))^{\frac{p+N}{N}},$$
$$K_{n+1} = (||u_0||_{L^1(\Omega)} + |\mu|(Q) + \lambda||\mathcal{G}(u_n)||_{L^1(\Omega)})^{\frac{p+N}{N}},$$

for any $n\geq 1$. Take $\varepsilon = \lambda + |\mu|(Q) + ||u_0||_{L^1(\Omega)} \leq 1$. Denoting by C_i some constants independent on n,ε, there holds $K_1 \leq C_2\varepsilon$, and for $n\geq 1$,

$$K_{n+1} \leq C_3\varepsilon(||\mathcal{G}(u_n)||^{1+\frac{p}{N}}_{L^1(\Omega)} + 1).$$

From (4.2) and (4.3), we find

$$\|\mathcal{G}(u_n)\|_{L^1(Q)} \leq |Q|\,G(2) + \int\limits_{\{u_n\geq 2\}|} G(u_n)dxdt \leq |Q|\,G(2) + C_4K_n\int_2^\infty G\left(s\right)s^{-1-p_c}ds.$$

Thus, $K_{n+1} \leq C_5\varepsilon(K_n^{1+\frac{p}{N}}+1)$. Therefore, if ε is small enough, $\{K_n\}$ is bounded. Since $\{u_n\}$ is nondecreasing, from (4.2) and the relation $\mathcal{G}(x,t,u_n) \leq G(u_n)$, we deduce that $\{\mathcal{G}(u_n)\}$ converges. Then by Theorem 3.4, up to a subsequence, $\{u_n\}$ converges to a R-solution u of (2.5). ■

Remark 4.1 *Theorems 2.1 and 2.2 are still valid for operators $\mathcal{A}$ also depending on t, satisfying conditions analogous to (1.2), (1.3).*

5 General case with absorption terms

In the sequel we combine the results of Theorem 3.4 with delicate techniques introduced in [9] for the elliptic problem (4.1), for proving Theorems 2.3 and 2.5. In these proofs the use of the elliptic Wolff potential is an essential tool.

We recall a first result obtained in [9, Corollary 3.4 and Theorem 3.8] for the elliptic problem without perturbation term, inspired from [27, Theorem 2.1]:

Theorem 5.1 *Let $1<p<N$, Ω be a bounded domain of $\mathbb{R}^N$ and $\omega\in\mathcal{M}_b(\Omega)$ with compact support in Ω. Suppose that u_n is a solution of problem*

$$\left\{\begin{array}{l}-\mathcal{A}(u_n) = \varphi_n * \omega \qquad \text{in } \Omega,\\ u_n = 0 \qquad \text{on } \partial\Omega,\end{array}\right.$$

where $\{\varphi_n\}$ is a sequence of mollifiers in $\mathbb{R}^N$. Then, up to subsequence, u_n converges a.e in Ω to a renormalized solution u of

$$\begin{cases} -\mathcal{A}(u) = \omega & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in the elliptic sense of [15], satisfying

$$-\kappa \mathbf{W}_{1,p}^{2D}[\omega^-] \leq u \leq \kappa \mathbf{W}_{1,p}^{2D}[\omega^+] \tag{5.1}$$

where κ is a constant which only depends of $N, p, \Lambda_1, \Lambda_2$.

Next we give a general result for the parabolic problem (1.5) with absorption:

Theorem 5.2 *Let $p < N$, and assume that $s \mapsto \mathcal{G}(x,t,s)$ is nondecreasing and odd, for a.e. (x,t) in Q. Let $\mu_1, \mu_2 \in \mathcal{M}_b^+(Q)$ such that there exist $\{\omega_n\} \subset \mathcal{M}_b^+(\Omega)$ and nondecreasing sequences $\{\mu_{1,n}\}, \{\mu_{2,n}\}$ in $\mathcal{M}_b^+(Q)$ with compact support in Q, converging to μ_1, μ_2, respectively in the narrow topology, and satisfying*

$$\mu_{1,n}, \mu_{2,n} \leq \omega_n \otimes \chi_{(0,T)}, \quad \text{and} \quad \mathcal{G}((n + \kappa \mathbf{W}_{1,p}^{2D}[\omega_n])) \in L^1(Q),$$

where the constant κ is given at Theorem 5.1. Let $u_0 \in L^1(\Omega)$, and $\mu = \mu_1 - \mu_2$.

Then there exists a R-solution u of problem (1.5). Moreover if $u_0 \in L^\infty(\Omega)$, and $\omega_n \leq \gamma$ for any $n \in \mathbb{N}$, for some $\gamma \in \mathcal{M}_b^+(\Omega)$, then a.e. in Q,

$$|u(x,t)| \leq \kappa \mathbf{W}_{1,p}^{2D}[\gamma](x) + ||u_0||_{L^\infty(\Omega)}. \tag{5.2}$$

For proving this result, we need two Lemmas:

Lemma 5.3 *Let $\mathcal{G}$ satisfy the assumptions of Theorem 5.2 and $\mathcal{G} \in L^\infty(Q \times \mathbb{R})$. For $i = 1, 2$, let $u_{0,i} \in L^\infty(\Omega)$ be nonnegative, and $\lambda_i = \lambda_{i,0} + \lambda_{i,s} \in \mathcal{M}_b^+(Q)$ with compact support in Q, $\gamma \in \mathcal{M}_b^+(\Omega)$ with compact support in Ω such that $\lambda_i \leq \gamma \otimes \chi_{(0,T)}$. Let $\lambda_{i,0} = (f_i, g_i, h_i)$ be a decomposition of $\lambda_{i,0}$ into functions with compact support in Q.*

Then, there exist R-solutions u, u_1, u_2, to problems

$$\begin{cases} u_t - \mathcal{A}(u) + \mathcal{G}(u) = \lambda_1 - \lambda_2 & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_{0,1} - u_{0,2}, & \text{in } \Omega, \end{cases} \tag{5.3}$$

$$\begin{cases} (u_i)_t - \mathcal{A}(u_i) + \mathcal{G}(u_i) = \lambda_i & \text{in } Q, \\ u_i = 0 & \text{on } \partial\Omega \times (0,T), \\ u_i(0) = u_{0,i}, & \text{in } \Omega, \end{cases} \tag{5.4}$$

relative to decompositions $(f_{1,n} - f_{2,n} - \mathcal{G}(u_n), g_{1,n} - g_{2,n}, h_{1,n} - h_{2,n})$, $(f_{i,n} - \mathcal{G}(u_{i,n}), g_{i,n}, h_{i,n})$, such that a.e. in Q,

$$-||u_{0,2}||_{L^\infty(\Omega)} - \kappa \mathbf{W}_{1,p}^{2D}[\gamma](x) \leq -u_2(x,t) \leq u(x,t) \leq u_1(x,t) \leq \kappa \mathbf{W}_{1,p}^{2D}[\gamma](x) + ||u_{0,1}||_{L^\infty(\Omega)}, \tag{5.5}$$

and

$$\int_Q |\mathcal{G}(u)|\, dxdt \leq \sum_{i=1,2} \left(\lambda_i(Q) + ||u_{0,i}||_{L^1(\Omega)}\right) \quad \text{and} \quad \int_Q \mathcal{G}(u_i) dxdt \leq \lambda_i(Q) + ||u_{0,i}||_{L^1(\Omega)}, \quad i = 1, 2. \tag{5.6}$$

Furthermore, assume that $\mathcal{H}, \mathcal{K}$ have the same properties as $\mathcal{G}$, and $\mathcal{H}(x,t,s) \leq \mathcal{G}(x,t,s) \leq \mathcal{K}(x,t,s)$ for any $s \in (0,+\infty)$ and a.e. in Q. Then, one can find solutions $u_i(\mathcal{H}), u_i(\mathcal{K})$, corresponding to $\mathcal{H}, \mathcal{K}$ with data λ_i, such that $u_i(\mathcal{H}) \geq u_i \geq u_i(\mathcal{K})$, $i = 1,2$.

Assume that ω_i, θ_i have the same properties as λ_i and $\omega_i \leq \lambda_i \leq \theta_i$, $u_{0,i,1}, u_{0,i,2} \in L^{\infty+}(\Omega)$, $u_{0,i,2} \leq u_{0,i} \leq u_{0,i,1}$. Then one can find solutions $u_i(\omega_i), u_i(\theta_i)$, corresponding to $(\omega_i, u_{0,i,2}), (\theta_i, u_{0,i,1})$, such that $u_i(\omega_i, u_{0,i,2}) \leq u_i \leq u_i(\theta_i, u_{0,i,1})$.

Proof. Let $\{\varphi_{1,n}\}, \{\varphi_{2,n}\}$ be sequences of mollifiers in $\mathbb{R}$ and $\mathbb{R}^N$, and $\varphi_n = \varphi_{1,n}\varphi_{2,n}$. Set $\gamma_n = \varphi_{2,n} * \gamma$, and for $i = 1,2$, $u_{0,i,n} = \varphi_{2,n} * u_{0,i}$,

$$\lambda_{i,n} = \varphi_n * \lambda_i = f_{i,n} - \operatorname{div}(g_{i,n}) + (h_{i,n})_t + \lambda_{i,s,n},$$

where $f_{i,n} = \varphi_n * f_i$, $g_{i,n} = \varphi_n * g_i$, $h_{i,n} = \varphi_n * h_i$, $\lambda_{i,s,n} = \varphi_n * \lambda_{i,s}$, and

$$\lambda_n = \lambda_{1,n} - \lambda_{2,n} = f_n - \operatorname{div}(g_n) + (h_n)_t + \lambda_{s,n},$$

where $f_n = f_{1,n} - f_{2,n}$, $g_n = g_{1,n} - g_{2,n}$, $h_n = h_{1,n} - h_{2,n}$, $\lambda_{s,n} = \lambda_{1,s,n} - \lambda_{2,s,n}$. Then for n large enough, $\lambda_{1,n}, \lambda_{2,n}, \lambda_n \in C_c^\infty(Q)$, $\gamma_n \in C_c^\infty(\Omega)$. Thus there exist unique solutions $u_n, u_{i,n}, v_{i,n}$, $i = 1,2$, of problems

$$\begin{cases} (u_n)_t - \mathcal{A}(u_n) + \mathcal{G}(u_n) = \lambda_{1,n} - \lambda_{2,n} & \text{in } Q, \\ u_n = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u_n(0) = u_{0,1,n} - u_{0,2,n} & \text{in } \Omega, \end{cases}$$

$$\begin{cases} (u_{i,n})_t - \mathcal{A}(u_{i,n}) + \mathcal{G}(u_{i,n}) = \lambda_{i,n} & \text{in } Q, \\ u_{i,n} = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u_{i,n}(0) = u_{0,i,n} & \text{in } \Omega, \end{cases}$$

$$-\mathcal{A}(w_n) = \gamma_n \quad \text{in } \Omega, \qquad w_n = 0 \quad \text{on } \partial\Omega,$$

such that

$$-||u_{0,2}||_{L^\infty(\Omega)} - w_n(x) \leq -u_{2,n}(x,t) \leq u_n(x,t) \leq u_{1,n}(x,t) \leq w_n(x) + ||u_{0,1}||_{L^\infty(\Omega)}, \quad a.e. \text{ in } Q.$$

Otherwise, as in the Proof of Theorem 2.1, (i), there holds

$$\int_Q |\mathcal{G}(u_n)| dxdt \leq \sum_{i=1,2} \left(\lambda_i(Q) + ||u_{0,i,n}||_{L^1(\Omega)}\right), \quad \text{and} \quad \int_Q \mathcal{G}(u_{i,n}) dxdt \leq \lambda_i(Q) + ||u_{0,i,n}||_{L^1(\Omega)}, \quad i = 1,2.$$

From Proposition 3.3, up to a common subsequence, $\{u_n, u_{1,n}, u_{2,n}\}$ converge to some (u, u_1, u_2), *a.e.* in Q. Since $\mathcal{G}$ is bounded, in particular, $\{\mathcal{G}(u_n)\}$ converges to $\mathcal{G}(u)$ and $\{\mathcal{G}(u_{i,n})\}$ converges to $\mathcal{G}(u_i)$ in $L^1(Q)$. Thus, (5.6) is satisfied. Moreover $\{\lambda_{i,n} - \mathcal{G}(u_{i,n}), f_{i,n} - \mathcal{G}(u_{i,n}), g_{i,n}, h_{i,n}, \lambda_{i,s,n}, u_{0,i,n}\}$ is an approximation of $(\lambda_i - \mathcal{G}(u_i), f_i - \mathcal{G}(u_i), g_i, h_i, \lambda_{i,s}, u_{0,i})$, and $\{\lambda_n - \mathcal{G}(u_n), f_n - \mathcal{G}(u_n), g_n, h_n, \lambda_{s,n}, u_{0,1,n} - u_{0,2,n}\}$ is an approximation of $(\lambda_1 - \lambda_2 - \mathcal{G}(u), f - \mathcal{G}(u), g, h, \lambda_s, u_{0,1} - u_{0,2})$, in the sense of Theorem 3.4. Thus, we can find (different) subsequences converging *a.e.* to u, u_1, u_2, R-solutions of (5.3) and (5.4). Furthermore, from Theorem 5.1, up to a subsequence, $\{w_n\}$ converges *a.e.* in Q to a renormalized solution of

$$-\mathcal{A}(w) = \gamma \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \partial\Omega,$$

such that $w \leq \kappa \mathbf{W}_{1,p}^{2D}[\gamma]$, *a.e.* in Ω. Hence, we get the inequality (5.5). The other conclusions follow in the same way. ∎

Lemma 5.4 *Let $\mathcal{G}$ satisfy the assumptions of Theorem 5.2. For $i = 1, 2$, let $u_{0,i} \in L^\infty(\Omega)$ be nonnegative, $\lambda_i \in \mathcal{M}_b^+(Q)$ with compact support in Q, and $\gamma \in \mathcal{M}_b^+(\Omega)$ with compact support in Ω, such that*

$$\lambda_i \leq \gamma \otimes \chi_{(0,T)}, \quad \text{and} \quad \mathcal{G}((||u_{0,i}||_{L^\infty(\Omega)} + \kappa \mathbf{W}_{1,p}^{2D}[\gamma])) \in L^1(Q). \tag{5.7}$$

Let $\lambda_{i,0} = (f_i, g_i, h_i)$ be a decomposition of $\lambda_{i,0}$ into functions with compact support in Q.

Then, there exist R-solutions u, u_1, u_2 of the problems (5.3) and (5.4), respectively relative to the decompositions $(f_1 - f_2 - \mathcal{G}(u), g_1 - g_2, h_1 - h_2)$, $(f_i - \mathcal{G}(u_i), g_i, h_i)$, satifying (5.5) and (5.6).

Moreover, assume that ω_i, θ_i have the same properties as λ_i and $\omega_i \leq \lambda_i \leq \theta_i$, $u_{0,i,1}, u_{0,i,2} \in L^\infty(\Omega)$, $0 \leq u_{0,i,2} \leq u_{0,i} \leq u_{0,i,1}$. Then, one can find solutions $u_i(\omega_i, u_{0,i,2})$, $u_i(\theta_i, u_{0,i,1})$, corresponding with $(\omega_i, u_{0,i,2})$, $(\theta_i, u_{0,i,1})$, such that $u_i(\omega_i, u_{0,i,2}) \leq u_i \leq u_i(\theta_i, u_{0,i,1})$.

Proof. From Lemma 5.3 there exist R-solutions u_n, $u_{i,n}$ to problems

$$\begin{cases} (u_n)_t - \mathcal{A}(u_n) + T_n(\mathcal{G}(u_n)) = \lambda_1 - \lambda_2 & \text{in } Q, \\ u_n = 0 & \text{on } \partial\Omega \times (0,T), \\ u_n(0) = u_{0,1} - u_{0,2} & \text{in } \Omega, \end{cases}$$

$$\begin{cases} (u_{i,n})_t - \mathcal{A}(u_{i,n}) + T_n(\mathcal{G}(u_{i,n})) = \lambda_i & \text{in } Q, \\ u_{i,n} = 0 & \text{on } \partial\Omega \times (0,T), \\ u_{i,n}(0) = u_{0,i}, & \text{in } \Omega, \end{cases}$$

relative to the decompositions $(f_1 - f_2 - T_n(\mathcal{G}(u_n)), g_1 - g_2, h_1 - h_2)$, $(f_i - T_n(\mathcal{G}(u_{i,n})), g_i, h_i)$; and they satisfy, *a.e.* in Q,

$$-||u_{0,2}||_{L^\infty(\Omega)} - \kappa \mathbf{W}_{1,p}^{2D}[\gamma](x) \leq -u_{2,n}(x,t) \leq u_n(x,t) \leq u_{1,n}(x,t) \leq \kappa \mathbf{W}_{1,p}^{2D}\gamma(x) + ||u_{0,1}||_{L^\infty(\Omega)}, \tag{5.8}$$

$$\int_Q |T_n(\mathcal{G}(u_n))| dxdt \leq \sum_{i=1,2} (\lambda_i(Q) + ||u_{0,i}||_{L^1(\Omega)}), \quad \text{and} \quad \int_Q T_n(\mathcal{G}(u_{i,n})) dxdt \leq \lambda_i(Q) + ||u_{0,i}||_{L^1(\Omega)}.$$

As in Lemma 5.3, up to a common subsequence, $\{u_n, u_{1,n}, u_{2,n}\}$ converges *a.e.* in Q to $\{u, u_1, u_2\}$ for which (5.5) is satisfied *a.e.* in Q. From (5.7), (5.8) and the dominated convergence Theorem, we deduce that $\{T_n(\mathcal{G}(u_n))\}$ converges to $\mathcal{G}(u)$ and $\{T_n(\mathcal{G}(u_{i,n}))\}$ converges to $\mathcal{G}(u_i)$ in $L^1(Q)$. Thus, from Theorem 3.4, u and u_i are respective R-solutions of (5.3) and (5.4) relative to the decompositions $(f_1 - f_2 - \mathcal{G}(u), g_1 - g_2, h_1 - h_2)$, $(f_i - \mathcal{G}(u_i), g_i, h_i)$, and (5.5) and (5.6) hold. The last statement follows from the same assertion in Lemma 5.3. ∎

Proof of Theorem 5.2. By Proposition 3.6, for $i = 1, 2$, there exist $f_{i,n}, f_i \in L^1(Q)$, $g_{i,n}, g_i \in (L^{p'}(Q))^N$ and $h_{i,n}, h_i \in L^p((0,T); W_0^{1,p}(\Omega))$, $\mu_{i,n,s}, \mu_{i,s} \in \mathcal{M}_s^+(Q)$ such that

$$\mu_i = f_i - \operatorname{div} g_i + (h_i)_t + \mu_{i,s}, \qquad \mu_{i,n} = f_{i,n} - \operatorname{div} g_{i,n} + (h_{i,n})_t + \mu_{i,n,s},$$

and $\{f_{i,n}\}, \{g_{i,n}\}, \{h_{i,n}\}$ strongly converge to f_i, g_i, h_i in $L^1(Q)$, $(L^{p'}(Q))^N$ and $L^p((0,T); W_0^{1,p}(\Omega))$ respectively, and $\{\mu_{i,n}\}, \{\mu_{i,n,s}\}$ converge to $\mu_i, \mu_{i,s}$ (strongly) in $\mathcal{M}_b(Q)$, and

$$||f_{i,n}||_{L^1(\Omega)} + ||g_{i,n}||_{L^{p'}(\Omega)} + ||h_{i,n}||_{L^p((0,T);W_0^{1,p}(\Omega))} + \mu_{i,n,s}(\Omega) \leq 2\mu(Q).$$

By Lemma 5.4, there exist R-solutions u_n, $u_{i,n}$ to problems

$$\begin{cases} (u_n)_t - \mathcal{A}(u_n) + \mathcal{G}(u_n) = \mu_{1,n} - \mu_{2,n} & \text{in } Q, \\ u_n = 0 & \text{on } \partial\Omega \times (0,T), \\ u_n(0) = T_n(u_0) & \text{in } \Omega, \end{cases}$$

$$\begin{cases} (u_{i,n})_t - \mathcal{A}(u_{i,n}) + \mathcal{G}(u_{i,n}) = \mu_{i,n} & \text{in } Q, \\ u_{i,n} = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u_{i,n}(0) = T_n(u_0^{\pm}) \quad \text{in } \Omega, \end{cases}$$

for $i = 1,2$, relative to the decompositions $(f_{1,n} - f_{2,n} - \mathcal{G}(u_n), g_{1,n} - g_{2,n}, h_{1,n} - h_{2,n})$, $(f_{i,n} - \mathcal{G}(u_{i,n}), g_{i,n}, h_{i,n})$, such that $\{u_{i,n}\}$ is nonnegative and nondecreasing, and $-u_{2,n} \le u_n \le u_{1,n}$; and

$$\int_Q |\mathcal{G}(u_n)| dxdt, \int_Q \mathcal{G}(u_{i,n}) dxdt \le \mu_1(Q) + \mu_2(Q) + ||u_0||_{L^1(\Omega)}. \tag{5.9}$$

As in the proof of Lemma 5.4, up to a common subsequence $\{u_n, u_{1,n}, u_{2,n}\}$ converge *a.e.* in Q to $\{u, u_1, u_2\}$. Since $\{\mathcal{G}(u_{i,n})\}$ is nondecreasing, and nonnegative, from the monotone convergence Theorem and (5.9), we obtain that $\{\mathcal{G}(u_{i,n})\}$ converges to $\mathcal{G}(u_i)$ in $L^1(Q)$, $i = 1,2$. Finally, $\{\mathcal{G}(u_n)\}$ converges to $\mathcal{G}(u)$ in $L^1(Q)$, since $|\mathcal{G}(u_n)| \le \mathcal{G}(u_{1,n}) + \mathcal{G}(u_{2,n})$. Thus, we can see that

$$\{\mu_{1,n} - \mu_{2,n} - \mathcal{G}(u_n), f_{1,n} - f_{2,n} - \mathcal{G}(u_n), g_{1,n} - g_{2,n}, h_{1,n} - h_{2,n}, \mu_{1,s,n} - \mu_{2,s,n}, T_n(u_0)\}$$

is an approximation of $(\mu_1 - \mu_2 - \mathcal{G}(u), f_1 - f_2 - \mathcal{G}(u), g_1 - g_2, h_1 - h_2, \mu_{1,s} - \mu_{2,s}, u_0)$, in the sense of Theorem 3.4. Therefore, u is a R-solution of (1.1), and (5.2) holds if $u_0 \in L^\infty(\Omega)$ and $\omega_n \le \gamma$ for any $n \in \mathbb{N}$ and some $\gamma \in \mathcal{M}_b^+(\Omega)$. ∎

As a consequence of Theorem 5.2, we get a result for problem (2.1), used in Section 6:

Corollary 5.5 *Let $u_0 \in L^\infty(\Omega)$, and $\mu \in \mathcal{M}_b(Q)$ such that $|\mu| \le \omega \otimes \chi_{(0,T)}$ for some $\omega \in \mathcal{M}_b^+(\Omega)$. Then there exist a R-solution u of (2.1), such that*

$$|u(x,t)| \le \kappa \mathbf{W}_{1,p}^{2D}[\omega](x) + ||u_0||_{L^\infty(\Omega)}, \qquad \text{for a.e. } (x,t) \in Q, \tag{5.10}$$

where κ is defined at Theorem 5.1.

Proof. Let $\{\phi_n\}$ be a nonnegative, nondecreasing sequence in $C_c^\infty(Q)$ which converges to 1, *a.e.* in Q. Since $\{\phi_n \mu^+\}, \{\phi_n \mu^-\}$ are nondecreasing sequences, the result follows from Theorem 5.2. ∎

5.1 The power case

First recall some results relative to the elliptic case for the model problem

$$-\Delta_p u + |u|^{q-1} u = \omega \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{5.11}$$

with $\omega \in \mathcal{M}_b(\Omega), q > p - 1 > 0$.

For $p = 2$, it is shown in [2] that (5.11) admits a solution if and only if ω does not charge the sets of Bessel $\text{Cap}_{\mathbf{G}_2, \frac{q}{q-1}}$-capacity zero. For $p \ne 2$, existence holds for any measure $\omega \in \mathcal{M}_b(\Omega)$ in the subcritical case

$$q < p_e := N(p-1)/(N-p) \tag{5.12}$$

from [4]. Some necessary conditions for existence have been given in [5, 6]. From [9, Theorem 1.1], a sufficient condition for existence is that ω does not charge the sets of $\text{Cap}_{\mathbf{G}_p, \frac{q}{q+1-p}}$–capacity zero, and it can be conjectured that this condition is also necessary.

Next we prove Theorem 2.3. We use the following result of [9]:

Proposition 5.6 *Let $q > p-1$ and $\nu \in \mathcal{M}_b^+(\Omega)$.*

If ν does not charge the sets of $\mathrm{Cap}_{\mathbf{G}_{p},\frac{q}{q+1-p}}$-capacity zero, there exists a nondecreasing sequence $\{\nu_n\} \subset \mathcal{M}_b^+(\Omega)$ with compact support in Ω which converges to ν strongly in $\mathcal{M}_b(\Omega)$ and such that $\mathbf{W}_{1,p}^R[\nu_n] \in L^q(\mathbb{R}^N)$, for any $n \in \mathbb{N}$ and $R > 0$.

Proof of Theorem 2.3. Let $f \in L^1(Q)$, $u_0 \in L^1(\Omega)$, and $\mu \in \mathcal{M}_b(Q)$ such that $|\mu| \leq \omega \otimes F$, where $F \in L^1((0,T))$ and ω does not charge the sets of $\mathrm{Cap}_{\mathbf{G}_{p},\frac{q}{q+1-p}}$-capacity zero. From Proposition 5.6, there exists a nondecreasing sequence $\{\omega_n\} \subset \mathcal{M}_b^+(\Omega)$ with compact support in Ω which converges to ω, strongly in $\mathcal{M}_b(\Omega)$, such that $\mathbf{W}_{1,p}^{2D}[\omega_n] \in L^q(\mathbb{R}^N)$. We can write

$$f + \mu = \mu_1 - \mu_2, \qquad \mu_1 = f^+ + \mu^+, \qquad \mu_2 = f^- + \mu^-, \tag{5.13}$$

and $\mu^+, \mu^- \leq \omega \otimes F$. We set

$$Q_n = \{(x,t) \in \Omega \times (\frac{1}{n}, T - \frac{1}{n}) : d(x, \partial\Omega) > \frac{1}{n}\}, \qquad F_n = T_n(\chi_{(\frac{1}{n}T - \frac{1}{n})}F), \tag{5.14}$$

$$\mu_{1,n} = T_n(\chi_{Q_n} f^+) + \inf\{\mu^+, \omega_n \otimes F_n\}, \qquad \mu_{2,n} = T_n(\chi_{Q_n} f^-) + \inf\{\mu^-, \omega_n \otimes F_n\}. \tag{5.15}$$

Then $\{\mu_{1,n}\}, \{\mu_{2,n}\}$ are nondecreasing sequences with compact support in Q, and

$$\mu_{1,n}, \mu_{2,n} \leq \tilde{\omega}_n \otimes \chi_{(0,T)}, \qquad \text{with } \tilde{\omega}_n = n(\chi_\Omega + \omega_n),$$

and $(n + \kappa \mathbf{W}_{1,p}^{2D}[\tilde{\omega}_n])^q \in L^1(Q)$. Besides, $\omega_n \otimes F_n$ converges to $\omega \otimes F$ strongly in $\mathcal{M}_b(Q)$. Indeed we easily check that

$$||\omega_n \otimes F_n - \omega \otimes F||_{\mathcal{M}_b(Q)} \leq ||F_n||_{L^1((0,T))} ||\omega_n - \omega||_{\mathcal{M}_b(\Omega)} + ||\omega||_{\mathcal{M}_b(\Omega)} ||F_n - F||_{L^1((0,T))}$$

Observe that for any measures $\nu, \theta, \eta \in \mathcal{M}_b(Q)$, there holds

$$|\inf\{\nu, \theta\} - \inf\{\nu, \eta\}| \leq |\theta - \eta|,$$

hence $\{\mu_{1,n}\}, \{\mu_{2,n}\}$ converge to μ_1, μ_2 respectively in $\mathcal{M}_b(Q)$. Therefore, the result follows from Theorem 5.2. ∎

Remark 5.7 *From Theorem 2.3, we deduce the existence for any measure $\omega \in \mathcal{M}_b(\Omega)$ for $p < p_e$, whre p_e is defined at (5.12), since p_e is the critical exponent of the elliptic problem (5.11). Note that $p_e > p_c$ since $p > p_1$. Let $\mathcal{M}_{0,e}(\Omega)$ be the set of Radon measures ω on that do not charge the sets of zero c_p^Ω-capacity, where, for any compact set $K \subset \Omega$,*

$$c_p^\Omega(K) = \inf\{\int_\Omega |\nabla\varphi|^p dx : \varphi \geq \chi_K, \varphi \in C_c^\infty(\Omega)\}.$$

From [16, Theorem 2.16], for any $F \in L^1((0,T))$ with $\int_0^T F(t)dt \neq 0$, and $\omega \in \mathcal{M}_b(\Omega)$,

$$\omega \in \mathcal{M}_{0,e}(\Omega) \Longleftrightarrow \omega \otimes F \in \mathcal{M}_0(Q).$$

If $q \geq p_e$, there exist measures $\omega \in \mathcal{M}_b^+(\Omega)$ which do not charge the sets of $\mathrm{Cap}_{\mathbf{G}_{p},\frac{q}{q+1-p}}$-capacity zero, such that $\omega \notin \mathcal{M}_{0,e}(\Omega)$. As a consequence, Theorem 2.3 shows the existence for some measures $\mu \notin \mathcal{M}_0(Q)$.

Remark 5.8 *Let $\mathcal{G} : Q\times\mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that the map $s \mapsto \mathcal{G}(x,t,s)$ is nondecreasing and odd, for a.e. (x,t) in Q. Let $\mu \in \mathcal{M}_b(Q), f \in L^1(Q), u_0 \in L^1(\Omega)$ and $\omega \in \mathcal{M}_b^+(\Omega)$ such that (2.7) holds.*

If $\omega(\{x : \mathbf{W}_{1,p}^{2D}[\omega](x) = \infty\}) = 0$, then, (1.5) has a R-solution with data $(f+\mu, u_0)$. The proof is similar to the one of Theorem 2.3, after replacing ω_n by $\chi_{W_{1,p}^{2D}[\omega]\le n}\omega$. Note that $\omega(\{x : \mathbf{W}_{1,p}^{2D}[\omega](x) = \infty\}) = 0$ if and only if $\omega \in \mathcal{M}_{0,e}(\Omega)$, see [21].

Remark 5.9 *As in [9], from Theorem 5.2, we can extend Theorem 2.3 given for $\mathcal{G}(u) = |u|^{q-1}u$, to the case of a function $\mathcal{G}(x,t,.)$, odd for a.e. $(x,t) \in Q$, such that*

$$|\mathcal{G}(x,t,u)| \le G(|u|), \qquad \int_1^\infty G(s)s^{-q-1}ds < \infty,$$

where G is a nondecreasing continuous, under the condition that ω does not charge the sets of zero $\mathrm{Cap}_{\mathbf{G}_p,\frac{q}{q+1-p},1}$-capacity, where for any Borel set $E \subset \mathbb{R}^N$,

$$\mathrm{Cap}_{\mathbf{G}_p,\frac{q}{q+1-p},1}(E) = \inf\{||\varphi||_{L^{\frac{q}{q-p+1},1}(\mathbb{R}^N)} : \varphi \in L^{\frac{q}{q-p+1},1}(\mathbb{R}^N), \quad \mathbf{G}_p * \varphi \ge \chi_E\}$$

where $L^{\frac{q}{q-p+1},1}(\mathbb{R}^N)$ is the Lorentz space of order $(q/(q-p+1),1)$.

5.2 The exponential case

Theorem 2.5 extends the elliptic result of [9, Theorem 1.2] to the parabolic case. For the proof, we use the following property of [9, Theorem 2.4]:

Proposition 5.10 *Suppose $1 < p < N$. Let $\nu \in \mathcal{M}_b^+(\Omega)$, $\beta > 1$, and $\delta_0 = ((12\beta)^{-1})^\beta p\ln 2$. There exists $C = C(N,p,\beta,D)$ such that, for any $\delta \in (0,\delta_0)$,*

$$\int_\Omega \exp(\delta\frac{(\mathbf{W}_{1,p}^{2D}[\nu])^\beta}{||\mathbf{M}_{p,2D}^{\frac{p-1}{\beta'}}[\nu]||_{L^\infty(\mathbb{R}^N)}^{\frac{\beta}{p-1}}})dx \le \frac{C}{\delta_0-\delta}.$$

Proof of Theorem 2.5. Let Q_n be defined at (5.14), and $\omega_n = \omega\chi_{\Omega_n}$, where $\Omega_n = \{x \in \Omega : d(x,\partial\Omega) > 1/n\}$. We still consider $\mu_1, \mu_2, F_n, \mu_{1,n}, \mu_{2,n}$ as in (5.13), (5.15).

Case (i): Assume that $||F||_{L^\infty((0,T))} \le 1$ and (2.12) holds. We have $\mu_{1,n}, \mu_{2,n} \le n\chi_\Omega + \omega$. For any $\varepsilon > 0$, there exists $c_\varepsilon = c_\varepsilon(\varepsilon, N, p, \beta, \kappa, D) > 0$ such that

$$(n + \kappa\mathbf{W}_{1,p}^{2D}[n\chi_\Omega + \omega])^\beta \le c_\varepsilon n^{\frac{\beta p}{p-1}} + (1+\varepsilon)\kappa^\beta(\mathbf{W}_{1,p}^{2D}[\omega])^\beta$$

$a.e.$ in Ω. Thus,

$$\exp\left(\tau(n + \kappa\mathbf{W}_{1,p}^{2D}[n\chi_\Omega + \omega])^\beta\right) \le \exp\left(\tau c_\varepsilon n^{\frac{\beta p}{p-1}}\right)\exp\left(\tau(1+\varepsilon)\kappa^\beta(\mathbf{W}_{1,p}^{2D}[\omega])^\beta\right).$$

If (2.12) holds with $M_0 = \left(\delta_0/\tau\kappa^\beta\right)^{(p-1)/\beta}$ then we can chose ε such that

$$\tau(1+\varepsilon)\kappa^\beta||\mathbf{M}_{p,2D}^{\frac{p-1}{\beta'}}[\nu]||_{L^\infty(\mathbb{R}^N)}^{\frac{\beta}{p-1}} < \delta_0.$$

From Proposition 5.10, we get $\exp(\tau(1+\varepsilon)\kappa^\beta \mathbf{W}_{1,p}^{2D}[\omega])^\beta) \in L^1(\Omega)$, which implies $\exp(\tau(n+\kappa^\beta \mathbf{W}_{1,p}^{2D}[n\chi_\Omega + \omega])^\beta) \in L^1(\Omega)$ for all n. We conclude from Theorem 5.2.

Case (ii): Assume that there exists $\varepsilon > 0$ such that $\mathbf{M}_{p,2D}^{(p-1)/(\beta+\varepsilon)'}[\omega] \in L^\infty(\mathbb{R}^N)$. Now we use the inequality $\mu_{1,n}, \mu_{2,n} \leq n(\chi_\Omega + \omega)$. For any $\varepsilon > 0$ and any $n \in \mathbb{N}$ there exists $c_{\varepsilon,n} > 0$ such that

$$(n + \kappa \mathbf{W}_{1,p}^{2D}[n(\chi_\Omega + \omega)])^\beta \leq c_{\varepsilon,n} + \varepsilon(\mathbf{W}_{1,p}^{2D}[\omega])^{\beta_0}.$$

Thus, from Proposition 5.10, we obtain that $\exp(\tau(n + \kappa^\beta \mathbf{W}_{1,p}^{2D}[n(\chi_\Omega + \omega)])^\beta) \in L^1(\Omega)$ for any $n \in \mathbb{N}$. We conclude from Theorem 5.2. ∎

6 General case with source term

The results of this Section are based on Corollary 5.5 and elliptic techniques of Wolff potential used in [27], [28] and [22, Theorem 2.5].

6.1 The power case

Recall some results of [27], [28] for the nonnegative solutions of equation

$$-\Delta_p u = u^q + \omega \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{6.1}$$

It was proved that if $\omega(E) \leq C\mathrm{Cap}_{\mathbf{G}_{p},\frac{q}{q+1-p}}(E)$,for any compact of $\mathbb{R}^N$, with C small enough, problem (6.1) has at least a solution, and conversely if there exists a solution, and ω has a compact support, then there exists a constant C' such that

$$\omega(E) \leq C'\mathrm{Cap}_{\mathbf{G}_{p},\frac{q}{q+1-p}}(E), \qquad \text{for any compact set } E \text{ of } \mathbb{R}^N.$$

For proving Theorem 2.4 we use the following property of Wolff potentials, shown in [27]:

Theorem 6.1 *Let $q > p-1$, $0 < p < N$, $\omega \in \mathcal{M}_b^+(\Omega)$. If for some $\lambda > 0$,*

$$\omega(E) \leq \lambda \mathrm{Cap}_{\mathbf{G}_{p},\frac{q}{q+1-p}}(E) \qquad \text{for any compact set } E \subset \mathbb{R}^N, \tag{6.2}$$

then $(\mathbf{W}_{1,p}^{2D}[\omega])^q \in L^1(\Omega)$, and there exists $M = M(N,p,q,\mathrm{diam}(\Omega))$ such that, a.e. in Ω,

$$\mathbf{W}_{1,p}^{2D}\left[(\mathbf{W}_{1,p}^{2D}[\omega])^q\right] \leq M\lambda^{\frac{q-p+1}{(p-1)^2}}\mathbf{W}_{1,p}^{2D}[\omega] < \infty. \tag{6.3}$$

We deduce the following:

Lemma 6.2 *Let $\omega \in \mathcal{M}_b^+(\Omega)$, and $b \geq 0$ and $K > 0$. Suppose that $\{u_m\}_{m\geq1}$ is a sequence of nonnegative functions in Ω that satisfies*

$$u_1 \leq K\mathbf{W}_{1,p}^{2D}[\omega] + b,$$
$$u_{m+1} \leq K\mathbf{W}_{1,p}^{2D}[u_m^q + \omega] + b \qquad \forall m \geq 1.$$

Assume that ω satisfies (6.2) for some $\lambda > 0$. Then there exist λ_0 and b_0, depending on N,p,q,K,D, such that, if $\lambda \leq \lambda_0$ and $b \leq b_0$, then $\mathbf{W}_{1,p}^{2D}[\omega] \in L^q(\Omega)$ and for any $m \geq 1$,

$$u_m \leq 2\beta_p K\mathbf{W}_{1,p}^{2D}[\omega] + 2b, \qquad \beta_p = \max(1, 3^{\frac{2-p}{p-1}}). \tag{6.4}$$

Proof. Clearly, (6.4) holds for $m = 1$. Now, assume that it holds at the order m. Then

$$u_m^q \leq 2^{q-1}(2\beta_p)^q K^q (\mathbf{W}_{1,p}^{2D}[\omega])^q + 2^{q-1}(2b)^q .$$

Using (6.3) we get

$$\begin{aligned} u_{m+1} &\leq K\mathbf{W}_{1,p}^{2D}\left[2^{q-1}(2\beta_p)^q K^q (W_{1,p}^{2D}[\omega])^q + 2^{q-1}(2b)^q + \omega\right] + b \\ &\leq \beta_p K\left(A_1 \mathbf{W}_{1,p}^{2D}\left[(W_{1,p}^{2D}[\omega])^q\right] + \mathbf{W}_{1,p}^{2D}\left[(2b)^q\right] + W_{1,p}^{2D}[\omega]\right) + b \\ &\leq \beta_p K (A_1 M \lambda^{\frac{q-p+1}{(p-1)^2}} + 1)\mathbf{W}_{1,p}^{2D}[\omega] + \beta_p K \mathbf{W}_{1,p}^{2D}\left[(2b)^q\right] + b \\ &= \beta_p K (A_1 M \lambda^{\frac{q-p+1}{(p-1)^2}} + 1)\mathbf{W}_{1,p}^{2D}[\omega] + A_2 b^{\frac{q}{p-1}} + b, \end{aligned}$$

where M is as in (6.3) and

$$A_1 = \left(2^{q-1}(2\beta_p)^q K^q\right)^{1/(p-1)}, \qquad A_2 = \beta_p K 2^{q/(p-1)} |B_1|^{1/(p-1)} (p')^{-1} (2D)^{p'}.$$

Thus, (6.4) holds for $m = n + 1$ if we prove that

$$A_1 M \lambda^{\frac{q-p+1}{(p-1)^2}} \leq 1 \text{ and } A_2 b^{\frac{q}{p-1}} \leq b,$$

which is equivalent to

$$\lambda \leq (A_1 M)^{-\frac{(p-1)^2}{q-p+1}} \text{ and } b \leq A_2^{-\frac{p-1}{q-p+1}}.$$

Therefore, we obtain the result with $\lambda_0 = (A_1 M)^{-(p-1)^2/(q-p+1)}$ and $b_0 = A_2^{-(p-1)/(q-p+1)}$. ■

Proof of Theorem 2.4. From Corollary 3.7 and 5.5, we can construct a sequence of nonnegative *nondecreasing* R-solutions $\{u_m\}_{m\geq 1}$, defined in the following way: u_1 is a R-solution of (2.1), and u_{m+1} is a nonnegative R-solution of

$$\begin{cases} (u_{m+1})_t - \mathcal{A}(u_{m+1}) = u_m^q + \mu & \text{in } Q, \\ u_{m+1} = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u_{m+1}(0) = u_0 & \text{in } \Omega. \end{cases}$$

Setting $\overline{u}_m = \sup_{t\in(0,T)} u_m(t)$ for all $m \geq 1$, there holds

$$\begin{aligned} \overline{u}_1 &\leq \kappa \mathbf{W}_{1,p}^{2D}[\omega] + ||u_0||_{L^\infty(\Omega)}, \\ \overline{u}_{m+1} &\leq \kappa \mathbf{W}_{1,p}^{2D}[\overline{u}_m^q + \omega] + ||u_0||_{L^\infty(\Omega)} \qquad \forall m \geq 1. \end{aligned}$$

From Lemma 6.2, we can find $\lambda_0 = \lambda_0(N,p,q,D)$ and $b_0 = b_0(N,p,q,D)$ such that if (2.9) is satisfied with λ_0 and b_0; then

$$u_m \leq \overline{u}_m \leq 2\beta_p \kappa \mathbf{W}_{1,p}^{2D}[\omega] + 2||u_0||_{L^\infty(\Omega)} \qquad \forall m \geq 1. \tag{6.5}$$

Thus $\{u_m\}$ converges *a.e.* in Q and in $L^q(Q)$ to some function u, for which (2.11) is satisfied in Ω with $c = 2\beta_p\kappa$. Finally, one can apply Theorem 3.4 to the sequence of measures $\{u_m^q + \mu\}$, and obtain that u is a R-solution of (2.10). ■

6.2 The exponential case

We end this Section by proving Theorem 2.6. We first recall an approximation property, which is a consequence of [22, Theorem 2.5]:

Theorem 6.3 *Let $\tau > 0$, $b \geq 0$, $K > 0$, $l \in \mathbb{N}$ and $\beta \geq 1$ such that $l\beta > p-1$. Let $\mathcal{E}$ be defined by (2.13). Let $\{v_m\}$ be a sequence of nonnegative functions in Ω such that, for some $K > 0$,*

$$v_1 \leq K\mathbf{W}_{1,p}^{2D}[\mu] + b,$$
$$v_{m+1} \leq K\mathbf{W}_{1,p}^{2D}[\mathcal{E}(\tau v_m^\beta) + \mu] + b, \quad \forall m \geq 1.$$

Then, there exist b_0 and M_0, depending on $N, p, \beta, \tau, l, K, D$, such that if $b \leq b_0$ and

$$||\mathbf{M}_{p,2D}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{\infty,\mathbb{R}^N} \leq M_0, \tag{6.6}$$

then, setting $c_p = 2\max(1, 2^{\frac{2-p}{p-1}})$,

$$\exp(\tau(Kc_p\mathbf{W}_{1,p}^{2D}[\mu] + 2b_0)^\beta) \in L^1(\Omega),$$

$$v_m \leq Kc_pW_{1,p}^{2D}[\mu] + 2b_0, \quad \forall m \geq 1. \tag{6.7}$$

Proof of Theorem 2.6. From Corollary 3.7 and 5.5 we can construct a sequence of nonnegative *nondecreasing* R-solutions $\{u_m\}_{m\geq 1}$ defined in the following way: u_1 is a R-solution of problem (2.1), and by induction, u_{m+1} is a R-solution of

$$\begin{cases} (u_{m+1})_t - \mathcal{A}(u_{m+1}) = \mathcal{E}(\tau u_m^\beta) + \mu & \text{in } Q, \\ u_{m+1} = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u_{m+1}(0) = u_0 \quad \text{in } \Omega. \end{cases} \tag{6.8}$$

And, setting $\overline{u}_m = \sup_{t\in(0,T)} u_m(t)$, there holds

$$\overline{u}_1 \leq \kappa W_{1,p}^{2D}[\omega] + ||u_0||_{\infty,\Omega},$$
$$\overline{u}_{m+1} \leq \kappa W_{1,p}^{2D}[\mathcal{E}(\tau\overline{u}_m^\beta) + \omega] + ||u_0||_{L^\infty(\Omega)}, \qquad \forall m \geq 1.$$

Thus, from Theorem 6.3, there exist $b_0 \in (0,1]$ and $M_0 > 0$, depending on N, p, β, τ, l, D, such that, if (6.6) holds, then (6.7) is satisfied with $v_m = \overline{u}_m$. As a consequence, u_m is well defined. Thus, $\{u_m\}$ converges *a.e.* in Q to some function u, for which (2.15) is satisfied in Ω. Furthermore, $\{\mathcal{E}(\tau u_m^\beta)\}$ converges to $\mathcal{E}(\tau u^\beta)$ in $L^1(Q)$. Finally, one can apply Theorem 3.4 to the sequence of measures $\{\mathcal{E}(\tau u_m^\beta) + \mu\}$, and obtain that u is a R-solution of (2.14). ■

Remark 6.4 *In [22, Theorem 1.1], when $\mathcal{A} = \Delta_p$, we showed that there exist $M = M(N, p, \beta, \tau, l, D)$ such that if*

$$||\mathbf{M}_{p,2D}^{\frac{(p-1)(\beta-1)}{\beta}}[\omega]||_{L^\infty(\mathbb{R}^N)} \leq M,$$

then the problem

$$\begin{cases} -\Delta_p v = \mathcal{E}(\tau v^\beta) + \omega & \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{6.9}$$

has a renormalized solution in the sense of [15]. We claim the following:

Let $\mathcal{A} = \Delta_p$ and $u_0 \equiv 0$. If (6.9) has a renormalized solution v and $\omega \in \mathcal{M}_{0,e}(\Omega)$, then the problem (2.14) in Theorem 2.6 admits a R-solution u, satisfying $u(x,t) \leq v(x)$ a.e in Q.

Indeed, since $\omega \in \mathcal{M}_{0,e}(\Omega)$, there holds $\mu \in \mathcal{M}_0(Q)$. Otherwise, for any measure $\eta \in \mathcal{M}_0(Q)$ the problem

$$\begin{cases} u_t - \Delta_p u = \eta & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u = 0 & \text{in } \Omega, \end{cases}$$

*has a (unique) R-solution, and the comparison principle is valid, see [26]. Thus, as in the proof of Theorem 2.6, we can construct a **unique** sequence of nonnegative nondecreasing R-solutions $\{u_m\}_{m\geq 1}$, defined in the following way: u_1 is a R-solution of problem (2.1) and satisfies $u_1 \leq v$ a.e in Q ; and by induction, u_{m+1} is a R-solution of (6.8) and satisfies $u_{m+1} \leq v$ a.e in Q. Then $\left\{\mathcal{E}(\tau u_m^\beta)\right\}$ converges to $\mathcal{E}(\tau u^\beta)$ in $L^1(Q)$. Finally, $u := \lim_{n\to\infty} u_n$ is a solution of (2.14). Clearly, this claim is also valid for power source term.*

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