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Abstract

In this paper, we study the existence and regularity of the quasilinear parabolic equations:

$$u_t - \mathrm{div}(A(x,t,\nabla u)) = B(u,\nabla u) + \mu$$

in $\mathbb{R}^{N+1}$, $\mathbb{R}^N \times (0,\infty)$ and a bounded domain $\Omega \times (0,T) \subset \mathbb{R}^{N+1}$. Here $N \geq 2$, the nonlinearity A fulfills standard growth conditions and B term is a continuous function and μ is a radon measure. Our first task is to establish the existence results with $B(u,\nabla u) = \pm|u|^{q-1}u$, for $q > 1$. We next obtain global weighted-Lorentz, Lorentz-Morrey and Capacitary estimates on gradient of solutions with $B \equiv 0$, under minimal conditions on the boundary of domain and on nonlinearity A. Finally, due to these estimates, we solve the existence problems with $B(u,\nabla u) = |\nabla u|^q$ for $q > 1$.

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Keywords: quasilinear parabolic equations; renormalized solutions; Wolff parabolic potential; Riesz parabolic potential; Bessel parabolic potential; maximal potential; heat kernel; Radon measures; uniformly thick domain; Reifenberg flat domain; decay estimates; Lorentz spaces; Riccati type equations; capacity

Contents

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1 Introduction

In this article, we study a class of quasilinear parabolic equations:

$$u_t - \operatorname{div}(A(x,t,\nabla u)) = B(x,t,u,\nabla u) + \mu \tag{1.1}$$

in $\mathbb{R}^{N+1}$ or $\mathbb{R}^N \times (0,\infty)$ or a bounded domain $\Omega_T := \Omega \times (0,T) \subset \mathbb{R}^{N+1}$. Where $N \geq 2$, $A : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function which satisfies

$$|A(x,t,\zeta)| \leq \Lambda_1 |\zeta| \quad \text{and} \tag{1.2}$$

$$\langle A(x,t,\zeta) - A(x,t,\lambda), \zeta - \lambda \rangle \geq \Lambda_2 |\zeta - \lambda|^2, \tag{1.3}$$

for every $(\lambda,\zeta) \in \mathbb{R}^N \times \mathbb{R}^N$ and a.e. $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, here Λ_1 and Λ_2 are positive constants, $B : \mathbb{R}^{N+1} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is also a Carathéodory function and μ is a Radon measure.

The existence and regularity theory, the Wiener criteria and Harnack inequalities, Blow-up at a finite time associated with above parabolic quasilinear operator was studied and developed intensely over the past 50 years, one can found in [58, 44, 30, 48, 49, 25, 50, 60, 83, 75, 73]. Moreover, we also refer to [19]-[22] for L^p-gradient estimates theory in non-smooth domains and [63] Wiener criteria for existence of large solutions of nonlinear parabolic equations with absorption in a non-cylindrical domain.

First, we are specially interested in the existence of solutions to quasilinear parabolic equations with absorption, source terms and data measure:

$$u_t - \operatorname{div}(A(x,t,\nabla u)) + |u|^{q-1}u = \mu, \tag{1.4}$$

$$u_t - \operatorname{div}(A(x,t,\nabla u)) = |u|^{q-1}u + \mu, \tag{1.5}$$

in $\mathbb{R}^{N+1}$ and

$$u_t - \operatorname{div}(A(x,t,\nabla u)) + |u|^{q-1}u = \mu, \quad u(0) = \sigma, \tag{1.6}$$

$$u_t - \operatorname{div}(A(x,t,\nabla u)) = |u|^{q-1}u + \mu, \quad u(0) = \sigma, \tag{1.7}$$

in $\mathbb{R}^N \times (0,\infty)$ or a bounded domain $\Omega_T \subset \mathbb{R}^{N+1}$, where $q > 1$ and μ, σ are Radon measures.

The linear case $A(x,t,\nabla u) = \nabla u$ was studied in detail by Fujita, Brezis and Friedman, Baras and Pierre.

In [18], showed that if $\mu = 0$ and σ is a Dirac mass in Ω, the problem (1.6) in Ω_T (with Dirichlet boundary condition) admits a (unique) solution if and only if $q < (N+2)/N$. Then, optimal results had been considered in [5], for any $\mu \in \mathfrak{M}_b(\Omega_T)$ and $\sigma \in \mathfrak{M}_b(\Omega)$:

there exists a (unique) solution of (1.6) in Ω_T if and only if μ, σ are absolutely continuous with respect to the capacity $\text{Cap}_{2,1,q'}, \text{Cap}_{\mathbf{G}_{2/q},q'}$ (in Ω_T, Ω) respectively, for simplicity we write $\mu << \text{Cap}_{2,1,q'}$ and $\sigma << \text{Cap}_{\mathbf{G}_{2/q},q'}$, with q' is the conjugate exponent of q, i.e $q' = \frac{q}{q-1}$. Where these two capacities will be defined in section 2.

For source case, in [6], showed that for any $\mu \in \mathfrak{M}_b^+(\Omega_T)$ and $\sigma \in \mathfrak{M}_b^+(\Omega)$, the problem (1.7) in bounded domain Ω_T has a nonnegative solution if

$$\mu(E) \leq C\text{Cap}_{2,1,q'}(E) \text{ and } \sigma(O) \leq C\text{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}(O)$$

hold for every compact sets $E \subset \mathbb{R}^{N+1}$, $O \subset \mathbb{R}^N$ here $C = C(N, \text{diam}(\Omega), T)$ is small enough. Conversely, the existence holds then for compact subset $K \subset\subset \Omega$, one find $C_K > 0$ such that

$$\mu(E \cap (K \times [0,T])) \leq C_K\text{Cap}_{2,1,q'}(E) \text{ and } \sigma(O \cap K) \leq C_K\text{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}(O)$$

hold for every compact sets $E \subset \mathbb{R}^{N+1}$, $O \subset \mathbb{R}^N$. In unbounded domain $\mathbb{R}^N \times (0,\infty)$, in [30] asserted that an inequality

$$u_t - \Delta u \geq u^q, u \geq 0 \text{ in } \mathbb{R}^N \times (0,\infty), \tag{1.8}$$

i. if $q < (N+2)/N$ then the only nonnegative global (in time) solution of above inequality is $u \equiv 0$,

ii. if $q > (N+2)/N$ then there exists global positive solution of above inequality.

More general, see [6], for $\mu \in \mathfrak{M}^+(\mathbb{R}^N \times (0,\infty))$ and $\sigma \in \mathfrak{M}^+(\mathbb{R}^N)$, (1.7) has a nonnegative solution in $\mathbb{R}^N \times (0,\infty)$ (with $A(x,t,\nabla u) = \nabla u$) if and only if

$$\mu(E) \leq C\text{Cap}_{\mathcal{H}_2,q'}(E) \text{ and } \sigma(O) \leq C\text{Cap}_{\mathbf{I}_{\frac{2}{q}},q'}(O) \tag{1.9}$$

hold for every compact sets $E \subset \mathbb{R}^{N+1}$, $O \subset \mathbb{R}^N$, here $C = C(N,q)$ is small enough, two capacities $\text{Cap}_{\mathcal{H}_2,q'}, \text{Cap}_{\mathbf{I}_{\frac{2}{q}},q'}$ will be defined in section 2. Note that a necessary and sufficient condition for (1.9) holding with $\mu \in \mathfrak{M}^+(\mathbb{R}^N \times (0,\infty))\backslash\{0\}$ or $\sigma \in \mathfrak{M}^+(\mathbb{R}^N)\backslash\{0\}$ is $q \geq (N+2)/N$. In particular, (1.8) has a (global) positive solution if and only if $q \geq (N+2)/N$. It is known that conditions for data μ, σ in problems with absorption are softer than source. Recently, in exponential case, i.e $|u|^{q-1}u$ is replaced by $P(u) \sim \exp(a|u|^q)$, for $a > 0$ and $q \geq 1$ was established in [61].

We consider (1.6) and (1.7) in Ω_T with Dirichlet boundary conditions when $\text{div}(A(x,t,\nabla u))$ is replaced by $\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$ for $p \in (2-1/N, N)$. In [66], showed that for any $q > p-1$, (1.6) admits a (unique renormalized) solution provided $\sigma \in L^1(\Omega)$ and $\mu \in \mathfrak{M}_b(\Omega_T)$ is diffuse measure i.e absolutely continuous with respect to C_p-parabolic capacity in Ω_T defined on a compact set $K \subset \Omega_T$:

$$C_p(K,\Omega_T) = \inf\left\{||\varphi||_X : \varphi \geq \chi_K, \varphi \in C_c^\infty(\Omega_T)\right\},$$

where $X = \{\varphi : \varphi \in L^p(0,T;W_0^{1,p}(\Omega)), \varphi_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))\}$ endowed with norm $||\varphi||_X = ||\varphi||_{L^p(0,T;W_0^{1,p}(\Omega))} + ||\varphi_t||_{L^{p'}(0,T;W^{-1,p'}(\Omega))}$ and χ_K is the characteristic function of K. An improving result was presented in [14] for measures that have good behavior in time, it is based on results of [16] relative to the elliptic case. That is, (1.6) has a (renormalized) solution for $q > p-1$ if $\sigma \in L^1(\Omega)$ and $|\mu| \leq f + \omega \otimes F$, where $f \in L^1_+(\Omega_T), F \in L^1_+((0,T))$ and $\omega \in \mathfrak{M}_b^+(\Omega)$ is absolutely continuous with respect to $\text{Cap}_{\mathbf{G}_p,\frac{q}{q-p+1}}$ in Ω. Also, (1.7) has a (renormalized) nonnegative solution if $\sigma \in L^\infty_+(\Omega)$, $0 \leq \mu \leq \omega \otimes \chi_{(0,T)}$ with $\omega \in \mathfrak{M}_b^+(\Omega)$ and

$$\omega(E) \leq C_1\text{Cap}_{\mathbf{G}_p,\frac{q}{q-p+1}}(E) \;\forall \text{ compact } E \subset \mathbb{R}^N, \quad ||\sigma||_{L^\infty(\Omega)} \leq C_2$$

for some C_1, C_2 small enough. Another improving results are also stated in [15], especially if $q > p-1$, $p > 2$, $\mu \in \mathfrak{M}_b(\Omega_T)$ and $\sigma \in \mathfrak{M}_b(\Omega)$ are absolutely continuous with respect to $\text{Cap}_{2,1,q'}$ in Ω_T and $\text{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}$ in Ω then (1.6) has a distribution solution.

In [15], we also obtain the existence of solutions for porous medium equation with absorption and data measure: for $q > m > \frac{N-2}{N}$, a sufficient condition for existence solution to the problem

$$u_t - \Delta(|u|^{m-1}u) + |u|^{q-1}u = \mu \;\text{ in }\; \Omega_T, \;\; u = 0 \text{ on } \;\partial\Omega \times (0,T), \quad \text{and }\; u(0) = \sigma \;\text{ in }\; \Omega,$$

is $\mu << \text{Cap}_{2,1,q'}$, $\sigma << \text{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}$ if $m \geq 1$ and $\mu << \text{Cap}_{\mathcal{G}_2,\frac{2q}{2(q-1)+N(1-m)}}$, $\sigma << \text{Cap}_{\mathbf{G}_{\frac{2-N(1-m)}{q}},\frac{2q}{2(q-1)+N(1-m)}}$ if $\frac{N-2}{N} < m \leq 1$. A necessary condition is $\mu << \text{Cap}_{2,1,\frac{q}{q-\max\{m,1\}}}$ and $\sigma << \text{Cap}_{\mathbf{G}_{\frac{2\max\{m,1\}}{q}},\frac{q}{q-\max\{m,1\}}}$. Moreover, if $\mu = \mu_1 \otimes \chi_{[0,T]}$ with $\mu_1 \in \mathfrak{M}_b(\Omega)$ and $\sigma \equiv 0$ then a condition $\mu_1 << \text{Cap}_{\mathbf{G}_2,\frac{q}{q-m}}$ is not only a sufficient but also a necessary for existence of solutions to above problem.

We would like to make a brief survey of quasilinear elliptic equations with absorption, source terms and data measure:

$$-\Delta_p u + |u|^{q-1}u = \omega, \tag{1.10}$$

$$-\Delta_p u = |u|^{q-1}u + \omega, u \geq 0, \tag{1.11}$$

in Ω with Dirichlet boundary conditions where $1 < p < N$, $q > p-1$. In [16], we proved that the existence solution of equation (1.10) holds if $\omega \in \mathfrak{M}_b(\Omega)$ is absolutely continuous with respect to $\text{Cap}_{\mathbf{G}_p,\frac{q}{q-p+1}}$. Moreover, a necessary condition for existence was also showed in [10, 11]. For problem with source term, it was solved in [68] (also see [69]). Exactly, if $\omega \in \mathfrak{M}_b^+(\Omega)$ has compact support in Ω, then a sufficient and necessary condition for the existence of solutions of problem (1.11) is

$$\omega(E) \leq C\text{Cap}_{\mathbf{G}_p,\frac{q}{q-p+1}}(E) \qquad \text{for all compact set } E \subset \Omega,$$

where C is a constant only depending on N, p, q and $d(\text{supp}(\omega), \partial\Omega)$. Their construction is based upon sharp estimates of solutions of the problem

$$-\Delta_p u = \omega \quad \text{in } \Omega, \quad u = 0 \;\text{ on } \partial\Omega,$$

for nonnegative Radon measures ω in Ω and a deep analysis of the Wolff potential. Corresponding results in case that u^q term is changed by $P(u) \approx \exp(au^\lambda)$ for $a > 0, \lambda > 0$, was given in [16, 62].

In [27], Duzaar and Mingione gave a local pointwise estimate from above of solutions to equation

$$u_t - \text{div}(A(x,t,\nabla u)) = \mu, \tag{1.12}$$

in Ω_T involving the Wolff parabolic potential $\mathbb{I}_2[|\mu|]$ defined by

$$\mathbb{I}_2[|\mu|](x,t) = \int_0^\infty \frac{|\mu|(\tilde{Q}_\rho(x,t))}{\rho^N}\frac{d\rho}{\rho} \quad \text{for all} \quad (x,t) \in \mathbb{R}^{N+1},$$

here $\tilde{Q}_\rho(x,t) := B_\rho(x) \times (t - \rho^2/2, t + \rho^2/2)$. Specifically if $u \in L^2(0,T; H^1(\Omega)) \cap C(\Omega_T)$ is a weak solution to above equation with data $\mu \in L^2(\Omega_T)$, then

$$|u(x,t)| \leq C \fint_{\tilde{Q}_R(x,t)} |u| dyds + C\int_0^{2R} \frac{|\mu|(\tilde{Q}_\rho(x,t))}{\rho^N}\frac{d\rho}{\rho}, \tag{1.13}$$

for any $Q_{2R}(x,t) := B_{2R}(x) \times (t-(2R)^2, t) \subset \Omega_T$, where a constant C only depends on N and the structure of operator A. Moreover, in this paper we show that if $u \geq 0, \mu \geq 0$ we also have local pointwise estimate from below:

$$u(y,s) \geq C^{-1} \sum_{k=0}^{\infty} \frac{\mu(Q_{r_k/8}(y, s - \frac{35}{128} r_k^2))}{r_k^N}, \tag{1.14}$$

for any $Q_r(y,s) \subset \Omega_T$, see section 5, where $r_k = 4^{-k} r$.

From preceding two inequalities, we obtain global pointwise estimates of solution to (1.12). For example, if $\mu \in \mathfrak{M}(\mathbb{R}^{N+1})$ with $\mathbb{I}_2[|\mu|](x_0,t_0) < \infty$ for some $(x_0,t_0) \in \mathbb{R}^{N+1}$ then there exists a distribution solution to (1.12) in $\mathbb{R}^{N+1}$ such that

$$-K\mathbb{I}_2[\mu^-](x,t) \leq u(x,t) \leq K\mathbb{I}_2[\mu^+](x,t) \quad \text{for a.e } (x,t) \in \mathbb{R}^{N+1}, \tag{1.15}$$

and we emphasize that if $u \geq 0, \mu \geq 0$ then

$$u(x,t) \geq K^{-1} \sum_{k=-\infty}^{\infty} \frac{\mu(Q_{2^{-2k-3}}(x, t - 35 \times 2^{-4k-7}))}{2^{-2Nk}} \quad \text{for a.e } (x,t) \in \mathbb{R}^{N+1},$$

and for $q > 1$,

$$||u||_{L^q(\mathbb{R}^{N+1})} \approx ||\mathbb{I}_2[\mu]||_{L^q(\mathbb{R}^{N+1})}.$$

Where a constant K only depends on N and the structure of operator A.

Our first aim is to verify that

i. problems (1.4) and (1.6) have solutions if μ, σ are absolutely continuous with respect to the capacity $\text{Cap}_{2,1,q'}$, $\text{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}$ respectively,

ii. problems (1.5) in $\mathbb{R}^{N+1}$ and (1.7) in $\mathbb{R}^N \times (0,\infty)$ with data signed measure μ, σ admit a solution if

$$|\mu|(E) \leq C\text{Cap}_{\mathcal{H}_2,q'}(E) \quad \text{and} \quad |\sigma|(O) \leq C\text{Cap}_{\mathbf{I}_{\frac{2}{q}},q'}(O) \tag{1.16}$$

hold for every compact sets $E \subset \mathbb{R}^{N+1}$, $O \subset \mathbb{R}^N$. Also, the equation (1.7) in a bounded domain Ω_T has a solution if (1.16) holds where capacities $\text{Cap}_{2,1,q'}$, $\text{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}$ are exploited instead of $\text{Cap}_{\mathcal{H}_2,q'}$, $\text{Cap}_{\mathbf{I}_{\frac{2}{q}},q'}$.

It is worth mention that solutions obtained of (1.5) in $\mathbb{R}^{N+1}$ and (1.7) in $\mathbb{R}^N \times (0,\infty)$ obey

$$\int_E |u|^q dxdt \leq C\text{Cap}_{\mathcal{H}_2,q'}(E) \quad \text{for all compact } E \subset \mathbb{R}^{N+1},$$

and we also have an analogous estimate for a solution of (1.7) in Ω_T;

$$\int_E |u|^q dxdt \leq C\text{Cap}_{2,1,q'}(E) \quad \text{for all compact } E \subset \mathbb{R}^{N+1},$$

for some a constant $C > 0$.

In case $\mu \equiv 0$, solutions (1.7) in $\mathbb{R}^N \times (0,\infty)$ and Ω_T are accepted the decay estimate

$$-Ct^{-\frac{1}{q-1}} \leq \inf_x u(x,t) \leq \sup_x u(x,t) \leq Ct^{-\frac{1}{q-1}} \text{ for any } \ t > 0.$$

The strategy for establishment above results that is, we rely upon the combination some techniques of quasilinear elliptic equations in two articles [16, 68] with the global pointwise estimate (1.15), delicate estimates on Wolff parabolic potential and the stability theorem see [13], Proposition 3.17 of this paper. They will be demonstrated in section 6.

We next are interested in global regularity of solutions to quasilinear parabolic equations

$$u_t - \operatorname{div}(A(x,t,\nabla u)) = \mu \text{ in } \Omega_T, \ u = 0 \text{ on } \partial\Omega \times (0,T) \text{ and } u(0) = \sigma \text{ in } \Omega, \quad (1.17)$$

where domain Ω_T and nonlinearity A are as mentioned at the beginning.

Our aim is to achieve minimal conditions on the boundary of Ω and on nonlinearity A so that the following statement holds

$$|||\nabla u|||_{\mathcal{K}} \leq C||\mathbb{M}_1[\omega]||_{\mathcal{K}}.$$

Here $\omega = |\mu| + |\sigma| \otimes \delta_{\{t=0\}}$ and $\mathbb{M}_1$ is the first order fractional Maximal parabolic potential defined by

$$\mathbb{M}_1[\omega](x,t) = \sup_{\rho>0} \frac{\omega(\tilde{Q}_\rho(x,t))}{\rho^{N+1}} \ \ \forall\ (x,t) \in \mathbb{R}^{N+1},$$

a constant C does not depend on u and $\mu \in \mathfrak{M}_b(\Omega_T), \sigma \in \mathfrak{M}_b(\Omega)$ and $\mathcal{K}$ is a function space. The same question is as above for the elliptic framework studied by N. C. Phuc in [70, 71, 72].

First, we take $\mathcal{K} = L^{p,s}(\Omega_T)$ for $1 \leq p \leq \theta$ and $0 < s \leq \infty$ under a capacity density condition on the domain Ω where $L^{p,s}(\Omega_T)$ is the Lorentz space and a constant $\theta > 2$ depends on the structure of this condition and of nonlinearity A. It follows the recent result in [7], see remark 2.18. The capacity density condition is that, the complement of Ω satisfies *uniformly* $2-$*thick*, see section 2. We remark that under this condition, the Sobolev embedding $H_0^1(\Omega) \subset L^{\frac{2N}{N-2}}(\Omega)$ for $N > 2$ is valid and it is fulfilled by any domain with Lipschitz boundary, or even of corkscrew type. This condition was used in two papers [70, 72]. Also, it is essentially sharp for higher integrability results, presented in [41, Remark 3.3]. Furthermore, we also assert that if $\frac{\gamma}{\gamma-1} < p < \theta$, $2 \leq \gamma < N+2$, $0 < s \leq \infty$ and $\sigma \equiv 0$ then

$$|||\nabla u|||_{L_*^{p,s;(\gamma-1)p}(\Omega_T)} \leq C||\mu||_{L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\Omega_T)},$$

for some a constant C where $L_*^{p,s;(\gamma-1)p}(\Omega_T), L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\Omega_T)$ are the Lorentz-Morrey spaces involving "calorie" introduced in section 2. We would like to refer to [55] as the first paper where Lorentz-Morrey estimates for solutions of quasilinear elliptic equations via fractional operators have been obtained.

Next, in order to obtain shaper results, we take $\mathcal{K} = L^{q,s}(\Omega_T, dw)$, the weighted Lorentz spaces with weight in the Muckenhoupht class A_∞ for $q \geq 1$, $0 < s \leq \infty$, we require some stricter conditions on the domain Ω and nonlinearity A. A condition on Ω is flat enough in the sense of Reifenberg, essentially, that at boundary point and every scale the boundary of domain is between two hyperplanes at both sides (inside and outside) of domain by a distance which depends on the scale. Conditions on A are that BMO type of A with respect to the $x-$variable is small enough and the derivative of $A(x,t,\zeta)$ with respect to ζ is uniformly bounded. By choosing an appropriate weight we can establish the following important estimates:

a. The Lorentz-Morrey estimates involving "calorie" for $0 < \kappa \leq N+2$ is obtained

$$|||\nabla u|||_{L_*^{q,s;\kappa}(\Omega_T)} \leq C||\mathbb{M}_1[|\omega|]||_{L_*^{q,s;\kappa}(\Omega_T)}.$$

b. Another Lorentz-Morrey estimates is also obtained for $0 < \vartheta \leq N$

$$||\mathbb{M}(|\nabla u|)||_{L_{**}^{q,s;\vartheta}(\Omega_T)} \leq C||\mathbb{M}_1[|\omega|]||_{L_{**}^{q,s;\vartheta}(\Omega_T)},$$

where $L_{**}^{q,s;\vartheta}(\Omega_T)$ is introduced in section 2. This estimate implies global Holder-estimate in space variable and L^q-estimate in time, that is for all ball $B_\rho \subset \mathbb{R}^N$

$$\left(\int_0^T |\text{osc}_{B_\rho\cap\overline{\Omega}} u(t)|^q dt\right)^{\frac{1}{q}} \le C\rho^{1-\frac{\vartheta}{q}} ||\mathbb{M}_1[|\omega|]||_{L_{**}^{q;\vartheta}(\Omega_T)} \text{ provided } \ 0<\vartheta<\min\{q,N\}.$$

In particular, there hold

$$\left(\int_0^T |\text{osc}_{B_\rho\cap\overline{\Omega}} u(t)|^q dt\right)^{\frac{1}{q}} \le C\rho^{1-\frac{\vartheta}{q}} ||\sigma||_{L^{\frac{\vartheta q}{\vartheta+2-q};\vartheta}(\Omega)} + C\rho^{1-\frac{\vartheta}{q}} ||\mu||_{L^{\frac{\vartheta q q_1}{(\vartheta+2+q)q_1-2q};\vartheta}(\Omega,L^{q_1}((0,T)))}$$

provided

$$\begin{aligned} &1<q_1\le q<2,\\ &\max\left\{\frac{2-q}{q-1},\frac{1}{q-1}\left(2+q-\frac{2q}{q_1}\right)\right\}<\vartheta\le N. \end{aligned}$$

Where $L^{\frac{\vartheta q}{\vartheta+2-q};\vartheta}(\Omega)$ is the standard Morrey space and

$$||\mu||_{L^{q_2;\vartheta}(\Omega,L^{q_1}((0,T)))} = \sup_{\rho>0,x\in\Omega} \rho^{\frac{\vartheta-N}{q_2}} \left(\int_{B_\rho(x)\cap\Omega}\left(\int_0^T |\mu(y,t)|^{q_1}dt\right)^{\frac{q_2}{q_1}} dy\right)^{\frac{1}{q_2}},$$

with $q_2=\frac{\vartheta q q_1}{(\vartheta+2+q)q_1-2q}$. Besides, we also find

$$\left(\int_0^T |\text{osc}_{B_\rho\cap\overline{\Omega}} u(t)|^q dt\right)^{\frac{1}{q}} \le C\rho^{1-\frac{\vartheta}{q}} ||\mu||_{L^{\frac{\vartheta q q_1}{(\vartheta+2+q)q_1-2q};\vartheta}(\Omega,L^{q_1}((0,T)))}$$

provided

$$\begin{aligned} &\sigma\equiv 0, \quad q\ge 2, 1<q_1\le q,\\ &\frac{1}{q-1}\left(2+q-\frac{2q}{q_1}\right)<\vartheta\le N. \end{aligned}$$

c. A global capacitary estimate is also given

$$\sup_{\substack{\text{compact } K\subset\mathbb{R}^{N+1}\\ \text{Cap}_{\mathcal{G}_1,q'}(K)>0}} \left(\frac{\int_K|\nabla u|^q dxdt}{\text{Cap}_{\mathcal{G}_1,q'}(K)}\right) \le C \sup_{\substack{\text{compact } K\subset\mathbb{R}^{N+1}\\ \text{Cap}_{\mathcal{G}_1,q'}(K)>0}} \left(\frac{|\omega|(K)}{\text{Cap}_{\mathcal{G}_1,q'}(K)}\right)^q.$$

To obtain this estimate we employ profound techniques in nonlinear potential theory, see section 4 and Theorem 2.22.

We utilize some ideas (in the quasilinear elliptic framework) in articles of N.C. Phuc [70, 72, 71] during we establish above estimates.

We would like to emphasize that above estimates is also true for solutions to equation (1.17) in $\mathbb{R}^{N+1}$ with data μ (of course still true for (1.17) in $\mathbb{R}^N\times(0,\infty)$) with data μ provided $\mathbb{I}_2[|\mu|](x_0,t_0)<\infty$ for some $(x_0,t_0)\in\mathbb{R}^{N+1}$ see Theorem 2.25 and 2.27. Moreover, a global pointwise estimates of gradient of solutions is obtained when A is independent of space variable x, that is

$$|\nabla u(x,t)|\le C\mathbb{I}_1[|\mu|](x,t) \quad \text{a.e } (x,t)\in\mathbb{R}^{N+1},$$

see Theorem 2.5.

Our final aim is to obtain existence results for the quasilinear Riccati type parabolic problems (1.1) where $B(x,t,u,\nabla u) = |\nabla u|^q$ for $q > 1$. The strategy we use in order to prove these existence results is that using Schauder Fixed Point Theorem and all above estimates and the stability Theorem see [13], Proposition 3.17 in section 3. They will be carried out in section 9. By our methods in the paper, we can treat general equations (1.1), where

$$|B(x,t,u,\nabla u)| \le C_1|u|^{q_1} + C_2|\nabla u|^{q_2},\ q_1,q_2 > 1,$$

with constant coefficients $C_1, C_2 > 0$.

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2 Main Results

Throughout the paper, we assume that Ω is a bounded open subset of $\mathbb{R}^N$, $N \ge 2$ and $T > 0$. Besides, we always denote $\Omega_T = \Omega \times (0,T)$, $T_0 = \mathrm{diam}(\Omega) + T^{1/2}$ and $Q_\rho(x,t) = B_\rho(x) \times (t-\rho^2, t)$ $\tilde{Q}_\rho(x,t) = B_\rho(x) \times (t-\rho^2/2, t+\rho^2/2)$ for $(x,t) \in \mathbb{R}^{N+1}$ and $\rho > 0$. We always assume that $A : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Caratheodory vector valued function, i.e. A is measurable in (x,t) and continuous with respect to ∇u for each fixed (x,t) and satisfies (1.2) and (1.3). This article is divided into three parts. First part, we study the existence problems for the quasilinear parabolic equations with absorption and source terms

$$\begin{cases} u_t - div(A(x,t,\nabla u)) + |u|^{q-1}u = \mu \text{ in } \Omega_T, \\ u = 0 \qquad \text{on } \ \partial\Omega \times (0,T), \\ u(0) = \sigma \quad \text{in } \ \Omega, \end{cases} \tag{2.1}$$

and

$$\begin{cases} u_t - div(A(x,t,\nabla u)) = |u|^{q-1}u + \mu \text{ in } \Omega_T, \\ u = 0 \qquad \text{on } \ \partial\Omega \times (0,T), \\ u(0) = \sigma \quad \text{in } \ \Omega, \end{cases} \tag{2.2}$$

where $q > 1$, and μ, σ are Radon measures.

In order to state our results, let us introduce some definitions and notations. If D is either a bounded domain or whole $\mathbb{R}^l$ for $l \in \mathbb{N}$, we denote by $\mathfrak{M}(D)$ (resp. $\mathfrak{M}_b(D)$) the set of Radon measure (resp. bounded Radon measures) in D. Their positive cones are $\mathfrak{M}^+(D)$ and $\mathfrak{M}_b^+(D)$ respectively. For $R \in (0,\infty]$, we define the $R-$truncated Riesz parabolic potential $\mathbb{I}_\alpha$ and Fractional Maximal parabolic potential $\mathbb{M}_\alpha$, $\alpha \in (0, N+2)$, on $\mathbb{R}^{N+1}$ of a measure $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ by

$$\mathbb{I}_\alpha^R[\mu](x,t) = \int_0^R \frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha}} \frac{d\rho}{\rho} \text{ and } \mathbb{M}_\alpha^R[\mu](x,t) = \sup_{0<\rho<R} \frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha}}, \tag{2.3}$$

for all (x,t) in $\mathbb{R}^{N+1}$. If $R = \infty$, we drop it in expressions of (2.3).
We denote by $\mathcal{H}_\alpha$ the Heat kernel of order $\alpha \in (0, N+2)$:

$$\mathcal{H}_\alpha(x,t) = C_\alpha \frac{\chi_{(0,\infty)}(t)}{t^{(N+2-\alpha)/2}} \exp\left(-\frac{|x|^2}{4t}\right) \quad \text{for} \quad (x,t) \quad \text{in} \quad \mathbb{R}^{N+1},$$

and $\mathcal{G}_\alpha$ the parabolic Bessel kernel of order $\alpha > 0$:

$$\mathcal{G}_\alpha(x,t) = C_\alpha \frac{\chi_{(0,\infty)}(t)}{t^{(N+2-\alpha)/2}} \exp\left(-t - \frac{|x|^2}{4t}\right) \quad \text{for} \quad (x,t) \quad \text{in} \quad \mathbb{R}^{N+1},$$

see [4], where $C_\alpha = \left((4\pi)^{N/2}\Gamma(\alpha/2)\right)^{-1}$. It is known that $\mathcal{F}(\mathcal{H}_\alpha)(x,t) = (|x|^2+it)^{-\alpha/2}$ and $\mathcal{F}(\mathcal{G}_\alpha)(x,t) = (1+|x|^2+it)^{-\alpha/2}$. We define the parabolic Riesz potential $\mathcal{H}_\alpha$ of a measure $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ by

$$\mathcal{H}_\alpha[\mu](x,t) = (\mathcal{H}_\alpha * \mu)(x,t) = \int_{\mathbb{R}^{N+1}} \mathcal{H}_\alpha(x-y,t-s)d\mu(y,s) \text{ for any } (x,t) \text{ in } \mathbb{R}^{N+1},$$

the parabolic Bessel potential $\mathcal{G}_\alpha$ of a measure $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ by

$$\mathcal{G}_\alpha[\mu](x,t) = (\mathcal{G}_\alpha * \mu)(x,t) = \int_{\mathbb{R}^{N+1}} \mathcal{G}_\alpha(x-y,t-s)d\mu(y,s) \text{ for any } (x,t) \text{ in } \mathbb{R}^{N+1}.$$

We also define $\mathbf{I}_\alpha, \mathbf{G}_\alpha, 0<\alpha<N$ the Riesz, Bessel potential of a measure $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ by

$$\mathbf{I}_\alpha[\mu](x) = \int_0^\infty \frac{\mu(B_\rho(x))}{\rho^{N-\alpha}}\frac{d\rho}{\rho} \text{ and } \mathbf{G}_\alpha[\mu](x) = \int_{\mathbb{R}^N} \mathbf{G}_\alpha(x-y)d\mu(y) \text{ for any } x \text{ in } \mathbb{R}^N,$$

where $\mathbf{G}_\alpha$ is the Bessel kernel of order α, see [2].
Several different capacities will be used over the paper. For $1<p<\infty$, the $(\mathcal{H}_\alpha,p)$-capacity, $(\mathcal{G}_\alpha,p)$-capacity of a Borel set $E \subset \mathbb{R}^{N+1}$ are defined by

$$\text{Cap}_{\mathcal{H}_\alpha,p}(E) = \inf\left\{\int_{\mathbb{R}^{N+1}} |f|^p dxdt : f \in L^p_+(\mathbb{R}^{N+1}), \mathcal{H}_\alpha * f \geq \chi_E\right\} \text{ and}$$
$$\text{Cap}_{\mathcal{G}_\alpha,p}(E) = \inf\left\{\int_{\mathbb{R}^{N+1}} |f|^p dxdt : f \in L^p_+(\mathbb{R}^{N+1}), \mathcal{G}_\alpha * f \geq \chi_E\right\}.$$

The $W^{2,1}_p$-capacity of compact set $E \subset \mathbb{R}^{N+1}$ is defined by

$$\text{Cap}_{2,1,p}(E) = \inf\left\{||\varphi||^p_{W^{2,1}_p(\mathbb{R}^{N+1})} : \varphi \in S(\mathbb{R}^{N+1}), \varphi \geq 1 \text{ in a neighborhood of } E\right\},$$

where

$$||\varphi||_{W^{2,1}_p(\mathbb{R}^{N+1})} = ||\varphi||_{L^p(\mathbb{R}^{N+1})} + ||\frac{\partial \varphi}{\partial t}||_{L^p(\mathbb{R}^{N+1})} + ||\nabla\varphi||_{L^p(\mathbb{R}^{N+1})} + \sum_{i,j=1,2,\ldots,N} ||\frac{\partial^2\varphi}{\partial x_i \partial x_j}||_{L^p(\mathbb{R}^{N+1})}.$$

We remark that thanks to Richard J. Bagby's result (see [4]) we obtain the equivalent of capacities $\text{Cap}_{2,1,p}$ and $\text{Cap}_{\mathcal{G}_2,p}$, i.e, for any compact set $K \subset \mathbb{R}^{N+1}$ there holds

$$C^{-1}\text{Cap}_{2,1,p}(K) \leq \text{Cap}_{\mathcal{G}_2,p}(K) \leq C\text{Cap}_{2,1,p}(K),$$

for some $C = C(N,p)$, see Corollary (4.18) in section 4.
The $(\mathbf{I}_\alpha,p)$-capacity, $(\mathbf{G}_\alpha,p)$-capacity of a Borel set $O \subset \mathbb{R}^N$ are defined by

$$\text{Cap}_{\mathbf{I}_\alpha,p}(O) = \inf\left\{\int_{\mathbb{R}^N} |g|^p dx : g \in L^p_+(\mathbb{R}^N), \mathbf{I}_\alpha * g \geq \chi_O\right\} \text{ and}$$
$$\text{Cap}_{\mathbf{G}_\alpha,p}(O) = \inf\left\{\int_{\mathbb{R}^N} |g|^p dx : g \in L^p_+(\mathbb{R}^N), \mathbf{G}_\alpha * g \geq \chi_O\right\}.$$

In our first three Theorems, we present global pointwise potential estimates for solutions to quasilinear parabolic problems

$$\begin{cases} u_t - div\,(A(x,t,\nabla u)) = \mu \text{ in } \Omega_T, \\ u = 0 \quad \text{ on } \partial\Omega \times (0,T), \\ u(0) = \sigma \quad \text{in } \Omega, \end{cases} \tag{2.4}$$

$$\begin{cases} u_t - div\,(A(x,t,\nabla u)) = \mu \text{ in } \mathbb{R}^N \times (0,\infty), \\ u(0) = \sigma \quad \text{in } \mathbb{R}^N, \end{cases} \tag{2.5}$$

and

$$u_t - div\,(A(x,t,\nabla u)) = \mu \text{ in } \mathbb{R}^{N+1}. \tag{2.6}$$

Theorem 2.1 *There exists a constant K depending on N, Λ_1, Λ_2 such that for any $\mu \in \mathfrak{M}_b(\Omega_T), \sigma \in \mathfrak{M}_b(\Omega)$ there is a distribution solution u of* (2.4) *which satisfies*

$$-K\mathbb{I}_2^{2T_0}[\mu^- + \sigma^- \otimes \delta_{\{t=0\}}] \le u \le K\mathbb{I}_2^{2T_0}[\mu^+ + \sigma^+ \otimes \delta_{\{t=0\}}] \text{ in } \Omega_T. \tag{2.7}$$

Remark 2.2 *Since $\sup_{x\in\mathbb{R}^N} \mathbb{I}_\alpha[\sigma^\pm \otimes \delta_{\{t=0\}}](x,t) \le \frac{\sigma^\pm(\Omega)}{(N+2-\alpha)(2|t|)^{\frac{N+2-\alpha}{2}}}$ for any $t \neq 0$ with $0 < \alpha < N+2$. Thus, if $\mu \equiv 0$, then we obtain the decay estimate:*

$$-\frac{K\sigma^-(\Omega)}{N(2t)^{\frac{N}{2}}} \le \inf_{x\in\Omega} u(x,t) \le \sup_{x\in\Omega} u(x,t) \le \frac{K\sigma^+(\Omega)}{N(2t)^{\frac{N}{2}}} \text{ for any } 0 < t < T.$$

Theorem 2.3 *There exists a constant C depending on N, Λ_1, Λ_2 such that for any $\mu \in \mathfrak{M}_b^+(\Omega_T), \sigma \in \mathfrak{M}_b^+(\Omega)$, there is a distribution solution u of* (2.4) *satisfying for a.e $(y,s) \in \Omega_T$ and $B_r(y) \subset \Omega$*

$$u(y,s) \ge C\sum_{k=0}^{\infty} \frac{\mu(Q_{r_k/8}(y, s - \frac{35}{128}r_k^2))}{r_k^N} + C\sum_{k=0}^{\infty} \frac{(\sigma \otimes \delta_{\{t=0\}})(Q_{r_k/8}(y, s - \frac{35}{128}r_k^2))}{r_k^N}, \tag{2.8}$$

where $r_k = 4^{-k}r$.

Remark 2.4 *The Theorem 2.3 is also true when we replace the assumption* (1.3) *by a weaker one*

$$\langle A(x,t,\zeta), \zeta\rangle \ge \Lambda_2|\zeta|^2, \quad \langle A(x,t,\zeta) - A(x,t,\lambda), \zeta - \lambda\rangle > 0,$$

for every $(\lambda, \zeta) \in \mathbb{R}^N \times \mathbb{R}^N$, $\lambda \neq \zeta$ and a.e. $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

Theorem 2.5 *Let K be the constant in Theorem 2.1. Let $\omega \in \mathfrak{M}(\mathbb{R}^{N+1})$ such that $I_2[|\omega|](x_0,t_0) < \infty$ for some $(x_0,t_0) \in \mathbb{R}^{N+1}$. Then, there is a distribution solution u to* (2.6) *with data $\mu = \omega$ satisfying*

$$-K\mathbb{I}_2[\omega^-] \le u \le K\mathbb{I}_2[\omega^+] \text{ in } \mathbb{R}^{N+1} \tag{2.9}$$

such that the following statements hold.

a. *If $\omega \ge 0$, there exists $C_1 = C_1(N, \Lambda_1, \Lambda_2)$ such that for a.e $(x,t) \in \mathbb{R}^{N+1}$*

$$u(x,t) \ge C_1 \sum_{k=-\infty}^{\infty} \frac{\omega(Q_{2^{-2k-3}}(x, t - 35 \times 2^{-4k-7}))}{2^{-2Nk}}. \tag{2.10}$$

In particular, for any $q > \frac{N+2}{N}$

$$C_2^{-1}||\mathcal{H}_2[\omega]||_{L^q(\mathbb{R}^{N+1})} \le ||u||_{L^q(\mathbb{R}^{N+1})} \le C_2||\mathcal{H}_2[\omega]||_{L^q(\mathbb{R}^{N+1})}, \tag{2.11}$$

with $C_2 = C_2(N, \Lambda_1, \Lambda_2)$.

b. *If A is independent of space variable x and satisfies* (2.27), *then there exists $C_2 = C_2(N, \Lambda_1, \Lambda_2)$ such that*

$$|\nabla u| \le C_2\mathbb{I}_1[|\omega|] \text{ in } \mathbb{R}^{N+1}. \tag{2.12}$$

c. *If $\omega = \mu + \sigma \otimes \delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}(\mathbb{R}^N \times (0,\infty))$ and $\sigma \in \mathfrak{M}(\mathbb{R}^N)$, then $u = 0$ in $\mathbb{R}^N \times (-\infty, 0)$ and $u|_{\mathbb{R}^N\times[0,\infty)}$ is a distribution solution to* (2.5).

Remark 2.6 *For $q > \frac{N+2}{N}$, we alway have the following claim:*

$$||\mathcal{H}_2[\mu + \omega \otimes \delta_{\{t=0\}}]||_{L^q(\mathbb{R}^{N+1})} \approx ||\mathcal{H}_2[\mu]||_{L^q(\mathbb{R}^{N+1})} + ||\mathbf{I}_{2/q}[\sigma]||_{L^q(\mathbb{R}^{N+1})},$$

for every $\mu \in \mathfrak{M}^+(\mathbb{R}^N \times (0,\infty))$ and $\sigma \in \mathfrak{M}^+(\mathbb{R}^N)$.

Remark 2.7 *For $\omega \in \mathfrak{M}^+(\mathbb{R}^{N+1})$, $0 < \alpha < N+2$ if $\mathbb{I}_\alpha[\omega](x_0,t_0) < \infty$ for some $(x_0,t_0) \in \mathbb{R}^{N+1}$ then for any $0 < \beta \le \alpha$, $\mathbb{I}_\beta[\omega] \in L^s_{loc}(\mathbb{R}^{N+1})$ for any $0 < s < \frac{N+2}{N+2-\beta}$. However, for $0 < \beta < \alpha < N+2$, one can find $\omega \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ such that $\mathbb{I}_\alpha[\omega] \equiv \infty$ and $\mathbb{I}_\beta[\omega] < \infty$ in $\mathbb{R}^{N+1}$, see Appendix section.*

The next four theorems provide the existence of solutions to quasilinear parabolic equations with absorption and source terms. For convenience, we always denote by q' the conjugate exponent of $q \in (1,\infty)$ i.e $q' = \frac{q}{q-1}$.

Theorem 2.8 *Let $q > 1$, $\mu \in \mathfrak{M}_b(\Omega_T)$ and $\sigma \in \mathfrak{M}_b(\Omega)$. Suppose that μ, σ are absolutely continuous with respect to the capacities $Cap_{2,1,q'}$, $Cap_{\mathbf{G}_{\frac{2}{q}},q'}$ in Ω_T, Ω respectively. Then there exists a distribution solution u of (2.1) satisfying*

$$-K\mathbb{I}_2[\mu^- + \sigma^- \otimes \delta_{\{t=0\}}] \le u \le K\mathbb{I}_2[\mu^+ + \sigma^+ \otimes \delta_{\{t=0\}}] \quad in\ \Omega_T.$$

Here the constant K is in Theorem 2.1.

Theorem 2.9 *Let K be the constant in Theorem 2.1. Let $q > 1$, $\mu \in \mathfrak{M}_b(\Omega_T)$ and $\sigma \in \mathfrak{M}_b(\Omega)$. There exists a constant $C_1 = C_1(N,q,\Lambda_1,\Lambda_2,diam(\Omega),T)$ such that if*

$$|\mu|(E) \le C_1\, Cap_{2,1,q'}(E) \quad and \quad |\sigma|(O) \le C_1\, Cap_{\mathbf{G}_{\frac{2}{q}},q'}(O). \tag{2.13}$$

hold for every compact sets $E \subset \mathbb{R}^{N+1}$, $O \subset \mathbb{R}^N$, then the problem (2.2) has a distribution solution u satisfying

$$-\frac{Kq}{q-1}\mathbb{I}_2[\mu^- + \sigma^- \otimes \delta_{\{t=0\}}] \le u \le \frac{Kq}{q-1}\mathbb{I}_2[\mu^+ + \sigma^+ \otimes \delta_{\{t=0\}}] \quad in\ \ \Omega_T. \tag{2.14}$$

Besides, for every compact set $E \subset \mathbb{R}^{N+1}$ there holds

$$\int_E |u|^q dxdt \le C_2\, Cap_{2,1,q'}(E), \tag{2.15}$$

where $C_2 = C_2(N,q,\Lambda_1,\Lambda_2,T_0)$.

Remark 2.10 *From (2.15) we get if $q > \frac{N+2}{N}$,*

$$\int_{\tilde{Q}_\rho(y,s)} |u|^q dxdt \le C\rho^{N+2-2q'} \quad for\ any\ \ \tilde{Q}_\rho(y,s) \subset \mathbb{R}^{N+1},$$

if $q = \frac{N+2}{N}$,

$$\int_{\tilde{Q}_\rho(y,s)} |u|^q dxdt \le C\left(\log(1/\rho)\right)^{-\frac{1}{q-1}} \quad for\ any\ \ \tilde{Q}_\rho(y,s) \subset \mathbb{R}^{N+1}, 0 < \rho < 1/2,$$

for some $C = C(N,q,\Lambda_1,\Lambda_2,T_0)$, see Remark 4.14.

Remark 2.11 *In the sub-critical case $1 < q < \frac{N+2}{N}$, since the capacity $Cap_{2,1,q'}$, $Cap_{\mathbf{G}_{\frac{2}{q}},q'}$ of a single are positive thus the conditions (2.13) hold for some constant $C_1 > 0$ provided $\mu \in \mathfrak{M}_b(\Omega_T), \sigma \in \mathfrak{M}_b(\Omega)$. Moreover, in the super-critical case $q > \frac{N+2}{N}$, we have*

$$Cap_{2,1,q'}(E) \ge c_1|E|^{1-\frac{2q'}{N+2}} \quad and \quad Cap_{\mathbf{G}_{\frac{2}{q}},q'}(O) \ge c_2|O|^{1-\frac{2}{(q-1)N}},$$

for every Borel sets $E \subset \mathbb{R}^{N+1}$, $O \subset \mathbb{R}^N$, thus if $\mu \in L^{\frac{N+2}{2q'},\infty}(\Omega_T)$ and $\sigma \in L^{\frac{(q-1)N}{2},\infty}(\Omega)$ then (2.13) holds for some constant $C_1 > 0$. In addition, if $\mu \equiv 0$, then (2.14) implies for any $0 < t < T$,

$$-c_3(T_0)t^{-\frac{1}{q-1}} \le \inf_{x\in\Omega} u(x,t) \le \sup_{x\in\Omega} u(x,t) \le c_3(T_0)t^{-\frac{1}{q-1}},$$

since $|\sigma|(B_\rho(x)) \le c_4(T_0)\rho^{N-\frac{2}{q-1}}$ for all $x \in \mathbb{R}^N$, $0 < \rho < 2T_0$.

Theorem 2.12 *Let K be the constant in Theorem 2.1 and $q > 1$. If $\omega \in \mathfrak{M}(\mathbb{R}^{N+1})$ is absolutely continuous with respect to the capacity $Cap_{2,1,q'}$ in $\mathbb{R}^{N+1}$, then there exists a distribution solution $u \in L^{\gamma}_{loc}(\mathbb{R}; W^{1,\gamma}_{loc}(\mathbb{R}^N))$ for any $1 \le \gamma < \frac{2q}{q+1}$ to problem*

$$u_t - div\,(A(x,t,\nabla u)) + |u|^{q-1}u = \omega \ \text{in}\ \mathbb{R}^{N+1}, \tag{2.16}$$

which satisfies

$$-K\mathbb{I}_2[\omega^-] \le u \le K\mathbb{I}_2[\omega^+] \quad \text{in}\ \mathbb{R}^{N+1}. \tag{2.17}$$

Furthermore, when $\omega = \mu + \sigma \otimes \delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}(\mathbb{R}^N \times (0,\infty))$, $\sigma \in \mathfrak{M}(\mathbb{R}^N)$ then $u = 0$ in $\mathbb{R}^N \times (-\infty, 0)$ and $u|_{\mathbb{R}^N \times [0,\infty)}$ is a distribution solution to problem

$$\begin{cases} u_t - div\,(A(x,t,\nabla u)) + |u|^{q-1}u = \mu \ \text{in}\ \mathbb{R}^N \times (0,\infty), \\ u(0) = \sigma \quad \text{in}\ \ \mathbb{R}^N. \end{cases} \tag{2.18}$$

Remark 2.13 *The measure $\omega = \mu + \sigma \otimes \delta_{\{t=0\}}$ is absolutely continuous with respect to the capacity $Cap_{2,1,q'}$ in $\mathbb{R}^{N+1}$ if and only if μ, σ are absolutely continuous with respect to the capacities $Cap_{2,1,q'}$, $Cap_{\mathbf{G}_{\frac{2}{q}},q'}$ in $\mathbb{R}^{N+1}, \mathbb{R}^N$ respectively.*

Existence result of the problem (2.2) on $\mathbb{R}^{N+1}$ or on $\mathbb{R}^N \times (0,\infty)$ is similar to Theorem 2.9 presented in the following Theorem, where the capacities $\text{Cap}_{\mathcal{H}_2,q'}, \text{Cap}_{\mathbf{I}_{\frac{2}{q}},q'}$ are used in place of respectively $\text{Cap}_{2,1,q'}, \text{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}$.

Theorem 2.14 *Let K be the constant in Theorem 2.1 and $q > \frac{N+2}{N}$, $\omega \in \mathfrak{M}(\mathbb{R}^{N+1})$. There exists a constant $C_1 = C_1(N, q, \Lambda_1, \Lambda_2)$ such that if*

$$|\omega|(E) \le C_1\, Cap_{\mathcal{H}_2,q'}(E), \tag{2.19}$$

for every compact set $E \subset \mathbb{R}^{N+1}$, then the problem

$$u_t - div\,(A(x,t,\nabla u)) = |u|^{q-1}u + \omega \ \text{in}\ \mathbb{R}^{N+1} \tag{2.20}$$

has a distribution solution $u \in L^{\gamma}_{loc}(\mathbb{R}; W^{1,\gamma}_{loc}(\mathbb{R}^N))$ for any $1 \le \gamma < \frac{2q}{q+1}$ satisfying

$$-\frac{Kq}{q-1}\mathbb{I}_2[\omega^-] \le u \le \frac{Kq}{q-1}\mathbb{I}_2[\omega^+] \quad \text{in} \ \ \mathbb{R}^{N+1}. \tag{2.21}$$

Moreover, when $\omega = \mu + \sigma \otimes \delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}(\mathbb{R}^N \times (0,\infty))$, $\sigma \in \mathfrak{M}(\mathbb{R}^N)$ then $u = 0$ in $\mathbb{R}^N \times (-\infty, 0)$ and $u|_{\mathbb{R}^N \times [0,\infty)}$ is a distribution solution to problem

$$\begin{cases} u_t - div\,(A(x,t,\nabla u)) = |u|^{q-1}u + \mu \ \text{in}\ \mathbb{R}^N \times (0,\infty), \\ u(0) = \sigma \quad \text{in}\ \ \mathbb{R}^N. \end{cases} \tag{2.22}$$

In addition, for any compact set $E \subset \mathbb{R}^{N+1}$ there holds

$$\int_E |u|^q dxdt \le C_2\, Cap_{\mathcal{H}_2,q'}(E), \tag{2.23}$$

for some $C_2 = C_2(N, q, \Lambda_1, \Lambda_2)$.

Remark 2.15 *The measure $\omega = \mu + \sigma \otimes \delta_{\{t=0\}}$ satisfies* (2.19) *if and only if*

$$|\mu|(E) \le C\, Cap_{\mathcal{H}_2,q'}(E) \quad \text{and} \quad |\sigma|(O) \le C\, Cap_{\mathbf{I}_{\frac{2}{q}},q'}(O),$$

for every compact sets $E \subset \mathbb{R}^{N+1}$ and $O \subset \mathbb{R}^N$, where $C = C_3 C_1$, $C_3 = C_3(N, q)$.

Remark 2.16 *If $\omega \in L^{\frac{N+2}{2q'},\infty}(\mathbb{R}^{N+1})$ then (2.19) holds for some constant $C_1 > 0$. Moreover, if $\omega = \sigma \otimes \delta_{\{t=0\}}$ with $\sigma \in \mathfrak{M}_b(\mathbb{R}^N)$, then from (2.21) we get the decay estimate:*

$$-c_1 t^{-\frac{1}{q-1}} \leq \inf_{x\in\mathbb{R}^N} u(x,t) \leq \sup_{x\in\mathbb{R}^N} u(x,t) \leq c_1 t^{-\frac{1}{q-1}} \text{ for any } t > 0,$$

since $|\sigma|(B_\rho(x)) \leq c_2\rho^{N-\frac{2}{q-1}}$ for any $B_\rho(x) \subset \mathbb{R}^N$.

Second part, we establish global regularity in weighted-Lorentz and Lorentz-Morrey on gradient of solutions to problem (2.4). For this purpose, we need a capacity density condition imposed on Ω. That is, the complement of Ω satisfies *uniformly p-thick with constants* c_0, r_0, i.e, for all $0 < r \leq r_0$ and all $x \in \mathbb{R}^N\backslash\Omega$ there holds

$$\text{Cap}_p(\overline{B_r(x)} \cap (\mathbb{R}^N\backslash\Omega), B_{2r}(x)) \geq c_0\text{Cap}_p(\overline{B_r(x)}, B_{2r}(x)), \tag{2.24}$$

where the involved capacity of a compact set $K \subset B_{2r}(x)$ is given as follows

$$\text{Cap}_p(K, B_{2r}(x)) = \inf\left\{\int_{B_{2r}(x)} |\nabla\phi|^p dy : \phi \in C_c^\infty(B_{2r}(x)), \phi \geq \chi_K\right\}. \tag{2.25}$$

In order to obtain better regularity we need a stricter condition on Ω which is expressed in the following way. We say that Ω is a $(\delta, R_0)-$Reifenberg flat domain for $\delta \in (0,1)$ and $R_0 > 0$if for every $x_0 \in \partial\Omega$ and every $r \in (0, R_0]$, there exists a system of coordinates $\{z_1, z_2, ..., z_n\}$, which may depend on r and x_0, so that in this coordinate system $x_0 = 0$ and that

$$B_r(0) \cap \{z_n > \delta r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{z_n > -\delta r\}. \tag{2.26}$$

We remark that this class of flat domains is rather wide since it includes C^1, Lipschitz domains with sufficiently small Lipschitz constants and fractal domains. Besides, it has many important roles in the theory of minimal surfaces and free boundary problems, this class was first appeared in a work of Reifenberg (see [74]) in the context of a Plateau problem. Its properties can be found in [37, 38, 78].

On the other hand, it is well-known that in general, conditions (1.2) and (1.3) on the nonlinearity $A(x,t,\zeta)$ are not enough to ensure higher integral of gradient of solutions to problem (2.4), we need to assume that A satisfies

$$\langle A_\zeta(x,t,\zeta)\xi, \xi\rangle \geq \Lambda_2|\xi|^2, \quad |A_\zeta(x,t,\zeta)| \leq \Lambda_1, \tag{2.27}$$

for every $(\xi,\zeta) \in \mathbb{R}^N \times \mathbb{R}^N\backslash\{(0,0)\}$ and a.e $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, where Λ_1, Λ_2 are constants in (1.2) and (1.3). We also require that the nonlinearity A satisfies a smallness condition of BMO type in the x-variable. We say that $A(x,t,\zeta)$ satisfies a (δ, R_0)-BMO condition for some $\delta, R_0 > 0$ with exponent $s > 0$ if

$$[A]_s^{R_0} := \sup_{(y,s)\in\mathbb{R}^N\times\mathbb{R}, 0<r\leq R_0} \left(\fint_{Q_r(y,s)} (\Theta(A, B_r(y))(x,t))^s \, dxdt\right)^{\frac{1}{s}} \leq \delta,$$

where

$$\Theta(A, B_r(y))(x,t) := \sup_{\zeta\in\mathbb{R}^N\backslash\{0\}} \frac{|A(x,t,\zeta) - \overline{A}_{B_r(y)}(t,\zeta)|}{|\zeta|},$$

and $\overline{A}_{B_r(y)}(t,\zeta)$ is denoted the average of $A(t,.,\zeta)$ over the cylinder $B_r(y)$, i.e,

$$\overline{A}_{B_r(y)}(t,\zeta) := \fint_{B_r(y)} A(x,t,\zeta)dx = \frac{1}{|B_r(y)|}\int_{B_r(y)} A(x,t,\zeta)dx.$$

The above condition was appeared in [21]. It is easy to see that the $(\delta, R_0)-$BMO condition on A is satisfied when A is continuous or has small jump discontinuities with respect to x.

In this paper, $\mathbb{M}$ denotes the Hardy-Littlewood maximal function defined for each locally integrable function f in $\mathbb{R}^{N+1}$ by

$$\mathbb{M}(f)(x,t) = \sup_{\rho>0} \fint_{\tilde{Q}_\rho(x,t)} |f(y,s)|dyds \quad \forall (x,t) \in \mathbb{R}^{N+1}.$$

We verify that $\mathbb{M}$ is bounded operator from $L^1(\mathbb{R}^{N+1})$ to $L^{1,\infty}(\mathbb{R}^{N+1})$ and $L^s(\mathbb{R}^{N+1})$ ($L^{s,\infty}(\mathbb{R}^{N+1})$) to itself for $s>1$, see [76, 77].

We recall that a positive function $w \in L^1_{\text{loc}}(\mathbb{R}^{N+1})$ is called an A_∞ if there are two positive constants C and ν such that

$$w(E) \le C \left(\frac{|E|}{|Q|} \right)^\nu w(Q),$$

for all cylinder $Q = \tilde{Q}_\rho(x,t)$ and all measurable subsets E of Q. The pair (C,ν) is called the A_∞ constant of w and is denoted by $[w]_{A_\infty}$.

For a weight function $w \in A_\infty$, the weighted Lorentz spaces $L^{q,s}(D,dw)$ with $0<q<\infty$, $0<s\le\infty$ and a Borel set $D \subset \mathbb{R}^{N+1}$, is the set of measurable functions g on D such that

$$||g||_{L^{q,s}(D,dw)} := \begin{cases} \left(q\int_0^\infty \left(\rho^q w\left(\{(x,t)\in D : |g(x,t)|>\rho\}\right)\right)^{\frac{s}{q}} \frac{d\rho}{\rho} \right)^{1/s} < \infty & \text{if } s<\infty, \\ \sup_{\rho>0} \rho w\left(\{(x,t)\in D : |g(x,t)|>\rho\}\right)^{1/q} < \infty & \text{if } s=\infty. \end{cases}$$

Here we write $w(E) = \int_E w(x,t)dxdt$ for a measurable set $E \subset \mathbb{R}^{N+1}$. Obviously, $||g||_{L^{q,q}(D,dw)} = ||g||_{L^q(D,dw)}$, thus we have $L^{q,q}(D,dw) = L^q(D,dw)$. As usual, when $w \equiv 1$ we simply write $L^{q,s}(D)$ instead of $L^{q,s}(D,dw)$.

We now state the next results of the paper.

Theorem 2.17 *Let $\mu \in \mathfrak{M}_b(\Omega_T)$, $\sigma \in \mathfrak{M}_b(\Omega)$, set $\omega = |\mu| + |\sigma| \otimes \delta_{\{t=0\}}$. There exists a distribution solution of* (2.4) *with data μ and σ such that if $\mathbb{R}^N \backslash \Omega$ satisfies uniformly $2-$thick with constants c_0, r_0 then for any $1 \le p < \theta$ and $0 < s \le \infty$,*

$$||\mathbb{M}(|\nabla u|)||_{L^{p,s}(\Omega_T)} \le C_1 ||\mathbb{M}_1[\omega]||_{L^{p,s}(Q)}. \tag{2.28}$$

Here $\theta = \theta(N,\Lambda_1,\Lambda_1,c_0) > 2$ and $C_1 = C_1(N,\Lambda_1,\Lambda_2,p,s,c_0,T_0/r_0)$ and $Q = B_{diam(\Omega)}(x_0) \times (0,T)$ which $\Omega \subset B_{diam(\Omega)}(x_0)$.
Especially, when $1<p<2$, then

$$||\mathbb{M}(|\nabla u|)||_{L^p(\Omega_T)} \le C_2 \left(||\mathcal{G}_1[|\mu|]||_{L^p(\mathbb{R}^{N+1})} + ||\mathbf{G}_{\frac{2}{p}-1}[|\sigma|]||_{L^p(\mathbb{R}^N)} \right), \tag{2.29}$$

where $C_2 = C_2(N,\Lambda_1,\Lambda_2,p,c_0,T_0/r_0)$.

Remark 2.18 *If $\frac{N+2}{N+1} < p < 2$, there hold*

$$||\mathcal{G}_1[|\mu|]||_{L^p(\mathbb{R}^{N+1})} \le C_1 ||\mu||_{L^{\frac{p(N+2)}{N+2+p}}(\Omega_T)} \quad \text{and} \quad ||\mathbf{G}_{\frac{2}{p}-1}[|\sigma|]||_{L^p(\mathbb{R}^N)} \le C_1 ||\sigma||_{L^{\frac{pN}{N+2-p}}(\Omega)},$$

for some $C_1 = C_1(N,p)$. From (2.29) *we obtain*

$$|||\nabla u|||_{L^p(\Omega_T)} \le C_2 ||\mu||_{L^{\frac{p(N+2)}{N+2+p}}(\Omega_T)} + C_2 ||\sigma||_{L^{\frac{pN}{N+2-p}}(\Omega)} \quad \text{provided } \frac{N+2}{N+1} < p < 2.$$

We should mention that if $\sigma \equiv 0$, then

$$||\mathbb{M}_1[\omega]||_{L^{p,s}(\mathbb{R}^{N+1})} \le C_2 ||\mu||_{L^{\frac{q(N+2)}{N+2+q},s}(\Omega_T)},$$

and we get [7, Theorem 1.2] from estimate (2.28).

In order to state the next results, we need to introduce Lorentz-Morrey spaces $L_*^{q,s;\theta}(D)$ involving "calorie" with a Borel set $D\subset\mathbb{R}^{N+1}$, is the set of measurable functions g on D such that

$$||g||_{L_*^{q,s;\kappa}(D)}:=\sup_{0<\rho<\operatorname{diam}(D),(x,t)\in D}\rho^{\frac{\kappa-N-2}{q}}||g||_{L^{q,s}(\tilde{Q}_\rho(x,t)\cap D)}<\infty,$$

where $0<\kappa\le N+2$, $0<q<\infty$, $0<s\le\infty$. Clearly, $L_*^{q,s;N+2}(D)=L^{q,s}(D)$. Moreover, when $q=s$ the space $L_*^{q,s;\theta}(D)$ will be denoted by $L_*^{q;\theta}(D)$.
The following theorem provides an estimate on gradient in Lorentz-Morrey spaces.

Theorem 2.19 *Let $\mu\in\mathfrak{M}_b(\Omega_T)$, $\sigma\in\mathfrak{M}_b(\Omega)$, set $\omega=|\mu|+|\sigma|\otimes\delta_{\{t=0\}}$. There exists a distribution solution of (2.4) with data μ and σ such that if $\mathbb{R}^N\backslash\Omega$ satisfies uniformly $2-$thick with constants c_0,r_0 then for any $1\le p<\theta$ and $0<s\le\infty$, $2-\gamma_0<\gamma<N+2$, $\gamma\le\frac{N+2}{p}+1$,*

$$\begin{aligned}||\mathbb{M}\left(|\nabla u|\right)||_{L_*^{p,s;p(\gamma-1)}(\Omega_T)}&\le C_1||\mathbb{M}_\gamma[\omega]||_{L^\infty(\Omega_T)}\\&+C_2\sup_{0<R\le T_0,(y_0,s_0)\in\Omega_T}\left(R^{\frac{p(\gamma-1)-N-2}{p}}||\mathbb{M}_1[\chi_{\tilde{Q}_R(y_0,s_0)}\omega]||_{L^{p,s}(\tilde{Q}_R(y_0,s_0))}\right).\end{aligned}\tag{2.30}$$

Here θ is in Theorem 2.17, $\gamma_0=\gamma_0(N,\Lambda_1,\Lambda_1,c_0)\in(0,1/2]$ and $C_1=C_1(N,\Lambda_1,\Lambda_2,p,s,\gamma,c_0,T_0/r_0)$, $C_2=C_2(N,\Lambda_1,\Lambda_2,p,s,\gamma,c_0)$. Besides, if $\frac{\gamma}{\gamma-1}<p<\theta$, $2-\gamma_0<\gamma<N+2$, $0<s\le\infty$ and $\mu\in L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\Omega_T)$, $\sigma\equiv0$, then u is a unique renormalized solution satisfied

$$||\mathbb{M}\left(|\nabla u|\right)||_{L_*^{p,s;(\gamma-1)p}(\Omega_T)}\le C_3||\mu||_{L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\Omega_T)},\tag{2.31}$$

where $C_3=C_3(N,\Lambda_1,\Lambda_2,p,s,\gamma,c_0,T_0/r_0)$.

Theorem 2.20 *Suppose that A satisfies (2.27). Let $\mu\in\mathfrak{M}_b(\Omega_T)$, $\sigma\in\mathfrak{M}_b(\Omega)$, set $\omega=|\mu|+|\sigma|\otimes\delta_{\{t=0\}}$. There exists a distribution solution of (2.4) with data μ,σ such that the following holds. For any $w\in A_\infty$, $1\le q<\infty$, $0<s\le\infty$ we find $\delta=\delta(N,\Lambda_1,\Lambda_2,q,s,[w]_{A_\infty})\in(0,1)$ and $s_0=s_0(N,\Lambda_1,\Lambda_2)>0$ such that if Ω is (δ,R_0)-Reifenberg flat domain Ω and $[A]_{s_0}^{R_0}\le\delta$ for some R_0 then*

$$||\mathbb{M}(|\nabla u|)||_{L^{q,s}(\Omega_T,dw)}\le C||\mathbb{M}_1[\omega]||_{L^{q,s}(\Omega_T,dw)}.\tag{2.32}$$

Here C depends on $N,\Lambda_1,\Lambda_2,q,s,[w]_{A_\infty}$ and T_0/R_0.

Next results are actually consequences of Theorem 2.20. For our purpose, we introduce another Lorentz-Morrey spaces $L_{**}^{q,s;\theta}(O_1\times O_2)$, is the set of measurable functions g on $O_1\times O_2$ such that

$$||g||_{L_{**}^{q,s;\vartheta}(O_1\times O_2)}:=\sup_{0<\rho<\operatorname{diam}(O_1),x\in O_1}\rho^{\frac{\vartheta-N}{q}}||g||_{L^{q,s}((B_\rho(x)\cap O_1)\times O_2))}<\infty,$$

where O_1,O_2 are Borel sets in $\mathbb{R}^N$ and $\mathbb{R}$ respectively, $0<\vartheta\le N$, $0<q<\infty$, $0<s\le\infty$. Obviously, $L_{**}^{q,s;N}(D)=L^{q,s}(D)$. For simplicity of notation, we write $L_{**}^{q;\vartheta}(D)$ instead of $L_{**}^{q,s;\vartheta}(D)$ when $q=s$. Moreover,

$$||g||_{L_{**}^{q,q;\vartheta}(O_1\times O_2)}=||G||_{L^{q;\vartheta}(O_1)},$$

where $G(x)=||g(x,.)||_{L^q(O_1)}$ and $L^{q;\vartheta}(O_1)$ is the usual Morrey space, i.e the spaces of all measurable functions f on O_1 with

$$||f||_{L^{q;\vartheta}(O_1)}:=\sup_{0<\rho<\operatorname{diam}(O_1),y\in O_1}\rho^{\frac{\vartheta-N}{q}}||f||_{L^q(B_\rho(y)\cap O_1)}<\infty.$$

Theorem 2.21 *Suppose that A satisfies (2.27). Let $\mu \in \mathfrak{M}_b(\Omega_T)$, $\sigma \in \mathfrak{M}_b(\Omega)$, set $\omega = |\mu| + |\sigma| \otimes \delta_{\{t=0\}}$. Let s_0 be in Theorem 2.20. There exists a distribution solution of (2.4) with data μ, σ such that the following holds.*

a. *For any $1 \le q < \infty$, $0 < s \le \infty$ and $0 < \kappa \le N+2$ we find $\delta = \delta(N, \Lambda_1, \Lambda_2, q, s, \kappa) \in (0,1)$ such that if Ω is (δ, R_0)-Reifenberg flat domain Ω and $[A]_{s_0}^{R_0} \le \delta$ for some R_0 then*

$$||\mathbb{M}(|\nabla u|)||_{L_*^{q,s;\kappa}(\Omega_T)} \le C_1 ||\mathbb{M}_1[|\omega|]||_{L_*^{q,s;\kappa}(\Omega_T)}. \tag{2.33}$$

Here C_1 depends on $N, \Lambda_1, \Lambda_2, q, s, \kappa$ and T_0/R_0.

b. *For any $1 \le q < \infty$, $0 < s \le \infty$ and $0 < \vartheta \le N$ we find $\delta = \delta(N, \Lambda_1, \Lambda_2, q, s, \vartheta) \in (0,1)$ such that if Ω is (δ, R_0)-Reifenberg flat domain Ω and $[A]_{s_0}^{R_0} \le \delta$ for some R_0 then*

$$||\mathbb{M}(|\nabla u|)||_{L_{**}^{q,s;\vartheta}(\Omega_T)} \le C_2 ||\mathbb{M}_1[|\omega|]||_{L_{**}^{q,s;\vartheta}(\Omega_T)}, \tag{2.34}$$

for some $C_2 = C_2(N, \Lambda_1, \Lambda_2, q, s, \vartheta, T_0/R_0)$. Especially, when $q = s$ and $0 < \vartheta < \min\{N, q\}$, there holds for any ball $B_\rho \subset \mathbb{R}^N$

$$\left(\int_0^T |osc_{B_\rho \cap \overline{\Omega}} u(t)|^q dt\right)^{\frac{1}{q}} \le C_3 \rho^{1-\frac{\vartheta}{q}} ||\mathbb{M}_1[|\omega|]||_{L_{**}^{q;\vartheta}(\Omega_T)}, \tag{2.35}$$

for some $C_3 = C_3(N, \Lambda_1, \Lambda_2, q, \vartheta, T_0/R_0)$.

The following global capacitary estimates on gradient.

Theorem 2.22 *Suppose that A satisfies (2.27). Let $\mu \in \mathfrak{M}_b(\Omega_T)$, $\sigma \in \mathfrak{M}_b(\Omega)$, set $\omega = |\mu| + |\sigma| \otimes \delta_{\{t=0\}}$. Let s_0 be in Theorem 2.20. There exists a distribution solution of (2.4) with data μ, σ such that following holds. For any $1 < q < \infty$, we find $\delta = \delta(N, \Lambda_1, \Lambda_2, q) \in (0,1)$ such that if Ω is a (δ, R_0)- Reifenberg flat domain and $[A]_{s_0}^{R_0} \le \delta$ for some R_0 then*

$$\sup_{\substack{compact\ K \subset \mathbb{R}^{N+1} \\ Cap_{\mathcal{G}_1,q'}(K)>0}} \left(\frac{\int_{K\cap\Omega_T} |\nabla u|^q dxdt}{Cap_{\mathcal{G}_1,q'}(K)}\right) \le C_1 \sup_{\substack{compact\ K \subset \mathbb{R}^{N+1} \\ Cap_{\mathcal{G}_1,q'}(K)>0}} \left(\frac{\omega(K)}{Cap_{\mathcal{G}_1,q'}(K)}\right)^q, \tag{2.36}$$

and if $q > \frac{N+2}{N+1}$,

$$\sup_{\substack{compact\ K \subset \mathbb{R}^{N+1} \\ Cap_{\mathcal{H}_1,q'}(K)>0}} \left(\frac{\int_{K\cap\Omega_T} |\nabla u|^q dxdt}{Cap_{\mathcal{H}_1,q'}(K)}\right) \le C_2 \sup_{\substack{compact\ K \subset \mathbb{R}^{N+1} \\ Cap_{\mathcal{H}_1,q'}(K)>0}} \left(\frac{\omega(K)}{Cap_{\mathcal{H}_1,q'}(K)}\right)^q. \tag{2.37}$$

Where $C_1 = C_1(N, \Lambda_1, \Lambda_2, q, T_0/R_0, T_0)$ and $C_2 = C_2(N, \Lambda_1, \Lambda_2, q, T_0/R_0)$.

Remark 2.23 *We have if $1 < q < 2$, then*

$$\begin{aligned} C^{-1} \sup_{\substack{compact\ K \subset \mathbb{R}^{N+1} \\ Cap_{\mathcal{G}_1,q'}(K)>0}} \left(\frac{(|\sigma| \otimes \delta_{\{t=0\}})(K)}{Cap_{\mathcal{G}_1,q'}(K)}\right) &\le \sup_{\substack{compact\ O \subset \mathbb{R}^{N} \\ Cap_{\mathbf{G}_{\frac{2}{q}-1},q'}(O)>0}} \left(\frac{|\sigma|(O)}{Cap_{\mathbf{G}_{\frac{2}{q}-1},q'}(O)}\right) \\ &\le C \sup_{\substack{compact\ K \subset \mathbb{R}^{N+1} \\ Cap_{\mathcal{G}_1,q'}(K)>0}} \left(\frac{(|\sigma| \otimes \delta_{\{t=0\}})(K)}{Cap_{\mathcal{G}_1,q'}(K)}\right), \end{aligned}$$

for $C = C(N,q)$, if $\frac{N+2}{N+1} < q < 2$, then above estimate is true when two capacities $Cap_{\mathcal{G}_1,q'}$, $,Cap_{\mathbf{G}_{\frac{2}{q}-1},q'}$ are replaced by $Cap_{\mathcal{H}_1,q'}$ $,Cap_{\mathbf{I}_{\frac{2}{q}-1},q'}$ respectively, see Remark 4.34.

Remark 2.24 *Above results also hold when $[A]_s^{R_0}$ is replaced by $\{A\}_s^{R_0}$:*

$$\{A\}_s^{R_0} := \sup_{(y,s)\in\mathbb{R}^N\times\mathbb{R}, 0<r\le R_0} \left(\fint_{Q_r(y,s)} (\Theta(A,Q_r(y,s))(x,t))^s \, dxdt \right)^{\frac{1}{s}} \le \delta,$$

where

$$\Theta(A,Q_r(y,s))(x,t) := \sup_{\zeta\in\mathbb{R}^N\setminus\{0\}} \frac{|A(x,t,\zeta) - \overline{A}_{Q_r(y,s)}(\zeta)|}{|\zeta|},$$

and $\overline{A}_{Q_r(y,s)}(\zeta)$ is denoted the average of $A(.,.,\zeta)$ over the cylinder $Q_r(y,s)$, i.e,

$$\overline{A}_{Q_r(y,s)}(\zeta) := \fint_{Q_r(y,s)} A(x,t,\zeta)dxdt = \frac{1}{|Q_r(y,s)|}\int_{Q_r(y,s)} A(x,t,\zeta)dxdt.$$

Next results are corresponding estimates of gradient for domain $\mathbb{R}^N\times(0,\infty)$ or whole $\mathbb{R}^{N+1}$.

Theorem 2.25 *Let $\theta\in(2,N+2)$ be in Theorem 2.17 and $\omega\in\mathfrak{M}(\mathbb{R}^{N+1})$. There exists a distribution solution u of (2.6) with data $\mu=\omega$ such that the following statements hold*

a. *For any $\frac{N+2}{N+1}<p<\theta$ and $0<s\le\infty$,*

$$|||\nabla u|||_{L^{p,s}(\mathbb{R}^{N+1})} \le C_1 ||\mathbb{M}_1[|\omega|]||_{L^{p,s}(\mathbb{R}^{N+1})}, \tag{2.38}$$

for some $C_1=C_1(N,\Lambda_1,\Lambda_2,p,s)$.

b. *For any $\frac{N+2}{N+1}<p<\theta$ and $0<s\le\infty$, $2-\gamma_0<\gamma<N+2$ and $\gamma\le\frac{N+2}{p}+1$,*

$$\begin{aligned}|||\nabla u|||_{L_*^{p,s;p(\gamma-1)}(\mathbb{R}^{N+1})} &\le C_2||\mathbb{M}_\gamma[|\omega|]||_{L^\infty(\mathbb{R}^{N+1})} \\ &+ C_2 \sup_{R>0,(y_0,s_0)\in\mathbb{R}^{N+1}} \left(R^{\frac{p(\gamma-1)-N-2}{p}} ||\mathbb{M}_1[\chi_{\tilde{Q}_R(y_0,s_0)}|\omega|]||_{L^{p,s}(\tilde{Q}_R(y_0,s_0))}\right),\end{aligned} \tag{2.39}$$

provided $\mathbb{I}_2[|\omega|](x_0,t_0)<\infty$ for some $(x_0,t_0)\in\mathbb{R}^{N+1}$.
Also, if $\omega\in L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\mathbb{R}^{N+1})$ with $p>\frac{\gamma}{\gamma-1}$ then

$$|||\nabla u|||_{L_*^{p,s;(\gamma-1)p}(\mathbb{R}^{N+1})} \le C_3||\omega||_{L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\mathbb{R}^{N+1})}, \tag{2.40}$$

for some $\gamma_0=\gamma_0(N,\Lambda_1,\Lambda_2)\in(0,\frac{1}{2}]$ and $C_i=C_i(N,\Lambda_1,\Lambda_2,p,s,\gamma)$, $i=2,3$.

c. *The statement **c** in Theorem 2.5 is true.*

Remark 2.26 *Let $s>1$. For $\omega\in\mathfrak{M}^+(\mathbb{R}^{N+1})$, $\mathbb{I}_1[\omega]\in L^{s,\infty}(\mathbb{R}^{N+1})$ implies $\mathbb{I}_2[|\omega|]<\infty$ a.e in $\mathbb{R}^{N+1}$ if and only if $s\le N+2$.*

Theorem 2.27 *Suppose that A satisfies (2.27). Let s_0 be in Theorem 2.20. Let $\omega\in\mathfrak{M}(\mathbb{R}^{N+1})$ with $\mathbb{I}_2[|\omega|](x_0,t_0)<\infty$ for some $(x_0,t_0)\in\mathbb{R}^{N+1}$. There exists a distribution solution of (2.6) with data $\mu=\omega$ such that following statements hold,*

a. *For any $w\in A_\infty$, $1\le q<\infty$, $0<s\le\infty$ we find $\delta=\delta(N,\Lambda_1,\Lambda_2,q,s,[w]_{A_\infty})\in(0,1)$ such that if $[A]_{s_0}^\infty\le\delta$ then*

$$|||\nabla u|||_{L^{q,s}(\mathbb{R}^{N+1},dw)} \le C_1||\mathbb{M}_1[|\omega|]||_{L^{q,s}(\mathbb{R}^{N+1},dw)}. \tag{2.41}$$

Here C_1 depends on $N,\Lambda_1,\Lambda_2,q,s,[w]_{A_\infty}$.

b. *For any* $\frac{N+2}{N+1} < q < \infty$, $0 < s \leq \infty$ *and* $0 < \kappa \leq N+2$ *we find* $\delta = \delta(N, \Lambda_1, \Lambda_2, q, s, \kappa) \in (0,1)$ *such that if* $[A]_{s_0}^{\infty} \leq \delta$ *then*

$$|||\nabla u|||_{L_*^{q,s;\kappa}(\mathbb{R}^{N+1})} \leq C_2 ||\mathbb{M}_1[|\omega|]||_{L_*^{q,s;\kappa}(\mathbb{R}^{N+1})}. \tag{2.42}$$

Here C_2 *depends on* $N, \Lambda_1, \Lambda_2, q, s, \kappa$.

c. *For any* $\frac{N+2}{N+1} < q < \infty$, $0 < s \leq \infty$ *and* $0 < \vartheta \leq N$ *one find* $\delta = \delta(N, \Lambda_1, \Lambda_2, q, s, \vartheta) \in (0,1)$ *such that if* $[A]_{s_0}^{\infty} \leq \delta$ *then*

$$|||\nabla u|||_{L_{**}^{q,s;\vartheta}(\mathbb{R}^{N+1})} \leq C_3 ||\mathbb{M}_1[|\omega|]||_{L_{**}^{q,s;\vartheta}(\mathbb{R}^{N+1})}. \tag{2.43}$$

Here C_3 *depends on* $N, \Lambda_1, \Lambda_2, q, s, \vartheta$. *Especially, when* $q = s$ *and* $0 < \vartheta < \min\{N, q\}$, *there holds for any ball* $B_\rho \subset \mathbb{R}^N$

$$\left(\int_{\mathbb{R}} |osc_{B_\rho} u(t)|^q dt\right)^{\frac{1}{q}} \leq C_4 \rho^{1-\frac{\vartheta}{q}} ||\mathbb{M}_1[|\omega|]||_{L_{**}^{q;\vartheta}(\mathbb{R}^{N+1})}, \tag{2.44}$$

for some $C_4 = C_4(N, \Lambda_1, \Lambda_2, q, \vartheta)$.

d. *For any* $\frac{N+2}{N+1} < q < \infty$, *one find* $\delta = \delta(N, \Lambda_1, \Lambda_2, q) \in (0,1)$ *such that if* $[\mathcal{A}]_{s_0}^{\infty} \leq \delta$ *then*

$$\sup_{\substack{compact\ K \subset \mathbb{R}^{N+1} \\ Cap_{\mathcal{H}_1,q'}(K) > 0}} \left(\frac{\int_K |\nabla u|^q dxdt}{Cap_{\mathcal{H}_1,q'}(K)}\right) \leq C_5 \sup_{\substack{compact\ K \subset \mathbb{R}^{N+1} \\ Cap_{\mathcal{H}_1,q'}(K) > 0}} \left(\frac{|\omega|(K)}{Cap_{\mathcal{H}_1,q'}(K)}\right)^q, \tag{2.45}$$

for some $C_5 = C_5(N, \Lambda_1, \Lambda_2, q)$.

e. *The statement* **c** *in Theorem 2.5 is true.*

The following some estimates for norms of $\mathbb{M}_1[\omega]$ in $L_*^{q;\kappa}(\mathbb{R}^{N+1})$ and $L_{**}^{q;\vartheta}(\mathbb{R}^{N+1})$

Proposition 2.28 *Let* $1 < \kappa \leq N+2$, $0 < \vartheta \leq N$ *and* $q, q_1 > 1$. *Suppose that* $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$. *Then* $\mathbb{M}_1[\mu] \leq 2^{N+2}\mathbb{I}_1[\mu]$ *and*

a. *If* $q > \frac{\kappa}{\kappa-1}$ *then*

$$||\mathbb{I}_1[\mu]||_{L_*^{q;\kappa}(\mathbb{R}^{N+1})} \leq C_1 ||\mu||_{L_*^{\frac{q\kappa}{q+\kappa};\kappa}(\mathbb{R}^{N+1})}. \tag{2.46}$$

Here C_1 *depends on* N, q, κ.

b. *If* $1 < q < 2$ *then*

$$||\mathbb{I}_1[\mu](x,.)||_{L^q(\mathbb{R})} \leq \mathbf{I}_{\frac{2}{q}-1}[\mu_1](x), \tag{2.47}$$

where μ_1 *is a nonnegative radon measure in* $\mathbb{R}^N$ *defined by* $\mu_1(A) = \mu(A \times \mathbb{R})$ *for every Borel set* $A \subset \mathbb{R}^N$. *In particular,*

$$||\mathbb{I}_1[\mu]||_{L_{**}^{q;\vartheta}(\mathbb{R}^{N+1})} \leq ||\mathbf{I}_{\frac{2}{q}-1}[\mu_1]||_{L^{q;\vartheta}(\mathbb{R}^N)}, \tag{2.48}$$

and if $\vartheta > \frac{2-q}{q-1}$ *there holds*

$$||\mathbb{I}_1[\mu]||_{L_{**}^{q;\vartheta}(\mathbb{R}^{N+1})} \leq C_2 ||\mu_1||_{L^{\frac{\vartheta q}{\vartheta+2-q};\vartheta}(\mathbb{R}^N)}, \tag{2.49}$$

for some $C_2 = C_2(N, q, \vartheta)$.

c. *If* $\frac{2q}{q+2} < q_1 \le q$ *then*

$$||\mathbb{I}_1[\mu](x,.)||_{L^q(\mathbb{R})} \le \mathbf{I}_{\frac{2}{q}+1-\frac{2}{q_1}}[\mu_2](x), \tag{2.50}$$

where $d\mu_2(x) = ||\mu(x,.)||_{L^{q_1}(\mathbb{R})}dx$. *In particular,*

$$||\mathbb{I}_1[\mu]||_{L_{**}^{q;\vartheta}(\mathbb{R}^{N+1})} \le ||\mathbf{I}_{\frac{2}{q}+1-\frac{2}{q_1}}[\mu_2]||_{L^{q;\vartheta}(\mathbb{R}^N)}, \tag{2.51}$$

and if $\vartheta > \frac{1}{q-1}\left(2+q-\frac{2q}{q_1}\right)$ *there holds*

$$||\mathbb{I}_1[\mu]||_{L_{**}^{q;\vartheta}(\mathbb{R}^{N+1})} \le C_3||\mu_2||_{L^{\frac{\vartheta q q_1}{(\vartheta+2+q)q_1-2q};\vartheta}(\mathbb{R}^N)} = C_3||\mu||_{L^{\frac{\vartheta q q_1}{(\vartheta+2+q)q_1-2q};\vartheta}(\mathbb{R}^N,L^{q_1}(\mathbb{R}))}, \tag{2.52}$$

for some $C_3 = C_3(N,q,\vartheta)$.

The proof of Proposition 2.28 will performed at the end of section 8.

Remark 2.29 *Let* $1 < q < 2$, $0 < \vartheta \le N$ *and* $\sigma \in \mathfrak{M}(\mathbb{R}^N)$. *From* (2.48) *and* (2.49) *in Proposition 2.28 we assert that*

$$||\mathbb{I}_1[|\sigma|\otimes\delta_{\{t=0\}}]||_{L_{**}^{q;\vartheta}(\mathbb{R}^{N+1})} \le ||\mathbf{I}_{\frac{2}{q}-1}[|\sigma|]||_{L^{q;\vartheta}(\mathbb{R}^N)},$$

and

$$||\mathbb{I}_1[|\sigma|\otimes\delta_{\{t=0\}}]||_{L_{**}^{q;\vartheta}(\mathbb{R}^{N+1})} \le C_1||\sigma||_{L^{\frac{\vartheta q}{\vartheta+2-q};\vartheta}(\mathbb{R}^N)} \quad \text{if } \vartheta > \frac{2-q}{q-1},$$

for some $C_1 = C_1(N,q,\vartheta)$.
Furthermore, from preceding inequality and (2.52) *in Proposition 2.28 we can state that*

$$||\mathbb{I}_1[|\sigma|\otimes\delta_{\{t=0\}}+|\mu|]||_{L_{**}^{q;\vartheta}(\mathbb{R}^{N+1})} \le C_2||\sigma||_{L^{\frac{\vartheta q}{\vartheta+2-q};\vartheta}(\mathbb{R}^N)} + C_2||\mu||_{L^{\frac{\vartheta q q_1}{(\vartheta+2+q)q_1-2q};\vartheta}(\mathbb{R}^N,L^{q_1}(\mathbb{R}))},$$

provided

$$\begin{array}{l} 1 < q_1 \le q < 2, \\ \max\left\{\dfrac{2-q}{q-1}, \dfrac{1}{q-1}\left(2+q-\dfrac{2q}{q_1}\right)\right\} < \vartheta \le N, \end{array}$$

for some $C_2 = C_2(N,q,\vartheta)$. *Where*

$$||\mu||_{L^{q_2;\vartheta}(\mathbb{R}^N,L^{q_1}(\mathbb{R}))} = \sup_{\rho>0,x\in\mathbb{R}^N} \rho^{\frac{\vartheta-N}{q_2}}\left(\int_{B_\rho(x)}\left(\int_{\mathbb{R}}|\mu(y,t)|^{q_1}dt\right)^{\frac{q_2}{q_1}}dy\right)^{\frac{1}{q_2}},$$

with $q_2 = \frac{\vartheta q q_1}{(\vartheta+2+q)q_1-2q}$.

Final part, we prove the existence solutions for the quasilinear Riccati type parabolic problems

$$\begin{cases} u_t - div(A(x,t,\nabla u)) = |\nabla u|^q + \mu & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial\Omega\times(0,T), \\ u(0) = \sigma & \text{in } \Omega, \end{cases} \tag{2.53}$$

$$\begin{cases} u_t - div\left(A(x,t,\nabla u)\right) = |\nabla u|^q + \mu \text{ in } \mathbb{R}^N\times(0,\infty), \\ u(0) = \sigma \quad \text{in} \quad \mathbb{R}^N, \end{cases} \tag{2.54}$$

and

$$u_t - div\left(A(x,t,\nabla u)\right) = |\nabla u|^q + \mu \text{ in } \mathbb{R}^{N+1}, \tag{2.55}$$

where $q > 1$.

The following result is considered in subcritical case this means $1 < q < \frac{N+2}{N+1}$, to obtain existence solutions in this case we need data μ, σ to be finite measures and small enough.

Theorem 2.30 *Let* $1 < q < \frac{N+2}{N+1}$ *and* $\mu \in \mathfrak{M}_b(\Omega_T)$, $\sigma \in \mathfrak{M}_b(\Omega)$. *There exists* $\varepsilon_0 = \varepsilon_0(N, \Lambda_1, \Lambda_2, q) > 0$ *such that if*

$$|\Omega_T|^{-1+\frac{q'}{N+2}} \left(|\mu|(\Omega_T) + |\omega|(\Omega)\right) \le \varepsilon_0,$$

the problem (2.53) *has a distribution solution* u, *satisfied*

$$|||\nabla u|||_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} \le C\left(|\mu|(\Omega_T) + |\omega|(\Omega)\right),$$

for some $C = C(N, \Lambda_1, \Lambda_2, q) > 0$.

In the next results are concerned in critical and supercritical case.

Theorem 2.31 *Suppose that* $\mathbb{R}^N \backslash \Omega$ *satisfies uniformly* $2-$*thick with constants* c_0, r_0. *Let* θ *be in Theorem 2.17,* $q \in \left(\frac{N+2}{N+1}, \frac{N+2+\theta}{N+2}\right)$, $\mu \in \mathfrak{M}_b(\Omega_T)$ *and* $\sigma \in \mathfrak{M}_b(\Omega)$. *Assume that* $\sigma \equiv 0$ *when* $q \ge \frac{N+4}{N+2}$. *There exists* $\varepsilon_0 = \varepsilon_0(N, \Lambda_1, \Lambda_2, q, c_0, T_0/r_0) > 0$ *such that if*

$$||\mathbb{I}_1[|\mu|]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})} + ||\mathbf{I}_{\frac{2}{(N+2)(q-1)}-1}[|\sigma|]||_{L^{(N+2)(q-1)}(\mathbb{R}^N)} \le \varepsilon_0,$$

then the problem (2.53) *has a distribution solution* u *satisfying*

$$|||\nabla u|||_{L^{(q-1)(N+2),\infty}(\Omega_T)} \le C||\mathbb{I}_1[|\mu|]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})} + C||\mathbf{I}_{\frac{2}{(N+2)(q-1)}-1}[|\sigma|]||_{L^{(N+2)(q-1)}(\mathbb{R}^N)}, \tag{2.56}$$

for some $C = C(N, \Lambda_1, \Lambda_2, q, c_0, T_0/r_0)$.

We remark that a necessary condition for existence $\sigma \in \mathfrak{M}_b(\Omega)\backslash\{0\}$ with $\mathbb{M}_1[|\sigma| \otimes \delta_{\{t=0\}}] \in L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})$ is $\frac{N+2}{N+1} \le q < \frac{N+4}{N+2}$.

Theorem 2.32 *Suppose that* A *satisfies* (2.27). *Let* s_0 *be the constant in Theorem 2.20. Let* $q \ge \frac{N+2}{N+1}$ *and* $\mu \in \mathfrak{M}_b(\Omega_T), \sigma \in \mathfrak{M}_b(\Omega)$, *set* $\omega = |\mu| + |\sigma| \otimes \delta_{\{t=0\}}$. *There exists* $\delta = \delta(N, \Lambda_1, \Lambda_2, q) \in (0,1)$ *such that* Ω *is* (δ, R_0)*-Reifenberg flat domain* Ω *and* $[A]_{s_0}^{R_0} \le \delta$ *for some* R_0 *and the following holds. The problem* (2.53) *has a distribution solution* u *if one of the following three cases is true:*

Case a. *A is a linear operator and*

$$\omega(K) \le C_1 Cap_{\mathcal{G}_1,q'}(K) \quad \text{for every compact subset } K \subset \mathbb{R}^{N+1}, \tag{2.57}$$

with a constant C_1 *small enough.*

Case b. *there holds*

$$\omega(K) \le C_2 Cap_{\mathcal{G}_1,(q+\varepsilon)'}(K) \quad \text{for every compact subset } K \subset \mathbb{R}^{N+1}, \tag{2.58}$$

where $\varepsilon > 0$ *and* C_2 *is a constant small enough.*

Case c. $\begin{cases} q > \frac{N+2}{N+1}, \\ q \ge \frac{N+4}{N+2} \quad \text{if } \sigma \equiv 0, \\ ||\mathbb{I}_1[|\mu|]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})}, ||\mathbf{I}_{\frac{2}{(N+2)(q-1)}-1}[|\sigma|]||_{L^{(N+2)(q-1)}(\mathbb{R}^N)} \\ \quad \text{is small enough.} \end{cases}$

A solution u *corresponds to* ***Case a**,* ***b*** *and* ***c*** *satisfying*

$$\int_K |\nabla u|^q dxdt \le C_3 C_1^q Cap_{\mathcal{G}_1,q'}(K) \quad \text{for every compact subset } K \subset \mathbb{R}^{N+1},$$

$$\int_K |\nabla u|^{q+\varepsilon} dxdt \le C_4 C_2^{q+\varepsilon} Cap_{\mathcal{G}_1,(q+\varepsilon)'}(K) \quad \text{for every compact subset } K \subset \mathbb{R}^{N+1},$$

and

$$|||\nabla u|||_{L^{(N+2)(q-1),\infty}(\Omega_T)}$$
$$\leq C_5||\mathbb{I}_1[|\mu|]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})} + C_5||\mathbf{I}_{\frac{2}{(N+2)(q-1)}-1}[|\sigma|]||_{L^{(N+2)(q-1)}(\mathbb{R}^N)},$$

respectively. Where C_3, C_4, C_5 are constants depended on $N, \Lambda_1, \Lambda_2, q, \varepsilon, T_0/R_0$, besides C_3, C_4 also depend on T_0.

Since $\text{Cap}_{\mathcal{G}_1,s}(B_r(0) \times \{t = 0\}) = 0$ for all $r > 0$ and $0 < s \leq 2$, see Remark 4.13 thus if there is $\sigma \in \mathfrak{M}_b(\Omega)\backslash\{0\}$ satisfying $(|\sigma| \otimes \delta_{\{t=0\}})(E) \leq \text{Cap}_{\mathcal{G}_1,s}(E)$ for every compact subsets $E \subset \mathbb{R}^{N+1}$ then we must have $s > 2$.

The above results are not sharp in the case A is a nonlinear operator. However, if A is Holder continuous with respect to x we can prove that problem (2.53) has a distribution solution with data having compact support in Ω_T.

Theorem 2.33 *Let Ω be a bounded open subset in $\mathbb{R}^N$ such that the boundary of Ω is in $C^{1,\beta}$ with $\beta \in (0,1)$. Suppose that A satisfies (2.27) and*

$$|A(x,t,\zeta) - A(y,t,\zeta)| \leq \Lambda_3|x-y|^\beta|\zeta|, \tag{2.59}$$

for every $x, y \in \Omega$ and $t > 0, \zeta \in \mathbb{R}^N$. Let $\Omega' \subset\subset \Omega$ and set $d = dist(\Omega', \Omega) > 0$. Then, there exist $C = C(N, q, \Lambda_1, \Lambda_2, \Lambda_3, \beta, d, \Omega, T) > 0$ and $\Lambda = \Lambda(N, q, \Lambda_1, \Lambda_2, \Lambda_3, \beta, d, \Omega, T) > 0$ such that for any $\mu \in \mathfrak{M}_b(\Omega_T)$, $\sigma \in \mathfrak{M}_b(\Omega)$ with $supp(\mu) \subset \Omega' \times [0,T]$, $supp(\sigma) \subset \Omega'$, the problem (2.53) has a distribution solution u, satisfying

$$|\nabla u(x,t)| \leq \Lambda\mathbb{I}_1[|\mu| + |\sigma| \otimes \delta_{\{t=0\}}](x,t) \quad a.e\ (x,t) \in \Omega_T, \tag{2.60}$$

provided that one of the following two cases is true:

Case a. $1 < q < 2$ *and*

$$|\mu|(E) \leq C\,Cap_{\mathcal{G}_1,q'}(E) \text{ and } |\sigma|(O) \leq C\,Cap_{\mathbf{G}_{\frac{2}{q}-1},q'}(O), \tag{2.61}$$

for all compact subsets $E \subset \mathbb{R}^{N+1}$ and $O \subset \mathbb{R}^N$.

Case b. $q \geq 2$ *and* $\sigma \equiv 0$,

$$|\mu|(E) \leq C\,Cap_{\mathcal{G}_1,q'}(E), \tag{2.62}$$

for all compact subsets $E \subset \mathbb{R}^{N+1}$.

Remark 2.34 *If $q > \frac{N+2}{N+1}$, $\mu \equiv 0$ and **Case a** is satisfied then (2.60) gives the decay estimate:*

$$\sup_{x\in\Omega} |\nabla u(x,t)| \leq c_1 t^{-\frac{1}{2(q-1)}} \ \forall\ 0 < t < T,$$

since $|\sigma|(B_\rho(x)) \leq c_2(T_0)\rho^{N-\frac{2-q}{q-1}}$ for any $B_\rho(x) \subset \mathbb{R}^N$.

We have an **important** Proposition.

Proposition 2.35 *All the existence results considered the bounded domain Ω_T have recently been presented in above Theorems, if $\sigma \in L^1(\Omega)$ then the solutions obtained in those Theorems are renormalized solutions.*

Theorem 2.36 *Let $\theta \in (2, N+2)$ be in Theorem 2.17, $q \in \left(\frac{N+2}{N+1}, \frac{N+2+\theta}{N+2}\right)$ and $\omega \in \mathfrak{M}(\mathbb{R}^{N+1})$. There exists $C_1 = C_1(N, \Lambda_1, \Lambda_2, q) > 0$ such that if*

$$||\mathbb{I}_1[|\omega|]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})} \leq C_1,$$

then the problem (2.55) has a distribution solution $u \in L^1_{loc}(\mathbb{R}; W^{1,1}_{loc}(\mathbb{R}^N))$ such that

$$|||\nabla u|||_{L^{(q-1)(N+2),\infty}(\mathbb{R}^{N+1})} \leq C_2||\mathbb{I}_1[|\omega|]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})}, \tag{2.63}$$

for some $C_2 = C_2(N, \Lambda_1, \Lambda_2, q)$. Furthermore, when $\omega = \mu + \sigma \otimes \delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}(\mathbb{R}^N \times (0, \infty))$ and $\sigma \in \mathfrak{M}(\mathbb{R}^N)$ then $u = 0$ in $\mathbb{R}^N \times (-\infty, 0)$ and $u|_{\mathbb{R}^N \times [0,\infty)}$ is a distribution solution to problem (2.54).

Theorem 2.37 *Suppose that A satisfies (2.27). Let $q > \frac{N+2}{N+1}$ and $\omega \in \mathfrak{M}(\mathbb{R}^{N+1})$ such that $\mathbb{I}_2[|\omega|](x_0, t_0) < \infty$ for some $(x_0, t_0) \in \mathbb{R}^{N+1}$. Let s_0 be the constant in Theorem 2.20, δ in Theorem 2.32. There exists $C_1 = C_1(N, \Lambda_1, \Lambda_2, q) > 0$ such that if $[A]^{\infty}_{s_0} \leq \delta$ and*

$$||\mathbb{I}_1[|\omega|]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})} \leq C_1, \tag{2.64}$$

then the problem (2.55) has a distribution solution u satisfying (2.63). Furthermore, when $\omega = \mu + \sigma \otimes \delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}(\mathbb{R}^N \times (0, \infty))$ and $\sigma \in \mathfrak{M}(\mathbb{R}^N)$ then $u = 0$ in $\mathbb{R}^N \times (-\infty, 0)$ and $u|_{\mathbb{R}^N \times [0,\infty)}$ is a distribution solution to problem (2.54).

From Remark 2.26, we see that if $q \leq 2$ then (2.64) follows the assumption $\mathbb{I}_2[|\omega|](x_0, t_0) < \infty$ for some $(x_0, t_0) \in \mathbb{R}^{N+1}$.

When A is independent of space variable, we can improve the result of Theorem 2.37 as follows:

Theorem 2.38 *Suppose that A is independent of space variable and satisfies (2.27). Let $q > \frac{N+2}{N+1}$ and $\omega \in \mathfrak{M}(\mathbb{R}^{N+1})$. Assume that $\mathbb{I}_2[|\omega|](x_0, t_0) < \infty$ for some $(x_0, t_0) \in \mathbb{R}^{N+1}$. There exist constants $\Lambda = \Lambda(N, \Lambda_1, \Lambda_2, q)$ and $C = C(N, \Lambda_1, \Lambda_2, q)$ such that the problem*

$$u_t - div\,(A(t, \nabla u)) = |\nabla u|^q + \omega \ \text{ in } \ \mathbb{R}^{N+1}, \tag{2.65}$$

has a distribution solution u, satisfying

$$|\nabla u| \leq \Lambda \mathbb{I}_1[\omega] \quad \text{in } \ \mathbb{R}^{N+1}, \tag{2.66}$$

provided that for all compact subset $E \subset \mathbb{R}^{N+1}$

$$|\omega|(E) \leq C\, Cap_{\mathcal{H}_1, q'}(E). \tag{2.67}$$

Furthermore, when $\omega = \mu + \sigma \otimes \delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}(\mathbb{R}^N \times (0, \infty))$ and $\sigma \in \mathfrak{M}(\mathbb{R}^N)$ then $u = 0$ in $\mathbb{R}^N \times (-\infty, 0)$ and $u|_{\mathbb{R}^N \times [0,\infty)}$ is a distribution solution to problem

$$\begin{cases} u_t - div\,(A(t, \nabla u)) = |\nabla u|^q + \mu \ \text{ in } \ \mathbb{R}^N \times (0, \infty), \\ u(0) = \sigma \quad \text{in} \quad \mathbb{R}^N. \end{cases} \tag{2.68}$$

Remark 2.39 *If $\frac{N+2}{N+1} < q < 2$, $\omega = \mu + \sigma \otimes \delta_{\{t=0\}}$ satisfies (2.67) if and only if*

$$|\mu|(E) \leq C' C\, Cap_{\mathcal{H}_1, q'}(E) \text{ and } |\sigma|(O) \leq C' C\, Cap_{\mathbf{I}_{\frac{2}{q}-1}, q'}(O), \tag{2.69}$$

for all compact subsets $E \subset \mathbb{R}^{N+1}$ and $O \subset \mathbb{R}^N$, where $C' = C'(N, q)$.

Remark 2.40 *If $\omega = \sigma \otimes \delta_{\{t=0\}}$ then (2.66) follows the decay estimate:*

$$\sup_{x \in \mathbb{R}^N} |\nabla u(x, t)| \leq c_1 t^{-\frac{1}{2(q-1)}} \ \forall \ 0 < t < T,$$

since $|\sigma|(B_\rho(x)) \leq c_2 \rho^{N - \frac{2-q}{q-1}}$ for any $B_\rho(x) \subset \mathbb{R}^N$.

3 The notion of solutions and some properties

Although the notion of renormalized solutions becomes more and more familiar in the theory of quasilinear parabolic equations with measure data, it is still necessary to present below some main aspects concerning this notion. Let Ω be a bounded domain in $\mathbb{R}^N$, $(a,b) \subset\subset \mathbb{R}$. If $\mu \in \mathfrak{M}_b(\Omega \times (a,b))$, we denote by μ^+ and μ^- respectively its positive and negative part. We denote by $\mathfrak{M}_0(\Omega \times (a,b))$ the space of measures in $\Omega \times (a,b)$ which are absolutely continuous with respect to the C_2-capacity defined on a compact set $K \subset \Omega \times (a,b)$ by

$$C_2(K, \Omega \times (a,b)) = \inf \{ ||\varphi||_W : \varphi \geq \chi_K, \varphi \in C_c^\infty(\Omega \times (a,b)) \}. \tag{3.1}$$

where $W = \{z : z \in L^2(a,b,H_0^1(\Omega)), z_t \in L^2(a,b,H^{-1}(\Omega))\}$ endowed with norm $||\varphi||_W = ||\varphi||_{L^2(a,b,H_0^1(\Omega))} + ||\varphi_t||_{L^2(a,b,H^{-1}(\Omega))}$ and χ_K is the characteristic function of K.

We also denote $\mathfrak{M}_s(\Omega \times (a,b))$ the space of measures in $\Omega \times (a,b)$ with support on a set of zero C_2-capacity. Classically, any $\mu \in \mathfrak{M}_b(\Omega \times (a,b))$ can be written in a unique way under the form $\mu = \mu_0 + \mu_s$ where $\mu_0 \in \mathfrak{M}_0(\Omega \times (a,b)) \cap \mathfrak{M}_b(\Omega \times (a,b))$ and $\mu_s \in \mathfrak{M}_s(\Omega \times (a,b))$. We recall that any $\mu_0 \in \mathfrak{M}_0(\Omega \times (a,b)) \cap \mathfrak{M}_b(\Omega \times (a,b))$ can be decomposed under the form $\mu_0 = f - \text{div} g + h_t$ where $f \in L^1(\Omega \times (a,b))$, $g \in L^2(\Omega \times (a,b), \mathbb{R}^N)$ and $h \in L^2(a,b,H_0^1(\Omega))$ and (f,g,h) is said to be decomposition of μ_0. Set $\widehat{\mu_0} = \mu_0 - h_t = f - \text{div} g$. In the general case $\widehat{\mu_0} \notin \mathfrak{M}(\Omega \times (a,b))$, but we write, for convenience,

$$\int_{\Omega \times (a,b)} w d\widehat{\mu_0} := \int_{\Omega \times (a,b)} (fw + g.\nabla w) dxdt, \qquad \forall w \in L^2(a,b,H_0^1(\Omega)) \cap L^\infty(\Omega \times (a,b)).$$

However, for $\sigma \in \mathfrak{M}_b(\Omega)$ and $t_0 \in (a,b)$ then $\sigma \otimes \delta_{\{t=t_0\}} \in \mathfrak{M}_0(\Omega \times (a,b))$ if and only if $\sigma \in L^1(\Omega)$, see [26]. We also have that for $\sigma \in \mathfrak{M}_b(\Omega)$, $\sigma \otimes \chi_{[a,b]} \in \mathfrak{M}_0(\Omega \times (a,b))$ if and only if σ is absolutely continuous with respect to the $\text{Cap}_{\mathbf{G}_1,2}$-capacity, see [26].

For $k > 0$ and $s \in \mathbb{R}$ we set $T_k(s) = \max\{\min\{s,k\}, -k\}$. We recall that if u is a measurable function defined and finite a.e. in $\Omega \times (a,b)$, such that $T_k(u) \in L^2(a,b,H_0^1(\Omega))$ for any $k > 0$, there exists a measurable function $v : \Omega \times (a,b) \to \mathbb{R}^N$ such that $\nabla T_k(u) = \chi_{|u| \leq k} v$ a.e. in $\Omega \times (a,b)$ and for all $k > 0$. We define the gradient ∇u of u by $v = \nabla u$.
We recall the definition of a renormalized solution given in [65].

Definition 3.1 *Suppose that $B \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$. Let $\mu = \mu_0 + \mu_s \in \mathfrak{M}_b(\Omega \times (a,b))$ and $\sigma \in L^1(\Omega)$. A measurable function u is a* ***renormalized solution*** *of*

$$\begin{cases} u_t - div(A(x,t,\nabla u)) = B(u, \nabla u) + \mu \text{ in } \Omega \times (a,b), \\ u = 0 \quad \text{ on } \partial\Omega \times (a,b), \\ u(a) = \sigma \quad \text{ in } \Omega, \end{cases} \tag{3.2}$$

if there exists a decomposition (f,g,h) of μ_0 such that

$$v = u - h \in L^s(a,b,W_0^{1,s}(\Omega)) \cap L^\infty(a,b,L^1(\Omega)) \ \forall s \in \left[1, \frac{N+2}{N+1}\right)$$
$$T_k(v) \in L^2(a,b,H_0^1(\Omega)) \ \forall k > 0, B(u,\nabla u) \in L^1(\Omega \times (a,b)) \tag{3.3}$$

and:

(i) for any $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support on $\mathbb{R}$, and $S(0) = 0$,

$$-\int_\Omega S(\sigma)\varphi(a)dx - \int_{\Omega \times (a,b)} \varphi_t S(v) dxdt + \int_{\Omega \times (a,b)} S'(v) A(x,t,\nabla u) \nabla \varphi dxdt$$
$$+ \int_{\Omega \times (a,b)} S''(v) \varphi A(x,t,\nabla u).\nabla v dxdt = \int_{\Omega \times (a,b)} S'(v) \varphi B(u,\nabla u) dxdt + \int_{\Omega \times (a,b)} S'(v) \varphi d\widehat{\mu_0}, \tag{3.4}$$

for any $\varphi \in L^2(a,b,H_0^1(\Omega)) \cap L^\infty(\Omega\times(a,b))$ *such that* $\varphi_t \in L^2(a,b,H^{-1}(\Omega)) + L^1(\Omega\times(a,b))$ *and* $\varphi(.,b)=0$*;*

(ii) for any $\phi \in C(\overline{\Omega}\times[a,b])$,

$$\lim_{m\to\infty}\frac{1}{m}\int\limits_{\{m\le v<2m\}}\phi A(x,t,\nabla u)\nabla v dxdt = \int_{\Omega\times(a,b)}\phi d\mu_s^+ \quad \text{and} \tag{3.5}$$

$$\lim_{m\to\infty}\frac{1}{m}\int\limits_{\{-m\ge v>-2m\}}\phi A(x,t,\nabla u)\nabla v dxdt = \int_{\Omega\times(a,b)}\phi d\mu_s^-. \tag{3.6}$$

Remark 3.2 *If* $\mu\in L^1(\Omega\times(a,b))$*, then we have the following estimates:*

$$\begin{aligned}
&||u||_{L^{\frac{N+2}{N},\infty}(\Omega\times(a,b))} \le C_1\left(||\sigma||_{L^1(\Omega)} + |\mu|(\Omega\times(a,b))\right) \quad \text{and}\\
&|||\nabla u|||_{L^{\frac{N+2}{N+1},\infty}(\Omega\times(a,b))} \le C_1\left(||\sigma||_{L^1(\Omega)} + |\mu|(\Omega\times(a,b))\right),
\end{aligned}$$

where $C_1 = C_1(N,\Lambda_1,\Lambda_2)$*, see [13, Remark 4.9].*
In particular,

$$\begin{aligned}
&||u||_{L^1(\Omega\times(a,b))} \le C_2(diam(\Omega)+(b-a)^{1/2})^2\left(||\sigma||_{L^1(\Omega)} + |\mu|(\Omega\times(a,b))\right) \quad \text{and}\\
&|||\nabla u|||_{L^1(\Omega\times(a,b))} \le C_2(diam(\Omega)+(b-a)^{1/2})\left(||\sigma||_{L^1(\Omega)} + |\mu|(\Omega\times(a,b))\right),
\end{aligned}$$

where $C_2 = C_2(N,\Lambda_1,\Lambda_2)$.

Remark 3.3 *It is easy to see that* u *is a weak solution of problem* (3.2) *in* $\Omega\times(a,b)$ *with* $\mu\in L^2(\Omega\times(a,b))$, $\sigma\in H_0^1(\Omega)$ *and* $B\equiv 0$ *then* $U=\chi_{[a,b]}u$ *is a unique renormalized solution of*

$$\begin{cases}
U_t - div\left(A(x,t,\nabla U)\right) = \chi_{(a,b)}\mu + (\chi_{[a,b)}\sigma)_t & \text{in } \Omega\times(c,b),\\
U=0 & \text{on } \partial\Omega\times(c,b),\\
U(c)=0 & \text{in } \Omega,
\end{cases}$$

for any $c<a$.

Remark 3.4 *Let* $\Omega'\subset\subset\Omega$ *and* $a<a'<b'<b$*. For a nonnegative function* $\eta\in C_c^\infty(\Omega'\times(a',b'))$*, from* (3.4) *we have*

$$\begin{aligned}
&(\eta S(v))_t - \eta_t S(v) + S'(v)A(x,t,\nabla u)\nabla\eta - div(S'(v)\eta A(x,t,\nabla u))\\
&\quad + S''(v)\eta A(x,t,\nabla u)\nabla v = S'(v)\eta f + \nabla\left(S'(v)\eta\right).g - div(S'(v)\eta g)
\end{aligned}$$

in $\mathcal{D}'(\Omega'\times(a',b'))$ *Thus,* $(\eta S(v))_t\in L^2(a',b',H^{-1}(\Omega'))+L^1(D)$ *and we have the following estimate*

$$\begin{aligned}
||\left(\eta S(v)\right)_t||_{L^2(a',b',H^{-1}(\Omega'))+L^1(D)} &\le C||S||_{W^{2,\infty}(\mathbb{R})}\big(||\eta_t v||_{L^1(D)}\\
&+ |||\nabla u||\nabla\eta|||_{L^1(D)} + ||\eta|\nabla u|\chi_{|v|\le M}||_{L^2(D)} + ||\eta|\nabla u||\nabla v|\chi_{|v|\le M}||_{L^2(D)}\\
&+ ||\eta f||_{L^1(D)} + ||\eta|\nabla u|^2\chi_{|v|\le M|}||_{L^1(D)} + ||\eta|g|^2||_{L^1(D)} + ||\eta|g|||_{L^2(D)}\big)
\end{aligned} \tag{3.7}$$

with $D=\Omega'\times(a',b')$ *and* $supp(S')\subset[-M,M]$.

We recall the following important results, see [13].

Proposition 3.5 *Let* $\{\mu_n\}$ *be a bounded in* $\mathfrak{M}_b(\Omega\times(a,b))$ *and* σ_n *be a bounded in* $L^1(\Omega)$*. Let* u_n *be a renormalized solution of* (2.4) *with data* $\mu_n=\mu_{n,0}+\mu_{n,s}$ *relative to a decomposition* (f_n,g_n,h_n) *of* $\mu_{n,0}$ *and initial data* σ_n*. If* $\{f_n\}$ *is bounded in* $L^1(\Omega_T)$*,* $\{g_n\}$ *bounded in* $L^2(\Omega\times(a,b),\mathbb{R}^N)$ *and* $\{h_n\}$ *convergent in* $L^2(a,b,H_0^1(\Omega))$*, then, up to a subsequence,* $\{u_n\}$ *converges to a function* u *in* $L^1(\Omega\times(a,b))$*. Moreover, if* $\{\mu_n\}$ *is a bounded in* $L^1(\Omega\times(a,b))$ *then* $\{u_n\}$ *is convergent in* $L^s(a,b,W_0^{1,s}(\Omega))$ *for any* $s\in\left[1,\frac{N+2}{N+1}\right)$.

We say that a sequence of bounded measures $\{\mu_n\}$ in $\Omega \times (a,b)$ converges to a bounded measure μ in $\Omega \times (a,b)$ in the *narrow topology* of measures if

$$\lim_{n\to\infty} \int_{\Omega\times(a,b)} \varphi d\mu_n = \int_{\Omega\times(a,b)} \varphi d\mu \quad \text{for all } \varphi \in C(\Omega \times (a,b)) \cap L^\infty(\Omega \times (a,b))).$$

We recall the following fundamental stability result of [13].

Theorem 3.6 *Suppose that $B \equiv 0$. Let $\sigma \in L^1(\Omega)$ and*

$$\mu = f - \operatorname{div} g + h_t + \mu_s^+ - \mu_s^- \in \mathfrak{M}_b(\Omega \times (a,b)),$$

with $f \in L^1(\Omega \times (a,b)), g \in L^2(\Omega \times (a,b), \mathbb{R}^N)$, $h \in L^2(a,b,H_0^1(\Omega))$ and $\mu_s^+, \mu_s^- \in \mathfrak{M}_s^+(\Omega \times (a,b))$. Let $\sigma_n \in L^1(\Omega)$ and

$$\mu_n = f_n - \operatorname{div} g_n + (h_n)_t + \rho_n - \eta_n \in \mathfrak{M}_b(\Omega \times (a,b))$$

with $f_n \in L^1(\Omega \times (a,b)), g_n \in L^2(\Omega \times (a,b), \mathbb{R}^N), h_n \in L^2(a,b,H_0^1(\Omega))$, and $\rho_n, \eta_n \in \mathfrak{M}_b^+(\Omega \times (a,b))$, such that

$$\rho_n = \rho_n^1 - \operatorname{div} \rho_n^2 + \rho_{n,s}, \qquad \eta_n = \eta_n^1 - \operatorname{div} \eta_n^2 + \eta_{n,s},$$

with $\rho_n^1, \eta_n^1 \in L^1(\Omega \times (a,b)), \rho_n^2, \eta_n^2 \in L^2(\Omega \times (a,b), \mathbb{R}^N)$ and $\rho_{n,s}, \eta_{n,s} \in \mathfrak{M}_s^+(\Omega \times (a,b))$.
Assume that $\{\mu_n\}$ is a bounded in $\mathfrak{M}_b(\Omega\times(a,b))$, $\{\sigma_n\}, \{f_n\}, \{g_n\}, \{h_n\}$ converge to σ, f, g, h in $L^1(\Omega)$,weakly in $L^1(\Omega \times (a,b))$,in $L^2(\Omega \times (a,b), \mathbb{R}^N)$,in $L^2(a,b,H_0^1(\Omega))$ respectively and $\{\rho_n\}, \{\eta_n\}$ converge to μ_s^+, μ_s^- in the narrow topology of measures; and $\{\rho_n^1\}, \{\eta_n^1\}$ are bounded in $L^1(\Omega \times (a,b))$, and $\{\rho_n^2\}, \{\eta_n^2\}$ bounded in $L^2(\Omega \times (a,b), \mathbb{R}^N)$.
Let $\{u_n\}$ be a sequence of renormalized solutions of

$$\begin{cases} (u_n)_t - \operatorname{div}(A(x,t,\nabla u_n)) = \mu_n & \text{in } \Omega \times (a,b), \\ u_n = 0 & \text{on } \partial\Omega \times (a,b), \\ u_n(a) = \sigma_n & \text{in } \Omega, \end{cases} \tag{3.8}$$

relative to the decomposition $(f_n + \rho_n^1 - \eta_n^1, g_n + \rho_n^2 - \eta_n^2, h_n)$ of $\mu_{n,0}$. Let $v_n = u_n - h_n$. Then up to a subsequence, $\{u_n\}$ converges a.e. in $\Omega \times (a,b)$ to a renormalized solution u of (3.2), and $\{v_n\}$ converges a.e. in $\Omega \times (a,b)$ to $v = u - h$. Moreover, $\{\nabla u_n\}, \{\nabla v_n\}$ converge respectively to $\nabla u, \nabla v$ a.e in $\Omega \times (a,b)$, and $\{T_k(v_n)\}$ converges to $T_k(v)$ strongly in $L^2(a,b,H_0^1(\Omega))$ for any $k > 0$.

In order to apply above Theorem, we need some the following properties concerning approximate measures of $\mu \in \mathfrak{M}_b^+(\Omega \times (a,b))$, see [13].

Proposition 3.7 *Let $\mu = \mu_0 + \mu_s \in \mathfrak{M}_b^+(\Omega \times (a,b))$ with $\mu_0 \in \mathfrak{M}_0(\Omega \times (a,b)) \cap \mathfrak{M}_b^+(\Omega \times (a,b))$ and $\mu_s \in \mathfrak{M}_s^+(\Omega \times (a,b))$. Let $\{\varphi_n\}$ be sequence of standard mollifiers in $\mathbb{R}^{N+1}$. Then, there exist a decomposition (f,g,h) of μ_0 and $f_n, g_n, h_n \in C_c^\infty(\Omega \times (a,b))$, $\mu_{n,s} \in C_c^\infty(\Omega \times (a,b)) \cap \mathfrak{M}_b^+(\Omega \times (a,b))$ such that $\{f_n\}, \{g_n\}, \{h_n\}$ strongly converge to f, g, h in $L^1(\Omega \times (a,b)), L^2(\Omega \times (a,b), \mathbb{R}^N)$ and $L^2(a,b,H_0^1(\Omega))$, $\mu_n = f_n - \operatorname{div} g_n + (h_n)_t + \mu_{n,s}, \mu_{n,s}$ converge to μ, μ_s in the narrow topology respectively, $0 \le \mu_n \le \varphi_n * \mu$ and*

$$||f_n||_{L^1(\Omega\times(a,b))} + \|g_n\|_{L^2(\Omega\times(a,b),\mathbb{R}^N)} + ||h_n||_{L^2(a,b,H_0^1(\Omega))} + \mu_{n,s}(\Omega \times (a,b)) \le 2\mu(\Omega \times (a,b)).$$

Proposition 3.8 *Let $\mu = \mu_0 + \mu_s, \mu_n = \mu_{n,0} + \mu_{n,s} \in \mathfrak{M}_b^+(\Omega \times (a,b))$ with $\mu_0, \mu_{n,0} \in \mathfrak{M}_0(\Omega \times (a,b)) \cap \mathfrak{M}_b^+(\Omega \times (a,b))$ and $\mu_{n,s}, \mu_s \in \mathfrak{M}_s^+(\Omega \times (a,b))$ such that $\{\mu_n\}$ nondecreasingly converges to μ in $\mathfrak{M}_b(\Omega \times (a,b))$. Then, $\{\mu_{n,s}\}$ is nondecreasing and converging to μ_s in $\mathfrak{M}_b(\Omega \times (a,b))$ and there exist decompositions (f,g,h) of μ_0, (f_n, g_n, h_n) of $\mu_{n,0}$ such that $\{f_n\}, \{g_n\}, \{h_n\}$ strongly converge to f, g, h in $L^1(\Omega \times (a,b)), L^2(\Omega \times (a,b), \mathbb{R}^N)$ and $L^2(a,b,H_0^1(\Omega))$ respectively satisfying*

$$||f_n||_{L^1(\Omega\times(a,b))} + \|g_n\|_{L^2(\Omega\times(a,b),\mathbb{R}^N)} + ||h_n||_{L^2(a,b,H_0^1(\Omega))} + \mu_{n,s}(\Omega \times (a,b)) \le 2\mu(\Omega \times (a,b)).$$

Remark 3.9 *For* $0 < \rho \leq \frac{1}{3}\min\{\sup_{x\in\Omega} d(x,\partial\Omega),(b-a)^{1/2}\}$*, set*

$$\Omega_\rho^j = \{x \in \Omega : d(x,\partial\Omega) > j\rho\} \times (a + (j\rho)^2, a + ((b-a)^{1/2} - j\rho)^2) \ \text{ for } \ j = 0, ..., k_\rho,$$

where $k_\rho = \left[\frac{\min\{\sup_{x\in\Omega} d(x,\partial\Omega),(b-a)^{1/2}\}}{2\rho}\right]$.
We can choose f_n, g_n, h_n *in above two Propositions such that for any* $j = 1, ..., k_\rho$,

$$||f_n||_{L^1(\Omega_\rho^j)} + \|g_n\|_{L^2(\Omega_\rho^j,\mathbb{R}^N)} + |||h_n| + |\nabla h_n|||_{L^2(\Omega_\rho^j)} \leq 2\mu(\Omega_\rho^{j-1}) \ \forall n \in \mathbb{N} \tag{3.9}$$

In fact, set $\mu_j = \chi_{\Omega_\rho^{k_\rho-j}\backslash\Omega_\rho^{k_\rho-j+1}}\mu$ *if* $j = 1, ..., k_\rho - 1$, $\mu_j = \chi_{\Omega\times(a,b)\backslash\Omega_\rho^1}\mu$ *if* $j = k_\rho$ *and* $\mu_j = \chi_{\Omega_\rho^{k_\rho}}\mu$ *if* $j = 0$*. From the proof of above two Propositions in [13], for any* $\varepsilon > 0$ *we can assume supports of* f_n, g_n, h_n *containing in* $supp(\mu) + \tilde{Q}_\varepsilon(0,0)$*. Thus, for any* $\mu = \mu_j$ *we have* f_n^j, g_n^j, h_n^j *correspondingly such that their supports contain in* $\Omega_{\rho,T}^{k_\rho-j-1/2}\backslash\Omega_{\rho,T}^{k_\rho-j+3/2}$ *if* $j = 1, ..., k_\rho - 1$ *and* $\Omega_T\backslash\Omega_{\rho,T}^{3/2}$ *if* $j = k_\rho$ *and* $\Omega_{\rho,T}^{k_\rho-1/2}$ *if* $j = 0$*. By* $\mu = \sum_{j=0}^{k_\rho}\mu_j$*, thus it is allowed to choose* $f_n = \sum_{j=0}^{k_\rho} f_n^j$, $f_n = \sum_{j=0}^{k_\rho} g_n^j$ *and* $h_n = \sum_{j=0}^{k_\rho} h_n^j$ *and* (3.9) *satisfies since*

$$\begin{aligned}
&||f_n||_{L^1(\Omega_\rho^j)} + \|g_n\|_{L^2(\Omega_\rho^j,\mathbb{R}^N)} + |||h_n| + |\nabla h_n|||_{L^2(\Omega_\rho^j)} \\
&\leq \sum_{i=0}^{k_\rho}\left(||f_n^i||_{L^1(\Omega_\rho^j)} + \|g_n^i\|_{L^2(\Omega_\rho^j,\mathbb{R}^N)} + |||h_n^i| + |\nabla h_n^i|||_{L^2(\Omega_\rho^j)}\right) \\
&= \sum_{i=0}^{k_\rho-j+1}\left(||f_n^i||_{L^1(\Omega_\rho^j)} + \|g_n^i\|_{L^2(\Omega_\rho^j,\mathbb{R}^N)} + |||h_n^i| + |\nabla h_n^i|||_{L^2(\Omega_\rho^j)}\right) \\
&\leq \sum_{i=j-1}^{k_\rho-j+1} 2\mu_j(\Omega \times (a,b)) = 2\mu(\Omega_\rho^{j-1}).
\end{aligned}$$

Definition 3.10 *Let* $\mu \in \mathfrak{M}_b(\Omega \times (a,b))$ *and* $\sigma \in \mathfrak{M}_b(\Omega)$*. A measurable function* u *is a distribution solution to problem* (3.2) *if* $u \in L^s(a,b,W_0^{1,s}(\Omega))$ *for any* $s \in \left[1, \frac{N+2}{N+1}\right)$ *and* $B(u,\nabla u) \in L^1(\Omega \times (a,b))$ *such that*

$$\begin{aligned}
&-\int_{\Omega\times(a,b)} u\varphi_t dxdt + \int_{\Omega\times(a,b)} A(x,t,\nabla u)\nabla\varphi dxdt \\
&\quad = \int_{\Omega\times(a,b)} B(u,\nabla u)\varphi dxdt + \int_{\Omega\times(a,b)} \varphi d\mu + \int_\Omega \varphi(a)d\sigma
\end{aligned}$$

for every $\varphi \in C_c^1(\Omega \times [a,b))$.

Remark 3.11 *Let* $\sigma' \in \mathfrak{M}_b(\Omega)$ *and* $a' \in (a,b)$*, set* $\omega = \mu + \sigma' \otimes \delta_{\{t=a'\}}$*. If* u *is a distribution solution to problem* (3.2) *with data* ω *and* $\sigma = 0$ *such that* $supp(\mu) \subset \overline{\Omega} \times [a',b]$*, and* $u = 0, B(u,\nabla u) = 0$ *in* $\Omega \times (a,a')$*, then* $\tilde{u} := u|_{\Omega\times[a',b)}$ *is a distribution solution to problem* (3.2) *in* $\Omega \times (a',b)$ *with data* μ *and* σ'*. Indeed, for any* $\varphi \in C_c^1(\Omega \times [a',b))$ *we defined*

$$\tilde{\varphi}(x,t) = \begin{cases} \varphi(x,t) \ \text{if } (x,t) \in \Omega \times [a',b), \\ (1+\varepsilon_0)(t-a')\varphi_t(x,a') + \varphi(x,(1+\varepsilon_0)a' - \varepsilon_0 t) \ \text{if } (x,t) \in \Omega \times [a,a'), \end{cases}$$

where $\varepsilon_0 \in \left(0, \frac{b-a'}{a'-a}\right)$.
Clearly, $\tilde{\varphi} \in C_c^1(\Omega \times [a,b))$*, thus we have*

$$\begin{aligned}
&-\int_{\Omega\times(a,b)} u\tilde{\varphi}_t dxdt + \int_{\Omega\times(a,b)} A(x,t,\nabla u)\nabla\tilde{\varphi} dxdt \\
&\quad = \int_{\Omega\times(a,b)} B(u,\nabla u)\tilde{\varphi} dxdt + \int_{\Omega\times(a,b)} \tilde{\varphi} d\omega,
\end{aligned}$$

which implies

$$-\int_{\Omega\times(a',b)}\tilde{u}\varphi_t dxdt+\int_{\Omega\times(a',b)}A(x,t,\nabla\tilde{u})\nabla\varphi dxdt$$
$$=\int_{\Omega\times(a',b)}B(\tilde{u},\nabla\tilde{u})\varphi dxdt+\int_{\Omega\times(a',b)}\varphi d\mu+\int_{\Omega}\varphi(a')d\sigma'.$$

Definition 3.12 *Let $\mu\in\mathfrak{M}(\mathbb{R}^N\times[a,+\infty))$, for $a\in\mathbb{R}$ and $\sigma\in\mathfrak{M}(\mathbb{R}^N)$. A measurable function u is a distribution solution to problem*

$$\begin{cases} u_t-div\,(A(x,t,\nabla u))=B(u,\nabla u)+\mu \text{ in } \mathbb{R}^N\times(a,+\infty)\\ u(a)=\sigma \quad \text{ in } \mathbb{R}^N\end{cases}\tag{3.10}$$

if $u\in L^s_{loc}(a,\infty,W^{1,s}_{loc}(\mathbb{R}^N))$ for any $s\in\left[1,\frac{N+2}{N+1}\right)$ and $B(u,\nabla u)\in L^1_{loc}(\mathbb{R}^N\times[a,\infty))$ such that

$$-\int_{\mathbb{R}^N\times(a,\infty)}u\varphi_t dxdt+\int_{\mathbb{R}^N\times(a,\infty)}A(x,t,\nabla u)\nabla\varphi dxdt$$
$$=\int_{\mathbb{R}^N\times(a,\infty)}B(u,\nabla u)\varphi dxdt+\int_{\mathbb{R}^N\times(a,\infty)}\varphi d\mu+\int_{\mathbb{R}^N}\varphi(a)d\sigma$$

for every $\varphi\in C^1_c(\mathbb{R}^N\times[a,\infty))$.

Definition 3.13 *Let $\omega\in\mathfrak{M}(\mathbb{R}^{N+1})$. A measurable function u is a distribution solution to problem*

$$u_t-div\,(A(x,t,\nabla u))=B(u,\nabla u)+\omega \text{ in } \mathbb{R}^{N+1}\tag{3.11}$$

if $u\in L^s_{loc}(\mathbb{R};W^{1,s}_{loc}(\mathbb{R}^N))$ for any $s\in\left[1,\frac{N+2}{N+1}\right)$ and $B(u,\nabla u)\in L^1_{loc}(\mathbb{R}^{N+1})$ such that

$$-\int_{\mathbb{R}^{N+1}}u\varphi_t dxdt+\int_{\mathbb{R}^{N+1}}A(x,t,\nabla u)\nabla\varphi dxdt=\int_{\mathbb{R}^{N+1}}B(u,\nabla u)\varphi dxdt+\int_{\mathbb{R}^{N+1}}\varphi d\omega,$$

for every $\varphi\in C^1_c(\mathbb{R}^{N+1})$.

Remark 3.14 *Let $\mu\in\mathfrak{M}(\mathbb{R}^N\times[a,+\infty))$, for $a\in\mathbb{R}$ and $\sigma\in\mathfrak{M}(\mathbb{R}^N)$. If u is a distribution solution to problem* (3.11) *with data $\omega=\mu+\sigma\otimes\delta_{\{t=a\}}$ such that $u=0, B(u,\nabla u)=0$ in $\mathbb{R}^N\times(-\infty,a)$, then $\tilde{u}:=u|_{\mathbb{R}^N\times[a,\infty)}$ is a distribution solution to problem* (3.10) *in $\mathbb{R}^N\times(a,\infty)$ with data μ and σ, see Remark 3.11.*

To prove the existence distribution solution of problem (3.10) we need the following results. First, we have local estimates of the renormalized solution which get from [13, Proposition 2.8].

Proposition 3.15 *Let u,v be in Definition 3.1. There exists $C=C(\Lambda_1,\Lambda_2)>0$ such that for $k\geq 1$ and $0\leq\eta\in C^\infty_c(\Omega\times(a,b))$*

$$\int_{|v|\leq k}\eta|\nabla u|^2dxdt+\int_{|v|\leq k}\eta|\nabla v|^2dxdt\leq CkA\tag{3.12}$$

where

$$A=||v\eta_t||_{L^1(\Omega\times(a,b))}+|||\nabla u||\nabla\eta|||_{L^1(\Omega\times(a,b))}+||\eta f||_{L^1(\Omega\times(a,b))}+||\eta|g|^2||_{L^1(\Omega\times(a,b))}$$
$$+|||\nabla\eta||g|||_{L^1(\Omega\times(a,b))}+||\eta|\nabla h|^2||_{L^1(\Omega\times(a,b))}+\int_{\Omega\times(a,b)}\eta d|\mu_s|.$$

For our purpose, we recall the Landes-time approximation of functions w belonging to $L^2(a,b,H_0^1(\Omega))$, introduced in [45], used in [24, 17, 8]. For $\nu>0$ we define

$$\langle w\rangle_\nu(x,t)=\nu\int_a^{\min\{t,b\}} w(x,s)e^{\nu(s-t)}ds \quad \text{for all} \quad (x,t)\in\Omega\times(a,b).$$

We have that $\langle w\rangle_\nu$ converges to w strongly in $L^2(a,b,H_0^1(\Omega))$ and $||\langle w\rangle_\nu||_{L^q(\Omega\times(a,b))}\leq ||w||_{L^q(\Omega\times(a,b))}$ for every $q\in[1,\infty]$. Moreover,

$$(\langle w\rangle_\nu)_t=\nu\left(w-\langle w\rangle_\nu\right) \quad \text{in the sense of distributions}$$

if $w\in L^\infty(\Omega\times(a,b))$ then

$$\int_{\Omega\times(a,b)}(\langle w\rangle_\nu)_t\varphi dxdt=\nu\int_{\Omega\times(a,b)}\left(w-\langle w\rangle_\nu\right)\varphi dxdt \quad \text{for all} \quad \varphi\in L^2(a,b,H_0^1(\Omega)).$$

Proposition 3.16 *Let $q_0>1$ and $0<\alpha<1/2$ such that $q_0>\alpha+1$. Let $L:\mathbb{R}\to\mathbb{R}$ be continuous and nondecreasing such that $L(0)=0$. If u is a solution of*

$$\begin{cases} u_t-div(A(x,t,\nabla u))+L(u)=\mu \ in\ \Omega\times(a,b),\\ u=0 \qquad on\ \ \partial\Omega\times(a,b),\\ u(a)=0 \quad in\ \ \Omega,\end{cases} \tag{3.13}$$

with $\mu\in C_c^\infty(\Omega\times(a,b))$ there exists $C_1>0$ depending on $\Lambda_1,\Lambda_2,\alpha,q_0$ such that for $0\leq\eta\in C_c^\infty(D)$ where $D=\Omega'\times(a',b')$, $\Omega'\subset\subset\Omega$ and $a<a'<b'<b$, then

$$\begin{aligned}&\frac{1}{k}\int_D|\nabla T_k(u)|^2\eta dxdt\\ &\qquad+\int_D\frac{|\nabla u|^2}{(|u|+1)^{\alpha+1}}\eta dxdt+|||\nabla u||\nabla\eta|||_{L^1(D)}+||L(u)\eta||_{L^1(D)}\leq C_1B,\end{aligned} \tag{3.14}$$

where $q_1=\frac{q_0-\alpha-1}{2q_0}$,

$$B=||\eta_t(|u|+1)||_{L^1(D)}+\int_D(|u|+1)^{q_0}\eta dxdt+\int_D|\nabla\eta^{1/q_1}|^{q_1}dxdt+\int_D\eta d|\mu|.$$

Furthermore, for $T_k(w)\in L^2(a',b',H_0^1(\Omega'))$, the Landes-time approximation $\langle T_k(w)\rangle_\nu$ of the truncate function $T_k(w)$ in D then for any $\varepsilon\in(0,1)$ and $\nu>0$

$$\begin{aligned}&\nu\int_D\eta\left(T_k(w)-\langle T_k(w)\rangle_\nu\right)T_\varepsilon(T_k(u)-\langle T_k(w)\rangle_\nu)dxdt\\ &\qquad+\int_D\eta A(x,t,\nabla T_k(u))\nabla T_\varepsilon(T_k(u)-\langle T_k(w)\rangle_\nu)dxdt\leq C_2\varepsilon(1+k)B,\end{aligned} \tag{3.15}$$

for some $C_2=C_2(\Lambda_1,\Lambda_2,\alpha,q_0)$.

Proposition 3.17 *Let $q_0>1$, $\mu_n=\mu_{n,0}+\mu_{n,s}\in\mathfrak{M}_b(B_n(0)\times(-n^2,n^2))$. Let u_n be a renormalized solution of*

$$\begin{cases}(u_n)_t-\operatorname{div}(A(x,t,\nabla u_n))=\mu_n \ \ in\ B_n(0)\times(-n^2,n^2),\\ u_n=0 \qquad on\ \ \partial B_n(0)\times(-n^2,n^2),\\ u_n(-n^2)=0\ \ in\ B_n(0),\end{cases} \tag{3.16}$$

relative to the decomposition (f_n,g_n,h_n) of $\mu_{n,0}$ satisfying (3.15) in Proposition 3.16 with $L\equiv 0$. Assume that for any $m\in\mathbb{N}$ and $\alpha\in(0,1/2)$, $D_m:=B_m(0)\times(-m^2,m^2)$

$$\begin{aligned}&\frac{1}{k}|||\nabla T_k(u)|^2||_{L^1(D_m)}+|||\nabla u|^2(|u|+1)^{-\alpha-1}||_{L^1(D_m)}+|||\nabla u|||_{L^1(D_m)}+|\mu_n|(D_m)\\ &\quad+||f_n||_{L^1(D_m)}+||g_n||_{L^2(D_m,\mathbb{R}^N)}+|||h_n|+|\nabla h_n|||_{L^2(D_m)}+||u_n||_{L^{q_0}(D_m)}\leq C(m,\alpha)\end{aligned}$$

for all $n \geq m$ and h_n is convergent in $L^1_{loc}(\mathbb{R}^{N+1})$. Then, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$ such that u_n converges to u a.e in $\mathbb{R}^{N+1}$ and in $L^s_{loc}(\mathbb{R}; W^{1,s}_{loc}(\mathbb{R}^N))$ for any $s \in [1, \frac{N+2}{N+1})$.

Proofs of above two Propositions are given in the Appendix section. The following result is as a consequence of Proposition 3.17.

Corollary 3.18 *Let $\mu_n \in L^1(B_n(0) \times (-n^2, n^2))$. Let u_n be a unique renormalized solution of problem 3.16. Assume that for any $m \in \mathbb{N}$,*

$$\sup_{n \geq m} |\mu_n|(B_m(0) \times (-m^2, m^2)) < \infty \quad \text{and} \quad \sup_{n \geq m} \int_{B_m(0)\times(-m^2,m^2)} |u_n|^{q_0} dxdt < \infty.$$

then there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$ such that u_n converges to u a.e in $\mathbb{R}^{N+1}$ and in $L^s_{loc}(\mathbb{R}; W^{1,s}_{loc}(\mathbb{R}^N))$ for any $s \in [1, \frac{N+2}{N+1})$.

Finally, we would like to present a technical lemma which will be used several times in the paper, specially in the proof of Theorem 2.17, 2.19 and 2.20. It is a consequence of Vitali Covering Lemma, a proof of lemma can be seen in [22, 21, 54].

Lemma 3.19 *Let Ω be a (R_0, δ)- Reifenberg flat domain with $\delta < 1/4$ and let w be an A_∞ weight. Suppose that the sequence of balls $\{B_r(y_i)\}_{i=1}^L$ with centers $y_i \in \overline{\Omega}$ and a common radius $r \leq R_0/4$ covers Ω. Set $s_i = T - ir^2/2$ for all $i = 0, 1, ..., [\frac{2T}{r^2}]$. Let $E \subset F \subset \Omega_T$ be measurable sets for which there exists $0 < \varepsilon < 1$ such that $w(E) < \varepsilon w(\tilde{Q}_r(y_i, s_j))$ for all $i = 1, ..., L$, $j = 0, 1, ..., [\frac{2T}{r^2}]$; and for all $(x, t) \in \Omega_T$, $\rho \in (0, 2r]$, we have $\tilde{Q}_\rho(x, t) \cap \Omega_T \subset F$ if $w(E \cap \tilde{Q}_\rho(x, t)) \geq \varepsilon w(\tilde{Q}_\rho(x, t))$. Then $w(E) \leq B\varepsilon w(F)$ for a constant B depending only on N and $[w]_{A_\infty}$.*

Clearly, the Lemma contains the following two Lemmas

Lemma 3.20 *Let $0 < \varepsilon < 1, R > 0$ and cylinder $\tilde{Q}_R := \tilde{Q}_R(x_0, t_0)$ for some $(x_0, t_0) \in \mathbb{R}^{N+1}$ and $w \in A_\infty$. let $E \subset F \subset \tilde{Q}_R$ be two measurable sets in $\mathbb{R}^{N+1}$ with $w(E) < \varepsilon w(\tilde{Q}_R)$ and satisfying the following property: for all $(x, t) \in \tilde{Q}_R$ and $r \in (0, R]$, we have $\tilde{Q}_r(x, t) \cap \tilde{Q}_R \subset F$ provided $w(E \cap \tilde{Q}_r(x, t)) \geq \varepsilon w(\tilde{Q}_r(x, t))$. Then $w(E) \leq B\varepsilon w(F)$ for some $B = B(N, [w]_{A_\infty})$.*

Lemma 3.21 *Let $0 < \varepsilon < 1$ and $R > R' > 0$ and let $E \subset F \subset Q = B_R(x_0) \times (a, b)$ be two measurable sets in $\mathbb{R}^{N+1}$ with $|E| < \varepsilon |\tilde{Q}_{R'}|$ and satisfying the following property: for all $(x, t) \in Q$ and $r \in (0, R']$, we have $Q_r(x, t) \cap Q \subset F$ if $|E \cap \tilde{Q}_r(x, t)| \geq \varepsilon |\tilde{Q}_r(x, t)|$. Then $|E| \leq B\varepsilon |F|$ for a constant B depending only on N.*

4 Estimates on Potential

In this section, we will develop nonlinear potential theory corresponding to quasilinear parabolic equations.

First we introduce the Wolff parabolic potential of $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ by

$$\mathbb{W}^R_{\alpha,p}[\mu](x,t) = \int_0^R \left(\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha p}} \right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} \quad \text{for any} \quad (x,t) \in \mathbb{R}^{N+1},$$

where $\alpha > 0, 1 < p < \alpha^{-1}(N + 2)$ and $0 < R \leq \infty$. For convenience, $\mathbb{W}_{\alpha,p}[\mu] := \mathbb{W}^\infty_{\alpha,p}[\mu]$.

The following result is an extension of [36, Theorem 1.1], [16, Proposition 2.2] to Parabolic potential.

Theorem 4.1 *Let $\alpha > 0$, $1 < p < \alpha^{-1}(N+2)$ and $w \in A_\infty$, $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$. There exist constants $C_1, C_2 > 0$ and $\varepsilon_0 \in (0,1)$ depending on $N, \alpha, p, [w]_{A_\infty}$ such that for any $\lambda > 0$ and $\varepsilon \in (0, \varepsilon_0)$*

$$w(\{\mathbb{W}^R_{\alpha,p}[\mu] > a\lambda, (\mathbb{M}^R_{\alpha p}[\mu])^{\frac{1}{p-1}} \le \varepsilon\lambda\}) \le C_1 \exp(-C_2\varepsilon^{-1}) w(\{\mathbb{W}^R_{\alpha,p}[\mu] > \lambda\}) \tag{4.1}$$

where $a = 2 + 3^{\frac{N+2-\alpha p}{p-1}}$.

Proof of Theorem 4.1. We only consider case $R < \infty$. Let $\{\tilde{Q}_R(x_j,t_j)\}$ be a cover of $\mathbb{R}^{N+1}$ such that $\sum_j \chi_{\tilde{Q}_R(x_j,t_j)} \le M$ in $\mathbb{R}^{N+1}$ for some constant $M = M(N) > 0$. It is enough to show that there exist constants $c_1, c_2 > 0$ and $\varepsilon_0 \in (0,1)$ depending on $N, \alpha, p, [w]_{A_\infty}$ such that for any $Q \in \{\tilde{Q}_R(x_j,t_j)\}$, $\lambda > 0$ and $\varepsilon \in (0, \varepsilon_0)$

$$w(Q \cap \{\mathbb{W}^R_{\alpha,p}[\mu] > a\lambda, (\mathbb{M}^R_{\alpha p}[\mu])^{\frac{1}{p-1}} \le \varepsilon\lambda\}) \le c_1 \exp(-c_2\varepsilon^{-1}) w(Q \cap \{\mathbb{W}^R_{\alpha,p}[\mu] > \lambda\}). \tag{4.2}$$

Fix $\lambda > 0$ and $0 < \varepsilon < 1/10$. We set

$$E = Q \cap \{\mathbb{W}^R_{\alpha,p}[\mu] > a\lambda, (\mathbb{M}^R_{\alpha p}[\mu])^{\frac{1}{p-1}} \le \varepsilon\lambda\} \quad \text{and} \quad F = Q \cap \{\mathbb{W}^R_{\alpha,p}[\mu] > \lambda\}.$$

Thanks to Lemma 3.20 we will get (4.2) if we verify the following two claims:

$$w(E) \le c_3 \exp(-c_4\varepsilon^{-1}) w(Q), \tag{4.3}$$

and for any $(x,t) \in Q$, $0 < r \le R$,

$$w(E \cap \tilde{Q}_r(x,t)) < c_5 \exp(-c_6\varepsilon^{-1}) w(\tilde{Q}_r(x,t)), \tag{4.4}$$

provided that $\tilde{Q}_r(x,t) \cap Q \cap F^c \ne \emptyset$ and $E \cap \tilde{Q}_r(x,t) \ne \emptyset$, where constants c_3, c_4, c_5 and c_6 depend on N, α, p and $[w]_{A_\infty}$.
Claim (4.3): Set

$$g_k(x,t) = \int_{2^{-k}R}^{2^{-k+1}R} \left(\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho}.$$

We have for $m \in \mathbb{N}$ and $(x,t) \in E$

$$\begin{aligned}
\mathbb{W}^R_{\alpha,p}[\mu](x,t) &= \sum_{k=m+1}^{\infty} g_k(x,t) + \int_{2^{-m}R}^{R} \left(\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} \\
&\le \sum_{k=m+1}^{\infty} g_k(x,t) + m(\mathbb{M}^R_{\alpha p}[\mu](x,t))^{\frac{1}{p-1}} \\
&\le \sum_{k=m+1}^{\infty} g_k(x,t) + m\varepsilon\lambda.
\end{aligned}$$

We deduce that for $\beta > 0$, $m \in \mathbb{N}$

$$\begin{aligned}
|E| &\le |Q \cap \{\sum_{k=m+1}^{\infty} g_k > (1-m\varepsilon)\lambda\}| \\
&= |Q \cap \{\sum_{k=m+1}^{\infty} g_k > \sum_{k=m+1}^{\infty} 2^{-\beta(k-m-1)}(1-2^{-\beta})(1-m\varepsilon)\lambda\}| \\
&\le \sum_{k=m+1}^{\infty} |Q \cap \{g_k > 2^{-\beta(k-m-1)}(1-2^{-\beta})(1-m\varepsilon)\lambda\}|.
\end{aligned}$$

We can assume that $(x_0,t_0) \in Q$, $(\mathbb{M}^R_{\alpha p}[\mu](x_0,t_0))^{\frac{1}{p-1}} \le \varepsilon\lambda$. Thus, by computing, see [16, Proof of Proposition 2.2] we have for any $k \in \mathbb{N}$

$$|Q \cap \{g_k > s\}| \le \frac{c_7}{s^{p-1}} 2^{-k\alpha p}|Q|(\varepsilon\lambda)^{p-1}.$$

Consequently,

$$\begin{aligned}|E| &\le \sum_{k=m+1}^{\infty} \frac{c_7}{\left(2^{-\beta(k-m-1)}(1-2^{-\beta})(1-m\varepsilon)\lambda\right)^{p-1}} 2^{-k\alpha p}|Q|(\varepsilon\lambda)^{p-1} \\ &\le c_7 2^{-(m+1)\alpha p}\left(\frac{\varepsilon}{1-m\varepsilon}\right)^{p-1} |Q| \left(1-2^{-\beta}\right)^{-p+1} \sum_{k=m+1}^{\infty} 2^{(\beta(p-1)-\alpha p)(k-m-1)}.\end{aligned}$$

If we choose $\varepsilon^{-1} - 2 < m \le \varepsilon^{-1} - 1$ and $\beta = \beta(\alpha,p)$ so that $\beta(p-1) - \alpha p < 0$, we obtain

$$|E| \le c_8 \exp(-\alpha p \ln(2)\varepsilon^{-1})|Q|.$$

Thus, we get (4.3).
Claim (4.4). Take $(x,t) \in Q$ and $0 < r \le R$. Now assume that $\tilde{Q}_r(x,t) \cap Q \cap F^c \ne \emptyset$ and $E \cap \tilde{Q}_r(x,t) \ne \emptyset$ i.e, there exist $(x_1,t_1),(x_2,t_2) \in \tilde{Q}_r(x,t) \cap Q$ such that $\mathbb{W}^R_{\alpha,p}[\mu](x_1,t_1) \le \lambda$ and $(\mathbb{M}^R_{\alpha p}[\mu](x_2,t_2))^{\frac{1}{p-1}} \le \varepsilon\lambda$. We need to prove that

$$w(E \cap \tilde{Q}_r(x,t)) < c_9 \exp(-c_{10}\varepsilon^{-1}) w(\tilde{Q}_r(x,t)).$$

To do this, for all $(y,s) \in E \cap \tilde{Q}_r(x,t)$. $\tilde{Q}_\rho(y,s) \subset \tilde{Q}_{3\rho}(x_1,t_1)$ if $\rho > r$.
If $r \le R/3$,

$$\begin{aligned}\mathbb{W}^R_{\alpha,p}[\mu](y,s) &= \mathbb{W}^r_{\alpha,p}[\mu](y,s) + \int_r^{R/3} \left(\frac{\mu(\tilde{Q}_\rho(y,s))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} + \int_{R/3}^{R} \left(\frac{\mu(\tilde{Q}_\rho(y,s))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} \\ &\le \mathbb{W}^r_{\alpha,p}[\mu](y,s) + \int_r^{R/3} \left(\frac{\mu(\tilde{Q}_{3\rho}(x_1,t_1))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} + 2(\mathbb{M}^R_{\alpha p}[\mu](y,s))^{\frac{1}{p-1}} \\ &\le \mathbb{W}^r_{\alpha,p}[\mu](y,s) + 3^{\frac{N+2-\alpha p}{p-1}}\lambda + 2\varepsilon\lambda.\end{aligned}$$

which follows $\mathbb{W}^r_{\alpha,p}[\mu](y,s) > \lambda$.
If $r \ge R/3$

$$\begin{aligned}\mathbb{W}^R_{\alpha,p}[\mu](y,s) &\le \mathbb{W}^r_{\alpha,p}[\mu](y,s) + \int_{R/3}^{R} \left(\frac{\mu(\tilde{Q}_\rho(y,s))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} \\ &\le \mathbb{W}^r_{\alpha,p}[\mu](y,s) + 2\varepsilon\lambda,\end{aligned}$$

which follows $\mathbb{W}^r_{\alpha,p}[\mu](y,s) > \lambda$.
Thus,

$$w(E \cap \tilde{Q}_r(x,t)) \le w(\tilde{Q}_r(x,t) \cap \{\mathbb{W}^r_{\alpha,p}[\mu] > \lambda\}).$$

Since $(x_2,t_2) \in \tilde{Q}_r(x,t)$, $(\mathbb{M}^R_{\alpha p}[\mu](x_2,t_2))^{\frac{1}{p-1}} \le \varepsilon\lambda$, so as above we also obtain

$$w(\tilde{Q}_r(x,t) \cap \{\mathbb{W}^r_{\alpha,p}[\mu] > \lambda\}) \le c_9 \exp(-c_{10}\varepsilon^{-1}) w(\tilde{Q}_r(x,t)),$$

which implies (4.4). This completes the proof of the Theorem. ∎

Theorem 4.2 *Let $\alpha > 0$, $1 < p < \alpha^{-1}(N+2)$, $p-1 < q < \infty$ and $0 < s \le \infty$ and $w \in A_\infty$. There holds*

$$C^{-1}||(\mathbb{M}^R_{\alpha p}[\mu])^{\frac{1}{p-1}}||_{L^{q,s}(\mathbb{R}^{N+1},dw)} \le ||\mathbb{W}^R_{\alpha,p}[\mu]||_{L^{q,s}(\mathbb{R}^{N+1},dw)} \le C||(\mathbb{M}^R_{\alpha p}[\mu])^{\frac{1}{p-1}}||_{L^{q,s}(\mathbb{R}^{N+1},dw)}, \tag{4.5}$$

for all $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ and $R \in (0,\infty]$ where C is a positive constant only depending on N, α, p, q, s and $[w]_{A_\infty}$.

Proof. From (4.1) in Theorem (4.1), we have for $0 < s < \infty$

$$\begin{aligned}
||\mathbb{W}^R_{\alpha,p}[\mu]||^s_{L^{q,s}(\mathbb{R}^{N+1},dw)} &= a^s q \int_0^\infty \lambda^s w(\{\mathbb{W}^R_{\alpha,p}[\mu] > a\lambda\})^{\frac{s}{q}} \frac{d\lambda}{\lambda} \\
&\le c_1 \exp(-c_2\varepsilon^{-1}) q \int_0^\infty \lambda^s w(\{\mathbb{W}^R_{\alpha,p}[\mu] > \lambda\})^{\frac{s}{q}} \frac{d\lambda}{\lambda} + c_3 s \int_0^\infty \lambda^s w(\{(\mathbb{M}^R_{\alpha p}[\mu])^{\frac{1}{p-1}} > \varepsilon\lambda\})^{\frac{s}{q}} \frac{d\lambda}{\lambda} \\
&= c_1 \exp(-c_2\varepsilon^{-1}) ||\mathbb{W}^R_{\alpha,p}[\mu]||^s_{L^{q,s}(\mathbb{R}^{N+1},dw)} + c_3\varepsilon^{-s}||(\mathbb{M}^R_{\alpha p}[\mu])^{\frac{1}{p-1}}||^s_{L^{q,s}(\mathbb{R}^{N+1},dw)}.
\end{aligned}$$

Choose $0 < \varepsilon < \varepsilon_0$ such that $c_1 \exp(-c_2\varepsilon^{-1}) < 1/2$ we get

$$||\mathbb{W}^R_{\alpha,p}[\mu]||^s_{L^{q,s}(\mathbb{R}^{N+1},dw)} \le c_4||(\mathbb{M}^R_{\alpha p}[\mu])^{\frac{1}{p-1}}||^s_{L^{q,s}(\mathbb{R}^{N+1},dw)}.$$

Similarly, we also get above inequality in case $s = \infty$. So, we proved the right-hand side inequality of (4.5).
To complete the proof, we prove the left-hand side inequality of (4.5). Since for every $(x,t) \in \mathbb{R}^{N+1}$

$$(\mathbb{W}^R_{\alpha p}[\mu](x,t))^{\frac{1}{p-1}} \le c_5 \left(\mathbb{W}^R_{\alpha,p}[\mu](x,t) + \left(\frac{\mu(\tilde{Q}_{2R}(x,t))}{R^{N+2-\alpha p}} \right)^{\frac{1}{p-1}} \right) \quad \text{and}$$

$$\left(\frac{\mu(\tilde{Q}_{R/2}(x,t))}{R^{N+2-\alpha p}} \right)^{\frac{1}{p-1}} \le c_6 \mathbb{W}^R_{\alpha,p}[\mu](x,t),$$

thus it is enough to show that for any $\lambda > 0$

$$w\left(\left\{ (x,t) : \left(\frac{\mu(\tilde{Q}_{2R}(x,t))}{R^{N+2-\alpha p}} \right)^{\frac{1}{p-1}} > \lambda \right\} \right) \le c_7 w \left(\left\{ (x,t) : \left(\frac{\mu(\tilde{Q}_{R/2}(x,t))}{R^{N+2-\alpha p}} \right)^{\frac{1}{p-1}} > c_8\lambda \right\} \right). \tag{4.6}$$

Let $\{Q_j\} = \{\tilde{Q}_{R/4}(x_j,t_j)\}$ be a cover of $\mathbb{R}^{N+1}$ such that for any $Q_j \in \{Q_j\}$, there exist $Q_{j,1}, ..., Q_{j,M_1} \in \{Q_j\}$ with $\sum_j \sum_{k=1}^{M_1} \chi_{Q_{j,k}} \le M_2$ and $Q_j + \tilde{Q}_{2R}(0,0) \subset \bigcup_{k=1}^{M_1} Q_{j,k}$ for some integer constants $M_i = M_i(N), i = 1,2$. Then,

$$\begin{aligned}
w\left(\left\{ (x,t) : \left(\frac{\mu(\tilde{Q}_{2R}(x,t))}{R^{N+2-\alpha p}} \right)^{\frac{1}{p-1}} > \lambda \right\} \right) &\le \sum_j w \left(\left\{ (x,t) : \left(\frac{\mu(\tilde{Q}_{2R}(x,t))}{R^{N+2-\alpha p}} \right)^{\frac{1}{p-1}} > \lambda \right\} \cap Q_j \right) \\
&\le \sum_j w \left(\left\{ (x,t) : \sum_{k=1}^{M_1} \frac{\mu(Q_{j,k})}{R^{N+2-\alpha p}} > \lambda^{p-1} \right\} \cap Q_j \right) \\
&\le \sum_j \sum_{k=1}^{M_1} w \left(\left\{ (x,t) : \left(\frac{\mu(Q_{j,k})}{R^{N+2-\alpha p}} \right)^{\frac{1}{p-1}} > M_1^{-1/(p-1)}\lambda \right\} \cap Q_j \right) \\
&= \sum_j \sum_{k=1}^{M_1} a_{j,k} w(Q_j),
\end{aligned}$$

where $a_{j,k}=1$ if $\left(\frac{\mu(Q_{j,k})}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} > M_1^{-1/(p-1)}\lambda$ and $a_{j,k}=0$ if otherwise.
Using the strong doubling property of w, there is $c_9=c_9(N,[w]_{A_\infty})$ such that $w(Q_j)\leq c_9 w(Q_{j,k})$. On the other hand, if $a_{j,k}=1$ then $Q_{j,k}\subset\left\{(x,t):\left(\frac{\mu(\tilde{Q}_{R/2}(x,t))}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} > M_1^{-1/(p-1)}\lambda\right\}$.
Therefore,

$$w\left(\left\{(x,t):\left(\frac{\mu(\tilde{Q}_{2R}(x,t))}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}>\lambda\right\}\right)\leq\sum_j\sum_{k=1}^{M_1}c_9 a_{j,k}w(Q_{j,k})$$
$$\leq\sum_j\sum_{k=1}^{M_1}c_9 w\left(\left\{(x,t):\left(\frac{\mu(\tilde{Q}_{R/2}(x,t))}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}>M_1^{-1/(p-1)}\lambda\right\}\cap Q_{j,k}\right),$$

which implies (4.6) since $\sum_j\sum_{k=1}^{M_1}\chi_{Q_{j,k}}\leq M_2$ in $\mathbb{R}^{N+1}$. ∎

Theorem 4.3 *Let $0<\alpha p<N+2$ and $w\in A_\infty$ There exist $C_1,C_2>0$ depending on N,α,p and $[w]_{A_\infty}$ such that for any $\mu\in\mathfrak{M}^+(\mathbb{R}^{N+1})$, any cylinder $\tilde{Q}_\rho\subset\mathbb{R}^{N+1}$ there holds*

$$\frac{1}{w(\tilde{Q}_{2\rho})}\int_{\tilde{Q}_{2\rho}}\exp\left(C_1\mathbb{W}^R_{\alpha,p}[\mu_{\tilde{Q}_\rho}](x,t)\right)dw(x,t)\leq C_2 \tag{4.7}$$

provided $||\mathbb{M}^R_{\alpha p}[\mu_{\tilde{Q}_\rho}]||_{L^\infty(\tilde{Q}_\rho)}\leq 1$, where $\mu_{\tilde{Q}_\rho}=\chi_{\tilde{Q}_\rho}\mu$.

Proof. Assume that $||\mathbb{M}^R_{\alpha p}[\mu_{\tilde{Q}_\rho}]||_{L^\infty(\tilde{Q}_\rho)}\leq 1$. We apply Theorem (4.1) to $\mu_{\tilde{Q}_\rho}$. Then, choose $\varepsilon=\lambda^{-1}$ for all $\lambda\geq\lambda_0:=\max\{\varepsilon_0^{-1},\frac{N+2-\alpha p}{p-1}\}$, we obtain

$$w(\{\mathbb{W}^R_{\alpha,p}[\mu_{\tilde{Q}_\rho}]>a\lambda\}\cap\tilde{Q}_{2\rho})\leq C_1\exp(-C_2\varepsilon^{-1})w(\{\mathbb{W}^R_{\alpha,p}[\mu_{\tilde{Q}_\rho}]>\lambda\})\;\;\forall\,\lambda\geq\lambda_0,$$

On the other hand, if $\rho>R$, clearly we have $\mathbb{W}^R_{\alpha,p}[\mu_{\tilde{Q}_\rho}]\equiv 0$ in $\mathbb{R}^{N+1}\backslash\tilde{Q}_{2\rho}$, if $\rho\leq R$, for any $(x,t)\in\mathbb{R}^{N+1}\backslash\tilde{Q}_{2\rho}$

$$\mathbb{W}^R_{\alpha,p}[\mu_{\tilde{Q}_\rho}](x,t)=\int_\rho^R\left(\frac{\mu_{\tilde{Q}_\rho}(\tilde{Q}_r(x,t))}{r^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dr}{r}\leq\frac{N+2-\alpha p}{p-1}\left(\frac{\mu(\tilde{Q}_\rho)}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}\leq\lambda_0.$$

So, we get $\{\mathbb{W}^R_{\alpha,p}[\mu_{\tilde{Q}_\rho}]>\lambda\}\subset\tilde{Q}_{2\rho}$ for all $\lambda\geq\lambda_0$. This can be written under the form

$$w(\{\mathbb{W}^R_{\alpha,p}[\mu_{\tilde{Q}_\rho}]>a\lambda\}\cap\tilde{Q}_{2\rho})\leq\left(\chi_{(0,t_0]}+C_1\exp(-C_2\lambda)\right)w(\tilde{Q}_{2\rho}),$$

for all $\lambda>0$. Therefore, we get (4.7). ∎

In what follows, we need some estimates on Wolff parabolic potential:

Proposition 4.4 *Let $p>1, 0<\alpha p<N+2$ and $q>1,\alpha pq<N+2$. There exist C_1,C_2 such that*

$$||\mathbb{W}_{\alpha,p}[\mu]||_{L^{\frac{(N+2)(p-1)}{N+2-\alpha p},\infty}(\mathbb{R}^{N+1})}\leq C_1(\mu(\mathbb{R}^{N+1}))^{\frac{1}{p-1}}\;\;\forall\,\mu\in\mathfrak{M}^+_b(\mathbb{R}^{N+1}), \tag{4.8}$$

$$||\mathbb{W}_{\alpha,p}[\mu]||_{L^{\frac{q(N+2)(p-1)}{N+2-\alpha pq},\infty}(\mathbb{R}^{N+1})}\leq C_2||\mu||^{\frac{1}{p-1}}_{L^{q,\infty}(\mathbb{R}^{N+1})}\;\;\forall\,\mu\in L^{q,\infty}(\mathbb{R}^{N+1}),\mu\geq 0, \tag{4.9}$$

and

$$||\mathbb{W}_{\alpha,p}[\mu]||_{L^{\frac{q(N+2)(p-1)}{N+2-\alpha pq}}(\mathbb{R}^{N+1})}\leq C_2||\mu||^{\frac{1}{p-1}}_{L^{q}(\mathbb{R}^{N+1})}\;\;\forall\,\mu\in L^{q}(\mathbb{R}^{N+1}),\mu\geq 0. \tag{4.10}$$

In particular, for $s > \frac{(p-1)(N+2)}{N+2-\alpha p}$, *we define* $F(\mu) := (\mathbb{W}_{\alpha,p}[\mu])^s$ *for all* $\mu \in \mathfrak{M}_b^+(\mathbb{R}^{N+1})$. *Then,*

$$||F(\mu)||_{L^{\frac{(N+2)(s-p+1)}{\alpha sp}}(\mathbb{R}^{N+1})} \le C_3 ||\mu||^{\frac{s}{p-1}}_{L^{\frac{(N+2)(s-p+1)}{\alpha sp}}(\mathbb{R}^{N+1})} \quad \text{and}$$
$$||F(\mu)||_{L^{\frac{(N+2)(s-p+1)}{\alpha sp},\infty}(\mathbb{R}^{N+1})} \le C_3 ||\mu||^{\frac{s}{p-1}}_{L^{\frac{(N+2)(s-p+1)}{\alpha sp},\infty}(\mathbb{R}^{N+1})},$$

for some constant $C_i = C_i(N,p,\alpha,s)$ *for* $i = 3,4$.

Proof. Let $s \ge 1$ such that $\alpha sp < N+2$. It is known that if $\mu \in L^{s,\infty}(\mathbb{R}^{N+1})$ then

$$|\mu|(\tilde{Q}_\rho(x,t)) \le c_1 ||\mu||_{L^{s,\infty}(\mathbb{R}^{N+1})} \rho^{\frac{N+2}{s'}} \quad \forall \quad \rho > 0.$$

Thus for $\delta = ||\mu||^{\frac{s}{N+2}}_{L^{s,\infty}(\mathbb{R}^{N+1})} (\mathbb{M}(\mu)(x,t))^{-\frac{s}{N+2}}$ we have

$$\begin{aligned}
\mathbb{W}_{\alpha,p}[\mu](x,t) &= \int_0^\delta \left(\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} + \int_\delta^\infty \left(\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} \\
&\le c_2 (\mathbb{M}(\mu)(x,t))^{\frac{1}{p-1}} \delta^{\frac{\alpha p}{p-1}} + c_2 ||\mu||^{\frac{1}{p-1}}_{L^{s,\infty}(\mathbb{R}^{N+1})} \delta^{-\frac{N+2-\alpha sp}{s(p-1)}} \\
&= c_3 (\mathbb{M}(\mu)(x,t))^{\frac{N+2-\alpha sp}{(p-1)(N+2)}} ||\mu||^{\frac{\alpha sp}{(p-1)(N+2)}}_{L^{s,\infty}(\mathbb{R}^{N+1})}.
\end{aligned}$$

So, for any $\lambda > 0$

$$|\{\mathbb{W}_{\alpha,p}[\mu] > \lambda\}| \le |\{\mathbb{M}(\mu) > c_4 ||\mu||^{-\frac{\alpha sp}{N+2-\alpha sp}}_{L^{s,\infty}(\mathbb{R}^{N+1})} \lambda^{\frac{(p-1)(N+2)}{N+2-\alpha sp}}\}|.$$

Hence, since $\mathbb{M}$ is bounded from $\mathfrak{M}_b^+(\mathbb{R}^{N+1})$ to $L^{1,\infty}(\mathbb{R}^{N+1})$ and $L^q(\mathbb{R}^{N+1})$ ($L^{q,\infty}(\mathbb{R}^{N+1})$ resp.) to itself, we get the result. ■

Remark 4.5 *Assume that* $\alpha p = N+2$ *and* $R > 0$. *As above we also have for any* $\varepsilon > 0$

$$\mathbb{W}^R_{\alpha,p}[\mu](x,t) \le C_{1,\varepsilon} \max\left\{(|\mu|(\mathbb{R}^{N+1}))^{\frac{1}{p-1}}, \left((\mathbb{M}(\mu)(x,t))^\varepsilon (|\mu|(\mathbb{R}^{N+1}))^{\frac{\alpha p}{p-1}} R^{\varepsilon\alpha p}\right)^{\frac{1}{\alpha p+\varepsilon(p-1)}}\right\}$$

where $C_{1,\varepsilon} = C_1(N,\alpha,p,\varepsilon)$.
Therefore, for any $\lambda > C_\varepsilon(|\mu|(\mathbb{R}^{N+1}))^{\frac{1}{p-1}}$,

$$|\{\mathbb{W}^R_{\alpha,p}[\mu] > \lambda\}| \le C_{2,\varepsilon} \left(\frac{(|\mu|(\mathbb{R}^{N+1}))^{\frac{1}{p-1}}}{\lambda}\right)^{\frac{\alpha p+\varepsilon(p-1)}{\varepsilon}} R^{\alpha p}, \tag{4.11}$$

where $C_{2,\varepsilon} = C_2(N,\alpha,p,\varepsilon)$. *In particular, if* $\mu \in \mathfrak{M}_b^+(\mathbb{R}^{N+1})$ *then* $\mathbb{W}^R_{\alpha,p}[\mu] \in L^s_{loc}(\mathbb{R}^{N+1})$ *for all* $s > 0$.

Remark 4.6 *Assume that* $p, q > 1, 0 < \alpha pq < N+2$. *As in [59, Theorem 3], it is easy to prove that if* $w \in A_{\frac{q(N+2-\alpha)}{N+2-\alpha pq}}$, *i.e,* $0 < w \in L^1_{loc}(\mathbb{R}^{N+1})$ *and for any* $\tilde{Q}_\rho(y,s) \subset \mathbb{R}^{N+1}$

$$\sup_{\tilde{Q}_\rho(y,s)\subset\mathbb{R}^{N+1}} \left(\left(\fint_{\tilde{Q}_\rho(y,s)} w\,dxdt\right)\left(\fint_{\tilde{Q}_\rho(y,s)} w^{-\frac{N+2-\alpha pq}{(q-1)(N+2)}}dxdt\right)^{\frac{(q-1)(N+2)}{N+2-\alpha pq}}\right) = C_1 < \infty,$$

then

$$\left(\int_{\mathbb{R}^{N+1}} (\mathbb{M}_{\alpha p}[|f|])^{\frac{(N+2)q}{N+2-\alpha pq}} w\,dxdt\right)^{\frac{N+2-\alpha pq}{(N+2)q}} \le C_2 \left(\int_{\mathbb{R}^{N+1}} |f|^q w^{1-\frac{\alpha pq}{N+2}}dxdt\right)^{\frac{1}{q}},$$

for some a constant $C_2 = C_2(N, \alpha p, q, C_1)$.
Therefore, from (4.5) in Theorem 4.2 we get a weighted version of (4.10)

$$\left(\int_{\mathbb{R}^{N+1}} (\mathbb{W}_{\alpha,p}[|f|])^{\frac{(N+2)(p-1)q}{N+2-\alpha pq}} w dxdt\right)^{\frac{N+2-\alpha pq}{(N+2)q}} \le C_2 \left(\int_{\mathbb{R}^{N+1}} |f|^p w^{1-\frac{\alpha p}{N+2}} dxdt\right)^{\frac{1}{p}}.$$

The following another version of (4.10) in the Lorentz-Morrey spaces involving calorie.

Proposition 4.7 *Let $p, q > 1$, and $0 < \alpha pq < \theta \le N+2$. There exists a constant $C > 0$ such that*

$$|| (\mathbb{W}_{\alpha,p}[|\mu|])^{p-1} ||_{L^{\frac{\theta q}{\theta-\alpha pq};\theta}(\mathbb{R}^{N+1})} \le C||\mu||_{L^{q;\theta}(\mathbb{R}^{N+1})} \quad \forall \mu \in L^{q;\theta}(\mathbb{R}^{N+1}). \tag{4.12}$$

Proof. As the proof of Proposition 4.4 we have

$$\mathbb{W}_{\alpha,p}[|\mu|] \le c_1 \left(\mathbb{M}_{\theta/q}[|\mu|]\right)^{\frac{\alpha pq}{\theta(p-1)}} (\mathbb{M}[|\mu|])^{\frac{\theta-\alpha pq}{\theta(p-1)}}.$$

Since $\mathbb{M}_{\theta/q}[|\mu|] \le c_2 (\mathbb{M}_\theta[|\mu|^q])^{1/q}$, above inequality becomes

$$\mathbb{W}_{\alpha,p}[\mu] \le c_3 (\mathbb{M}_\theta[|\mu|^q])^{\frac{\alpha p}{\theta(p-1)}} (\mathbb{M}[\mu])^{\frac{\theta-\alpha pq}{\theta(p-1)}}. \tag{4.13}$$

Take $\tilde{Q}_\rho(y,s) \subset \mathbb{R}^{N+1}$, we have

$$\begin{aligned}\int_{\tilde{Q}_\rho(y,s)} (\mathbb{W}_{\alpha,p}[\mu])^{\frac{\theta q(p-1)}{\theta-\alpha pq}} dxdt &\le c_4 \Bigg(\int_{\tilde{Q}_\rho(y,s)} \left(\mathbb{W}_{\alpha,p}[\chi_{\tilde{Q}_{2\rho}(y,s)}\mu]\right)^{\frac{\theta q(p-1)}{\theta-\alpha pq}} dxdt \\ &\quad + \int_{\tilde{Q}_\rho(y,s)} \left(\mathbb{W}_{\alpha,p}[\chi_{(\tilde{Q}_{2\rho}(y,s))^c}\mu]\right)^{\frac{\theta q(p-1)}{\theta-\alpha pq}} dxdt\Bigg) \\ &= A + B.\end{aligned}$$

Using inequality (4.13) and boundless $\mathbb{M}$ from $L^q(\mathbb{R}^{N+1})$ to itself, yield

$$\begin{aligned}A &\le c_5 \int_{\mathbb{R}^{N+1}} (\mathbb{M}_\theta[|\mu|^q])^{\frac{\alpha q}{\theta-\alpha pq}} \left(\mathbb{M}[\chi_{\tilde{Q}_{2\rho}(y,s)}\mu]\right)^q dxdt \\ &\le c_6 ||\mu||_{L^{q;\theta}(\mathbb{R}^{N+1})}^{\frac{\alpha q^2}{\theta-\alpha pq}} \int_{\chi_{\tilde{Q}_{2\rho}(y,s)}} |\mu|^q dxdt \\ &\le c_7 ||\mu||_{L^{q;\theta}(\mathbb{R}^{N+1})}^{\frac{\theta q}{\theta-\alpha pq}} \rho^{N+2-\theta}.\end{aligned}$$

On the other hand, since $|\mu|(\tilde{Q}_r(x,t)) \le c_8 ||\mu||_{L^{q;\theta}(\mathbb{R}^{N+1})} r^{N+2-\frac{\theta}{q}}$ for all $\tilde{Q}_r(x,t) \subset \mathbb{R}^{N+1}$,

$$\begin{aligned}B &\le \int_{\tilde{Q}_\rho(y,s)} \left(\int_\rho^\infty \left(\frac{|\mu|(\tilde{Q}_r(x,t))}{r^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dr}{r}\right)^{\frac{\theta q(p-1)}{\theta-\alpha pq}} dxdt \\ &\le c_9 \int_{\tilde{Q}_\rho(y,s)} \left(\int_\rho^\infty \left(||\mu||_{L^{q;\theta}(\mathbb{R}^{N+1})} r^{-\frac{\theta}{q}+\alpha}\right)^{\frac{1}{p-1}} \frac{dr}{r}\right)^{\frac{\theta q(p-1)}{\theta-\alpha pq}} dxdt \\ &\le c_{10} ||\mu||_{L^{q;\theta}(\mathbb{R}^{N+1})}^{\frac{\theta q}{\theta-\alpha pq}} \rho^{N+2-\theta}.\end{aligned}$$

Therefore,

$$\int_{\tilde{Q}_\rho(y,s)} (\mathbb{W}_{\alpha,p}[\mu])^{\frac{\theta q(p-1)}{\theta-\alpha pq}} dxdt \le c_{11} ||\mu||_{L^{q;\theta}(\mathbb{R}^{N+1})}^{\frac{\theta q}{\theta-\alpha pq}} \rho^{N+2-\theta},$$

which follows (4.12). ■

In the next result we state a series of equivalent norms concerning potentials $\mathbb{I}_\alpha[\mu], \mathbb{I}_\alpha^R[\mu], \mathcal{H}_\alpha[\mu], \mathcal{G}_\alpha[\mu]$.

Proposition 4.8 *Let $q > 1$, $0 < \alpha < N + 2$ and $R > 0$. There exist constants $C_1 = C_1(N, \alpha, q)$ and $C_2 = C_2(N, \alpha, q, R)$ such that the following statements hold*

a. *for any $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$*

$$C_1^{-1}||\mathbb{I}_\alpha[\mu]||_{L^q(\mathbb{R}^{N+1})} \le ||\mathcal{H}_\alpha[\mu]||_{L^q(\mathbb{R}^{N+1})} \le C_1||\mathbb{I}_\alpha[\mu]||_{L^q(\mathbb{R}^{N+1})} \quad and \tag{4.14}$$

$$C_1^{-1}||\mathbb{I}_\alpha[\mu]||_{L^q(\mathbb{R}^{N+1})} \le ||\overset{\vee}{\mathcal{H}}_\alpha[\mu]||_{L^q(\mathbb{R}^{N+1})} \le C_1||\mathbb{I}_\alpha[\mu]||_{L^q(\mathbb{R}^{N+1})}. \tag{4.15}$$

b. *for any $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$*

$$C_2^{-1}||\mathbb{I}_\alpha^R[\mu]||_{L^q(\mathbb{R}^{N+1})} \le ||\mathcal{G}_\alpha[\mu]||_{L^q(\mathbb{R}^{N+1})} \le C_2||\mathbb{I}_\alpha^R[\mu]||_{L^q(\mathbb{R}^{N+1})} \quad and \tag{4.16}$$

$$C_2^{-1}||\mathbb{I}_\alpha^R[\mu]||_{L^q(\mathbb{R}^{N+1})} \le ||\overset{\vee}{\mathcal{G}}_\alpha[\mu]||_{L^q(\mathbb{R}^{N+1})} \le C_2||\mathbb{I}_\alpha^R[\mu]||_{L^q(\mathbb{R}^{N+1})}. \tag{4.17}$$

where $\overset{\vee}{\mathcal{H}}_\alpha[\mu]$ is the backward parabolic Riesz potential, defined by

$$\overset{\vee}{\mathcal{H}}_\alpha[\mu](x,t) = (\overset{\vee}{\mathcal{H}}_\alpha * \mu)(x,t) = \int_{\mathbb{R}^{N+1}} \mathcal{H}_\alpha(x-y, s-t)d\mu(y,s),$$

and $\overset{\vee}{\mathcal{G}}_\alpha[\mu]$ is the backward parabolic Bessel potential:

$$\overset{\vee}{\mathcal{G}}_\alpha[\mu](x,t) = (\overset{\vee}{\mathcal{G}}_\alpha * \mu)(x,t) = \int_{\mathbb{R}^{N+1}} \mathcal{G}_\alpha(y-x, s-t)d\mu(y,s).$$

Proof. a. We have:

$$\frac{c_1^{-1}}{t^{\frac{N+2-\alpha}{2}}}\chi_{t>0}\chi_{|x|\le 2\sqrt{t}} \le \mathcal{H}_\alpha(x,t) \le \frac{c_1}{\max\{|x|, \sqrt{2|t|}\}^{N+2-\alpha}},$$

which implies

$$c_2^{-1}\int_0^\infty \frac{\chi_{B_r(0)\times(\frac{r^2}{4},r^2)}(x,t)}{r^{N+2-\alpha}}\frac{dr}{r} \le \mathcal{H}_\alpha(x,t) \le c_2\int_0^\infty \frac{\chi_{\tilde{Q}_r(0,0)}(x,t)}{r^{N+2-\alpha}}\frac{dr}{r}.$$

Thus,

$$c_2^{-1}\int_0^\infty \frac{\mu\left(B(x,r)\times(t-r^2, t-\frac{r^2}{4})\right)}{r^{N+2-\alpha}}\frac{dr}{r} \le \mathcal{H}_\alpha[\mu](x,t) \le c_2\mathbb{I}_\alpha[\mu](x,t). \tag{4.18}$$

Thanks to Theorem 4.2 we will finish the proof of (4.14) when we show that

$$\int_{\mathbb{R}}\left(\int_0^\infty \frac{\mu\left(B(x,r)\times(t-r^2,t-\frac{r^2}{4})\right)}{r^{N+2-\alpha}}\frac{dr}{r}\right)^q dt \ge c_3\int_{\mathbb{R}}\int_0^{+\infty}\left(\frac{\mu(\tilde{Q}_r(x,t))}{r^{N+2-\alpha}}\right)^q\frac{dr}{r}dt.$$

Indeed, we have for $r_k = (\frac{2}{\sqrt{3}})^{-k}$,

$$\begin{aligned}
&\left(\int_0^\infty \frac{\mu\left(B(x,r)\times(t-r^2,t-r^2/4)\right)}{r^{N+2-\alpha}}\frac{dr}{r}\right)^q\\
&\qquad \ge c_4\left(\sum_{k=-\infty}^{\infty}\frac{\mu\left(B(x,r_k)\times(t-r_k^2, t-\frac{1}{3}r_k^2)\right)}{r_k^{N+2-\alpha}}\right)^q\\
&\qquad \ge c_4\sum_{k=-\infty}^{\infty}\left(\frac{\mu\left(B(x,r_k)\times(t-r_k^2,t-\frac{1}{3}r_k^2)\right)}{r_k^{N+2-\alpha}}\right)^q.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{R}}\left(\int_0^\infty \frac{\mu\left(B(x,r)\times(t-r^2,t-\frac{1}{4}r^2)\right)}{r^{N+2-\alpha}}\frac{dr}{r}\right)^q dt\\
&\geq c_4\sum_{k=-\infty}^{\infty}\int_{\mathbb{R}}\left(\frac{\mu\left(B(x,r_k)\times(t-r_k^2,t-\frac{1}{3}r_k^2)\right)}{r_k^{N+2-\alpha}}\right)^q dt\\
&= c_4\sum_{k=-\infty}^{\infty}\int_{\mathbb{R}}\left(\frac{\mu\left(B(x,r_k)\times(t-\frac{1}{3}r_k^2,t+\frac{1}{3}r_k^2)\right)}{r_k^{N+2-\alpha}}\right)^q dt\\
&\geq c_5\int_{\mathbb{R}}\int_0^{+\infty}\left(\frac{\mu(\tilde{Q}_r(x,t))}{r^{N+2-\alpha}}\right)^q\frac{dr}{r}dt.
\end{aligned}$$

Similarly, we also can prove (4.15).

b. Obviously

$$\begin{aligned}
&\frac{c_6^{-1}\exp(-4R^2)}{t^{\frac{N+2-\alpha}{2}}}\chi_{0<t<4R^2}\chi_{|x|\leq 2\sqrt{t}}\leq \mathcal{G}_\alpha(x,t)\\
&\leq \frac{c_6}{\max\{|x|,\sqrt{2|t|}\}^{N+2-\alpha}}\chi_{\tilde{Q}_{R/2}(0,0)}(x,t)+\frac{c_6}{R^{N+2-\alpha}}\exp\left(-\max\{|x|,\sqrt{2|t|}\}\right).
\end{aligned}$$

Thus, we can assert that

$$\begin{aligned}
c_7(R)\int_0^{2R}\frac{\chi_{B_r(0)\times(\frac{r^2}{4},r^2)}(x,t)}{r^{N+2-\alpha}}\frac{dr}{r}\leq \mathcal{G}_\alpha(x,t)\leq c_8\int_0^R\frac{\chi_{\tilde{Q}_r(0,0)}(x,t)}{r^{N+2-\alpha}}\frac{dr}{r}\\
+c_9(R)\int_{\mathbb{R}^{N+1}}\exp\left(-\max\{|y|,\sqrt{2|s|}\}\right)\chi_{\tilde{Q}_{R/2}(0,0)}(x-y,t-s)dyds.
\end{aligned}$$

Immediately, we get

$$c_7(R)\int_0^{2R}\frac{\mu\left(B(x,r)\times(t-r^2,t-\frac{r^2}{4})\right)}{r^{N+2-\alpha}}\frac{dr}{r}\leq \mathcal{G}_\alpha[\mu](x,t)\leq c_8\mathbb{I}_\alpha^R[\mu](x,t)+c_9(R)F(x,t),\tag{4.19}$$

where $F(x,t)=\int_{\mathbb{R}^{N+1}}\exp\left(-\max\{|y|,\sqrt{2|s|}\}\right)\mu\left(\tilde{Q}_{R/2}(x-y,t-s)\right)dyds.$

As above, we can show that

$$\int_0^\infty\left(\int_0^{2R}\frac{\mu\left(B(x,r)\times(t-r^2,t-\frac{r^2}{4})\right)}{r^{N+2-\alpha}}\frac{dr}{r}\right)^q dt\geq c_{10}\int_0^\infty\int_0^R\left(\frac{\mu(\tilde{Q}_r(x,t))}{r^{N+2-\alpha}}\right)^q\frac{dr}{r}.$$

Thus, thanks to Theorem 4.2 we get the left-hand side inequality of (4.16).

To show the right-hand side of (4.16), we use $\mu\left(\tilde{Q}_{R/2}(x-y,t-s)\right)\leq c_{10}R^{-(N+2-\alpha)}\mathbb{I}_\alpha^R[\mu](x-y,t-s)$ and Young inequality

$$\begin{aligned}
||\mathcal{G}_\alpha[\mu]||_{L^q(\mathbb{R}^{N+1})}&\leq c_8||\mathbb{I}_\alpha^R[\mu]||_{L^q(\mathbb{R}^{N+1})}+c_9(R)||F||_{L^q(\mathbb{R}^{N+1})}\\
&\leq c_8||\mathbb{I}_\alpha^R[\mu]||_{L^q(\mathbb{R}^{N+1})}+c_{11}(R)||\mathbb{I}_\alpha^R[\mu]||_{L^q(\mathbb{R}^{N+1})}\int_{\mathbb{R}^{N+1}}\exp\left(-\max\{|x|,\sqrt{2|t|}\}\right)dxdt\\
&=c_{12}(R)||\mathbb{I}_\alpha^R[\mu]||_{L^q(\mathbb{R}^{N+1})}.
\end{aligned}$$

Similarly, we also can prove (4.17). This completes the proof of the Proposition. ■

Remark 4.9 *Assume that* $0 < \alpha < N+2$. *From* (4.8) *in Proposition 4.4 and* $||\mathcal{G}_\alpha[\mu]||_{L^1(\mathbb{R}^{N+1})} \leq c_1\mu(\mathbb{R}^{N+1})$ *we deduce that for* $1 \leq s < \frac{N+2}{N+2-\alpha}$

$$||\mathcal{G}_\alpha[\mu]||_{L^s(\mathbb{R}^{N+1})} \leq c_2\mu(\mathbb{R}^{N+1}) \ \ \forall\ \mu \in \mathfrak{M}_b^+(\mathbb{R}^{N+1})$$

Next, we introduce the following kernel:

$$E_\alpha^R(x,t) = \max\{|x|, \sqrt{2|t|}\}^{-(N+2-\alpha)}\chi_{\tilde{Q}_R(0,0)}(x,t)$$

where $0 < \alpha < N+2$ and $0 < R \leq \infty$. We denote E_α^∞ by E_α. It is easy to see that $E_\alpha * \mu = (N+2-\alpha)\mathbb{I}_\alpha[\mu]$ and $||E_\alpha^R * \mu||_{L^s(\mathbb{R}^{N+1})}$ is equivalent to $||\mathbb{I}_\alpha^R[\mu]||_{L^s(\mathbb{R}^{N+1})}$ for every $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ where $1 \leq s < \infty$.
We obtain equivalences of capacities $\text{Cap}_{E_\alpha,p}, \text{Cap}_{E_\alpha^R,p}, \text{Cap}_{\mathcal{H}_\alpha,p}$ and $\text{Cap}_{\mathcal{G}_\alpha,p}$.

Corollary 4.10 *Let* $p > 1$, $1 < \alpha < N+2$ *and* $R > 0$. *There exist constants* $C_1 = C_1(N,\alpha,p)$ *and* $C_2 = C_2(N,\alpha,p,R)$ *such that the following statements hold*

a. *for any compact* $E \subset \mathbb{R}^{N+1}$

$$C_1^{-1}\,Cap_{\mathcal{H}_\alpha,p}(E) \leq Cap_{E_\alpha,p}(E) \leq C_1\,Cap_{\mathcal{H}_\alpha,p}(E) \tag{4.20}$$

b. *for any compact* $E \subset \mathbb{R}^{N+1}$

$$C_2^{-1}\,Cap_{\mathcal{G}_\alpha,p}(E) \leq Cap_{E_\alpha^R,p}(E) \leq C_2\,Cap_{\mathcal{G}_\alpha,p}(E) \tag{4.21}$$

c. *for any compact* $E \subset \mathbb{R}^{N+1}$

$$Cap_{\mathcal{H}_\alpha,p}(E) \leq Cap_{\mathcal{G}_\alpha,p}(E) \leq C_1\left(Cap_{\mathcal{H}_\alpha,p}(E) + \left(Cap_{\mathcal{H}_\alpha,p}(E)\right)^{\frac{N+2}{N+2-\alpha p}}\right) \tag{4.22}$$

provided $1 < \alpha p < N+2$.

Proof. By [2, Chapter 2], we have

$$\begin{aligned}
\text{Cap}_{E_\alpha,p}(E)^{1/p} &= \sup\{\mu(E) : \mu \in \mathfrak{M}^+(E), ||E_\alpha * \mu||_{L^{p'}(\mathbb{R}^{N+1})} \leq 1\},\\
\text{Cap}_{E_\alpha^R,p}(E)^{1/p} &= \sup\{\mu(E) : \mu \in \mathfrak{M}^+(E), ||E_\alpha^R * \mu||_{L^{p'}(\mathbb{R}^{N+1})} \leq 1\},\\
\text{Cap}_{\mathcal{H}_\alpha,p}(E)^{1/p} &= \sup\{\mu(E) : \mu \in \mathfrak{M}^+(E), ||\check{\mathcal{H}}_\alpha[\mu]||_{L^{p'}(\mathbb{R}^{N+1})} \leq 1\} \quad \text{and}\\
\text{Cap}_{\mathcal{G}_\alpha,p}(E)^{1/p} &= \sup\{\mu(E) : \mu \in \mathfrak{M}^+(E), ||\check{\mathcal{G}}_\alpha[\mu]||_{L^{p'}(\mathbb{R}^{N+1})} \leq 1\}.
\end{aligned}$$

Thanks to (4.15), (4.17) in Proposition 4.8 and $\mathbb{I}_\alpha[\mu] = E_\alpha * \mu$ and $||E_\alpha^R * \mu||_{L^s(\mathbb{R}^{N+1})}$ is equivalent to $||\mathbb{I}_\alpha^R[\mu]||_{L^s(\mathbb{R}^{N+1})}$, we get (4.20) and (4.21).
Since $\mathcal{G}_\alpha \leq \mathcal{H}_\alpha$, thus $\text{Cap}_{\mathcal{H}_\alpha,p}(E) \leq \text{Cap}_{\mathcal{G}_\alpha,p}(E)$ for any compact $E \subset \mathbb{R}^{N+1}$. Put $\text{Cap}_{E_\alpha,p}(E) = a > 0$. We need to prove that

$$\text{Cap}_{E_\alpha^1,p}(E) \leq c_1\left(a + a^{\frac{N+2}{N+2-\alpha p}}\right). \tag{4.23}$$

We will follow a proof of Yu.V. Netrusov in [2, Chapter 5]. First, we can find $f \in L_+^p(\mathbb{R}^{N+1})$ such that $||f||_{L^p(\mathbb{R}^{N+1})} \leq 2a$ and $E_\alpha * f \geq \chi_E$. Set $F_\alpha = E_\alpha - E_\alpha^1$, we have $c_2F_\alpha \leq E_\alpha^1 * F_\alpha$ for some $c_1 > 0$. Thus, $E \subset \{E_\alpha^1 * f \geq 1/2\} \cup \{E_\alpha^1 * (F_\alpha * f) \geq c_2/2\}$.
Since $||E_\alpha^1||_{L^1(\mathbb{R}^{N+1})} < \infty$, for $c_3 = c_2(4||E_\alpha^1||_{L^1(\mathbb{R}^{N+1})})^{-1}$

$$E_\alpha^1 * (F_\alpha * f) \leq c_2/4 + E_\alpha^1 * g \text{ with } g = \chi_{F_\alpha * f \geq c_3}F_\alpha * f,$$

which follows $E \subset \{E_\alpha^1 * f \ge 1/2\} \cup \{E_\alpha^1 * g \ge c_2/4\}$.
Using the subadditivity of capacity, we have

$$\begin{aligned}\mathrm{Cap}_{E_\alpha^1,p}(E) &\le \mathrm{Cap}_{E_\alpha^1,p}(\{E_\alpha^1 * f \ge 1/2\}) + \mathrm{Cap}_{E_\alpha^1,p}(\{E_\alpha^1 * g \ge c_1/4\})\\ &\le 2^p ||f||^p_{L^p(\mathbb{R}^{N+1})} + (4/c_1)^p ||g||^p_{L^p(\mathbb{R}^{N+1})}\\ &\le 2^p ||f||^p_{L^p(\mathbb{R}^{N+1})} + (4/c_1)^p c_3^{p*-p} ||E_\alpha * f||^{p*}_{L^{p*}(\mathbb{R}^{N+1})}, \text{ with } p* = \frac{(N+2)p}{N+2-\alpha p}.\end{aligned}$$

On the other hand, from (4.10) in Proposition 4.4 we have

$$||E_\alpha * f||_{L^{p*}(\mathbb{R}^{N+1})} \le c_4 ||f||_{L^p(\mathbb{R}^{N+1})}.$$

Hence, we get (4.23). ■

Remark 4.11 *Since* $\mathcal{G}_\alpha \in L^1(\mathbb{R}^{N+1})$,

$$\int_{\mathbb{R}^{N+1}} (\mathcal{G}_\alpha * f)^p \, dxdt \le ||\mathcal{G}_\alpha||^p_{L^1(\mathbb{R}^{N+1})} \int_{\mathbb{R}^{N+1}} f^p dxdt \quad \forall f \in L^p_+(\mathbb{R}^{N+1})$$

Thus, for any Borel set $E \subset \mathbb{R}^{N+1}$

$$Cap_{\mathcal{G}_\alpha,p}(E) \ge C|E| \text{ with } C = ||\mathcal{G}_\alpha||^{-p}_{L^1(\mathbb{R}^{N+1})}. \tag{4.24}$$

Remark 4.12 *It is well-known that* $\mathcal{H}_2$ *is the fundamental solution of the heat operator* $\frac{\partial}{\partial t} - \Delta$. *In [31], R. Gariepy and W. P. Ziemer introduced the following capacity:*

$$C_{\mathcal{H}_2}(K) = \sup\{\mu(K) : \mu \in \mathfrak{M}^+(K), \mathcal{H}_2[\mu] \le 1\},$$

whenever $K \subset \mathbb{R}^{N+1}$ *is compact. Thanks to [2, Theorem 2.5.5], we obtain*

$$Cap_{\mathcal{H}_1,2}(K) = C_{\mathcal{H}_2}(K).$$

Remark 4.13 *For any Borel set* $E \subset \mathbb{R}^N$, *then we always have* $Cap_{\mathcal{G}_1,2}(E \times \{t = 0\}) = 0$ *In fact,*

$$Cap_{E_1^1,2}(B_1(0) \times \{t = 0\}) = \sup\{\omega(B_1(0)) : \omega \in \mathfrak{M}^+(B_1(0)), ||E_1^1 * (\omega \otimes \delta_0)||_{L^2(\mathbb{R}^{N+1})} \le 1\}.$$

Since $||E_1^1 * (\omega \otimes \delta_0)||_{L^2(\mathbb{R}^{N+1})} = \infty$ *if* $\omega \ne 0$, *thus* $Cap_{\mathcal{G}_1,2}(B_1(0) \times \{t = 0\}) = Cap_{E_1^1,2}(B_1(0) \times \{t = 0\}) = 0$. *In particular,* $Cap_{\mathcal{G}_1,2}$ *is not absolutely continuous with respect to capacity* $C_{1,2}(., \Omega \times (a,b))$. *This capacity will be defined in next section.*

Remark 4.14 *Let* $p > 1$ *and* $\alpha > 0$. *Case* $\alpha p \ge p+1$, *we always have* $||\mathcal{H}_\alpha[\mu]||_{L^{p'}(\mathbb{R}^N)} = \infty$ *for any* $\mu \in \mathfrak{M}^+(\mathbb{R}^N)\backslash\{0\}$ *which implies* $Cap_{\mathcal{H}_\alpha,p}(\tilde{Q}_1(0,0)) = 0$. *If* $0 < \alpha p < N+2$, $Cap_{\mathcal{H}_\alpha,p}(\tilde{Q}_\rho(0,0)) = c\rho^{N+2-\alpha p}$ *for some constant c. From (4.22) in Corollary 4.10 we get* $Cap_{\mathcal{G}_\alpha,p}(\tilde{Q}_\rho(0,0)) \approx \rho^{N+2-\alpha p}$ *for* $0 < \rho < 1$ *if* $\alpha p < N+2$. *Since* $||\mathcal{G}_\alpha[\delta_{(0,0)}]||_{L^{p'}(\mathbb{R}^{N+1})} < \infty$ *thus* $Cap_{\mathcal{G}_\alpha,p}((0,0)) > 0$ *if* $\alpha p > N+2$.
If $\alpha p = N+2$, $Cap_{\mathcal{G}_\alpha,p}(\tilde{Q}_\rho(0,0)) \approx (\log(1/\rho))^{1-p}$ *for any* $0 < \rho < 1/2$. *In fact, we can prove that* $||\mathbb{I}_\alpha^{1/2}[\mu]||_{L^{p'}(\mathbb{R}^N)} \le c_1$ *for any* $d\mu(x,t) = (log(1/\rho))^{-1/p'} \rho^{-N-2} \chi_{\tilde{Q}_\rho(0,0)} dxdt$ *it follows* $Cap_{\mathcal{G}_\alpha,p}(\tilde{Q}_\rho(0,0)) \ge c_2 (\log(1/\rho))^{1-p}$. *Moreover, for* $\mu \in \mathfrak{M}^+(\tilde{Q}_\rho)$, *if* $||\mathbb{I}_\alpha^3[\mu]||^{p'}_{L^{p'}(\mathbb{R}^{N+1})} \le 1$,

$$\begin{aligned}1 &\ge \int_{\tilde{Q}_1(0,0)\backslash\tilde{Q}_\rho(0,0)} \left(\int_{2\max\{|x|,|2t|^{1/2}\}}^3 \frac{\mu(\tilde{Q}_r(x,t))}{r^{N+2-\alpha}} \frac{dr}{r}\right)^{p'} dxdt\\ &\ge \int_{\tilde{Q}_1(0,0)\backslash\tilde{Q}_\rho(0,0)} \left(\int_{2\max\{|x|,|2t|^{1/2}\}}^3 \frac{1}{r^{N+2-\alpha}} \frac{dr}{r}\right)^{p'} dxdt \mu(\tilde{Q}_\rho(0,0))^{p'}\\ &\ge c_3 \log(1/\rho) \mu(\tilde{Q}_\rho(0,0))^{p'}.\end{aligned}$$

So $Cap_{\mathcal{G}_\alpha,p}(\tilde{Q}_\rho(0,0)) \le c_4 \mu(\tilde{Q}_\rho(0,0))^p \le c_5 (\log(1/\rho))^{1-p}$.

Definition 4.15 *The parabolic Bessel potential $\mathcal{L}^p_\alpha(\mathbb{R}^{N+1})$, $\alpha>0$ and $p>1$ is defined by*

$$\mathcal{L}^p_\alpha(\mathbb{R}^{N+1})=\{f: f=\mathcal{G}_\alpha*g, g\in L^p(\mathbb{R}^{N+1})\} \tag{4.25}$$

with the norm $||f||_{\mathcal{L}^p_\alpha(\mathbb{R}^{N+1})}:=||g||_{L^p(\mathbb{R}^{N+1})}$. We denote its dual space by $\left(\mathcal{L}^p_\alpha(\mathbb{R}^{N+1})\right)^*$.

Definition 4.16 *For k a positive integer, the Sobolev space $W^{2k,k}_p(\mathbb{R}^{N+1})$ is defined by*

$$W^{2k,k}_p(\mathbb{R}^{N+1})=\{\varphi:\frac{\partial^{i_1+...+i_N+i}\varphi}{\partial x_1^{i_1}...\partial x_N^{i_N}\partial t^i}\in L^p(\mathbb{R}^{N+1}) \text{ for any } i_1+...+i_N+2i\le 2k\}$$

with the norm

$$||\varphi||_{W^{2k,k}_p(\mathbb{R}^{N+1})}=\sum_{i_1+...+i_N+2i\le 2k}||\frac{\partial^{i_1+...+i_N+i}\varphi}{\partial x_1^{i_1}...\partial x_N^{i_N}\partial t^i}||_{L^p(\mathbb{R}^{N+1})}.$$

We denote its dual space by $\left(W^{2k,k}_p(\mathbb{R}^{N+1})\right)^*$. We also define a corresponding capacity on compact set $E\subset\mathbb{R}^{N+1}$,

$$Cap_{2k,k,p}(E)=\inf\{||\varphi||^p_{W^{2k,k}_p(\mathbb{R}^{N+1})}:\varphi\in S(\mathbb{R}^{N+1}),\varphi\ge 1 \text{ in a neighborhood of } E\}.$$

Let us recall Richard J. Bagby's result, proved in [4].

Theorem 4.17 *Let $p>1$ and k be a positive integer. Then, there exists a constant C depending on N,k,p such that for any $u\in\mathcal{L}^p_{2k}(\mathbb{R}^{N+1})$,*

$$C^{-1}||u||_{W^{2k,k}_p(\mathbb{R}^{N+1})}\le||u||_{\mathcal{L}^p_{2k}(\mathbb{R}^{N+1})}\le C||u||_{W^{2k,k}_p(\mathbb{R}^{N+1})}.$$

Above Theorem gives the assertion of equivalence of capacity $\text{Cap}_{2k,k,p}$, $\text{Cap}_{\mathcal{G}_{2k},p}$.

Corollary 4.18 *Let $p>1$ and k be a positive integer. There exists a constant C depending on N,k,p such that for any compact set $E\subset\mathbb{R}^{N+1}$*

$$C^{-1}Cap_{2k,k,p}(E)\le Cap_{\mathcal{G}_{2k},p}(E)\le C\,Cap_{2k,k,p}(E). \tag{4.26}$$

Next result provides some relations of Riesz, Bessel parabolic potential and Riesz, Bessel potential.

Proposition 4.19 *Let $q>1$ and $\frac{2}{q'}<\alpha<N+\frac{2}{q'}$. There exists a constant C depending on N,q,α such that for any $\omega\in\mathfrak{M}^+(\mathbb{R}^N)$*

$$\begin{aligned}&C^{-1}||\mathbf{I}_{\alpha-\frac{2}{q'}}[\omega]||_{L^q(\mathbb{R}^N)}\\&\quad\le||\mathcal{H}_\alpha[\omega\otimes\delta_{\{t=0\}}]||_{L^q(\mathbb{R}^{N+1})},||\check{\mathcal{H}}_\alpha[\omega\otimes\delta_{\{t=0\}}]||_{L^q(\mathbb{R}^{N+1})}\le C||\mathbf{I}_{\alpha-\frac{2}{q'}}[\omega]||_{L^q(\mathbb{R}^N)}\end{aligned} \tag{4.27}$$

and

$$\begin{aligned}&C^{-1}||\mathbf{G}_{\alpha-\frac{2}{q'}}[\omega]||_{L^q(\mathbb{R}^N)}\\&\quad\le||\mathcal{G}_\alpha[\omega\otimes\delta_{\{t=0\}}]||_{L^q(\mathbb{R}^{N+1})},||\check{\mathcal{G}}_\alpha[\omega\otimes\delta_{\{t=0\}}]||_{L^q(\mathbb{R}^{N+1})}\le C||\mathbf{G}_{\alpha-\frac{2}{q'}}[\omega]||_{L^q(\mathbb{R}^N)}\end{aligned} \tag{4.28}$$

where $\delta_{\{t=0\}}$ is the Dirac mass in time at 0.

Proof. We have

$$\mathbb{I}_\alpha[\omega\otimes\delta_{\{t=0\}}](x,t)=\int_{\sqrt{2|t|}}^\infty\frac{\omega(B(x,r))}{r^{N+2-\alpha}}\frac{dr}{r},\ \mathbb{I}^1_\alpha[\omega\otimes\delta_{\{t=0\}}](x,t)=\int_{\min\{1,\sqrt{2|t|}\}}^1\frac{\omega(B(x,r))}{r^{N+2-\alpha}}\frac{dr}{r}.$$

By [16, Theorem 2.3] and Proposition 4.8, thus it is enough to show that

$$c_1^{-1}\int_0^\infty\left(\frac{\omega(B(x,r))}{r^{N+2-\alpha-2/q}}\right)^q\frac{dr}{r}\le\int_{\mathbb{R}}\left(\int_{\sqrt{2|t|}}^\infty\frac{\omega(B(x,r))}{r^{N+2-\alpha}}\frac{dr}{r}\right)^q dt\le c_1\int_0^\infty\left(\frac{\omega(B(x,r))}{r^{N+2-\alpha-2/q}}\right)^q\frac{dr}{r},\tag{4.29}$$

and

$$c_1^{-1}\int_0^{1/2}\left(\frac{\omega(B(x,r))}{r^{N+2-\alpha-2/q}}\right)^q\frac{dr}{r}\le\int_{\mathbb{R}}\left(\int_{\min\{1,\sqrt{2|t|}\}}^1\frac{\omega(B(x,r))}{r^{N+2-\alpha}}\frac{dr}{r}\right)^q dt\le c_1\int_0^1\left(\frac{\omega(B(x,r))}{r^{N+2-\alpha-2/q}}\right)^q\frac{dr}{r}\tag{4.30}$$

Indeed, by changing of variables

$$\int_{-\infty}^\infty\left(\int_{\sqrt{2|t|}}^\infty\frac{\omega(B(x,r))}{r^{N+2-\alpha}}\frac{dr}{r}\right)^q dt=2\int_0^\infty t\left(\int_t^\infty\frac{\omega(B(x,r))}{r^{N+2-\alpha}}\frac{dr}{r}\right)^q dt.\tag{4.31}$$

Using Hardy's inequality, we have

$$\int_0^\infty t\left(\int_t^\infty\frac{\omega(B(x,r))}{r^{N+2-\alpha}}\frac{dr}{r}\right)^q dt\le c_2\int_0^\infty r\left(\frac{\omega(B(x,r))}{r^{N+2-\alpha}}\right)^q dr$$

and using the fact that

$$\int_t^\infty\frac{\omega(B(x,r))}{r^{N+2-\alpha}}\frac{dr}{r}\ge c_3\frac{\omega(B(x,r))}{r^{N+2-\alpha}},$$

we get

$$\int_0^\infty t\left(\int_t^\infty\frac{\omega(B(x,r))}{r^{N+2-\alpha}}\frac{dr}{r}\right)^q dt\ge c_3\int_0^\infty r\left(\frac{\omega(B(x,r))}{r^{N+2-\alpha}}\right)^q dr.$$

Thus, we get (4.29). Likewise, we also obtain (4.30). ■

We have comparisons of $\mathrm{Cap}_{\mathcal{H}_\alpha,p},\mathrm{Cap}_{\mathcal{G}_\alpha,p},\mathrm{Cap}_{\mathbf{I}_{\alpha-\frac{2}{p}},p},\mathrm{Cap}_{\mathbf{G}_{\alpha-\frac{2}{p}},p}$.

Corollary 4.20 *Let $p>1$ and $\frac{2}{p}<\alpha<N+\frac{2}{p}$. There exists a constant C depending on N,q,α such that for any compact $K\subset\mathbb{R}^N$*

$$C^{-1}\,Cap_{\mathbf{I}_{\alpha-\frac{2}{p}},p}(K)\le Cap_{\mathcal{H}_\alpha,p}(K\times\{0\})\le C\,Cap_{\mathbf{I}_{\alpha-\frac{2}{p}},p}(K)\tag{4.32}$$

and

$$C^{-1}\,Cap_{\mathbf{G}_{\alpha-\frac{2}{p}},p}(K)\le Cap_{\mathcal{G}_\alpha,p}(K\times\{0\})\le C\,Cap_{\mathbf{G}_{\alpha-\frac{2}{p}},p}(K)\tag{4.33}$$

Proof. By [2, Chapter 2], we have

$$\begin{aligned}\mathrm{Cap}_{\mathcal{H}_\alpha,p}(K\times\{0\})^{1/p}&=\sup\{\mu(K\times\{0\}):\mu\in\mathfrak{M}^+(K\times\{0\}),||\overset{\vee}{\mathcal{H}}_\alpha[\mu]||_{L^{p'}(\mathbb{R}^{N+1})}\le1\}\\&=\sup\{\omega(K):\omega\in\mathfrak{M}^+(K),||\overset{\vee}{\mathcal{H}}_\alpha[\omega\otimes\delta_{\{t=0\}}]||_{L^{p'}(\mathbb{R}^{N+1})}\le1\},\\\mathrm{Cap}_{\mathcal{G}_\alpha,p}(K\times\{0\})^{1/p}&=\sup\{\omega(K):\omega\in\mathfrak{M}^+(K),||\overset{\vee}{\mathcal{G}}_\alpha[\omega\otimes\delta_0]||_{L^{p'}(\mathbb{R}^{N+1})}\le1\},\\\mathrm{Cap}_{\mathbf{I}_{\alpha-\frac{2}{p}},p}(K)^{1/p}&=\sup\{\omega(K):\omega\in\mathfrak{M}^+(K),||\mathbf{I}_{\alpha-\frac{2}{p}}[\omega]||_{L^{p'}(\mathbb{R}^{N+1})}\le1\},\\\mathrm{Cap}_{\mathbf{G}_{\alpha-\frac{2}{p}},p}(K)^{1/p}&=\sup\{\omega(K):\omega\in\mathfrak{M}^+(K),||\mathbf{G}_{\alpha-\frac{2}{p}}[\omega]||_{L^{p'}(\mathbb{R}^{N+1})}\le1\}.\end{aligned}$$

Therefore, thanks to Proposition (4.19) we get the results. ■

Corollary 4.21 *Let $p > 1$ and k be a positive integer such that $2k < N + 2/p$. There exists a constant C depending on N, k, p such that for any compact set $K \subset \mathbb{R}^N$*

$$C^{-1} Cap_{\mathbf{G}_{2k-\frac{2}{p}},p}(K) \le Cap_{2k,k,p}(K \times \{0\}) \le C\, Cap_{\mathbf{G}_{2k-\frac{2}{p}},p}(K). \tag{4.34}$$

We also have comparisons of $\text{Cap}_{\mathcal{G}_\alpha,p}, \text{Cap}_{\mathbf{G}_\alpha,p}$.

Proposition 4.22 *Let $0 < \alpha < N$, $p > 1$. For $a > 0$ there exists a constant C depending on N, α, p, a such that for any compact $K \subset \mathbb{R}^N$,*

$$C^{-1} Cap_{\mathbf{G}_\alpha,p}(K) \le Cap_{\mathcal{G}_\alpha,p}(K \times [-a, a]) \le C\, Cap_{\mathbf{G}_\alpha,p}(K).$$

Proof. By [2], we have

$$\text{Cap}_{\mathbf{I}_\alpha^{\frac{\sqrt{a}}{2}},p}(K) \le c_1 \text{Cap}_{\mathbf{G}_\alpha,p}(K),$$

for some $c_1 = c_1(N, \alpha, p, a) > 0$. So, we can find $f \in L^p_+(\mathbb{R}^N)$ such that $\mathbf{I}_\alpha^{\frac{\sqrt{a}}{2}} * f \ge \chi_K$ and

$$\int_{\mathbb{R}^N} |f|^p dx \le 2c_1 \text{Cap}_{\mathbf{G}_\alpha,p}(K).$$

Note that $(E_\alpha^{\sqrt{a}} * \tilde{f})(x, t) \ge c_2(\mathbf{I}_\alpha^{\frac{\sqrt{a}}{2}} * f)(x, t)$ for all $(x, t) \in \mathbb{R}^N \times [-a, a]$ where $\tilde{f}(x, t) = f(x)\chi_{[-2a,2a]}(t)$ and constant $c_2 = c_2(N, \alpha, p)$. So,

$$\begin{aligned} \text{Cap}_{E_\alpha^{\sqrt{a}},p}(K \times [-a, a]) &\le c_2^{-p} \int_{\mathbb{R}^{N+1}} |\tilde{f}|^p dxdt \\ &= 4c_2^{-p} a \int_{\mathbb{R}^N} |f|^p dx. \end{aligned}$$

By Corollary 4.10, there is $c_1 = c_1(N, \alpha, p, a) > 0$ such that

$$\text{Cap}_{\mathcal{G}_\alpha,p}(K \times [-a, a]) \le c_1 \text{Cap}_{E_\alpha^{\sqrt{a}},p}(K \times [-a, a]).$$

Thus, we get

$$\text{Cap}_{\mathcal{G}_\alpha,p}(K \times [-a, a]) \le c_3 \text{Cap}_{\mathbf{G}_\alpha,p}(K),$$

for some $c_3 = c_3(N, \alpha, p, a)$.
Finally, we prove other one. It is easy to see that

$$||\mathbb{I}_\alpha^{\sqrt{\frac{a}{2}}}[\omega \otimes \chi_{[-a,a]}]||_{L^{p'}(\mathbb{R}^{N+1})} \le c_4 ||\mathbf{I}_\alpha^{\sqrt{\frac{a}{2}}}[\omega]||_{L^{p'}(\mathbb{R}^N)} \ \ \forall\, \omega \in \mathfrak{M}^+(\mathbb{R}^N),$$

for some $c_4 = c_4(N, \alpha, p)$, which implies

$$||\mathcal{G}_\alpha[\omega \otimes \chi_{[-a,a]}]||_{L^{p'}(\mathbb{R}^{N+1})} \le c_5 ||\mathbf{G}_\alpha[\omega]||_{L^{p'}(\mathbb{R}^N)} \ \ \forall\, \omega \in \mathfrak{M}^+(\mathbb{R}^{N+1})$$

for some $c_4 = c_4(N, \alpha, p, a)$.
It follows,

$$\text{Cap}_{\mathcal{G}_\alpha,p}(K \times [-a, a]) \ge c_6 \text{Cap}_{\mathbf{G}_\alpha,p}(K),$$

for some $c_6 = c_6(N, \alpha, p, a)$. ■

The following proposition is useful for proving that many operators of classical analysis are bounded in the space the space of functions f such that

$$\int_K |f|^p dxdt \le C\text{Cap}(K)$$

for every compact set $K \subset \mathbb{R}^{N+1}$, $(1 < p < \infty)$, if they are bounded in $L^q(\mathbb{R}^{N+1}, dw)$ with $w \in A_\infty$.

Proposition 4.23 *Let $0 < R \le \infty$, $1 < p \le \alpha^{-1}(N+2)$, $0 < \delta < \alpha$ and $f, g \in L^1_{loc}(\mathbb{R}^{N+1})$. Suppose that*

1. *There exists a positive constant C_1 such that*

$$\int_K |f| dxdt \le C_1 \, Cap_{E_\alpha^{R,\delta}, p}(K) \ \text{ for any compact sets } K \subset \mathbb{R}^{N+1}. \tag{4.35}$$

2. *For all weights $w \in A_1$,*

$$\int_{\mathbb{R}^{N+1}} |g| w dxdt \le C_2 \int_{\mathbb{R}^{N+1}} |f| w dxdt, \tag{4.36}$$

where the constant C_2 depends only on N and $[w]_{A_1}$.

Then,

$$\int_K |g| dxdt \le C_3 \, Cap_{E_\alpha^{R,\delta}, p}(K) \ \text{ for any compact set } K \subset \mathbb{R}^{N+1}, \tag{4.37}$$

where the constant C_3 depends only on N, α, p, δ and C_1, C_2.

The capacity is mentioned in the Proposition (4.23), that is $(E_\alpha^{R,\delta}, p)$-capacity defined by

$$\text{Cap}_{E_\alpha^{R,\delta}, p}(E) = \inf \left\{ \int_{\mathbb{R}^{N+1}} |f|^p dxdt : f \in L^p_+(\mathbb{R}^{N+1}), E_\alpha^{R,\delta} * f \ge \chi_E \right\},$$

for all measurable sets $E \subset \mathbb{R}^{N+1}$, where $0 < R \le \infty$, $0 < \delta < \alpha < N+2$,

$$E_\alpha^{R,\delta}(x,t) = \max\{|x|, \sqrt{2|t|}\}^{-(N+2-\alpha)} \min \left\{ 1, \left(\frac{\max\{|x|, \sqrt{2|t|}\}}{R} \right)^{-\delta} \right\}.$$

Remark 4.24 *For $0 < \alpha q < N+2$, the inequality* (4.10) *in Proposition 4.4 implies*

$$\left(\int_{\mathbb{R}^{N+1}} \left(E_\alpha^{R,\delta} * f \right)^{\frac{q(N+2)}{N+2-\alpha q}} dxdt \right)^{1 - \frac{\alpha q}{N+2}} \le C \int_{\mathbb{R}^{N+1}} f^q dxdt \ \ \forall f \in L^q(\mathbb{R}^{N+1}), f \ge 0. \tag{4.38}$$

Hence, we get the isoperimetric inequality:

$$|E|^{1 - \frac{\alpha p}{N+2}} \le C \, Cap_{E_\alpha^{R,\delta}, p}(E), \tag{4.39}$$

for all measurable sets $E \subset \mathbb{R}^{N+1}$.

Also, we recall that a positive function $w \in L^1_{loc}(\mathbb{R}^{N+1})$ is called an A_1 weight, if the quality

$$[w]_{A_1} := \sup \left(\left(\fint_Q w dyds \right) ess \sup_{(x,t) \in Q} \frac{1}{w(x,t)} \right) < \infty,$$

where the supremum is taken over all cylinder $Q = \tilde{Q}_R(x,t) \subset \mathbb{R}^{N+1}$. The constant $[w]_{A_1}$ is called the A_1 constant of w.

To prove the Proposition (4.23), we need to introduce the $(R,\delta)-$Wolff parabolic potential,

$$\mathbb{W}_{\alpha,p}^{R,\delta}[\mu](x,t) = \int_0^\infty \left(\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha p}} \right)^{\frac{1}{p-1}} \min \left\{ 1, \left(\frac{\rho}{R} \right)^{-\delta} \right\} \frac{d\rho}{\rho} \ \text{ for any } (x,t) \in \mathbb{R}^{N+1},$$

where $p > 1$, $0 < \alpha p \le N+2$, $0 < \delta < \alpha p'$ and $R \in (0,\infty]$ and $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$.
It is easy to see that

$$\mathbb{W}^{R,\delta}_{\alpha,p}[\mu](x,t) \le C \sup_{(y,s)\in \mathrm{supp}\mu} \mathbb{W}^{R,\delta}_{\alpha,p}[\mu](y,s). \tag{4.40}$$

for some a constant $C = C(N,\alpha,p,\delta) > 0$.

Remark 4.25 *We easily verify that the Theorem 4.1 also holds for $\mathbb{W}^{R,\delta,R_1}_{\alpha,p}[\mu]$ and $\mathbb{M}^{R,\delta,R_1}_{\alpha p}[\mu]$:*

$$\mathbb{W}^{R,\delta,R_1}_{\alpha,p}[\mu](x,t) = \int_0^{R_1} \left(\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \min\left\{1, \left(\frac{\rho}{R}\right)^{-\delta}\right\} \frac{d\rho}{\rho},$$

$$\mathbb{M}^{R,\delta/(p-1),R_1}_{\alpha,p}[\mu](x,t) = \sup_{0<\rho<R_1} \left(\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha p}} \min\left\{1, \left(\frac{\rho}{R}\right)^{-\delta(p-1)}\right\}\right) \quad \text{for any } (x,t) \in \mathbb{R}^{N+1},$$

where $0 < \delta < \alpha p'$, $1 < p < \alpha^{-1}(N+2)$ and $R_1 > R > 0$. This means, for $w \in A_\infty$, $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$, there exist constants $C_1, C_2 > 0$ and $\varepsilon_0 \in (0,1)$ depending on $N, \alpha, p, \delta, [w]_{A_\infty}$ such that for any $\lambda > 0$ and $\varepsilon \in (0,\varepsilon_0)$

$$w(\{\mathbb{W}^{R,\delta,R_1}_{\alpha,p}[\mu] > a\lambda, (\mathbb{M}^{R,\delta(p-1),R_1}_{\alpha p}[\mu])^{\frac{1}{p-1}} \le \varepsilon\lambda\}) \le C_1 \exp(-C_2\varepsilon^{-1}) w(\{\mathbb{W}^{R,\delta,R_1}_{\alpha,p}[\mu] > \lambda\}), \tag{4.41}$$

where $a = 2 + 3^{\frac{N+2-\alpha p+\delta(p-1)}{p-1}}$.
Therefore, for $q > p-1$

$$||\mathbb{W}^{R,\delta,R_1}_{\alpha,p}[\mu]||_{L^q(\mathbb{R}^{N+1},dw)} \le C_3 ||(\mathbb{M}^{R,\delta(p-1),R_1}_{\alpha p}[\mu])^{\frac{1}{p-1}}||_{L^q(\mathbb{R}^{N+1},dw)},$$

where $C_3 = C_3(N,\alpha,p,\delta,q)$. Letting $R_1 \to \infty$, we get

$$||\mathbb{W}^{R,\delta}_{\alpha,p}[\mu]||_{L^q(\mathbb{R}^{N+1},dw)} \le C_3 ||(\mathbb{M}^{R,\delta(p-1)}_{\alpha p}[\mu])^{\frac{1}{p-1}}||_{L^q(\mathbb{R}^{N+1},dw)}, \tag{4.42}$$

where $\mathbb{M}^{R,\delta(p-1)}_{\alpha p}[\mu] := \mathbb{M}^{R,\delta(p-1),\infty}_{\alpha p}[\mu]$.

We will need the following three Lemmas to prove the Proposition (4.23).

Lemma 4.26 *Let $0 < p \le \alpha^{-1}(N+2)$ and $0 < \beta < \frac{(N+2)(p-1)}{N+2-\alpha p+\delta(p-1)}$. There exists a constant c depending on δ such that for each $\tilde{Q}_r = \tilde{Q}_r(x,t)$*

$$⨍_{\tilde{Q}_r} (\mathbb{W}^{R,\delta}_{\alpha,p}[\mu](y,s))^\beta dyds \le c(\mathbb{W}^{R,\delta}_{\alpha,p}[\mu](x,t))^\beta. \tag{4.43}$$

Proof. We set

$$U^r_{\alpha,p}[\mu](y,s) = \int_r^\infty \left(\frac{|\mu|(\tilde{Q}_\rho(y,s))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \min\left\{1, \left(\frac{\rho}{R}\right)^{-\delta}\right\} \frac{d\rho}{\rho} \quad \text{and}$$

$$L^r_{\alpha,p}[\mu](y,s) = \int_0^r \left(\frac{\mu(\tilde{Q}_\rho(y,s))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \min\left\{1, \left(\frac{\rho}{R}\right)^{-\delta}\right\} \frac{d\rho}{\rho}.$$

Thus,

$$⨍_{\tilde{Q}_r} (\mathbb{W}^{R,\delta}_{\alpha,p}[\mu](y,s))^\delta dyds \le c_1 ⨍_{\tilde{Q}_r} (U^r_{\alpha,p}[\mu](y,s))^\delta dyds + c_1 ⨍_{\tilde{Q}_r} (L^r_{\alpha,p}[\mu](y,s))^\delta dyds.$$

Since for each $(y,s)\in\tilde{Q}_r$ and $\rho\geq r$ we have $\tilde{Q}_\rho(y,s)\subset\tilde{Q}_{2\rho}(x,t)$, thus for each $(y,s)\in\tilde{Q}_r$,

$$\begin{aligned} U^r_{\alpha,p}[\mu](y,s) &\leq \int_r^\infty \left(\frac{\mu(\tilde{Q}_{2\rho}(x,t))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \left(\max\{1,\frac{\rho}{R}\}\right)^{-\delta}\frac{d\rho}{\rho} \\ &\leq c_2\mathbb{W}^{R,\delta}_{\alpha,p}[\mu](x,t), \end{aligned}$$

which implies

$$\fint_{\tilde{Q}_r}(U^r_{\alpha,p}[\mu](y,s))^\delta dyds\leq c_2(\mathbb{W}^{R,\delta}_{\alpha,p}[\mu](x,t))^\delta.$$

Since for each $(y,s)\in\tilde{Q}_r$ and $\rho\leq r$ we have $\tilde{Q}_\rho(y,s)\subset\tilde{Q}_{2r}(x,t)$ thus, $L^r_{\alpha,p}[\mu]=L^r_{\alpha,p}[\mu\chi_{\tilde{Q}_{2r}(x,t)}]\leq\mathbb{W}^{R,\delta}_{\alpha,p}[\mu\chi_{\tilde{Q}_{2r}(x,t)}]$ in $\tilde{Q}_r(x,t)$. We now consider two cases.
Case 1: $r\leq R$. We have for $a>0$,

$$\begin{aligned} \fint_{\tilde{Q}_r}(L^r_{\alpha,p}[\mu](y,s))^\beta dyds &\leq \fint_{\tilde{Q}_r}(\mathbb{W}^r_{\alpha,p}[\mu\chi_{\tilde{Q}_{2r}(x,t)}](y,s))^\beta dyds \\ &= \frac{1}{|\tilde{Q}_r|}\beta\int_0^\infty\lambda^{\beta-1}|\{\mathbb{W}^r_{\alpha,p}[\mu\chi_{\tilde{Q}_{2r}(x,t)}]>\lambda\}\cap\tilde{Q}_r|d\lambda \\ &\leq a^\beta+c_2r^{-N-2}\int_a^\infty\lambda^{\beta-1}|\{\mathbb{W}^r_{\alpha,p}[\mu\chi_{\tilde{Q}_{2r}(x,t)}]>\lambda\}|d\lambda. \end{aligned}$$

If $\alpha p=N+2$, we use (4.11) in Remark 4.5 with $\varepsilon=\frac{\alpha p}{\beta}$ and take $a=(\mu(\tilde{Q}_{2r}(x,t)))^{\frac{1}{p-1}}$

$$\begin{aligned} \fint_{\tilde{Q}_r}(L^r_{\alpha,p}[\mu](y,s))^\beta dyds &\leq a^\beta+c_3r^{-N-2}\int_a^\infty\lambda^{\beta-1}\left(\frac{(\mu(\tilde{Q}_{2r}(x,t)))^{\frac{1}{p-1}}}{\lambda}\right)^{\frac{\alpha p+\varepsilon(p-1)}{\varepsilon}} r^{\alpha p}d\lambda \\ &\leq c_4(\mu(\tilde{Q}_{2r}(x,t)))^{\frac{\beta}{p-1}} \\ &\leq c_5(\mathbb{W}^{R,\delta}_{\alpha,p}[\mu](x,t))^\beta. \end{aligned}$$

If $\alpha p<N+2$, we use (4.8) in Proposition 4.4 and take $a=\mu(\tilde{Q}_{2r}(x,t))^{\frac{1}{p-1}}r^{-\frac{N+2-\alpha p}{p-1}}$, we get

$$\begin{aligned} \fint_{\tilde{Q}_r}(L^r_{\alpha,p}[\mu](y,s))^\beta dyds &\leq c_6\left(\mu(\tilde{Q}_{2r}(x,t))^{\frac{1}{p-1}}r^{-\frac{N+2-\alpha p}{p-1}}\right)^\beta \\ &\leq c_7(\mathbb{W}^{R,\delta}_{\alpha,p}[\mu](x,t))^\beta. \end{aligned}$$

Case 2: $r\geq R$. As above case, we have

$$\fint_{\tilde{Q}_r}(\mathbb{W}_{\alpha-\frac{\delta}{p'},p}[\mu\chi_{\tilde{Q}_{2r}(x,t)}](y,s))^\beta dyds\leq c_6\left(\mu(\tilde{Q}_{2r}(x,t))^{\frac{1}{p-1}}r^{-\frac{N+2-\alpha p+\delta(p-1)}{p-1}}\right)^\beta.$$

Since $\mathbb{W}^{R,\delta}_{\alpha,p}[\mu\chi_{\tilde{Q}_{2r}(x,t)}]\leq R^\delta\mathbb{W}_{\alpha-\frac{\delta}{p'},p}[\mu\chi_{\tilde{Q}_{2r}(x,t)}]$, thus

$$\begin{aligned} \fint_{\tilde{Q}_r}(L^r_{\alpha,p}[\mu](y,s))^\beta dyds &\leq c_6\left(\mu(\tilde{Q}_{2r}(x,t))^{\frac{1}{p-1}}r^{-\frac{N+2-\alpha p+\delta(p-1)}{p-1}}R^\delta\right)^\beta \\ &\leq c_5(\mathbb{W}^{R,\delta}_{\alpha,p}[\mu](x,t))^\beta. \end{aligned}$$

Therefore, we get (4.43). The proof completes. ∎

Remark 4.27 *It is easy to see that the inequality* (4.43) *does not true for* $\mathbb{W}^R_{\alpha,p}[\delta_{(0,0)}]$ *where* $\delta_{(0,0)}$ *is the Dirac mass at* $(x,t)=(0,0)$.

Remark 4.28 *From Lemma (4.26), we have, if there exists* $(x_0,t_0)\in\mathbb{R}^{N+1}$ *such that* $\mathbb{W}^{R,\delta}_{\alpha,p}[\mu](x_0,t_0)<\infty$ *then* $\mathbb{W}^{R,\delta}_{\alpha,p}[\mu]\in L^{\beta}_{loc}(\mathbb{R}^{N+1})$ *for any* $0<\beta<\frac{(N+2)(p-1)}{N+2-\alpha p+\delta(p-1)}$.

Lemma 4.29 *Let* $R\in(0,\infty]$, $1<p\le\alpha^{-1}(N+2)$ *and* $0<\delta<\alpha p'$. *Assume that* $\alpha p<N+2$ *if* $R=\infty$. *Then, for any compact set* $K\subset\mathbb{R}^{N+1}$ *there exists a* $\mu\in\mathfrak{M}^+(K)$, *called a capacitary measure of* K *such that*

$$C_1^{-1}\,Cap_{E_\alpha^{R,\delta/p'},p}(K)\le\mu(K)\le C_1\,Cap_{E_\alpha^{R,\delta/p'},p}(K)$$

and $\mathbb{W}^{R,\delta}_{\alpha,p}[\mu](x,t)\ge C_2$ *a.e in* K *and* $\mathbb{W}^{R,\delta}_{\alpha,p}[\mu]\le C_3$ *a.e in* $\mathbb{R}^{N+1}$ *for some constants* $C_i=C_i(N,\alpha,p)$, $i=1,2,3$.

Proof. We consider a measure ν on $M=\mathbb{R}^{N+1}\times\mathbb{Z}$ as follows

$$\nu=m\otimes\sum_{n=-\infty}^{\infty}\delta_n,$$

where m is Lebesgue measure, and δ_n denotes unit mass at n. Thus, $f\in L^p(M,d\nu)$, means $f=\{f_n\}_{-\infty}^{\infty}$, with

$$||f||^p_{L^p(M,d\nu)}=\sum_{n=-\infty}^{\infty}||f_n||^p_{L^p(\mathbb{R}^{N+1})}.$$

Let $n_R\in\mathbb{Z}\cup\{+\infty\}$ such that $2^{-n_R}\le R<2^{-n_R+1}$ if $R<+\infty$ and $n_R-\infty$ if $R=+\infty$. We define a kernel $\mathbb{P}_\alpha$ in $\mathbb{R}^{N+1}\times M=\mathbb{R}^{N+1}\times\mathbb{R}^{N+1}\times\mathbb{Z}$ by

$$\mathbb{P}_\alpha(x,t,x',t',n)=\min\{1,2^{(n-n_R)\delta/p'}\}2^{n(N+2-\alpha)}\chi_{\tilde{Q}_{2^{-n}}}(x-x',t-t').$$

If f is $\nu-$measurable and nonnegative and $\mu\in\mathfrak{M}^+(\mathbb{R}^{N+1})$, the corresponding potentials $\mathcal{P}_\alpha f$, $\overset{\vee}{\mathcal{P}}_\alpha\mu$ and $V^\mu_{\mathbb{P}_\alpha,p}$ are everywhere well defined and given by

$$\begin{aligned}(\mathcal{P}_\alpha f)(x,t)&=\int_M\mathbb{P}_\alpha(x,t,x',t',n)f(x',t',n)d\nu(x',t',n)\\&=\sum_{n=-\infty}^{\infty}\min\{1,2^{(n-n_R)\delta/p'}\}2^{n(N+2-\alpha)}(\chi_{\tilde{Q}_{2^{-n}}}*f_n)(x,t),\\(\overset{\vee}{\mathcal{P}}_\alpha\mu)(x',t',n)&=\int_{\mathbb{R}^{N+1}}\mathbb{P}_\alpha(x,t,x',t',n)d\mu(x,t)\\&=\min\{1,2^{(n-n_R)\delta/p'}\}2^{n(N+2-\alpha)}(\chi_{\tilde{Q}_{2^{-n}}}*\mu)(x',t'),\\V^\mu_{P_\alpha,p}(x,t)&=(\mathcal{P}_\alpha(\overset{\vee}{\mathcal{P}}_\alpha\mu)^{p'-1})(x,t)\\&=\sum_{n=-\infty}^{\infty}\min\{1,2^{(n-n_R)\delta}\}2^{np'(N+2-\alpha)}\left(\chi_{\tilde{Q}_{2^{-n}}}*\left(\chi_{\tilde{Q}_{2^{-n}}}*\mu\right)^{p'-1}\right)(x,t).\end{aligned}$$

for any $(x,t,x',t',n)\in\mathbb{R}^{N+1}\times M$.
Since for all $(x,t)\in\mathbb{R}^{N+1}$,

$$\begin{aligned}|\tilde{Q}_1|2^{-(n+1)(N+2)}(\mu(\tilde{Q}_{2^{-n-1}}(x,t)))^{p'-1}&\le\left(\chi_{\tilde{Q}_{2^{-n}}}*\left(\chi_{\tilde{Q}_{2^{-n}}}*\mu\right)^{p'-1}\right)(x,t)\\&\le|\tilde{Q}_1|2^{-n(N+2)}(\mu(\tilde{Q}_{2^{-n+1}}(x,t)))^{p'-1},\end{aligned}$$

thus,

$$c_1^{-1}V^\mu_{\mathbb{P}_\alpha,p}\le\mathbb{W}^{R,\delta}_{\alpha,p}[\mu]\le c_1V^\mu_{\mathbb{P}_\alpha,p},\tag{4.44}$$

for some a positive constant c_1.
We now define the L^p-capacity with $1 < p < \infty$

$$\text{Cap}_{\mathbb{P}_\alpha,p}(E) = \inf\{||f||^p_{L^p(M,d\nu)} : f \in L^p_+(M,d\nu), \mathcal{P}_\alpha f \geq \chi_E\}.$$

for any Borel set $E \subset \mathbb{R}^{N+1}$. By [2, Theorem 2.5.1], for any compact set $K \subset \mathbb{R}^{N+1}$

$$\text{Cap}_{\mathbb{P}_\alpha,p}(K)^{1/p} = \sup\{\mu(K) : \mu \in \mathfrak{M}^+(K), ||\overset{\vee}{\mathcal{P}}_\alpha\mu||_{L^{p'}(M,d\nu)} \leq 1\}.$$

By [2, Theorem 2.5.6], for any compact set K in $\mathbb{R}^{N+1}$, there exists $\mu \in \mathfrak{M}^+(K)$, called a capacitary measure for K, such that $V^\mu_{\mathbb{P}_\alpha,p} \geq 1$ $\text{Cap}_{\mathbb{P}_\alpha,p}-$ q.e. in K, $V^\mu_{P_\alpha,p} \leq 1$ a.e in supp(μ) and $\mu(K) = \text{Cap}_{\mathbb{P}_\alpha,p}(K)$. Thanks to (4.44) and (4.40), we have $\mathbb{W}^{R,\delta}_{\alpha,p}[\mu] \geq c_1^{-1}$ $\text{Cap}_{\mathbb{P}_\alpha,p}-$ q.e. in K, $\mathbb{W}^{R,\delta}_{\alpha,p}[\mu] \leq c_2$ a.e in $\mathbb{R}^{N+1}$ and $\mu(K) = \text{Cap}_{\mathbb{P}_\alpha,p}(K)$.
On the other hand,

$$\begin{aligned}
||\overset{\vee}{\mathcal{P}}_\alpha\mu||^{p'}_{L^{p'}(M,d\nu)} &= \sum_{n=-\infty}^{\infty} ||\min\{1, 2^{(n-n_R)\delta/p'}\}2^{n(N+2-\alpha)}\chi_{\tilde{Q}_{2^{-n}}} * \mu||^{p'}_{L^{p'}(\mathbb{R}^{N+1})} \\
&= \sum_{n=-\infty}^{\infty} \min\{1, 2^{(n-n_R)\delta}\}2^{np'(N+2-\alpha)} \int_{\mathbb{R}^{N+1}} (\chi_{\tilde{Q}_{2^{-n}}} * \mu)^{p'} dxdt,
\end{aligned}$$

this quantity is equivalent to

$$\int_{\mathbb{R}^{N+1}} \int_0^\infty \left(\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha}}\right)^{p'} \min\{1, \left(\frac{\rho}{R}\right)^{-\delta}\} \frac{d\rho}{\rho} dxdt.$$

So, thanks to (4.42) in Remark 4.25, we obtain

$$c_2^{-1}||E^{R,\delta/p'}_\alpha * \mu||^{p'}_{L^{p'}(\mathbb{R}^{N+1})} \leq ||\overset{\vee}{\mathcal{P}}_\alpha\mu||^{p'}_{L^{p'}(M,d\nu)} \leq c_2||E^{R,\delta/p'}_\alpha * \mu||^{p'}_{L^{p'}(\mathbb{R}^{N+1})}.$$

for $c_2 = c_2(N,p,\alpha,\delta)$. It follows that two capacities $\text{Cap}_{\mathbb{P}_\alpha,p}$and $\text{Cap}_{E^{R,\delta/p'}_\alpha,p}$ are equivalent. Therefore, we obtain the desired results. ■

Lemma 4.30 *Let $R \in (0,\infty]$, $1 < p \leq \alpha^{-1}(N+2)$ and $0 < \delta < \alpha p'$. Assume that $\alpha p < N+2$ if $R = \infty$. Then there exists $C = C(N,\alpha,p,\delta)$ such that for any $\mu \in \mathfrak{M}^+_b(\mathbb{R}^{N+1})$*

$$Cap_{E^{R,\delta/p'}_\alpha,p}(\{\mathbb{W}^{R,\delta}_{\alpha,p}[\mu] > \lambda\}) \leq C\lambda^{-p+1}\mu(\mathbb{R}^{N+1}) \quad \forall\ \lambda > 0. \tag{4.45}$$

In particular, $\mathbb{W}^{R,\delta}_{\alpha,p}[\mu] < \infty$ $Cap_{E^{R,\delta/p'}_\alpha,p}-$q.e. in $\mathbb{R}^{N+1}$.

Proof. By Lemma 4.29, there is a capacitary measure σ for a compact subset K of $\{\mathbb{W}^{R,\delta}_{\alpha,p}[\mu] > \lambda\}$ such that $\mathbb{W}^{R,\delta}_{\alpha,p}[\sigma](x,t) \leq c_1$ on suppσ and $\text{Cap}_{E^{R,\delta/p'}_\alpha,p}(K) \approx \sigma(K)$ where $c_1 = c_1(N,\alpha,p,\delta)$.
Set $\mathbb{M}[\mu,\sigma](x,t) = \sup_{\rho>0} \frac{\mu(\tilde{Q}_\rho(x,t))}{\sigma(\tilde{Q}_{3\rho}(x,t))}$ for any $(x,t) \in$ suppσ. Then, for any $(x,t) \in$ suppσ

$$\begin{aligned}
\lambda < \mathbb{W}^{R,\delta}_{\alpha,p}[\mu](x,t) &\leq (\mathbb{M}[\mu,\sigma](x,t))^{\frac{1}{p-1}} \int_0^\infty \left(\frac{\sigma(\tilde{Q}_{3\rho}(x,t))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \min\{1, \left(\frac{\rho}{R}\right)^{-\delta}\} \frac{d\rho}{\rho} \\
&\leq c_2 \left(\mathbb{M}[\mu,\sigma](x,t)\right)^{\frac{1}{p-1}}.
\end{aligned}$$

Thus, for any $\lambda > 0$, supp$\sigma \subset \{c_2 \left(\mathbb{M}[\mu,\sigma]\right)^{\frac{1}{p-1}} > \lambda\} = \{\mathbb{M}[\mu,\sigma] > \left(\frac{\lambda}{c_2}\right)^{p-1}\}$. By Vitali Covering Lemma one can cover suppσ with a union of $\tilde{Q}_{3\rho_i}(x_i,t_i)$ for $i = 1,...,m(K)$ so that

$\tilde{Q}_{\rho_i}(x_i,t_i)$ are disjoint and $\sigma(\tilde{Q}_{3\rho_i}(x_i,t_i)) < (\lambda/c_2)^{-p+1}\mu(\tilde{Q}_{\rho_i}(x_i,t_i))$. It follows that

$$\begin{aligned}
\text{Cap}_{E_\alpha^R,p}(K) &\le c_3 \sum_{i=1}^{m(K)} \sigma(\tilde{Q}_{3\rho_i}(x_i,t_i)) \\
&\le c_3 c_2^{p-1}\lambda^{-p+1} \sum_{i=1}^{m(K)} \mu(\tilde{Q}_{\rho_i}(x_i,t_i)) \\
&\le c_3 c_2^{p-1}\lambda^{-p+1}\mu(\mathbb{R}^{N+1}).
\end{aligned}$$

So, for all compact subset K of $\{\mathbb{W}_{\alpha,p}^{R,\delta}[\mu] > \lambda\}$,

$$\text{Cap}_{E_\alpha^{R,\delta/p'},p}(K) \le c_1 c_2^{p-1}\lambda^{-p+1}\mu(\mathbb{R}^{N+1}).$$

Therefore we obtain (4.45). ■

Remark 4.31 *Let $0 < \delta < \alpha < N+2$ and $\delta \le 1$. From the following inequality*

$$\begin{aligned}
&|\max\{|x_1-z|,\sqrt{2|t_1-s|}\}^{-N-2+\alpha} - \max\{|x_2-z|,\sqrt{2|t_2-s|}\}^{-N-2+\alpha}| \\
&\le c_1\left(\max\{|x_1-z|,\sqrt{2|t_1-s|}\}^{-N-2+\alpha-\delta} + \max\{|x_2-z|,\sqrt{2|t_2-s|}\}^{-N-2+\alpha-\delta}\right) \\
&\quad \times \left(|x_1-x_2| + |t_1-t_2|^{1/2}\right)^\delta,
\end{aligned}$$

for all $(x_1,t_1),(x_2,t_2),(z,s) \in \mathbb{R}^{N+1}$, where c_1 is a constant depending on N,α,δ. Thus, for $\mu \in \mathfrak{M}_b^+(\mathbb{R}^{N+1})$

$$|\mathbb{I}_\alpha[\mu](x_1,t_1) - \mathbb{I}_\alpha[\mu](x_2,t_2)| \le c_2\left(\mathbb{I}_{\alpha-\delta}[\mu](x_1,t_1) + \mathbb{I}_{\alpha-\delta}[\mu](x_2,t_2)\right)\left(|x_1-x_2| + |t_1-t_2|^{1/2}\right)^\delta,$$

for all $(x_1,t_1),(x_2,t_2) \in \mathbb{R}^{N+1}$ and $c_2 = c_1\frac{N+2-\alpha+\delta}{N+2-\alpha}$.
Consequently, for any $\mu \in \mathfrak{M}_b^+(\mathbb{R}^{N+1})$, $\mathbb{I}_\alpha[\mu]$ is $\delta-$Holder $Cap_{E_{\frac{\alpha-\delta}{2}},2}$-quasicontinuous this means, for any $\varepsilon > 0$ there exists a Borel set $O_\varepsilon \subset \mathbb{R}^{N+1}$ and $c_\varepsilon > 0$ such that

$$|\mathbb{I}_\alpha[\mu](x_1,t_1) - \mathbb{I}_\alpha[\mu](x_2,t_2)| \le c_\varepsilon\left(|x_1-x_2| + |t_1-t_2|^{1/2}\right)^\delta \quad \forall (x_1,t_1),(x_2,t_2) \in O_\varepsilon$$

and $Cap_{E_{\frac{\alpha-\delta}{2}},2}(\mathbb{R}^{N+1}\backslash O_\varepsilon) < \varepsilon$.

Now we are ready to prove Proposition 4.23.
Proof of Proposition 4.23. By Lemma 4.26, 4.29 and 4.30, there is the capacitary measure μ of a compact subset $K \subset \mathbb{R}^{N+1}$ such that $\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu] \ge c_1$ a.e in K, $\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu] \le c_2$ a.e in $\mathbb{R}^{N+1}$ and $\text{Cap}_{E_\alpha^{R,\delta},p}(\{\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu] > \lambda\}) \le c_2\lambda^{-p+1}Cap_{E_\alpha^{R,\delta},p}(K)$ for all $\lambda > 0$, $(\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu])^\beta \in A_1$ for any $0 < \beta < \frac{(N+2)(p-1)}{N+2-\alpha p+\delta p}$. From second assumption we have

$$\int_{\mathbb{R}^{N+1}} |g|(\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu])^\beta dxdt \le C_2 \int_{\mathbb{R}^{N+1}} |f|(\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu])^\beta dxdt.$$

Thus

$$\begin{aligned}
\int_K |g|dxdt &\le c_1^{-\delta} \int_{\mathbb{R}^{N+1}} |g|(\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu])^\beta dxdt \\
&\le c_3 \int_{\mathbb{R}^{N+1}} |f|(\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu])^\beta dxdt \\
&= c_3\beta \int_0^{c_1} \int_{\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu]>\lambda} |f|dxdt\lambda^{\beta-1}d\lambda.
\end{aligned}$$

By first assumption we get

$$\int_{\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu]>\lambda} |f|dxdt \leq C_1 \mathrm{Cap}_{E_\alpha^{R,\delta},p}(\{\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu] > \lambda\}) \leq c_4 \lambda^{-p+1} \mathrm{Cap}_{E_\alpha^{R,\delta},p}(K).$$

Therefore,

$$\int_K |g|dxdt \leq c_5 \delta \int_0^{c_1} \lambda^{-p+1} \mathrm{Cap}_{E_\alpha^{R,\delta},p}(K) \lambda^{\delta-1} d\lambda = c_6 \mathrm{Cap}_{E_\alpha^{R,\delta},p}(K),$$

since one can choose $\delta > p-1$. This completes the proof of the Proposition. ∎

Definition 4.32 *Let $s > 1$, $\alpha > 0$. We define the space $\mathfrak{M}^{\mathcal{H}_\alpha,s}(\mathbb{R}^{N+1})$ ($\mathfrak{M}^{\mathcal{G}_\alpha,s}(\mathbb{R}^{N+1})$ resp.) to be the set of all measure $\mu \in \mathfrak{M}(\mathbb{R}^{N+1})$ such that*

$$[\mu]_{\mathfrak{M}^{\mathcal{H}_\alpha,s}(\mathbb{R}^{N+1})} := \sup\left\{|\mu|(K)/Cap_{\mathcal{H}_\alpha,s}(K) : \ Cap_{\mathcal{H}_\alpha,s}(K) > 0\right\} < \infty,$$
$$\left([\mu]_{\mathfrak{M}^{\mathcal{G}_\alpha,s}(\mathbb{R}^{N+1})} := \sup\left\{|\mu|(K)/Cap_{\mathcal{G}_\alpha,s}(K) : Cap_{\mathcal{G}_\alpha,s}(K) > 0\right\} < \infty \ resp.\right)$$

where the supremum is taken all compact sets $K \subset \mathbb{R}^{N+1}$.
For simplicity, we will write $\mathfrak{M}^{\mathcal{H}_\alpha,s}, \mathfrak{M}^{\mathcal{G}_\alpha,s}$ to denote $\mathfrak{M}^{\mathcal{H}_\alpha,s}(\mathbb{R}^{N+1}), \mathfrak{M}^{\mathcal{G}_\alpha,s}(\mathbb{R}^{N+1})$ resp.

We see that if $\alpha s \geq N+2$, $\mathfrak{M}^{\mathcal{H}_\alpha,s}(\mathbb{R}^{N+1}) = \{0\}$, if $\alpha s < N+2$, $\mathfrak{M}^{\mathcal{H}_\alpha,s}(\mathbb{R}^{N+1}) \subset \mathfrak{M}^{\mathcal{G}_\alpha,s}(\mathbb{R}^{N+1})$. On the other hand, $\mathfrak{M}^{\mathcal{G}_\alpha,s}(\mathbb{R}^{N+1}) \supset \mathfrak{M}_b(\mathbb{R}^{N+1})$ if $\alpha s > N+2$.
We now have the following two remarks:

Remark 4.33 *For $s > 1$, there is $C = C(N, \alpha, s) > 0$ such that*

$$[f]_{\mathfrak{M}^{\mathcal{G}_\alpha,p}} \leq C[|f|^s]^{1/s}_{\mathfrak{M}^{\mathcal{G}_\alpha,p}} \quad \text{for all function } f. \tag{4.46}$$

Indeed, set $a = [|f|^s]_{\mathfrak{M}^{\mathcal{G}_\alpha,p}}$, so for any compact set K in $\mathbb{R}^{N+1}$,

$$\int_K |f|^s dxdt \leq a Cap_{\mathcal{G}_\alpha,p}(K).$$

This gives $2a Cap_{\mathcal{G}_\alpha,p}(K) \geq \int_K (|f|^s + c_1 a)\, dxdt \geq c_2 a^{1-1/s} \int_K |f| dxdt$, here we used (4.24) in Remark 4.11 at the first inequality and Holder's inequality at the second one. It follows (4.46).

Remark 4.34 *Assume that $p > 1$ and $\frac{2}{p} < \alpha < N + \frac{2}{p}$. Clearly, from Corollary 4.20 we assert that for $\omega \in \mathfrak{M}^+(\mathbb{R}^N)$*

$$C_1^{-1} [\omega]_{\mathfrak{M}^{\mathbf{I}_{\alpha-2/p},p}} \leq \left[\omega \otimes \delta_{\{t=0\}}\right]_{\mathfrak{M}^{\mathcal{H}_\alpha,p}} \leq C_1 [\omega]_{\mathfrak{M}^{\mathbf{I}_{\alpha-2/p},p}},$$
$$C_2^{-1} [\omega]_{\mathfrak{M}^{\mathbf{G}_{\alpha-2/p},p}} \leq \left[\omega \otimes \delta_{\{t=0\}}\right]_{\mathfrak{M}^{\mathcal{G}_\alpha,p}} \leq C_2 [\omega]_{\mathfrak{M}^{\mathbf{G}_{\alpha-2/p},p}},$$

for some $C_i = C_i(N,p,\alpha)$, $i = 1,2$. Where $\mathfrak{M}^{\mathbf{I}_{\alpha-2/p},p} := \mathfrak{M}^{\mathbf{I}_{\alpha-2/p},p}(\mathbb{R}^N)$, $\mathfrak{M}^{\mathbf{G}_{\alpha-2/p},p} := \mathfrak{M}^{\mathbf{G}_{\alpha-2/p},p}(\mathbb{R}^N)$ and

$$[\omega]_{\mathfrak{M}^{\mathbf{I}_{\alpha-2/p},p}(\mathbb{R}^N)} := \sup\left\{\omega(K)/Cap_{\mathbf{I}_{\alpha-2/p},p}(K) : \ Cap_{\mathbf{I}_{\alpha-2/p},p}(K) > 0\right\},$$
$$[\omega]_{\mathfrak{M}^{\mathbf{G}_{\alpha-2/p},p}(\mathbb{R}^N)} := \sup\left\{\omega(K)/Cap_{\mathbf{G}_{\alpha-2/p},p}(K) : \ Cap_{\mathbf{G}_{\alpha-2/p},p}(K) > 0\right\},$$

where the supremum is taken all compact sets $K \subset \mathbb{R}^N$.

Clearly, Theorem 4.2 and Proposition 4.23 lead to the following result.

Proposition 4.35 *Let $q > p-1$, $s > 1$ and $0 < \alpha p < N+2$. Then the following quantities are equivalent*

$$\left[(\mathbb{W}_{\alpha,p}^{R}[\mu])^{q}\right]_{\mathfrak{M}^{\mathcal{H}_{\alpha,s}}},\quad \left[(\mathbb{I}_{\alpha p}^{R}[\mu])^{\frac{q}{p-1}}\right]_{\mathfrak{M}^{\mathcal{H}_{\alpha,s}}}\quad and \quad \left[(\mathbb{M}_{\alpha p}^{R}[\mu])^{\frac{q}{p-1}}\right]_{\mathfrak{M}^{\mathcal{H}_{\alpha,s}}},$$

for every $\mu \in \mathfrak{M}^{+}(\mathbb{R}^{N+1})$ and $0 < R \le \infty$.

In the next result, we present a characterization of the following trace inequality:

$$||E_{\alpha}^{R,\delta} * f||_{L^{p}(\mathbb{R}^{N+1},d\mu)} \le C_1||f||_{L^{p}(\mathbb{R}^{N+1})} \quad \forall f \in L^{p}(\mathbb{R}^{N+1}). \tag{4.47}$$

Theorem 4.36 *Let $0 < R \le \infty$,$1 < p < \alpha^{-1}(N+2)$, $0 < \delta < \alpha$ and μ be a nonnegative Radon measure on $\mathbb{R}^{N+1}$. Then the following statements are equivalent.*

1. *The trace inequality* (4.47) *holds.*

2. *There holds*

$$||E_{\alpha}^{R,\delta} * f||_{L^{p}(\mathbb{R}^{N+1},d\omega)} \le C_2||f||_{L^{p}(\mathbb{R}^{N+1})} \quad \forall f \in L^{p}(\mathbb{R}^{N+1}), \tag{4.48}$$

where $d\omega = (\mathbb{I}_{\alpha}^{R,\delta}\mu)^{p'}dxdt$.

3. *There holds*

$$||E_{\alpha}^{R,\delta} * f||_{L^{p,\infty}(\mathbb{R}^{N+1},d\mu)} \le C_3||f||_{L^{p}(\mathbb{R}^{N+1})} \quad \forall f \in L^{p}(\mathbb{R}^{N+1}). \tag{4.49}$$

4. *For every compact set $E \subset \mathbb{R}^{N+1}$,*

$$\mu(E) \le C_4 Cap_{E_{\alpha}^{R,\delta},p}(E). \tag{4.50}$$

5. *$\mathbb{I}_{\alpha}^{R,\delta}[\mu] < \infty$ a.e and*

$$\mathbb{I}_{\alpha}^{R,\delta}[(\mathbb{I}_{\alpha}^{R,\delta}[\mu])^{p'}] \le C_5\mathbb{I}_{\alpha}^{R,\delta}[\mu] \quad a.e. \tag{4.51}$$

6. *For every compact set $E \subset \mathbb{R}^{N+1}$,*

$$\int_{E}(\mathbb{I}_{\alpha}^{R,\delta}[\mu])^{p'}dxdt \le C_6 Cap_{E_{\alpha}^{R,\delta},p}(E). \tag{4.52}$$

7. *For every compact set $E \subset \mathbb{R}^{N+1}$,*

$$\int_{\mathbb{R}^{N+1}}(\mathbb{I}_{\alpha}^{R,\delta}[\mu\chi_E])^{p'}dxdt \le C_7\mu(E). \tag{4.53}$$

8. *For every compact set $E \subset \mathbb{R}^{N+1}$,*

$$\int_{E}(\mathbb{I}_{\alpha}^{R,\delta}[\mu\chi_E])^{p'}dxdt \le C_8\mu(E). \tag{4.54}$$

We can find a simple sufficient condition on μ so that trace inequality (4.47) is satisfied from the isoperimetric inequality (4.39).

Proof of Theorem 4.36. As in [80] we can show that $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4 \Leftrightarrow 6 \Leftrightarrow 7$ and $7 \Rightarrow 8, 5 \Rightarrow 2$. Thus, it is enough to show that $8. \Rightarrow 5$. First, we need to show that

$$\left(\int_{r}^{\infty}\frac{\mu(\tilde{Q}_{\rho}(x,t))}{\rho^{N+2-\alpha}}\min\{1,\left(\frac{\rho}{R}\right)^{-\delta}\}\frac{d\rho}{\rho}\right)^{p'-1} \le c_1 r^{-\alpha}\left(\min\{1,\left(\frac{r}{R}\right)^{-\delta}\}\right)^{-1} \tag{4.55}$$

We have for any $(y,s)\in \tilde{Q}_r(x,t)$

$$\begin{aligned}\mathbb{I}_\alpha^{R,\delta}[\mu\chi_{\tilde{Q}_r(x,t)}](y,s) &= \int_0^\infty \frac{\mu(\tilde{Q}_r(x,t)\cap\tilde{Q}_\rho(y,s))}{\rho^{N+2-\alpha}}\min\{1,\Big(\frac{\rho}{R}\Big)^{-\delta}\}\frac{d\rho}{\rho}\\ &\geq \int_{2r}^{4r} \frac{\mu(\tilde{Q}_r(x,t)\cap\tilde{Q}_\rho(y,s))}{\rho^{N+2-\alpha}}\min\{1,\Big(\frac{\rho}{R}\Big)^{-\delta}\}\frac{d\rho}{\rho}\\ &\geq c_2\frac{\mu(\tilde{Q}_r(x,t))}{r^{N+2-\alpha}}\min\{1,\Big(\frac{r}{R}\Big)^{-\delta}\}.\end{aligned}$$

In (4.54), we take $E=\tilde{Q}_r(x,t)$

$$\begin{aligned}c\mu(\tilde{Q}_r(x,t)) &\geq \int_{\tilde{Q}_r(x,t)}(\mathbb{I}_\alpha[\mu\chi_{\tilde{Q}_r(x,t)}])^{p'}\\ &\geq c_2^{p'}\left(\frac{\mu(\tilde{Q}_r(x,t))}{r^{N+2-\alpha}}\min\{1,\Big(\frac{r}{R}\Big)^{-\delta}\}\right)^{p'}|\tilde{Q}_r(x,t)|.\end{aligned}$$

So $\mu(\tilde{Q}_r(x,t))\leq c_3 r^{N+2-\alpha p}\left(\min\{1,\big(\frac{r}{R}\big)^{-\delta}\}\right)^{-p}$ which implies (4.55).
Next we set

$$L_r[\mu](x,t)=\int_r^{+\infty}\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho}\min\{1,\Big(\frac{\rho}{R}\Big)^{-\delta}\}\frac{d\rho}{\rho},$$

$$U_r[\mu](x,t)=\int_0^{r}\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho}\min\{1,\Big(\frac{\rho}{R}\Big)^{-\delta}\}\frac{d\rho}{\rho},$$

and

$$d\omega=(I_\alpha\mu)^{p'}dxdt,\quad d\sigma_{1,r}=(L_r[\mu])^{p'}dxdt,\quad d\sigma_{2,r}=(U_r[\mu])^{p'}dxdt.$$

We have $d\omega\leq 2^{p'-1}\left(d\sigma_{1,r}+d\sigma_{2,r}\right)$. To prove (4.51) we need to show that

$$\int_0^\infty\frac{\sigma_{1,r}(\tilde{Q}_r(x,t))}{r^{N+2-\alpha}}\min\{1,\Big(\frac{r}{R}\Big)^{-\delta}\}\frac{dr}{r}\leq c_4\mathbb{I}_\alpha^{R,\delta}[\mu](x,t), \tag{4.56}$$

$$\int_0^\infty\frac{\sigma_{2,r}(\tilde{Q}_r(x,t))}{r^{N+2-\alpha}}\min\{1,\Big(\frac{r}{R}\Big)^{-\delta}\}\frac{dr}{r}\leq c_5\mathbb{I}_\alpha^{R,\delta}[\mu](x,t). \tag{4.57}$$

Since, for all $r>0$, $0<\rho<r$ and $(y,s)\in\tilde{Q}_r(x,t)$ we have $\tilde{Q}_\rho(y,s)\subset\tilde{Q}_{2r}(x,t)$. So,

$$\sigma_{2,r}(\tilde{Q}_r(x,t))=\int_{\tilde{Q}_r(x,t)}(U_r[\mu](y,s))^{p'}dyds=\int_{\tilde{Q}_r(x,t)}\left(U_r[\mu\chi_{\tilde{Q}_{2r}(x,t)}](y,s)\right)^{p'}dyds.$$

Thus, from (4.54) we get

$$\begin{aligned}\sigma_{2,r}(\tilde{Q}_r(x,t)) &\leq \int_{\tilde{Q}_{2r}(x,t)}\left(U_r[\mu\chi_{\tilde{Q}_{2r}(x,t)}](y,s)\right)^{p'}dyds\\ &\leq \int_{\tilde{Q}_{2r}(x,t)}\left(\mathbb{I}_\alpha^{R,\delta}[\mu\chi_{\tilde{Q}_{2r}(x,t)}](y,s)\right)^{p'}dyds\\ &\leq c_6\mu(\tilde{Q}_{2r}(x,t)).\end{aligned}$$

Therefore, (4.57) follows.
Since, for all $r>0$, $\rho\geq r$ and $(y,s)\in\tilde{Q}_r(x,t)$ we have $\tilde{Q}_\rho(y,s)\subset\tilde{Q}_{2\rho}(x,t)$. So, for all $(y,s)\in\tilde{Q}_r(x,t)$ we have

$$\begin{aligned}L_r[\mu](y,s) &\leq \int_r^{+\infty}\frac{\mu(\tilde{Q}_{2\rho}(x,t))}{\rho^{N+2-\alpha}}\min\{1,\Big(\frac{\rho}{R}\Big)^{-\delta}\}\frac{d\rho}{\rho}\\ &\leq c_7 L_r[\mu](x,t).\end{aligned}$$

Hence,

$$\sigma_{1,r}(\tilde{Q}_r(x,t)) = \int_{\tilde{Q}_r(x,t)} (L_r[\mu](y,s))^{p'} dyds$$
$$\leq c_8 r^{N+2} (L_r[\mu](x,t))^{p'}.$$

Since $r^{\alpha-1}\min\{1,\left(\frac{r}{R}\right)^{-\delta}\} \leq \frac{1}{\alpha-\delta}\frac{d}{dr}\left(r^\alpha \min\{1,\left(\frac{r}{R}\right)^{-\delta}\}\right)$, we deduce that

$$\int_0^\infty \frac{\sigma_{1,r}(\tilde{Q}_r(x,t))}{r^{N+2-\alpha}} \min\{1,\left(\frac{r}{R}\right)^{-\delta}\}\frac{dr}{r} \leq c_7 \int_0^\infty r^{\alpha-1} (L_r[\mu](x,t))^{p'} \min\{1,\left(\frac{r}{R}\right)^{-\delta}\}dr$$
$$\leq \frac{c_7}{\alpha-\delta}\int_0^\infty \frac{d}{dr}\left(r^\alpha \min\{1,\left(\frac{r}{R}\right)^{-\delta}\}\right)(L_r[\mu](x,t))^{p'} dr$$
$$\leq c_8 \int_0^\infty r^\alpha (L_r[\mu](x,t))^{p'-1} \frac{\mu(\tilde{Q}_r(x,t))}{r^{N+2-\alpha}} \min\{1,\left(\frac{r}{R}\right)^{-\delta}\}^2 \frac{dr}{r}.$$

Therefore, we get (4.56) from (4.55). This completes the proof of Theorem. ■

Remark 4.37 *It is easy to assert that if* ***8.*** *holds then for any* $0<\beta<N+2$

$$\mathbb{I}_\beta\left[\left(\mathbb{I}_\alpha^{R,\delta}[\mu]\right)^{p'}\right] \leq C\mathbb{I}_\beta[\mu], \tag{4.58}$$

for some $C = C(N,\alpha,\beta,\delta,p) > 0$.

Corollary 4.38 *Let* $p>1, \alpha>0$ *such that* $0<\alpha p<N+2$. *There holds*

$$C_1^{-1}[\mu]^{p'}_{\mathfrak{M}^{\mathcal{H}_{\alpha,p}}} \leq \left[(\mathbb{I}_\alpha[\mu])^{p'}\right]_{\mathfrak{M}^{\mathcal{H}_{\alpha,p}}} \leq C_1 [\mu]^{p'}_{\mathfrak{M}^{\mathcal{H}_{\alpha,p}}} \tag{4.59}$$

for all $\mu\in\mathfrak{M}^+(\mathbb{R}^{N+1})$. *Furthermore,*

$$[\varphi_n * \mu]_{\mathfrak{M}^{\mathcal{H}_{\alpha,p}}} \leq C_2 [\mu]_{\mathfrak{M}^{\mathcal{H}_{\alpha,p}}} \tag{4.60}$$

for $n\in\mathbb{N}$, $\mu\in\mathfrak{M}^+(\mathbb{R}^{N+1})$ *where* $\{\varphi_n\}$ *is a sequence of mollifiers in* $\mathbb{R}^{N+1}$. *Here* $C_i = C_i(N,p,\alpha)$, $i=1,2$.

Proof. For $R=\infty$ we have $\mathbb{I}_\alpha^{R,\delta}[\mu] = \mathbb{I}_\alpha[\mu]$ and $E_\alpha^{R,\delta} = E_\alpha$. Thus, by (4.20) in Corollary 4.10 and Theorem 4.36 we get for every compact set $E\subset\mathbb{R}^{N+1}$,

$$\mu(E) \leq c_1 \text{Cap}_{\mathcal{H}_{\alpha,p}}(E)$$

if and only if for every compact set $E\subset\mathbb{R}^{N+1}$,

$$\int_E (\mathbb{I}_\alpha[\mu])^{p'} dxdt \leq c_2 \text{Cap}_{\mathcal{H}_{\alpha,p}}(E).$$

It follows (4.59).
Since $\mathbb{I}_\alpha[\varphi_n*\mu] = \varphi_n * \mathbb{I}_\alpha[\mu] \leq \mathbb{M}(\mathbb{I}_\alpha[\mu])$ and $\mathbb{M}$ is bounded in $L^{p'}(\mathbb{R}^{N+1},dw)$ with $w\in A_{p'}$ yield

$$\int_{\mathbb{R}^{N+1}} (\mathbb{I}_\alpha[\varphi_n*\mu])^{p'} dw \leq c_3([w]_{A_{p'}}) \int_{\mathbb{R}^{N+1}} (\mathbb{I}_\alpha[\mu])^{p'} dw.$$

Thanks to Proposition 4.23 we have

$$\left[(\mathbb{I}_\alpha[\varphi_n*\mu])^{p'}\right]_{\mathfrak{M}^{\mathcal{H}_{\alpha,p}}} \leq c_4 \left[(\mathbb{I}_\alpha[\mu])^{p'}\right]_{\mathfrak{M}^{\mathcal{H}_{\alpha,p}}},$$

which implies (4.60). ■

Corollary 4.39 *Let $p>1$, $\alpha>0$ with $0<\alpha p\le N+2$, $0<\delta<\alpha$ and $R,d>0$. There holds*

$$\left[\left(\mathbb{I}_\alpha^{R,\delta}[\mu]\right)^{p'}\right]_{\mathfrak{M}^{\mathcal{G}_{\alpha,p}}}\le C_1(d/R,R)\,[\mu]^{p'}_{\mathfrak{M}^{\mathcal{G}_{\alpha,p}}} \tag{4.61}$$

for all $\mu\in\mathfrak{M}^+(\mathbb{R}^{N+1})$ with diam(supp(μ)) $\le d$. Furthermore,

$$[\varphi_n*\mu]_{\mathfrak{M}^{\mathcal{G}_{\alpha,p}}}\le C_2(d)\,[\mu]_{\mathfrak{M}^{\mathcal{G}_{\alpha,p}}} \tag{4.62}$$

for $n\in\mathbb{N}$, $\mu\in\mathfrak{M}^+(\mathbb{R}^{N+1})$ with diam(supp(μ)) $\le d$ where $\{\varphi_n\}$ is a sequence of standard mollifiers in $\mathbb{R}^{N+1}$.

Proof. It is easy to see that

$$(c_1(d/R))^{-1}||E_\alpha^R[\mu]||_{L^{p'}(\mathbb{R}^{N+1})}\le||E_\alpha^{R,\delta}*\mu||_{L^{p'}(\mathbb{R}^{N+1})}\le c_1(d/R)||E_\alpha^R[\mu]||_{L^{p'}(\mathbb{R}^{N+1})}$$

for any $\mu\in\mathfrak{M}^+(\mathbb{R}^{N+1})$ with diam(supp(μ)) $\le d$, thus two quantities $\text{Cap}_{E_\alpha^{R,\delta},p}(E)$ and $\text{Cap}_{E_\alpha^R,p}(E)$ are equivalent for every compact set $E\subset\mathbb{R}^{N+1}$, diam$(E)\le d$ where equivalent constants depend only on N,p,α and d/R. Therefore, by Corollary 4.10 we get $\text{Cap}_{E_\alpha^{R,\delta},p}(E)\approx\text{Cap}_{\mathcal{G}_\alpha,p}(E)$ for every compact set $E\subset\mathbb{R}^{N+1}$, diam$(E)\le d$ where equivalent constants depend on d/R and R. Thus, by Theorem 4.36 and diam(supp(μ)) $\le d$ we get, if for every compact set $E\subset\mathbb{R}^{N+1}$,

$$\mu(E)\le c_2(d/R,R)\text{Cap}_{\mathcal{G}_\alpha,p}(E),$$

then for every compact set $E\subset\mathbb{R}^{N+1}$,

$$\int_E\left(\mathbb{I}_\alpha^{R,\delta}[\mu]\right)^{p'}dxdt\le c_3(d/R,R)\text{Cap}_{E_\alpha^{R,\delta},p}(E)\le c_4(d/R,R)\text{Cap}_{\mathcal{G}_\alpha,p}(E).$$

It follows (4.61). As in the Proof of Corollary 4.38 we also have for $w\in A_{p'}$

$$\int_{\mathbb{R}^{N+1}}\left(\mathbb{I}_\alpha^{1,\delta}[\varphi_n*\mu]\right)^{p'}dw\le c_5([w]_{A_{p'}})\int_{\mathbb{R}^{N+1}}\left(\mathbb{I}_\alpha^{1,\delta}[\mu]\right)^{p'}dw.$$

Thanks to Proposition 4.23 and Theorem 4.36 we obtain (4.62). ■

Remark 4.40 *Likewise (see [71, Lemma 5.7]), we can verify that if $\frac{2}{p}<\alpha<N+\frac{2}{p}$,*

$$[\varphi_{1,n}*\omega_1]_{\mathfrak{M}^{\mathbf{I}_{\alpha-2/p},p}}\le C_1\,[\omega_1]_{\mathfrak{M}^{\mathbf{I}_{\alpha-2/p},p}}\quad\textit{and}$$
$$[\varphi_{1,n}*\omega_2]_{\mathfrak{M}^{\mathbf{G}_{\alpha-2/p},p}}\le C_2(d)\,[\omega_2]_{\mathfrak{M}^{\mathbf{G}_{\alpha-2/p},p}},$$

for $n\in\mathbb{N}$ and $\omega_1,\omega_2\in\mathfrak{M}^+(\mathbb{R}^N)$ with diam(supp(ω_2)) $\le d$ where $C_1=C_1(N,\alpha,p)$, $C_2(d)=C_2(N,\alpha,p,d)$, $\{\varphi_{1,n}\}$ is a sequence of standard mollifiers in $\mathbb{R}^N$ and $[.]_{\mathfrak{M}^{\mathbf{I}_{\alpha-2/p},p}}$, $[.]_{\mathfrak{M}^{\mathbf{G}_{\alpha-2/p},p}}$ was defined in Remark 4.34. Hence, we obtain

$$\left[(\varphi_{1,n}*\omega_1)\otimes\delta_{\{t=0\}}\right]_{\mathfrak{M}^{\mathcal{H}_{\alpha,p}}}\le C_3\left[\omega_1\otimes\delta_{\{t=0\}}\right]_{\mathfrak{M}^{\mathcal{H}_{\alpha,p}}},$$
$$\left[(\varphi_{1,n}*\omega_2)\otimes\delta_{\{t=0\}}\right]_{\mathfrak{M}^{\mathcal{G}_{\alpha,p}}}\le C_4(d)\left[\omega_2\otimes\delta_{\{t=0\}}\right]_{\mathfrak{M}^{\mathcal{G}_{\alpha,p}}},$$

for $n\in\mathbb{N}$ and $\omega_1,\omega_2\in\mathfrak{M}^+(\mathbb{R}^{N+1})$, diam(supp($\mu$)) $\le d$ where $C_3=C_3(N,\alpha,p)$, $C_4(d)=C_4(N,\alpha,p,d)$.

Proposition 4.41 *Let $q>1$, $0<\alpha q<N+2$, $0<R\le\infty, 0<\delta<\alpha$ and $K>0$. Let $0\le f\in L^q_{loc}(\mathbb{R}^{N+1})$. Let C_4,C_5 be constants in inequalities (4.50) and (4.51) in Theorem 4.36 with $p=q'$. Suppose that $\{u_n\}$ is a sequence of nonnegative measurable functions in $\mathbb{R}^{N+1}$ satisfying*

$$\begin{aligned}u_{n+1}&\le K\mathbb{I}_\alpha^{R,\delta}[u_n^q]+f\quad\forall n\in\mathbb{N}\\ u_0&\le f\end{aligned} \tag{4.63}$$

Then, if for every compact set $E \subset \mathbb{R}^{N+1}$,

$$\int_E f^q dxdt \le C\, Cap_{E_\alpha^{R,\delta},q'}(E) \tag{4.64}$$

with

$$C \le C_4 \left(\frac{2^{-q+1}}{C_5(q-1)} \left(\frac{q-1}{qK2^{q-1}} \right)^q \right)^{q-1}, \tag{4.65}$$

then

$$u_n \le \frac{Kq2^{q-1}}{q-1} \mathbb{I}_\alpha^{R,\delta}[f^q] + f \quad \forall n \in \mathbb{N}. \tag{4.66}$$

Proof. From (4.50) and (4.51) in Theorem 4.36, we see that (4.64) implies

$$\mathbb{I}_\alpha^{R,\delta}[(I_\alpha^{R,\delta}[f^q])^q] \le \left(\frac{C}{C_4} \right)^{\frac{1}{q-1}} C_5 \mathbb{I}_\alpha^{R,\delta}[f^q]. \tag{4.67}$$

Now we prove (4.66) by induction. Clearly, (4.66) holds with $n = 0$. Next we assume that (4.66) holds with $n = m$. Then, by (4.65), (4.67) and (4.63) we have

$$\begin{aligned} u_{m+1} &\le K\mathbb{I}_\alpha^{R,\delta}[u_n^q] + f \\ &\le K2^{q-1} \left(\frac{Kq2^{q-1}}{q-1} \right)^q \mathbb{I}_\alpha^{R,\delta}[(\mathbb{I}_\alpha^{R,\delta}[f^q])^q] + K2^{q-1}\mathbb{I}_\alpha^{R,\delta}[f^q] + f \\ &\le K2^{q-1} \left(\frac{Kq2^{q-1}}{q-1} \right)^q \left(\frac{C}{C_4} \right)^{\frac{1}{q-1}} C_5 \mathbb{I}_\alpha^{R,\delta}[f^q] + K2^{q-1}\mathbb{I}_\alpha^{R,\delta}[f^q] + f \\ &\le \frac{Kq2^{q-1}}{q-1} \mathbb{I}_\alpha^{R,\delta}[f^q] + f. \end{aligned}$$

Therefore (4.66) also holds true with $n = m + 1$. This completes the proof of the Theorem. ■

Corollary 4.42 *Let* $q > \frac{N+2}{N+2-\alpha}$, $\alpha > 0$ *and* $f \in L_+^q(\mathbb{R}^{N+1})$. *There exists a constant* $C > 0$ *depending on* N, α, q *such that if for every compact set* $E \subset \mathbb{R}^{N+1}$, $\int_E f^q dxdt \le C\, Cap_{\mathcal{H}_\alpha,q'}(E)$, *then* $u = \mathcal{H}_\alpha[u^q] + f$ *admits a positive solution* $u \in L_{loc}^q(\mathbb{R}^{N+1})$.

Proof. Consider the sequence $\{u_n\}$ of nonnegative functions defined by $u_0 = f$ and $u_{n+1} = \mathcal{H}_\alpha[u_n^q] + f \ \ \forall\, n \ge 0$. It is easy to see that $u_{n+1} \le c_1\mathbb{I}_2[u_n^q] + f \ \ \forall n \ge 0$. By Proposition 4.41 and Corollary 4.38, there exists a constant $c_2 = c_2(N, \alpha, q) > 0$ such that if for every compact set $E \subset \mathbb{R}^{N+1}$, $\int_E f^q dxdt \le c_2 \mathrm{Cap}_{\mathcal{H}_\alpha,q'}(E)$ then u_n is well defined and

$$u_n \le \frac{c_1 q 3^{q-1}}{q-1} \mathbb{I}_\alpha[f^q] + f \quad \forall n \ge 0.$$

Since $\{u_n\}$ is nondecreasing, thus thanks to the dominated convergence theorem we obtain $u(x,t) = \lim_{n\to\infty} u_n(x,t)$ is a solution of $u = \mathcal{H}_\alpha[u^q] + f$ which $u \in L_{\mathrm{loc}}^q(\mathbb{R}^{N+1})$. This completes the proof of the Corollary. ■

Corollary 4.43 *Let* $q > 1$, $\alpha > 0$, $0 < R \le \infty, 0 < \delta < \alpha$ *and* $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$. *The following two statements are equivalent.*

a. *for every compact set* $E \subset \mathbb{R}^{N+1}$, $\int_E f^q \le C\, Cap_{E_\alpha^{R,\delta},q'}(E)$ *for some a constant* $C > 0$

b. *There exists a function* $u \in L_{loc}^q(\mathbb{R}^{N+1})$ *such that* $u = \mathbb{I}_\alpha^{R,\delta}[u^q] + \varepsilon f$ *for some* $\varepsilon > 0$.

Proof. We will prove *b.* $\Rightarrow$ *a.* Set $d\omega(x,t) = \left(\left(\mathbb{I}_\alpha^{R,\delta}[u^q]\right)^q + \varepsilon^q f^q\right) dxdt$, thus we have $dw(x,t) \geq \left(I_\alpha^{R,\delta}[\omega]\right)^q dxdt$. Let $\mathbb{M}_\omega$ denote the centered Hardy-littlewoood maximal function which is defined for $g \in L^1_{loc}(\mathbb{R}^{N+1}, d\omega)$,

$$\mathbb{M}_\omega g(x,t) = \sup_{\rho>0} \frac{1}{\omega(\tilde{Q}_\rho(x,t))} \int_{\tilde{Q}_\rho(x,t)} |g| d\omega(x,t).$$

For $E \subset \mathbb{R}^{N+1}$ is a compact set, we have

$$\int_{\mathbb{R}^{N+1}} (\mathbb{M}_\omega \chi_E)^q \left(\mathbb{I}_\alpha^{R,\delta}[\omega]\right)^q dxdt \leq \int_{\mathbb{R}^{N+1}} (\mathbb{M}_\omega \chi_E)^q d\omega(x,t).$$

Since $\mathbb{M}_\omega$ is bounded on $L^s(\mathbb{R}^{N+1}, d\omega)$ for $s > 1$ and $(\mathbb{M}_\omega \chi_E)^q \left(\mathbb{I}_\alpha^{R,\delta}[\omega]\right)^q \geq \left(\mathbb{I}_\alpha^{R,\delta}[\omega\chi_E]\right)^q$, thus

$$\int_{\mathbb{R}^{N+1}} \left(\mathbb{I}_\alpha^{R,\delta}[\omega\chi_E]\right)^q dxdt \leq c_1 \omega(E).$$

By Theorem 4.36, we get for any compact set $E \subset \mathbb{R}^{N+1}$

$$\omega(E) \leq c_2 \mathrm{Cap}_{E_\alpha^{R,\delta}, q'}(E).$$

It follows the results. ∎

Remark 4.44 *In [64], we also use Theorem* (4.36) *to show existence of mild solutions to the Navier-Stokes Equations*

$$\begin{cases} \partial_t u - \Delta u + \mathbb{P} div(u \otimes u) = \mathbb{P}F & in\ \mathbb{R}^N \times (0,\infty), \\ u(0) = u_0 & in\ \mathbb{R}^N. \end{cases} \tag{4.68}$$

where $u, F \in \mathbb{R}^N$, $\mathbb{P} = id - \nabla\Delta^{-1}\nabla$. *is the Helmholtz Leray projection onto the vector fields of zero divergence, i.e, for* $f \in \mathbb{R}^N$, $\mathbb{P}f = f - \nabla u$ *and* $\Delta u = divf$. *Namely, there exists* $C = C(N) > 0$ *such that if* $div(u_0) = 0$ *and*

$$\int_K |D(x,t)|^2 dxdt \leq C\, Cap_{\mathcal{H}_1,2}(K), \tag{4.69}$$

for any compact set $K \subset \mathbb{R}^{N+1}$, *where if* $(x,t) \in \mathbb{R}^N \times [0,+\infty)$,

$$D(x,t) = (e^{t\Delta}u_0)(x) + \int_0^t (e^{(t-s)\Delta}\mathbb{P}F)(x)ds,$$

and $D(x,t) = 0$ *otherwise. Then, the equation* (4.68) *has global solution* u *satisfying*

$$|u(x,t)| \leq |D(x,t)| + c\mathbb{I}_1[|D|^2](x,t), \tag{4.70}$$

for all $(x,t) \in \mathbb{R}^N \times (0,\infty)$ *for some* $c = c(N)$.

5 Global point wise estimates of solutions to the parabolic equations

First, we recall Duzzar and Mingione's result [27], also see [42, 43] which involves local pointwise estimates for solutions of equations (2.4).

Theorem 5.1 *Then, there exists a constant* C *depending only* N, Λ_1, Λ_2 *such that if* $u \in L^2(0,T,H^1(\Omega)) \cap C(\Omega_T)$ *is a weak solution to* (2.4) *with* $\mu \in L^2(\Omega_T)$ *and* $u(0) = 0$

$$|u(x,t)| \leq C \fint_{\tilde{Q}_R(x,t)} |u| dyds + C\mathbb{I}_2^{2R}[|\mu|](x,t) \tag{5.1}$$

for all $Q_{2R}(x,t) \subset \Omega \times (-\infty, T)$.
Furthermore, if A *is independent of space variable* x, (2.27) *is satisfied and* $\nabla u \in C(\Omega_T)$ *then*

$$|\nabla u(x,t)| \le C \fint_{\tilde{Q}_R(x,t)} |\nabla u| dyds + C\mathbb{I}_1^{2R}[|\mu|](x,t) \tag{5.2}$$

for all $Q_{2R}(x,t) \subset \Omega \times (-\infty, T)$.

Proof of Theorem 2.1. Let $\mu = \mu_0 + \mu_s \in \mathfrak{M}_b(\Omega_T)$, with $\mu_0 \in \mathfrak{M}_0(\Omega_T), \mu_s \in \mathfrak{M}_s(\Omega_T)$. By Proposition 3.7, there exist sequences of nonnegative measures $\mu_{n,0,i} = (f_{n,i}, g_{n,i}, h_{n,i})$ and $\mu_{n,s,i}$ such that $f_{n,i}, g_{n,i}, h_{n,i} \in C_c^\infty(\Omega_T)$ and strongly converge to some f_i, g_i, h_i in $L^1(\Omega_T), L^2(\Omega_T, \mathbb{R}^N)$ and $L^2(0,T,H_0^1(\Omega))$ respectively and $\mu_{n,1}, \mu_{n,2}, \mu_{n,s,1}, \mu_{n,s,2} \in C_c^\infty(\Omega_T)$ converge to $\mu^+, \mu^-, \mu_s^+, \mu_s^-$ resp. in the narrow topology with $\mu_{n,i} = \mu_{n,0,i} + \mu_{n,s,i}$, for $i = 1,2$ and satisfying $\mu_0^+ = (f_1, g_1, h_1)$, $\mu_0^- = (f_2, g_2, h_2)$ and $0 \le \mu_{n,1} \le \varphi_n * \mu^+, 0 \le \mu_{n,2} \le \varphi_n * \mu^-$, where $\{\varphi_n\}$ is a sequence of standard mollifiers in $\mathbb{R}^{N+1}$.

Let $\sigma_{1,n}, \sigma_{2,n} \in C_c^\infty(\Omega)$ be convergent to σ^+ and σ^- in the narrow topology and in $L^1(\Omega)$ if $\sigma \in L^1(\Omega)$ resp. such that $0 \le \sigma_{1,n} \le \varphi_{1,n} * \sigma^+, 0 \le \sigma_{2,n} \le \varphi_{1,n} * \sigma^-$ where $\{\varphi_{1,n}\}$ is a sequence of standard mollifiers in $\mathbb{R}^N$. Set $\mu_n = \mu_{n,1} - \mu_{n,2}$ and $\sigma_n = \sigma_{1,n} - \sigma_{2,n}$.
Let $u_n, u_{n,1}, u_{n,2}$ be solutions of equations

$$\begin{cases} (u_n)_t - div(A(x,t,\nabla u_n)) = \mu_n & \text{in } \Omega_T, \\ u_n = 0 & \text{on } \partial\Omega \times (0,T), \\ u_n(0) = \sigma_n & \text{on } \Omega, \end{cases} \tag{5.3}$$

$$\begin{cases} (u_{n,1})_t - div(A(x,t,\nabla u_{n,1})) = \chi_{\Omega_T}\mu_{n,1} & \text{in } B_{2T_0}(x_0) \times (0, 2T_0^2), \\ u_{n,1} = 0 & \text{on } \partial B_{2T_0}(x_0) \times (0, 2T_0^2), \\ u_{n,1}(0) = \sigma_{1,n} & \text{on } B_{2T_0}(x_0), \end{cases} \tag{5.4}$$

$$\begin{cases} (u_{n,2})_t + div(A(x,t,-\nabla u_{n,2})) = \chi_{\Omega_T}\mu_{n,2} & \text{in } B_{2T_0}(x_0) \times (0, 2T_0^2), \\ u_{n,2} = 0 & \text{on } \partial B_{2T_0}(x_0) \times (0, 2T_0^2), \\ u_{n,2}(0) = \sigma_{2,n} & \text{on } B_{2T_0}(x_0), \end{cases} \tag{5.5}$$

where $\Omega \subset B_{T_0}(x_0)$ for $x_0 \in \Omega$.
We see that $u_{n,1}, u_{n,2} \ge 0$ in $B_{2T_0}(x_0) \times (0, 2T_0^2)$ and $-u_{n,2} \le u_n \le u_{n,1}$ in Ω_T.
Now, we estimate $u_{n,1}$. By Remark 3.3 and Theorem 3.6, a sequence $\{u_{n,1,m}\}$ of solutions to equations

$$\begin{cases} (u_{n,1,m})_t - div(A(x,t,\nabla u_{n,1,m})) = (g_{n,m})_t + \chi_{\Omega_T}\mu_{n,1} & \text{in } B_{2T_0}(x_0) \times (-2T_0^2, 2T_0^2), \\ u_{n,1,m} = 0 & \text{on } \partial B_{2T_0}(x_0) \times (-2T_0^2, 2T_0^2), \\ u_{n,1,m}(-2T_0^2) = 0 & \text{on } B_{2T_0}(x_0), \end{cases} \tag{5.6}$$

converges to $u_{n,1}$ in $B_{2T_0}(x_0) \times (0, 2T_0^2)$, where $g_{n,m}(x,t) = \sigma_{1,n}(x) \int_{-2T_0^2}^t \varphi_{2,m}(s) ds$ and $\{\varphi_{2,m}\}$ is a sequence of mollifiers in $\mathbb{R}$.
By Remark 3.2, we have

$$||u_{n,1,m}||_{L^1(\tilde{Q}_{2T_0}(x_0,0))} \le c_1 T_0^2 A_{n,m}, \tag{5.7}$$

where $A_{n,m} = \mu_{n,1}(\Omega_T) + \int_{\tilde{Q}_{2T_0}(x_0,0)} \sigma_{1,n}(x)\varphi_{2,m}(t) dxdt$.
Hence, thanks to Theorem 5.1 we have for $(x,t) \in \Omega_T$

$$\begin{aligned} u_{n,1,m}(x,t) &\le c_8 T_0^{-N-2} ||u_{n,1,m}||_{L^1(\tilde{Q}_{2T_0}(x_0,0))} + c_8 \mathbb{I}_2[\mu_{n,1}](x,t) + c_8 \mathbb{I}_2[\sigma_{1,n}\varphi_m](x,t) \\ &\le c_9 \mathbb{I}_2[\mu_{n,1}](x,t) + c_9 \mathbb{I}_2[\sigma_{1,n}\varphi_m](x,t). \end{aligned}$$

Since $0 \le \mu_{n,1} \le \varphi_n * \mu^+$, $\sigma_{1,n} \le \varphi_{1,n} * \sigma^+$,

$$u_{n,1,m}(x,t) \le c_9 \varphi_n * \mathbb{I}_2[\mu^+](x,t) + c_9 (\varphi_{1,n}\varphi_{2,m}) * \mathbb{I}_2[\sigma^+ \otimes \delta_{\{t=0\}}](x,t) \ \forall\ (x,t) \in \Omega_T.$$

Letting $m \to \infty$, we get

$$u_{n,1}(x,t) \le c_9 \varphi_n * \mathbb{I}_2[\mu^+](x,t) + c_9 \varphi_{1,n} * \left(\mathbb{I}_2[\sigma^+ \otimes \delta_{\{t=0\}}](.,t)\right)(x) \ \forall\ (x,t) \in \Omega_T.$$

Similarly, we also get

$$u_{n,2}(x,t) \le c_9 \varphi_n * \mathbb{I}_2[\mu^-](x,t) + c_9 \varphi_{1,n} * \left(\mathbb{I}_2[\sigma^- \otimes \delta_{\{t=0\}}](.,t)\right)(x) \ \forall\ (x,t) \in \Omega_T.$$

Consequently, by Proposition 3.5 and Theorem 3.6 , up to a subsequence, $\{u_n\}$ converges to a distribution solution (a renormalized solution if $\sigma \in L^1(\Omega)$) u of (2.4) and satisfied (2.7). ■

Remark 5.2 *Obviously, if* $\sigma \equiv 0$ *and* $supp(\mu) \subset \overline{\Omega} \times [a,T]$, $a > 0$ *then* $u = 0$ *in* $\Omega \times (0,a)$.

Remark 5.3 *If* A *is independent of space variable* x, (2.27) *is satisfied then*

$$|\nabla u(x,t)| \le C(N,\Lambda_1,\Lambda_2,T_0/d)\mathbb{I}_1^{2T_0}[|\mu| + |\sigma| \otimes \delta_{\{t=0\}}](x,t) \tag{5.8}$$

for any $(x,t) \in \Omega^d \times (0,T)$ *and* $0 < d \le \frac{1}{2}\min\{\sup_{x\in\Omega} d(x,\partial\Omega), T_0^{1/2}\}$ *where* $\Omega^d = \{x \in \Omega : d(x,\partial\Omega) > d\}$. *Indeed, by Remark 3.3 and Theorem 3.6, a sequence* $\{v_{n,m}\}$ *of solutions to equations*

$$\begin{cases} (v_{n,m})_t - div(A(t,\nabla u_{n,m})) = (g_{n,m})_t + \chi_{\Omega_T}\mu_n & in\ \Omega \times (-2T_0^2,T), \\ v_{n,m} = 0 \quad on\ \partial\Omega \times (-2T_0^2,T), \\ v_{n,m}(-2T_0^2) = 0 \quad on\ \Omega, \end{cases} \tag{5.9}$$

converges to u_n *in* $L^1(0,T,W_0^{1,1}(\Omega))$, *where* $g_{n,m}(x,t) = \sigma_n(x)\int_{-2T_0^2}^t \varphi_{2,m}(s)ds$ *and* $\{\varphi_{2,m}\}$ *is a sequence of mollifiers in* $\mathbb{R}$.
By Theorem 5.1, we have for any $(x,t) \in \Omega^d \times (0,T)$

$$|\nabla v_{n,m}(x,t)| \le c_1 \fint_{\tilde{Q}_{d/2}(x,t)} |\nabla v_{n,m}|dyds + c_1\mathbb{I}_1^d[|\mu_n| + |\sigma_n| \otimes \varphi_{2,m}](x,t).$$

On the other hand, by remark 3.2,

$$|||\nabla v_{n,m}|||_{L^1(\Omega\times(-T_0^2,T))} \le c_2 T_0(|\mu_n| + |\sigma_n| \otimes \varphi_{2,m})(\Omega \times (-T_0^2,T)).$$

Therefore, for any $(x,t) \in \Omega^d \times (0,T)$

$$|\nabla v_{n,m}(x,t)| \le c_3\mathbb{I}_1[|\mu_n| + |\sigma_n| \otimes \varphi_{2,m}](x,t),$$

where c_3 *depends on* T_0/d.
Finally, letting $m \to \infty$ *and* $n \to \infty$ *we get for any* $(x,t) \in \Omega^d \times (0,T)$

$$|\nabla u(x,t)| \le c_3\mathbb{I}_1[|\mu| + |\sigma| \otimes \delta_{\{t=0\}}](x,t).$$

We conclude (5.8) *since* $\mathbb{I}_1[|\mu| + |\sigma| \otimes \delta_{\{t=0\}}] \le c_4\mathbb{I}_1^{2T_0}[|\mu| + |\sigma| \otimes \delta_{\{t=0\}}]$ *in* Ω_T.

Next, we will establish pointwise estimates from below for solutions of equations (2.4).

Theorem 5.4 *If* $u \in C(Q_r(y,s)) \cap L^2(s-r^2,s,H^1(B_r(y)))$ *is a nonnegative weak solution of* (2.4) *with data* $\mu \in \mathfrak{M}^+(Q_r(y,s))$ *and* $u(s-r^2) \ge 0$, *then there exists a constant* C *depending on* N,Λ_1,Λ_2 *such that*

$$u(y,s) \ge C\sum_{k=0}^{\infty} \frac{\mu(Q_{r_k/8}(y,s-\frac{35}{128}r_k^2))}{r_k^N}, \tag{5.10}$$

where $r_k = 4^{-k}r$.

Proof. It is enough to show that for $\rho \in (0, r)$

$$\frac{\mu(Q_{\rho/8}(y, s - \frac{35}{128}\rho^2))}{\rho^N} \le c_1(\inf_{Q_{\rho/4}(y,s)} u - \inf_{Q_\rho(y,s)} u). \tag{5.11}$$

By [50, Theorem 6.18, p. 122], we have for any $\theta \in (0, 1 + 2/N)$,

$$\left(\fint_{Q_{\rho/4}(y,s-\rho^2/4)} (u-a)^\theta \right)^{1/\theta} \le c_2(b-a), \tag{5.12}$$

where $b = \inf_{Q_{\rho/4}(y,s)} u$, $a = \inf_{Q_\rho(y,s)} u$ and a constant c_2 depends on $N, \Lambda_1, \Lambda_2, \theta$.

Let $\eta \in C_c^\infty(Q_\rho(y,s))$ such that $0 \le \eta \le 1$, supp$\eta \subset Q_{\rho/4}(y, s - \frac{1}{4}\rho^2)$, $\eta = 1$ in $Q_{\rho/8}(y, s - \frac{35}{128}\rho^2)$ and $|\nabla \eta| \le c_3/\rho^2$, $|\eta_t| \le c_3/\rho^2$ where $c_3 = c_3(N)$. We have

$$\begin{aligned}
\mu(Q_{\rho/8}(y, s - \frac{35}{128}\rho^2)) &\le \int_{Q_\rho(y,s)} \eta^2 d\mu(x,t) \\
&= \int_{Q_\rho(y,s)} u_t \eta^2 dxdt + 2\int_{Q_\rho(y,s)} \eta A(x,t,\nabla u)\nabla \eta dxdt \\
&= -2\int_{Q_\rho(y,s)} (u-a)\eta_t \eta dxdt + 2\int_{Q_\rho(y,s)} \eta A(x,t,\nabla u)\nabla \eta dxdt \\
&\le c_3 r^{-2} \int_{Q_{\rho/4}(y,s-\frac{1}{4}\rho^2)} (u-a) dxdt + 2\Lambda_1 \int_{Q_\rho(y,s)} \eta |\nabla u||\nabla \eta| dxdt \\
&\le c_4 r^N (b-a) + c_4 \int_{Q_\rho(y,s)} \eta |\nabla u||\nabla \eta| dxdt.
\end{aligned}$$

Here we used (5.12) with $\theta = 1$ in the last inequality. It remains to show that

$$\int_{Q_r(y,s)} \eta |\nabla u||\nabla \eta| dxdt \le c_5 r^N (b-a). \tag{5.13}$$

First, we verify that for $\varepsilon \in (0,1)$

$$\int_{Q_\rho(y,s)} |\nabla u|^2 (u-a)^{-\varepsilon-1} \eta^2 dxdt \le c_6 \int_{Q_\rho(y,s)} (u-a)^{1-\varepsilon} \left(\eta|\eta_t| + |\nabla \eta|^2 \right) dxdt. \tag{5.14}$$

Indeed, for $\delta \in (0,1)$ we choose $\varphi = (u - a + \delta)^{-\varepsilon}\eta^2$ as test function in (2.4),

$$\begin{aligned}
0 \le& \int_{Q_\rho(y,s)} u_t (u-a+\delta)^{-\varepsilon} \eta^2 dxdt + \int_{Q_\rho(y,s)} A(x,t,\nabla u)\nabla \left((u-a+\delta)^{-\varepsilon}\eta^2 \right) dxdt \\
\le& 2(1-\varepsilon)\int_{Q_\rho(y,s)} (u-a+\delta)^{1-\varepsilon} |\eta_t| \eta dxdt - \varepsilon \Lambda_2 \int_{Q_\rho(y,s)} |\nabla u|^2 (u-a+\delta)^{-\varepsilon-1}\eta^2 dxdt \\
&+ 2\Lambda_1 \int_{Q_\rho(y,s)} \eta |\nabla u| (u-a+\delta)^{-\varepsilon} |\nabla \eta| dxdt.
\end{aligned}$$

So, we deduce (5.14) from using the Holder inequality and letting $\delta \to 0$.
Therefore, for $\varepsilon \in (0, 2/N)$ using the Holder inequality and we get

$$\begin{aligned}
&\int_{Q_r(y,s)} \eta |\nabla u||\nabla \eta| dxdt \\
&\leq \left(\int_{Q_\rho(y,s)} |\nabla u|^2 (u-a)^{-\varepsilon-1} \eta^2 dxdt \right)^{1/2} \left(\int_{Q_\rho(y,s)} (u-a)^{\varepsilon+1} |\nabla \eta|^2 dxdt \right)^{1/2} \\
&\leq c_7 \left(\int_{Q_\rho(y,s)} (u-a)^{1-\varepsilon} \left(\eta |\eta_t| + |\nabla \eta|^2 \right) dxdt \right)^{1/2} \left(\int_{Q_\rho(y,s)} (u-a)^{\varepsilon+1} |\nabla \eta|^2 dxdt \right)^{1/2} \\
&\leq c_8 \rho^{-2} \left(\int_{Q_{\rho/4}(y,s-\frac{1}{4}\rho^2)} (u-a)^{1-\varepsilon} dxdt \right)^{1/2} \left(\int_{Q_{\rho/4}(y,s-\frac{1}{4}\rho^2)} (u-a)^{\varepsilon+1} dxdt \right)^{1/2}.
\end{aligned}$$

Consequently, we get (5.11) from (5.12). ∎

Proof of Theorem 2.3. Let $\mu_n \in (C_c^\infty(\Omega_T))^+, \sigma_n \in (C_c^\infty(\Omega))^+$ be in the proof of Theorem 2.1. Let u_n be a weak solution of equation

$$\begin{cases} (u_n)_t - div(A(x,t,\nabla u_n)) = \mu_n & \text{in } \Omega_T, \\ u_n = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u_n(0) = \sigma_n \quad \text{on } \Omega. \end{cases}$$

As the proof of Theorem 2.1, thanks to Theorem 5.4 we get By Remark for any $Q_r(y,s) \subset \Omega \times (-\text{diam}(\Omega), T)$ and $r_k = 4^{-k} r$

$$u_n(y,s) \geq c_1 \sum_{k=0}^{\infty} \frac{\mu_n(Q_{r_k/8}(y, s - \frac{35}{128} r_k^2))}{r_k^N} + c_1 \sum_{k=0}^{\infty} \frac{(\sigma_n \otimes \delta_{\{t=0\}})(Q_{r_k/8}(y, s - \frac{35}{128} r_k^2))}{r_k^N}.$$

Finally, by Proposition 3.5 and Theorem 3.6 we get the results. ∎

Remark 5.5 *If $u \in L^q(\Omega_T)$ satisfies (2.8) then $\mathcal{G}_2[\chi_E \mu] \in L^q(\mathbb{R}^{N+1})$ and $\mathbf{G}_{\frac{2}{q}}[\chi_F \sigma] \in L^q(\mathbb{R}^N)$ for every $E \subset\subset \Omega \times [0,T)$ and $F \subset\subset \Omega$. Indeed, for $E \subset\subset \Omega \times [0,T)$, $\varepsilon = dist(E, (\Omega \times (0,T)) \cup (\Omega \times \{t = T\})) > 0$, we can see that for any $(y,s) \in \Omega_T$, $r_k = 4^{-k}\varepsilon/4$*

$$u(y,s) \geq c_1 \sum_{k=0}^{\infty} \frac{\tilde{\mu}(E \cap Q_{r_k/8}(y, s - \frac{35}{128} r_k^2))}{r_k^N}, \tag{5.15}$$

where $\tilde{\mu} = \mu + \sigma \otimes \delta_{\{t=0\}}$.
Moreover, for any $(y,s) \notin \Omega_T$

$$\sum_{k=0}^{\infty} \frac{\tilde{\mu}(E \cap Q_{r_k/8}(y, s - \frac{35}{128} r_k^2))}{r_k^N} = 0.$$

Thus,

$$\infty > \int_{\mathbb{R}^{N+1}} \sum_{k=0}^{\infty} \left(\frac{\tilde{\mu}(E \cap Q_{r_k/8}(y, s - \frac{35}{128} r_k^2))}{r_k^N} \right)^q dyds$$
$$= \int_{\mathbb{R}^{N}} \sum_{k=0}^{\infty} \int_{\mathbb{R}} \left(\frac{\tilde{\mu}(E \cap Q_{r_k/8}(y, s - \frac{35}{128} r_k^2))}{r_k^N} \right)^q dsdy$$
$$\geq \int_{\mathbb{R}^{N}} \sum_{k=0}^{\infty} \int_{\mathbb{R}} \left(\frac{\tilde{\mu}(E \cap \tilde{Q}_{r_k/8}(y, s))}{r_k^N} \right)^q dsdy$$
$$\geq c_2 \int_{\mathbb{R}^{N+1}} \int_0^{\varepsilon/64} \left(\frac{\tilde{\mu}(E \cap \tilde{Q}_{\rho}(y, s))}{\rho^N} \right)^q \frac{d\rho}{\rho} dsdy$$
$$\geq c_3(\varepsilon) \int_{\mathbb{R}^{N+1}} \left(\mathcal{G}_2[\tilde{\mu}\chi_E]\right)^q dsdy.$$

Thus, from Proposition 4.19, we get the results.

Proof of Theorem 2.5. Set $D_n = B_n(0) \times (-n^2, n^2)$. For $n \geq 4$, by Theorem 2.1, there exists a renormalized solution u_n to problem

$$\begin{cases} (u_n)_t - div(A(x,t,\nabla u_n)) = \chi_{D_{n-1}}\omega \ \text{ in } D_n, \\ u_n = 0 \quad \text{on } \partial B_n(0) \times (-n^2, n^2), \\ u_n(-n^2) = 0 \ \text{ on } B_n(0). \end{cases}$$

relative to a decomposition (f_n, g_n, h_n) of $\chi_{D_{n-1}}\omega_0$ satisfying

$$-K\mathbb{I}_2[\omega^-](x,t) \leq u_n(x,t) \leq K\mathbb{I}_2[\omega^+](x,t) \ \ \forall \ (x,t) \in D_n. \tag{5.16}$$

From the proof of Theorem 2.1 and Remark 3.9, we can assume that u_n satisfies (3.14) and (3.15) in Proposition 3.16 with $1 < q_0 < \frac{N+2}{N}$, $L \equiv 0$ and

$$||f_n||_{L^1(D_i)} + ||g_n||_{L^2(D_i)} + |||h_n| + |\nabla h_n|||_{L^2(D_i)} \leq 2|\omega|(D_{i+1}) \tag{5.17}$$

for any $i = 1, ..., n-1$ and h_n is convergent in $L^1_{\text{loc}}(\mathbb{R}^{N+1})$.
On the other hand, by Lemma 4.26 we have for any $s \in (1, \frac{N+2}{N})$

$$\begin{aligned} \int_{D_m} |u_n|^s dxdt &\leq K^s \int_{D_m} (I_2[|\omega|])^s dxdt \\ &\leq K^s \int_{\tilde{Q}_{4m}(x_0,t_0)} (I_2[|\omega|])^s dxdt \\ &\leq c_1 M m^{N+2}, \end{aligned} \tag{5.18}$$

for $n \geq m \geq |x_0| + |t_0|^{1/2}$. Consequently, we can apply Proposition 3.17 and obtain that u_n converges to some u in $L^1_{loc}(\mathbb{R}; W^{1,1}_{loc}(\mathbb{R}^N))$.
Since for any $\alpha \in (0, 1/2)$

$$\int_{D_m} \frac{|\nabla u_n|^2}{(|u_n|+1)^{\alpha+1}} dxdt \leq C_m(\alpha) \ \ \forall \ n \geq m,$$

thus using (5.18) and Holder inequality, we get for any $1 \leq s_1 < \frac{N+2}{N+1}$

$$\int_{D_m} |\nabla u_n|^{s_1} dxdt \leq C_m(s_1) \quad \text{for all} \ \ n \geq m \geq |x_0| + |t_0|^{1/2}.$$

This yields $u_n \to u$ in $L^{s_1}_{\text{loc}}(\mathbb{R}; W^{1,s_1}_{\text{loc}}(\mathbb{R}^N))$.
Take $\varphi \in C_c^\infty(\mathbb{R}^{N+1})$ and $m_0 \in \mathbb{N}$ with $\text{supp}(\varphi) \subset D_{m_0}$, we have for $n \geq m_0 + 1$

$$-\int_{\mathbb{R}^{N+1}} u_n \varphi_t dxdt + \int_{\mathbb{R}^{N+1}} A(x,t,\nabla u_n)\nabla\varphi dxdt = \int_{\mathbb{R}^{N+1}} \varphi d\omega$$

Letting $n \to \infty$, we conclude that u is a distribution solution to problem (2.6) with data $\mu = \omega$ which satisfies (2.9).
Claim 1. If $\omega \geq 0$. By Theorem 2.3, we have for $n \geq 4^{k_0+1}$, $(y,s) \in B_{4^{k_0}} \times (0, n^2)$

$$u_n(y,s) \geq c_2 \sum_{k=0}^{\infty} \frac{\omega(Q_{r_k/8}(y, s - \frac{35}{128} r_k^2) \cap D_{n-1})}{r_k^N},$$

where $r_k = 4^{-k+k_0}$. This gives

$$u_n(y,s) \geq c_2 \sum_{k=-k_0}^{\infty} \frac{\omega(Q_{2^{-2k-3}}(y, s - 35 \times 2^{-4k-7}) \cap B_{n-1}(0) \times (0,(n-1)^2))}{2^{-2Nk}}.$$

Letting $n \to \infty$ and $k_0 \to \infty$ we have (2.10). Finally, thanks to Proposition 4.8 and Theorem 4.2, we will assert (2.11) if we show that for $q > \frac{N+2}{N}$

$$\int_{\mathbb{R}} \left(\sum_{k=-\infty}^{\infty} \frac{\omega(Q_{2^{-2k-3}}(x, t - 35 \times 2^{-4k-7}))}{2^{-2Nk}} \right)^q dxdt \geq c_3 \int_{\mathbb{R}} \int_0^{+\infty} \left(\frac{\omega(\tilde{Q}_\rho(x,t))}{\rho^N} \right)^q \frac{d\rho}{\rho} dxdt.$$

Indeed,

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\sum_{k=-\infty}^{\infty} \frac{\omega(Q_{2^{-2k-3}}(x, t - 35 \times 2^{-4k-7}))}{2^{-2Nk}} \right)^q dxdt \\
&\geq \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}} \left(\frac{\omega(Q_{2^{-2k-3}}(x, t - 35 \times 2^{-4k-7}))}{2^{-2Nk}} \right)^q dtdx \\
&= \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}} \left(\frac{\omega(\tilde{Q}_{2^{-2k-3}}(x,t))}{2^{-2Nk}} \right)^q dt \\
&\geq c_4 \int_{\mathbb{R}^{N+1}} \int_0^{+\infty} \left(\frac{\omega(\tilde{Q}_\rho(x,t))}{\rho^N} \right)^q \frac{d\rho}{\rho} dxdt.
\end{aligned}$$

Claim 2. If A is independent of space variable x and (2.27) is satisfied. By Remark 5.3 we get for any $(x,t) \in D_{n/4}$

$$|\nabla u_n(x,t)| \leq c_5 \mathbb{I}_1[|\omega|](x,t).$$

Letting $n \to \infty$, we get (2.12).
Claim 3. If $\omega = \mu + \sigma \otimes \delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}(\mathbb{R}^N \times (0,\infty))$ and $\sigma \in \mathfrak{M}(\mathbb{R}^N)$, then by Remark (5.2) we can assume that $u_n = 0$ in $B_n(0) \times (-n^2, 0)$. So, $u = 0$ in $\mathbb{R}^N \times (-\infty, 0)$. Therefore, clearly $u|_{\mathbb{R}^N \times [0,\infty)}$ is a distribution solution to (2.5). The proof is complete. ■

Remark 5.6 *If $\omega \in \mathfrak{M}_b(\mathbb{R}^{N+1})$ then u satisfies*

$$|||\nabla u|||_{L^{\frac{N+2}{N+1},\infty}(\mathbb{R}^{N+1})} \leq C(N, \Lambda_1, \Lambda_2)|\omega|(\mathbb{R}^{N+1}).$$

Moreover, $I_2[|\omega|] \in L^{\frac{N+2}{N},\infty}(\mathbb{R}^{N+1})$ and $I_2[|\omega|] < \infty$ a.e in $\mathbb{R}^{N+1}$.

6 Quasilinear Lane-Emden Type Parabolic Equations

6.1 Quasilinear Lane-Emden Parabolic Equations in Ω_T

To prove Theorem 2.8 we need the following proposition which was proved in [6].

Proposition 6.1 *Assume O is an open subset of $\mathbb{R}^{N+1}$. Let $p>1$ and $\mu \in \mathfrak{M}^+(O)$. If μ is absolutely continuous with respect to $Cap_{2,1,p}$ in O, there exists a nondecreasing sequence $\{\mu_n\} \subset \mathfrak{M}_b^+(O) \cap \left(W_p^{2,1}(\mathbb{R}^{N+1})\right)^*$, with compact support in O which converges to μ weakly in $\mathfrak{M}(O)$. Moreover, if $\mu \in \mathfrak{M}_b^+(O)$ then $||\mu_n - \mu||_{\mathfrak{M}_b(O)} \to 0$ as $n \to \infty$.*

Remark 6.2 *By Theorem 4.17, $W_p^{2,1}(\mathbb{R}^{N+1}) = \mathcal{L}_2^p(\mathbb{R}^{N+1})$, it follows $\{\mu_n\} \subset \mathfrak{M}_b^+(O) \cap \left(\mathcal{L}_2^p(\mathbb{R}^{N+1})\right)^*$. Note that $||\mu_n||_{\left(\mathcal{L}_2^p(\mathbb{R}^{N+1})\right)^*} = ||\overset{\vee}{\mathcal{G}}_2[\mu_n]||_{L^{p'}(\mathbb{R}^{N+1})}$. So $\overset{\vee}{\mathcal{G}}_2[\mu_n] \in L^{p'}(\mathbb{R}^{N+1})$. Consequently, from (4.17) in Proposition 4.8, we obtain $\mathbb{I}_2^R[\mu_n] \in L^{p'}(\mathbb{R}^{N+1})$ for any $n \in \mathbb{N}$ and $R>0$. In particular, $\mathbb{I}_2[\mu_n] \in L^{p'}_{loc}(\mathbb{R}^{N+1})$ for any $n \in \mathbb{N}$.*

Remark 6.3 *As in the proof of Theorem 2.5 in [16], we can prove a general version of Proposition 6.1, that is: for $p>1$, if μ is absolutely continuous with respect to $Cap_{\mathcal{G}_\alpha,p}$ in O, there exists a nondecreasing sequence $\{\mu_n\} \subset \mathfrak{M}_b^+(O) \cap \left(\mathcal{L}_\alpha^p(\mathbb{R}^{N+1})\right)^*$, with compact support in O which converges to μ weakly in $\mathfrak{M}(O)$. Furthermore, $\mathbb{I}_\alpha[\mu_n] \in L^{p'}_{loc}(\mathbb{R}^{N+1})$ for all $n \in \mathbb{N}$. Besides, we also obtain that for $\mu \in \mathfrak{M}_b(O)$ is absolutely continuous with respect to $Cap_{\mathcal{G}_\alpha,p}$ in O if and only if $\mu = f + \nu$ where $f \in L^1(O)$ and $\nu \in \left(\mathcal{L}_\alpha^p(\mathbb{R}^{N+1})\right)^*$.*

Proof of Theorem 2.8. First, assume that $\sigma \in L^1(\Omega)$. Because μ is absolutely continuous with respect to the capacity $\text{Cap}_{2,1,q'}$, so are μ^+ and μ^-. Applying Proposition 6.1 there exist two nondecreasing sequences $\{\mu_{1,n}\}$ and $\{\mu_{2,n}\}$ of positive bounded measures with compact support in Ω_T which converge to μ^+ and μ^- in $\mathfrak{M}_b(\Omega_T)$ respectively and such that $\mathbb{I}_2[\mu_{1,n}], \mathbb{I}_2[\mu_{2,n}] \in L^q(\Omega_T)$.

For $i=1,2$, set $\tilde{\mu}_{i,1} = \mu_{i,1}$ and $\tilde{\mu}_{i,j} = \mu_{i,j} - \mu_{i,j-1} \geq 0$, so $\mu_{i,n} = \sum_{j=1}^n \tilde{\mu}_{i,j}$. We write $\mu_{i,n} = \mu_{i,n,0} + \mu_{i,n,s}$, $\tilde{\mu}_{i,j} = \tilde{\mu}_{i,j,0} + \tilde{\mu}_{i,j,s}$ with $\mu_{i,n,0}, \tilde{\mu}_{i,n,0} \in \mathfrak{M}_0(\Omega_T)$, $\mu_{i,n,s}, \tilde{\mu}_{i,n,s} \in \mathfrak{M}_s(\Omega_T)$. As in the proof of Theorem 2.1, for any $j \in \mathbb{N}$ and $i=1,2$, there exist sequences of nonnegative measures $\tilde{\mu}_{m,i,j,0} = (f_{m,i,j}, g_{m,i,j}, h_{m,i,j})$ and $\tilde{\mu}_{m,i,j,s}$ such that $f_{m,i,j}, g_{m,i,j}, h_{m,i,j} \in C_c^\infty(\Omega_T)$ and strongly converge to some $f_{i,j}, g_{i,j}, h_{i,j}$ in $L^1(\Omega_T), L^2(\Omega_T, \mathbb{R}^N)$ and $L^2(0,T,H_0^1(\Omega))$ respectively and $\tilde{\mu}_{m,i,j}, \tilde{\mu}_{m,i,j,s} \in C_c^\infty(\Omega_T)$ converge to $\tilde{\mu}_{i,j}, \tilde{\mu}_{i,j,s}$ resp. in the narrow topology with $\tilde{\mu}_{m,i,j} = \tilde{\mu}_{m,i,j,0} + \tilde{\mu}_{m,i,j,s}$ which satisfy $\tilde{\mu}_{i,j,0} = (f_{i,j}, g_{i,j}, h_{i,j})$ and $0 \leq \tilde{\mu}_{m,i,j} \leq \varphi_m * \tilde{\mu}_{i,j}$ and

$$||f_{m,i,j}||_{L^1(\Omega_T)} + \|g_{m,i,j}\|_{L^2(\Omega_T,\mathbb{R}^N)} + ||h_{m,i,j}||_{L^2(0,T,H_0^1(\Omega))} + \mu_{m,i,j,s}(\Omega_T) \leq 2\tilde{\mu}_{i,j}(\Omega_T). \quad (6.1)$$

Here $\{\varphi_m\}$ is a sequence of mollifiers in $\mathbb{R}^{N+1}$.
For any $n,k,m \in \mathbb{N}$, let $u_{n,k,m}, u_{1,n,k,m}, u_{2,n,k,m} \in W$ with $W = \{z : z \in L^2(0,T,H_0^1(\Omega)), z_t \in L^2(0,T,H^{-1}(\Omega))\}$ be solutions of problems

$$\begin{cases} (u_{n,k,m})_t - div(A(x,t,\nabla u_{n,k,m})) + T_k(|u_{n,k,m}|^{q-1}u_{n,k,m}) = \sum_{j=1}^n (\tilde{\mu}_{m,1,j} - \tilde{\mu}_{m,2,j}) \text{ in } \Omega_T, \\ u_{n,k,m} = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u_{n,k,m}(0) = T_n(\sigma^+) - T_n(\sigma^-) \quad \text{on } \Omega, \end{cases} \quad (6.2)$$

$$\begin{cases} (u_{1,n,k,m})_t - div(A(x,t,\nabla u_{1,n,k,m})) + T_k(u_{1,n,k,m}^q) = \sum_{j=1}^n \tilde{\mu}_{m,1,j} \text{ in } \Omega_T, \\ u_{1,n,k,m} = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u_{1,n,k,m}(0) = T_n(\sigma^+) \quad \text{in } \Omega, \end{cases} \quad (6.3)$$

$$\begin{cases} (u_{2,n,k,m})_t - div(\tilde{A}(x,t,\nabla u_{2,n,k,m})) + T_k(u_{2,n,k,m}^q) = \sum_{j=1}^n \tilde{\mu}_{m,2,j} \text{ in } \Omega_T, \\ u_{2,n,k,m} = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u_{2,n,k,m}(0) = T_n(\sigma^-) \quad \text{in } \Omega, \end{cases} \quad (6.4)$$

where $\tilde{A}(x,t,\xi) = -A(x,t,-\xi)$.
By Comparison Principle Theorem and Theorem 2.1, there holds, for any m,k the sequences $\{u_{1,n,k,m}\}_n$ and $\{u_{2,n,k,m}\}_n$ are increasing and

$$\begin{aligned}-K\mathbb{I}_2[T_n(\sigma^-)\otimes\delta_{\{t=0\}}] - K\mathbb{I}_2[\mu_{2,n}*\varphi_m] &\le -u_{2,n,k,m} \le u_{n,k,m} \le u_{1,,n,k,m}\\ &\le K\mathbb{I}_2[\mu_{1,n}*\varphi_m] + K\mathbb{I}_2[T_n(\sigma^+)\otimes\delta_{\{t=0\}}],\end{aligned}$$

where a constant K is in Theorem 2.1. Thus,

$$\begin{aligned}-K\mathbb{I}_2[T_n(\sigma^-)\otimes\delta_{\{t=0\}}] - K\mathbb{I}_2[\mu_{2,n}]*\varphi_m &\le -u_{2,n,k,m} \le u_{n,k,m} \le u_{1,,n,k,m}\\ &\le K\mathbb{I}_2[\mu_{1,n}]*\varphi_m + K\mathbb{I}_2[T_n(\sigma^+)\otimes\delta_{\{t=0\}}].\end{aligned}$$

Moreover,

$$\int_{\Omega_T} T_k(u^q_{i,n,k,m})dxdt \le \int_{\Omega_T} \varphi_m * \mu_{i,n}dxdt + |\sigma|(\Omega) \le |\mu|(\Omega_T) + |\sigma|(\Omega).$$

As in [14, Proof of Lemma 5.3], thanks to Proposition 3.5 and Theorem 3.6, there exist subsequences of $\{u_{n,k,m}\}_m$ $\{u_{1,n,k,m}\}_m$, $\{u_{2,n,k,m}\}_m$, still denoted them, converging to renormalized solutions $u_{n,k}$ $u_{1,n,k}$, $u_{2,n,k}$ of equations (6.2) with data $\mu_{1,n}-\mu_{2,n}$, $u_{n,k}(0) = T_n(\sigma^+) - T_n(\sigma^-)$ and the decomposition $(\sum_{j=1}^n f_{1,j} - \sum_{j=1}^n f_{2,j}, \sum_{j=1}^n g_{1,j} - \sum_{j=1}^n g_{2,j}, \sum_{j=1}^n h_{1,j} - \sum_{j=1}^n h_{2,j})$ of $\mu_{1,n,0}-\mu_{2,n,0}$, (6.3) with data $\mu_{1,n}$, $u_{1,n,k}(0) = T_n(\sigma^+)$ and the decomposition $(\sum_{j=1}^n f_{1,j}, \sum_{j=1}^n g_{1,j}, \sum_{j=1}^n h_{1,j})$ of $\mu_{1,n,0}$, (6.4) with data $\mu_{2,n}$, $u_{2,n,k}(0) = T_n(\sigma^-)$ and the decomposition $(\sum_{j=1}^n f_{2,j}, \sum_{j=1}^n g_{2,j}, \sum_{j=1}^n h_{2,j})$ of $\mu_{2,n,0}$ respectively, which satisfy

$$\begin{aligned}-K\mathbb{I}_2[T_n(\sigma^-)\otimes\delta_{\{t=0\}}] - K\mathbb{I}_2[\mu_{2,n}] &\le -u_{2,n,k} \le u_{n,k} \le u_{1,n,k}\\ &\le K\mathbb{I}_2[\mu_{1,n}] + K\mathbb{I}_2[T_n(\sigma^+)\otimes\delta_{\{t=0\}}].\end{aligned}$$

Next, as in [14, Proof of Lemma 5.4] since $I_2[\mu_{i,n}] \in L^q(\Omega_T)$ for any n, thanks to Proposition 3.5 and Theorem 3.6, there exist subsequences of $\{u_{n,k}\}_k$ $\{u_{1,n,k}\}_k$, $\{u_{2,n,k}\}_k$, still denoted them, converging to renormalized solutions u_n $u_{1,n}$, $u_{2,n}$ of equations

$$\begin{cases} (u_n)_t - div(A(x,t,\nabla u_n)) + |u_n|^{q-1}u_n = \mu_{1,n} - \mu_{2,n} & \text{in } \Omega_T,\\ u_n = 0 & \text{on } \partial\Omega\times(0,T),\\ u_n(0) = T_n(\sigma^+) - T_n(\sigma^-) & \text{in } \Omega, \end{cases} \tag{6.5}$$

$$\begin{cases} (u_{1,n})_t - div(A(x,t,\nabla u_{1,n})) + u^q_{1,n} = \mu_{1,n} & \text{in } \Omega_T,\\ u_{1,n} = 0 & \text{on } \partial\Omega\times(0,T),\\ u_{1,n}(0) = T_n(\sigma^+) & \text{in } \Omega, \end{cases} \tag{6.6}$$

$$\begin{cases} (u_{2,n})_t - div(\tilde{A}(x,t,\nabla u_{2,n})) + u^q_{2,n} = \mu_{2,n} & \text{in } \Omega_T,\\ u_{2,n} = 0 & \text{on } \partial\Omega\times(0,T),\\ u_{2,n}(0) = T_n(\sigma^-) & \text{in } \Omega, \end{cases} \tag{6.7}$$

relative to the decomposition $(\sum_{j=1}^n f_{1,j} - \sum_{j=1}^n f_{2,j}, \sum_{j=1}^n g_{1,j} - \sum_{j=1}^n g_{2,j}, \sum_{j=1}^n h_{1,j} - \sum_{j=1}^n h_{2,j})$ of $\mu_{1,n,0}-\mu_{2,n,0}$, $(\sum_{j=1}^n f_{1,j}, \sum_{j=1}^n g_{1,j}, \sum_{j=1}^n h_{1,j})$ of $\mu_{1,n,0}$ and $(\sum_{j=1}^n f_{2,j}, \sum_{j=1}^n g_{2,j}, \sum_{j=1}^n h_{2,j})$ of $\mu_{2,n,0}$ respectively, which satisfy

$$\begin{aligned}-K\mathbb{I}_2[T_n(u_0^-)\otimes\delta_{\{t=0\}}] - K\mathbb{I}_2[\mu_{2,n}] &\le -u_{2,n} \le u_n \le u_{1,n}\\ &\le K\mathbb{I}_2[\mu_{1,n}] + K\mathbb{I}_2[T_n(u_0^+)\otimes\delta_{\{t=0\}}].\end{aligned}$$

and the sequences $\{u_{1,n}\}_n$ and $\{u_{2,n}\}_n$ are increasing and

$$\int_{\Omega_T} u^q_{i,n}dxdt \le |\mu|(\Omega_T) + |\sigma|(\Omega).$$

Note that from (6.1) we have

$$||f_{i,j}||_{L^1(\Omega_T)} + \|g_{i,j}\|_{L^2(\Omega_T,\mathbb{R}^N)} + ||h_{i,j}||_{L^2(0,T,H_0^1(\Omega))} \le 2\tilde{\mu}_{i,j}(\Omega_T)$$

which implies

$$\sum_{j=1}^{n} ||f_{i,j}||_{L^1(\Omega_T)} + \sum_{j=1}^{n} ||g_{i,j}||_{L^2(\Omega_T,\mathbb{R}^N)} + \sum_{j=1}^{n} ||h_{i,j}||_{L^2(0,T,H_0^1(\Omega))} \le 2\mu_{i,n}(\Omega_T) \le 2|\mu|(\Omega_T).$$

Finally, as in [14, Proof of Theorem 5.2] thanks to Proposition 3.5, Theorem 3.6 and Monotone Convergence Theorem there exist subsequences of $\{u_n\}_n$, $\{u_{1,n}\}_n$, $\{u_{2,n}\}_n$, still denoted them, converging to renormalized solutions u, u_1, u_2 of equations (6.5) with data μ, $u(0) = \sigma$ and the decomposition $(\sum_{j=1}^{\infty} f_{1,j} - \sum_{j=1}^{\infty} f_{2,j}, \sum_{j=1}^{\infty} g_{1,j} - \sum_{j=1}^{\infty} g_{2,j}, \sum_{j=1}^{\infty} h_{1,j} - \sum_{j=1}^{\infty} h_{2,j})$ of μ_0, (6.6) with data μ^+, $u_1(0) = \sigma^+$ and the decomposition $(\sum_{j=1}^{\infty} f_{1,j}, \sum_{j=1}^{\infty} g_{1,j},$ $\sum_{j=1}^{\infty} h_{1,j})$ of μ_0^+, (6.7) with data μ^-, $u_2(0) = \sigma^-$ and the decomposition $(\sum_{j=1}^{\infty} f_{2,j}, \sum_{j=1}^{\infty} g_{2,j},$ $\sum_{j=1}^{\infty} h_{2,j})$ of μ_0^-, respectively and

$$-K\mathbb{I}_2[\sigma^- \otimes \delta_{\{t=0\}}] - K\mathbb{I}_2[\mu^-] \le -u_2 \le u \le u_1 \le K\mathbb{I}_2[\mu^+] + K\mathbb{I}_2[\sigma^+ \otimes \delta_{\{t=0\}}].$$

We now have remark: if $\sigma \equiv 0$ and $\text{supp}(\mu) \subset \overline{\Omega} \times [a, T]$, $a > 0$, then $u = u_1 = u_2 = 0$ in $\Omega \times (0, a)$ since $u_{n,k} = u_{1,n,k} = u_{2,n,k} = 0$ in $\Omega \times (0, a)$.
Next, we will consider $\sigma \in \mathfrak{M}_b(\Omega)$ such that σ is absolutely continuous with respect to the capacity $\text{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}$ in Ω. So, $\chi_{\Omega_T}\mu + \sigma \otimes \delta_{\{t=0\}}$ is absolutely continuous with respect to the capacity $\text{Cap}_{2,1,q'}$ in $\Omega \times (-T, T)$. As above, we verify that there exists a renormalized solution u of

$$\begin{cases} u_t - div(A(x, t, \nabla u)) + |u|^{q-1}u = \chi_{\Omega_T}\mu + \sigma \otimes \delta_{\{t=0\}} & \text{in } \Omega \times (-T, T), \\ u = 0 & \text{on } \partial\Omega \times (-T, T), \\ u(-T) = 0 & \text{on } \Omega, \end{cases} \tag{6.8}$$

satisfying $u = 0$ in $\Omega \times (-T, 0)$ and

$$-K\mathbb{I}_2[\sigma^- \otimes \delta_{\{t=0\}}] - K\mathbb{I}_2[\mu^-] \le u \le K\mathbb{I}_2[\mu^+] + K\mathbb{I}_2[\sigma^+ \otimes \delta_{\{t=0\}}].$$

Finally, from remark 3.11 we get the result. This completes the proof of the theorem. ■

Proof of Theorem 2.9. Let $\{\mu_{n,i}\} \subset C_c^{\infty}(\Omega_T), \sigma_{i,n} \in C_c^{\infty}(\Omega)$ for $i = 1, 2$ be as in the proof of Theorem 2.1. We have $0 \le \mu_{n,1} \le \varphi_n * \mu^+, 0 \le \mu_{n,2} \le \varphi_n * \mu^-, 0 \le \sigma_{1,n} \le \varphi_{1,n} * \sigma^+, 0 \le \sigma_{2,n} \le \varphi_{1,n} * \sigma^-$ for any $n \in \mathbb{N}$ where $\{\varphi_n\}$ and $\{\varphi_{1,n}\}$ are sequences of standard mollifiers in $\mathbb{R}^{N+1}, \mathbb{R}^N$ respectively.
We prove that the problem (2.2) has a solution with data $\mu = \mu_{n_0} = \mu_{n_0,1} - \mu_{n_0,2}, \sigma = \sigma_{n_0} = \sigma_{1,n_0} - \sigma_{2,n_0}$ for $n_0 \in \mathbb{N}$. Put

$$J = \Big\{u \in L^q(\Omega_T) : u^+ \le \frac{qK}{q-1}\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,1} + \sigma_{1,n_0} \otimes \delta_{\{t=0\}}] \\ \text{and} \quad u^- \le \frac{qK}{q-1}\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,2} + \sigma_{2,n_0} \otimes \delta_{\{t=0\}}] \Big\}.$$

where $\max\{-\frac{N+2}{q'} + 2, 0\} < \delta < 2$.
Clearly, J is closed under the strong topology of $L^q(\Omega_T)$ and convex.
We consider a map $S : J \to J$ defined for each $v \in J$ by $S(v) = u$, where $u \in L^1(\Omega_T)$ is the unique renormalized solution of

$$\begin{cases} u_t - div(A(x, t, \nabla u)) = |v|^{q-1}v + \mu_{n_0,1} - \mu_{n_0,2} \text{ in } \Omega_T, \\ u = 0 \quad \text{on } \partial\Omega \times (0, T), \\ u(0) = \sigma_{1,n_0} - \sigma_{2,n_0} \quad \text{in } \Omega. \end{cases} \tag{6.9}$$

By Theorem 2.1, we have

$$u^+ \le K\mathbb{I}_2^{2T_0}[(v^+)^q] + K\mathbb{I}_2^{2T_0}[\mu_{n_0,1} + \sigma_{1,n_0} \otimes \delta_{\{t=0\}}],$$
$$u^- \le K\mathbb{I}_2^{2T_0}[(v^-)^q] + K\mathbb{I}_2^{2T_0}[\mu_{n_0,2} + \sigma_{2,n_0} \otimes \delta_{\{t=0\}}],$$

where K is the constant in Theorem 2.1. Thus,

$$u^+ \le K\left(\frac{qK}{q-1}\right)^q \mathbb{I}_2^{2T_0,\delta}\left[\left(\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,1} + \sigma_{1,n_0} \otimes \delta_{\{t=0\}}]\right)^q\right] + K\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,1} + \sigma_{1,n_0} \otimes \delta_{\{t=0\}}],$$
$$u^- \le K\left(\frac{qK}{q-1}\right)^q \mathbb{I}_2^{2T_0,\delta}\left[\left(\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,2} + \sigma_{2,n_0} \otimes \delta_{\{t=0\}}]\right)^q\right] + K\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,2} + \sigma_{2,n_0} \otimes \delta_{\{t=0\}}].$$

Thus, thanks to Theorem 4.36 there exists $c_1 = c_1(N, K, \delta, q)$ such that if for every compact sets $E \subset \mathbb{R}^{N+1}$,

$$|\mu_{n_0,i}|(E) + (|\sigma_{i,n_0}| \otimes \delta_{\{t=0\}})(E) \le c_1 \text{Cap}_{E_2^{2T_0,\delta},q'}(E). \tag{6.10}$$

then $\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,i} + \sigma_{i,n_0} \otimes \delta_{\{t=0\}}] \in L^q(\mathbb{R}^{N+1})$ and

$$\mathbb{I}_2^{2T_0,\delta}\left[\left(\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,i} + \sigma_{i,n_0} \otimes \delta_{\{t=0\}}]\right)^q\right] \le \frac{(q-1)^{q-1}}{(Kq)^q}\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,i} + \sigma_{i,n_0} \otimes \delta_{\{t=0\}}] \ i = 1, 2.$$

which implies $u \in L^q(\Omega_T)$ and

$$u^+ \le \frac{qK}{q-1}\mathbb{I}_2^{2T_0}[\mu_{n_0,1} + \sigma_{1,n_n} \otimes \delta_{\{t=0\}}] \quad \text{and}$$
$$u^- \le \frac{qK}{q-1}\mathbb{I}_2^{2T_0}[\mu_{n_0,2} + \sigma_{2,n_0} \otimes \delta_{\{t=0\}}].$$

Now we assume that (6.10) is satisfied, so S is well defined. Therefore, if we can show that the map $S : J \to J$ is continuous and $S(J)$ is pre-compact under the strong topology of $L^q(\Omega_T)$ then by Schauder Fixed Point Theorem, S has a fixed point on J. Hence the problem (2.2) has a solution with data $\mu = \mu_{n_0}, \sigma = \sigma_{n_0}$.
Now we show that **S is continuous**. Let $\{v_n\}$ be a sequence in J such that v_n converges strongly in $L^q(\Omega_T)$ to a function $v \in J$. Set $u_n = S(v_n)$. We need to show that $u_n \to S(v)$ in $L^q(\Omega_T)$.

By Proposition 3.5, there exists a subsequence of $\{u_n\}$, still denoted by it, converging to u a.e in Ω_T. Since

$$|u_n| \le \sum_{i=1,2} \frac{qK}{q-1}\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,i} + \sigma_{i,n_0} \otimes \delta_{\{t=0\}}] \in L^q(\Omega_T) \ \ \forall\, n \in \mathbb{N}$$

Applying Dominated Convergence Theorem, we have $u_n \to u$ in $L^q(\Omega_T)$. Hence, thanks to Theorem 3.6 we get $u = S(v)$.
Next we show that **S is pre-compact**. Indeed if $\{u_n\} = \{S(v_n)\}$ is a sequence in $S(J)$. By Proposition 3.5, there exists a subsequence of $\{u_n\}$, still denoted by it, converging to u a.e in Ω_T. Again, using get Dominated Convergence Theorem we get $u_n \to u$ in $L^q(\Omega_T)$. So **S is pre-compact**.
Next, thanks to Corollary 4.39 and Remark 4.40 we have

$$[\mu_{n,i} + \sigma_{i,n} \otimes \delta_{\{t=0\}}]_{\mathfrak{M}^{\mathcal{G}_2,q'}} \le c_2[|\mu| + |\sigma| \otimes \delta_{\{t=0\}}]_{\mathfrak{M}^{\mathcal{G}_2,q'}} \ \ \forall\, n \in \mathbb{N}, i = 1, 2,$$

for some $c_2 = c_2(N, q)$.
In addition, by the proof of Corollary 4.39 we get

$$(c_3(T_0))^{-1}\text{Cap}_{\mathcal{G}_2,q'}(E) \le \text{Cap}_{E_2^{2T_0,\delta},q'}(E) \le c_3(T_0)\text{Cap}_{\mathcal{G}_2,q'}(E)$$

for every compact set E with $\mathrm{diam}(E) \leq 2T_0$. Thus, there is $c_4 = c_4(N, K, \delta, q, T_0)$ such that if

$$[|\mu| + |\sigma| \otimes \delta_{\{t=0\}}]_{\mathfrak{M}^{\mathcal{G}_2,q'}} \leq c_4, \tag{6.11}$$

then (6.10) holds for any $n_0 \in \mathbb{N}$.
Now we suppose that (6.11) holds, it is equivalent to (2.13) holding for some constant $C_1 = C_1(T_0)$ by Remark 4.34. Therefore, for any $n \in \mathbb{N}$ there exists a renormalized solution u_n of

$$\begin{cases} (u_n)_t - div(A(x,t,\nabla u_n)) = |u_n|^{q-1}u_n + \mu_{n,1} - \mu_{n,2} \text{ in } \Omega_T, \\ u_n = 0 \quad \text{ on } \partial\Omega \times (0,T), \\ u_n(0) = \sigma_{1,n} - \sigma_{2,n} \quad \text{in } \Omega, \end{cases} \tag{6.12}$$

which satisfies

$$-\frac{qK}{q-1}\mathbb{I}_2^{2T_0,\delta}[\mu_{n,2} + \sigma_{2,n} \otimes \delta_{\{t=0\}}] \leq u_n \leq \frac{qK}{q-1}\mathbb{I}_2^{2T_0,\delta}[\mu_{n,1} + \sigma_{1,n} \otimes \delta_{\{t=0\}}].$$

Thus, for every $(x,t) \in \Omega_T$,

$$\begin{aligned} -\frac{qK}{q-1}\varphi_n * \mathbb{I}_2^{2T_0,\delta}[\mu^-](x,t) - \frac{qK}{q-1}\varphi_{1,n} * (\mathbb{I}_2^{2T_0,\delta}[\sigma^- \otimes \delta_{\{t=0\}}](.,t))(x) \leq u_n(x,t) \\ \leq \frac{qK}{q-1}\varphi_n * (\mathbb{I}_2^{2T_0,\delta}[\mu^-])(x,t) + \frac{qK}{q-1}\varphi_{1,n} * (\mathbb{I}_2^{2T_0,\delta}[\sigma^- \otimes \delta_{\{t=0\}}](.,t))(x). \end{aligned}$$

Since $\varphi_n * \mathbb{I}_2^{2T_0,\delta}[\mu^\pm](x,t), \varphi_{1,n} * (\mathbb{I}_2^{2T_0,\delta}[\sigma^\pm \otimes \delta_{\{t=0\}}](.,t))(x)$ converge to $\mathbb{I}_2^{2T_0,\delta}[\mu^\pm](x,t), \mathbb{I}_2^{2T_0,\delta}[\sigma^\pm \otimes \delta_{\{t=0\}}](x,t)$ in $L^q(\mathbb{R}^{N+1})$ as $n \to \infty$, respectively, so $|u_n|^q$ is equi-integrable.
By Proposition 3.5, there exists a subsequence of $\{u_n\}$, still denoted by its, converging to u a.e in Ω_T. It follows $|u_n|^{q-1}u_n \to |u|^{q-1}u$ in $L^1(\Omega_T)$.
Consequently, by Proposition 3.5 and Theorem 3.6, we obtain that u is a distribution (a renormalized solution if $\sigma \in L^1(\Omega)$) of (2.2) with data μ, σ, and satisfies (2.14). Furthermore, by Corollary 4.39 we have

$$\begin{aligned} (c_5(T_0))^{-1} \left[|\mu| + |\sigma| \otimes \delta_{\{t=0\}}\right]^q_{\mathfrak{M}^{\mathcal{G}_2,q'}} \\ \leq \left[\left(\mathbb{I}_2^{2T_0,\delta}[|\mu| + |\sigma| \otimes \delta_{\{t=0\}}]\right)^q\right]_{\mathfrak{M}^{\mathcal{G}_2,q'}} \leq c_5(T_0)\left[|\mu| + |\sigma| \otimes \delta_{\{t=0\}}\right]^q_{\mathfrak{M}^{\mathcal{G}_2,q'}} \end{aligned}$$

which implies $[|u|^q]_{\mathfrak{M}^{\mathcal{G}_2,q'}} \leq c_4(T_0)$ and we get (2.15). This completes the proof of the Theorem. ■

Remark 6.4 *In view of above proof, we can see that*

i. *The Theorem 2.9 also holds when we replace assumption* (2.13) *by*

$$|\mu|(E) \leq C\,Cap_{\mathcal{H}_2,q'}(E) \quad and \quad |\sigma|(F) \leq C\,Cap_{\mathbf{I}_{\frac{2}{q}},q'}(F).$$

for every compact sets $E \subset \mathbb{R}^{N+1}, F \subset \mathbb{R}^N$ where $C = C(N\Lambda_1, \Lambda_2, q)$ is some a constant.

ii. *If $\sigma \equiv 0$ and $supp(\mu) \subset \overline{\Omega} \times [a,T]$, $a > 0$, then we can show that a solution u in Theorem 2.9 satisfies $u = 0$ in $\Omega \times (0,a)$ since we can replace the set E by E':*

$$\begin{aligned} E' = \Big\{ u \in L^q(\Omega_T) : u = 0 \text{ in } \Omega \times (0,a) \text{ and } u^+ \leq \frac{qK}{q-1}\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,1} + \sigma_{1,n_0} \otimes \delta_{\{t=0\}}] \\ \text{and} \quad u^- \leq \frac{qK}{q-1}\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,2} + \sigma_{2,n_0} \otimes \delta_{\{t=0\}}] \Big\}. \end{aligned}$$

6.2 Quasilinear Lane-Emden Parabolic Equations in $\mathbb{R}^N \times (0,\infty)$ and $\mathbb{R}^{N+1}$

This section is devoted to proofs of Theorem 2.12 and 2.14.

Proof of the Theorem 2.12. Since ω is absolutely continuous with respect to the capacity $\text{Cap}_{2,1,q'}$ in $\mathbb{R}^{N+1}$, $|\omega|$ is too. Set $D_n = B_n(0) \times (-n^2, n^2)$. From the proof of Theorem 2.8, there exist renormalized solutions u_n, v_n of

$$\begin{cases} (u_n)_t - \text{div}(A(x,t,\nabla u_n)) + |u_n|^{q-1}u_n = \chi_{D_n}\omega & \text{in } D_n, \\ u_n = 0 \quad \text{on } \partial B_n(0) \times (-n^2, n^2), \\ u_n(-n^2) = 0 \quad \text{in } B_n(0), \end{cases}$$

and

$$\begin{cases} (v_n)_t - \text{div}(A(x,t,\nabla v_n)) + v_n^q = \chi_{D_n}|\omega| & \text{in } D_n, \\ v_n = 0 \quad \text{on } \partial B_n(0) \times (-n^2, n^2), \\ v_n(-n^2) = 0 \quad \text{in } B_n(0), \end{cases}$$

relative to decompositions (f_n, g_n, h_n) of $\chi_{D_n}\omega_0$ and $(\overline{f}_n, \overline{g}_n, \overline{h}_n)$ of $\chi_{B_n(0)\times(0,n^2)}|\omega_0|$, satisfied (3.14), (3.15) in Proposition 3.16 with $1 < q_0 < q$, $L(u_n) = |u_n|^{q-1}u_n$, $L(v_n) = v_n^q$ and μ is replaced by $\chi_{D_n}\omega$ and $\chi_{D_n}|\omega|$ respectively. Moreover, there hold

$$-KI_2[\omega^-] \le u_n \le KI_2[\omega^+], \; 0 \le v_n \le KI_2[|\omega|] \;\text{ in }\; D_n, \tag{6.13}$$

and $v_{n+1} \ge v_n$, $|u_n| \le v_n$ in D_n.
By Remark 3.9, we can assume that

$$||f_n||_{L^1(D_i)} + ||g_n||_{L^2(D_i,\mathbb{R}^N)} + |||h_n| + |\nabla h_n|||_{L^2(D_i)} \le 2|\omega|(D_{i+1}) \text{ and}$$
$$||\overline{f}_n||_{L^1(D_i)} + ||\overline{g}_n||_{L^2(D_i,\mathbb{R}^N)} + |||\overline{h}_n| + |\nabla \overline{h}_n|||_{L^2(D_i)} \le 2|\omega|(D_{i+1}),$$

for any $i = 1, ..., n-1$ and $h_n, \overline{h}_n$ are convergent in $L^1_{\text{loc}}(\mathbb{R}^{N+1})$. On the other hand, since u_n, v_n satisfy (3.14) in Proposition 3.16 with $1 < q_0 < q$, $L(u_n) = |u_n|^{q-1}u_n$, $L(v_n) = v_n^q$ and thanks to Holder inequality: for any $\varepsilon \in (0,1)$

$$(|u_n|+1)^{q_0} \le \varepsilon|u_n|^q + c_1(\varepsilon) \quad \text{and} \quad (|v_n|+1)^{q_0} \le \varepsilon|v_n|^q + c_1(\varepsilon).$$

Thus we get

$$\int_{D_i} |u_n|^q dxdt + \int_{D_i} |u_n|^{q_0} dxdt + \int_{D_i} v_n^q dxdt + \int_{D_i} v_n^{q_0} dxdt \le C(i) + c_2|\omega|(D_{i+1}). \tag{6.14}$$

for $i = 1, ..., n-1$, where the constant $C(i)$ depends on $N, \Lambda_1, \Lambda_2, q_0, q$ and i.
Consequently, we can apply Proposition 3.17 with $\mu_n = -|u_n|^{q-1}u_n + \chi_{D_n}\omega, -v_n^q + \chi_{D_n}|\omega|$ and obtain that there are subsequences of u_n, v_n, still denoted by them, converging to some u, v in $L^1_{loc}(\mathbb{R}; W^{1,1}_{loc}(\mathbb{R}^N))$. So, $\frac{|\nabla u|^2}{(|u|+1)^{\alpha+1}} \in L^1_{loc}(\mathbb{R}^{N+1})$ for all $\alpha > 0$ and $u \in L^q_{loc}(\mathbb{R}^{N+1})$ satisfies (2.17). In addition, using Holder inequality we get $u \in L^\gamma_{\text{loc}}(\mathbb{R}; W^{1,\gamma}_{\text{loc}}(\mathbb{R}^N))$ for any $1 \le \gamma < \frac{2q}{q+1}$.
Thanks to (6.14) and Monotone Convergence Theorem we get $v_n \to v$ in $L^q_{loc}(\mathbb{R}^{N+1})$. After, we also have $u_n \to u$ in $L^q_{loc}(\mathbb{R}^{N+1})$ by $|u_n| \le v_n$ and Dominated Convergence Theorem. Consequently, u is a distribution solution of problem (2.16) which satisfies (2.17).
If $\omega = \mu + \sigma \otimes \delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}(\mathbb{R}^N \times (0,\infty))$ and $\sigma \in \mathfrak{M}(\mathbb{R}^N)$, then by the proof of Theorem 2.8 we can assume that $u_n = 0$ in $B_n(0) \times (-n^2, 0)$. So, $u = 0$ in $\mathbb{R}^N \times (-\infty, 0)$. Therefore, clearly $u|_{\mathbb{R}^N \times [0,\infty)}$ is a distribution solution to (2.18).
This completes the proof of the theorem. ■

Proof of the Theorem 2.14. By the proof of Theorem 2.9 and Remark 6.4, 4.34, there exists a constant $c_1 = c_1(N, q, \Lambda_1, \Lambda_2)$ such that if ω satisfy for every compact set $E \subset \mathbb{R}^{N+1}$,

$$|\omega|(E) \le c_1 \text{Cap}_{\mathcal{H}_2,q'}(E), \tag{6.15}$$

then there is a renormalized solution u_n of

$$\begin{cases} (u_n)_t - \operatorname{div}(A(x,t,\nabla u_n)) = |u_n|^{q-1}u_n + \chi_{D_n}\omega \text{ in } D_n \\ u_n = 0 \quad \text{ on } \partial B_n(0)\times(-n^2,n^2), \\ u_n(-n^2) = 0 \quad \text{in } B_n(0), \end{cases}$$

relative to a decomposition (f_n, g_n, h_n) of $\chi_{D_n}\omega_0$, satisfying (3.14), (3.15) in Proposition 3.16 with $q_0 = q$, $L \equiv 0$ and μ is replaced by $|u_n|^{q-1}u_n + \chi_{D_n}\omega$ and

$$-\frac{qK}{q-1}\mathbb{I}_2[\omega^-](x,t) \le u_n \le \frac{qK}{q-1}\mathbb{I}_2[\omega^+](x,t) \tag{6.16}$$

for a.e (x,t) in D_n and $I_2[\omega^\pm] \in L^q_{loc}(\mathbb{R}^{N+1})$.
Besides, thanks to Remark 3.9, we can assume that f_n, g_n, h_n satisfies (5.17) in proof of Theorem (2.5) and h_n is convergent in $L^1_{\text{loc}}(\mathbb{R}^{N+1})$.
Consequently, we can apply Proposition 3.17 and obtain that there exist a subsequence of u_n, still denoted by it, converging to some u a.e in $\mathbb{R}^{N+1}$ and in $L^1_{loc}(\mathbb{R}; W^{1,1}_{loc}(\mathbb{R}^N))$. Also, $u_n \to u$ in $L^q_{\text{loc}}(\mathbb{R}^{N+1})$ by Dominated Convergence Theorem, $\frac{|\nabla u|^2}{(|u|+1)^{\alpha+1}} \in L^1_{loc}(\mathbb{R}^{N+1})$ for all $\alpha > 0$. Using Holder inequality we get $u \in L^\gamma_{\text{loc}}(\mathbb{R}; W^{1,\gamma}_{\text{loc}}(\mathbb{R}^N))$ for any $1 \le \gamma < \frac{2q}{q+1}$.
Thus we obtain that u is a distribution solution of (2.20) which satisfies (2.21). Since (6.15) holds, thus by Theorem 4.36 we get

$$c_2^{-1}\left[|\omega|\right]^q_{\mathfrak{M}^{\mathcal{H}_2,q'}} \le \left[(\mathbb{I}_2[|\omega|])^q\right]_{\mathfrak{M}^{\mathcal{H}_2,q'}} \le c_2\left[|\omega|\right]^q_{\mathfrak{M}^{\mathcal{H}_2,q'}},$$

so we have $[|u|^q]_{\mathfrak{M}^{\mathcal{H}_2,q'}} \le c_3$. It follows (2.23).
If $\omega = \mu + \sigma\otimes\delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}(\mathbb{R}^N\times(0,\infty))$ and $\sigma \in \mathfrak{M}(\mathbb{R}^N)$, then by Remark 6.4 we can assume that $u_n = 0$ in $B_n(0)\times(-n^2,0)$. So, $u = 0$ in $\mathbb{R}^N\times(-\infty,0)$. Therefore, clearly $u|_{\mathbb{R}^N\times[0,\infty)}$ is a distribution solution to (2.22).
This completes the proof of the theorem. ∎

7 Interior Estimates and Boundary Estimates for Parabolic Equations

In this section we always assume that $u \in C(-T,T,L^2(\Omega)) \cap L^2(-T,T,H^1_0(\Omega))$ is a solution to equation (2.4) in $\Omega\times(-T,T)$ with $\mu \in L^2(\Omega\times(-T,T))$ and $u(-T) = 0$. We extend u by zero to $\Omega\times(-\infty,-T)$, clearly u is a solution to equation

$$\begin{cases} u_t - div\,(A(x,t,\nabla u)) = \chi_{(-T,T)}(t)\mu \quad \text{in } \Omega\times(-\infty,T), \\ u = 0 \quad \text{on } \partial\Omega\times(-\infty,T). \end{cases} \tag{7.1}$$

7.1 Interior Estimates

For each ball $B_{2R} = B_{2R}(x_0) \subset\subset \Omega$ and $t_0 \in (-T,T)$, one considers the unique solution

$$w \in C(t_0-4R^2,t_0;L^2(B_{2R})) \cap L^2(t_0-4R^2,t_0;H^1(B_{2R})) \tag{7.2}$$

to the following equation

$$\begin{cases} w_t - div\,(A(x,t,\nabla w)) = 0 \; in \; Q_{2R}, \\ w = u \quad on \;\; \partial_p Q_{2R}, \end{cases} \tag{7.3}$$

where $Q_{2R} = B_{2R}\times(t_0-4R^2,t_0)$ and $\partial_p Q_{2R} = \left(\partial B_{2R}\times(t_0-4R^2,t_0)\right)\cup\left(B_{2R}\times\left\{t = t_0-4R^2\right\}\right)$.

Theorem 7.1 *There exist constants $\theta_1 > 2$, $\beta_1 \in (0, \frac{1}{2}]$ and C_1, C_2, C_3 depending on N, Λ_1, Λ_2 such that the following estimates are true*

$$\fint_{Q_{2R}} |\nabla u - \nabla w| dxdt \le C_1 \frac{|\mu|(Q_{2R})}{R^{N+1}}, \tag{7.4}$$

$$\left(\fint_{Q_{\rho/2}(y,s)} |\nabla w|^{\theta_1} dxdt \right)^{\frac{1}{\theta_1}} \le C_2 \fint_{Q_\rho(y,s)} |\nabla w| dxdt, \tag{7.5}$$

$$\left(\fint_{Q_{\rho_1}(y,s)} |w - \overline{w}_{Q_{\rho_1}(y,s)}|^2 dxdt \right)^{1/2} \le C_3 \left(\frac{\rho_1}{\rho_2} \right)^{\beta_1} \left(\fint_{Q_{\rho_2}(y,s)} |w - \overline{w}_{Q_{\rho_2}(y,s)}|^2 dxdt \right)^{1/2}, \tag{7.6}$$

and

$$\left(\fint_{Q_{\rho_1}(y,s)} |\nabla w|^2 dxdt \right)^{1/2} \le C_3 \left(\frac{\rho_1}{\rho_2} \right)^{\beta_1 - 1} \left(\fint_{Q_{\rho_2}(y,s)} |\nabla w|^2 dxdt \right)^{1/2} \tag{7.7}$$

for any $Q_\rho(y,s) \subset Q_{2R}$, and $Q_{\rho_1}(y,s) \subset Q_{\rho_2}(y,s) \subset Q_{2R}$.

Proof. Inequalities (7.4), (7.5) and (7.6) were proved by Duzaar and Mingione in [27]. So, it remains to prove (7.7) in case $\rho_1 \le \rho_2/2$. By interior Caccioppoli inequality we have

$$\left(\fint_{Q_{\rho_1}(y,s)} |\nabla w|^2 dxdt \right)^{1/2} \le \frac{c_1}{\rho_1} \left(\fint_{Q_{2\rho_1}(y,s)} |w - \overline{w}_{Q_{2\rho_1}(y,s)}|^2 dxdt \right)^{1/2}.$$

On the other hand, by a Sobolev inequality there holds

$$\left(\fint_{Q_{\rho_2}(y,s)} |w - \overline{w}_{Q_{\rho_2}(y,s)}|^2 dxdt \right)^{1/2} \le c_2 \rho_2 \left(\fint_{Q_{\rho_2}(y,s)} |\nabla w|^2 dxdt \right)^{1/2}.$$

Therefore, (7.7) follows from (7.6). ∎

Corollary 7.2 *Let β_1 be the constant in Theorem 7.1. For $2 - \beta_1 < \theta < N + 2$, there exists a constant $C = C(N, \Lambda_1, \Lambda_2, \theta) > 0$ such that for any $B_\rho(y) \subset B_{\rho_0}(y) \subset\subset \Omega$, $s \in (-T, T)$*

$$\int_{Q_\rho(y,s)} |\nabla u| dxdt \le C \rho^{N+3-\theta} \left(\left(\frac{T_0}{\rho_0} \right)^{N+3-\theta} + 1 \right) ||\mathbb{M}_\theta[\mu]||_{L^\infty(\Omega \times (-T,T))}. \tag{7.8}$$

Proof. Take $B_{\rho_2}(y) \subset\subset \Omega$ and $s \in (-T, T)$. For any $Q_{\rho_1}(y,s) \subset Q_{\rho_2}(y,s)$ with $\rho_1 \le \rho_2/2$, we take w as in Theorem 7.1 with $Q_{2R} = Q_{\rho_2}(y,s)$. Thus,

$$\int_{Q_{\rho_1}(y,s)} |\nabla w| dxdt \le c_1 \left(\frac{\rho_1}{\rho_2} \right)^{N+\beta_1+1} \int_{Q_{\rho_2}(y,s)} |\nabla w| dxdt,$$

$$\int_{Q_{\rho_2}(y,s)} |\nabla u - \nabla w| dxdt \le c_2 \rho_2 |\mu|(Q_{\rho_2}(y,s)),$$

and we also have

$$c_3^{-1} \int_{Q_{\rho_2}(y,s)} |\nabla u| dxdt \le \int_{Q_{\rho_2}(y,s)} |\nabla w| dxdt \le c_3 \int_{Q_{\rho_2}(y,s)} |\nabla u| dxdt.$$

It follows that

$$\begin{aligned}\int_{Q_{\rho_1}(y,s)}|\nabla u|dxdt &\le \int_{Q_{\rho_1}(y,s)}|\nabla w|dxdt+\int_{Q_{\rho_1}(y,s)}|\nabla u-\nabla w|dxdt\\ &\le c_4\left(\frac{\rho_1}{\rho_2}\right)^{N+\beta_1+1}\int_{Q_{\rho_2}(y,s)}|\nabla w|dxdt+\int_{Q_{\rho_2}(y,s)}|\nabla u-\nabla w|dxdt\\ &\le c_5\left(\frac{\rho_1}{\rho_2}\right)^{N+\beta_1+1}\int_{Q_{\rho_2}(y,s)}|\nabla u|dxdt+c_5\rho_2|\mu|(Q_{\rho_2}(y,s)).\end{aligned}$$

This implies

$$\int_{Q_{\rho_1}(y,s)}|\nabla u|dxdt\le c_5\left(\frac{\rho_1}{\rho_2}\right)^{N+\beta_1+1}\int_{Q_{\rho_2}(y,s)}|\nabla u|dxdt+c_5\rho_2^{N+3-\theta}||\mathbb{M}_\theta[\mu]||_{L^\infty(\Omega\times(-T,T))}.$$

Since $N+3-\beta<N+\beta_1+1$, applying [50, Lemma 4.6, page 54] we obtain

$$\int_{Q_\rho(y,s)}|\nabla u|dxdt\le c_6\left(\frac{\rho}{\rho_0}\right)^{N+3-\theta}||\nabla u||_{L^1(\Omega_\times(-T,T))}+c_6\rho^{N+3-\theta}||\mathbb{M}_\theta[\mu]||_{L^\infty(\Omega\times(-T,T))},$$

for any $B_\rho(y)\subset B_{\rho_0}(y)\subset\subset\Omega$, $s\in(-T,T)$. On the other hand, by Remark 3.2

$$||\nabla u||_{L^1(\Omega\times(-T,T))}\le c_7T_0|\mu|(\Omega\times(-T,T))\le c_8T_0^{N+3-\theta}||\mathbb{M}_\theta[\mu]||_{L^\infty(\Omega\times(-T,T))}.$$

Hence, we get the desired result. ■

To continue, we consider the unique solution

$$v\in C(t_0-R^2,t_0;L^2(B_R))\cap L^2(t_0-R^2,t_0;H^1(B_R)) \tag{7.9}$$

to the following equation

$$\begin{cases} v_t-div\left(\overline{A}_{B_R(x_0)}(t,\nabla v)\right)=0 \ in\ Q_R,\\ v=w \qquad on\ \ \partial_pQ_R,\end{cases} \tag{7.10}$$

where $Q_R=B_R(x_0)\times(t_0-R^2,t_0)$ and $\partial_pQ_R=\big(\partial B_R\times(t_0-R^2,t_0)\big)\cup\big(B_R\times\{t=t_0-R^2\}\big)$.

Lemma 7.3 *Let θ_1 be the constant in Theorem 7.1. There exist constants $C_1=C_1(N,\Lambda_1,\Lambda_2)$ and $C_2=C_2(\Lambda_1,\Lambda_2)$ such that*

$$\left(\fint_{Q_R}|\nabla w-\nabla v|^2dxdt\right)^{1/2}\le C_1[A]_{s_1}^R\fint_{Q_{2R}}|\nabla w|dxdt, \tag{7.11}$$

with $s_1=\frac{2\theta_1}{\theta_1-2}$ and

$$C_2^{-1}\int_{Q_R}|\nabla v|^2dxdt\le\int_{Q_R}|\nabla w|^2dxdt\le C_2\int_{Q_R}|\nabla v|^2dxdt. \tag{7.12}$$

Proof. We can choose $\varphi=w-v$ as a test function for equations (7.3), (7.10) and since

$$\int_{Q_R}w_t(w-v)dxdt-\int_{Q_R}v_t(w-v)dxdt=\frac{1}{2}\int_{B_R}(w-v)^2(t_0)dx\ge0,$$

we find

$$-\int_{Q_R}\overline{A}_{B_R(x_0)}(t,\nabla v)\nabla(w-v)dxdt\le-\int_{Q_R}A(x,t,\nabla w)\nabla(w-v)dxdt.$$

By using inequalities (1.2) and (1.3) together with Holder's inequality we get

$$c_1^{-1}\int_{Q_R}|\nabla v|^2dxdt \le \int_{Q_R}|\nabla w|^2dxdt \le c_1\int_{Q_R}|\nabla v|^2dxdt, \tag{7.13}$$

and we also have

$$\begin{aligned}\Lambda_2\int_{Q_R}|\nabla w-\nabla v|^2dxdt &\le \int_{Q_R}\left(\overline{A}_{B_R(x_0)}(t,\nabla w)-\overline{A}_{B_R(x_0)}(t,\nabla v)\right)(\nabla w-\nabla v)\,dxdt\\ &\le \int_{Q_R}\left(\overline{A}_{B_R(x_0)}(t,\nabla w)-A(x,t,\nabla w)\right)(\nabla w-\nabla v)\,dxdt\\ &\le \int_{Q_R}\Theta(A,B_R(x_0))(x,t)|\nabla w||\nabla w-\nabla v|dxdt.\end{aligned}$$

Here we used the definition of $\Theta(A,B_R(x_0))$ in the last inequality. Using Holder's inequality with exponents $s_1=\frac{2\theta_1}{\theta_1-2},\theta_1$ and 2 gives

$$\begin{aligned}\Lambda_2 \fint_{Q_R}|\nabla w-\nabla v|^2 &\le \left(\fint_{Q_R}\Theta(A,B_R(x_0))(x,t)^{s_1}dxdt\right)^{1/s_1}\left(\fint_{Q_R}|\nabla w|^{\theta_1}dxdt\right)^{1/\theta_1}\\ &\quad\times\left(\fint_{Q_R}|\nabla w-\nabla v|^2dxdt\right)^{1/2}.\end{aligned}$$

In other words,

$$\left(\fint_{Q_R}|\nabla w-\nabla v|^2dxdt\right)^{1/2}\le \Lambda_2^{-1}[A]_{s_1}^R\left(\fint_{Q_R}|\nabla w|^{\theta_1}dxdt\right)^{1/\theta_1}.$$

After using the inequality (7.5) in Theorem 7.1 we get (7.11). ■

Lemma 7.4 *Let θ_1 be the constant in Theorem 7.1. There exists a functions $v\in C(t_0-R^2,t_0;L^2(B_R))\cap L^2(t_0-R^2,t_0;H^1(B_R))\cap L^\infty(t_0-\frac{1}{4}R^2,t_0;W^{1,\infty}(B_{R/2}))$ such that*

$$||\nabla v||_{L^\infty(Q_{R/2})}\le C\fint_{Q_{2R}}|\nabla u|dxdt+C\frac{|\mu|(Q_{2R})}{R^{N+1}}, \tag{7.14}$$

and

$$\fint_{Q_R}|\nabla u-\nabla v|dxdt\le C\frac{|\mu|(Q_{2R})}{R^{N+1}}+C[A]_{s_1}^R\left(\fint_{Q_{2R}}|\nabla u|dxdt+\frac{|\mu|(Q_{2R})}{R^{N+1}}\right), \tag{7.15}$$

where $s_1=\frac{2\theta_1}{\theta_1-2}$ and $C=C(N,\Lambda_1,\Lambda_2)$.

Proof. Let w and v be in equations (7.3) and (7.10). By standard interior regularity and inequality (7.5) in Theorem 7.1 and (7.12) in Lemma 7.3 we have

$$\begin{aligned}||\nabla v||_{L^\infty(Q_{R/2})} &\le c_1\left(\fint_{Q_R}|\nabla v|^2dxdt\right)^{1/2}\\ &\le c_2\left(\fint_{Q_R}|\nabla w|^2dxdt\right)^{1/2}\\ &\le c_3\fint_{Q_{2R}}|\nabla w|dxdt.\end{aligned}$$

Thus, we get (7.14) from (7.4) in Theorem 7.1.
On the other hand, (7.11) in Lemma 7.3 and Holder's inequality yield

$$\fint_{Q_R}|\nabla w-\nabla v|dxdt\le c_4[A]_{s_1}^R\fint_{Q_{2R}}|\nabla w|dxdt.$$

It leads

$$\fint_{Q_R} |\nabla u - \nabla v| dxdt \leq \fint_{Q_R} |\nabla u - \nabla w| dxdt + c_4 [A]_{s_1}^R \fint_{Q_{2R}} |\nabla w| dxdt.$$

Consequently, we get (7.15) from (7.4) in Theorem 7.1. The proof is complete. ∎

7.2 Boundary Estimates

In this subsection, we focus on the corresponding estimates near the boundary.
Let $x_0 \in \partial\Omega$ be a boundary point and for $R > 0$ and $t_0 \in (-T, T)$, we set $\tilde{\Omega}_{6R} = \tilde{\Omega}_{6R}(x_0, t_0) = (\Omega \cap B_{6R}(x_0)) \times (t_0 - (6R)^2, t_0)$ and $Q_{6R} = Q_{6R}(x_0, t_0)$.
We consider the unique solution w to the equation

$$\begin{cases} w_t - div\,(A(x,t,\nabla w)) = 0 \; in \; \tilde{\Omega}_{6R}, \\ w = u \qquad on \;\; \partial_p \tilde{\Omega}_{6R}. \end{cases} \tag{7.16}$$

In what follows we extend μ and u by zero to $(\Omega \times (-\infty, T))^c$ and then extend w by u to $\mathbb{R}^{N+1} \backslash \tilde{\Omega}_{6R}$.

In order to obtain estimates for w as in Theorem 7.1 we require the domain Ω to be satisfied $2-$Capacity uniform thickness condition.

7.2.1 2-Capacity uniform thickness domain

It is well known that if $\mathbb{R}^N \backslash \Omega$ satisfies uniformly $2-$thick with constants $c_0, r_0 > 0$, there exist $p_0 \in (\frac{2N}{N+2}, 2)$ and $C = C(N, c_0) > 0$ such that

$$\text{Cap}_{p_0}(\overline{B_r(x)} \cap (\mathbb{R}^N \backslash \Omega), B_{2r}(x)) \geq C r^{N - p_0}, \tag{7.17}$$

for all $0 < r \leq r_0$ and all $x \in \mathbb{R}^N \backslash \Omega$, see [47, 57].

Theorem 7.5 *Suppose that $\mathbb{R}^N \backslash \Omega$ satisfies uniformly $2-$thick with constants c_0, r_0. Let w be in (7.16) with $0 < 6R \leq r_0$. There exist constants $\theta_2 > 2$, $\beta_2 \in (0, \frac{1}{2}]$, C_2, C_3 depending on $N, \Lambda_1, \Lambda_2, c_0$ and C_1 depending on N, Λ_1, Λ_2 such that*

$$\fint_{Q_{6R}} |\nabla u - \nabla w| dxdt \leq C_1 \frac{|\mu|(\tilde{\Omega}_{6R})}{R^{N+1}}, \tag{7.18}$$

$$\left(\fint_{Q_{\rho/2}(z,s)} |\nabla w|^{\theta_2} dxdt \right)^{\frac{1}{\theta_2}} \leq C_2 \fint_{Q_{3\rho}(z,s)} |\nabla w| dxdt, \tag{7.19}$$

$$\left(\fint_{Q_{\rho_1}(y,s)} |w|^2 dxdt \right)^{1/2} \leq C_3 \left(\frac{\rho_1}{\rho_2} \right)^{\beta_2} \left(\fint_{Q_{\rho_2}(y,s)} |w|^2 dxdt \right)^{1/2}, \tag{7.20}$$

and

$$\left(\fint_{Q_{\rho_1}(z,s)} |\nabla w|^2 dxdt \right)^{1/2} \leq C_3 \left(\frac{\rho_1}{\rho_2} \right)^{\beta_2 - 1} \left(\fint_{Q_{\rho_2}(z,s)} |\nabla w|^2 dxdt \right)^{1/2}, \tag{7.21}$$

for any $Q_{3\rho}(z,s) \subset Q_{6R}$, $y \in \partial\Omega$, $Q_{\rho_1}(y,s) \subset Q_{\rho_2}(y,s) \subset Q_{6R}$ and $Q_{\rho_1}(z,s) \subset Q_{\rho_2}(z,s) \subset Q_{6R}$

Proof. 1. For $\eta \in C_c^\infty([t_0-(6R)^2,t_0))$, $0 \le \eta \le 1$, $\eta_t \le 0$ and $\eta(t_0-(6R)^2)=1$. Using $\varphi = T_k(u-w)\eta$, for any $k>0$, as a test function for (7.1) and (7.16), we get

$$\int_{\tilde{\Omega}_{6R}} (u-w)_t T_k(u-w)\eta dxdt + \int_{\tilde{\Omega}_{6R}} (A(x,t,\nabla u)-A(x,t,\nabla w))\nabla T_k(u-w)\eta dxdt = \int_{\tilde{\Omega}_{6R}} T_k(u-w)\eta d\mu.$$

Thanks to (1.3), we obtain

$$-\int_{\tilde{\Omega}_{6R}} \overline{T}_k(u-w)\eta_t dxdt + \Lambda_2 \int_{\tilde{\Omega}_{6R}} |\nabla T_k(u-w)|^2 \eta dxdt \le k|\mu|(\tilde{\Omega}_{6R}),$$

where $\overline{T}_k(s) = \int_0^s T_k(\tau)d\tau$. As in [13, Proposition 2.8], we also verify that

$$|||\nabla(u-w)|||_{L^{\frac{N+2}{N+1},\infty}(\tilde{\Omega}_{6R})} \le c_1|\mu|(\tilde{\Omega}_{6R}).$$

Hence we get (7.18).

2. We need to prove that

$$\fint_{Q_{r/4}(z,s)} |\nabla w|^2 dxdt \le \frac{1}{2}\fint_{Q_{\frac{26}{10}r}(z,s)} |\nabla w|^2 dxdt + c_7\left(\fint_{Q_{\frac{26}{10}r}(z,s)} |\nabla w|^{p_0} dxdt\right)^{\frac{2}{p_0}}, \quad (7.22)$$

for all $Q_{\frac{26}{10}r}(z,s) \subset Q_{6R} = Q_{6R}(x_0,t_0)$. Here the constant p_0 is in inequality (7.17).
Suppose that $B_r(z) \subset \Omega$. Take $\rho \in (0,r]$. Let $\varphi \in C_c^\infty(B_\rho(z))$, $\eta \in C_c^\infty((s-\rho^2,s])$ be such that $0 \le \varphi, \eta \le 1$, $\varphi = 1$ in $B_{\rho/2}(z)$, $\eta = 1$ in $[s-\rho^2/4,s]$ and $|\nabla\varphi| \le c_1/\rho$, $|\eta_t| \le c_1/\rho^2$. We denote

$$\tilde{w}_{B_\rho(z)}(t) = \left(\int_{B_\rho(z)} \varphi(x)^2 dx\right)^{-1} \int_{B_\rho(z)} w(x,t)\varphi(x)^2 dx.$$

Using $\varphi = (w-\tilde{w}_{B_\rho(z)})\varphi^2\eta^2$ as a test function for the equation (7.16) we have for all $s' \in [s-\rho^2/4,s]$

$$\int_{B_\rho(z)\times(s-\rho^2,s')} (w-\tilde{w}_{B_\rho(z)})_t (w-\tilde{w}_{B_\rho(z)})\varphi^2\eta^2 dxdt + \int_{B_\rho(z)\times(s-\rho^2,s')} A(x,t,\nabla w)\nabla\left((w-\tilde{w}_{B_\rho(z)})\varphi^2\eta^2\right) dxdt = 0.$$

Here we used the equality $\int_{B_\rho(z)\times(s-\rho^2,s')} \left(\tilde{w}_{B_\rho(z)}\right)_t (w-\tilde{w}_{B_\rho(z)})\varphi^2\eta^2 dxdt = 0$.
Thus, we can write

$$\begin{aligned}\frac{1}{2}\int_{B_\rho(z)} (w(s')-\tilde{w}_{B_\rho(z)}(s'))^2\varphi^2 dx &+ \int_{B_\rho(z)\times(s-\rho^2,s')} A(x,t,\nabla w)\nabla w\varphi^2\eta^2 dxdt \\ &= -2\int_{B_\rho(z)\times(s-\rho^2,s')} A(x,t,\nabla w)\nabla\varphi\varphi\eta^2(w-\tilde{w}_{B_\rho(z)})dxdt \\ &\quad + \int_{B_\rho(z)\times(s-\rho^2,s')} (w-\tilde{w}_{B_\rho(z)})^2\varphi^2\eta\eta_t dxdt.\end{aligned}$$

From conditions (1.2) and (1.3), we get

$$\begin{aligned}&\frac{1}{2}\int_{B_\rho(z)} (w(s')-\tilde{w}_{B_\rho(z)}(s'))^2\varphi^2 dx + \Lambda_2\int_{B_\rho(z)\times(s-\rho^2,s')} |\nabla w|^2\varphi^2\eta^2 dxdt \\ &\le 2\Lambda_1\int_{B_\rho(z)\times(s-\rho^2,s')} |\nabla w||\nabla\varphi|\varphi\eta^2|w-\tilde{w}_{B_\rho(z)}|dxdt + \frac{c_8}{\rho^2}\int_{Q_\rho(z,s)} (w-\tilde{w}_{B_\rho(z)})^2 dxdt.\end{aligned}$$

Using Holder inequality we can verify that

$$\sup_{s'\in[s-\rho^2/4,s]}\int_{B_\rho(z)}(w(s')-\tilde{w}_{B_\rho(z)}(s'))^2\varphi^2dx \\ +\int_{Q_{\rho/2}(z,s)}|\nabla w|^2dxdt\le\frac{c_9}{\rho^2}\int_{Q_\rho(z,s)}|w-\tilde{w}_{B_\rho(z)}|^2dxdt. \quad (7.23)$$

On the other hand, for any $s'\in[s-\rho^2/4,s]$

$$\int_{B_{\rho/2}(z)}(w(s')-\tilde{w}_{B_{\rho/2}(z)}(s'))^2dx\le 2(1+2^{N+2})\int_{B_\rho(z)}(w(s')-\tilde{w}_{B_\rho(z)}(s'))^2\varphi^2dx, \quad (7.24)$$

where $\varphi_1(x)=\varphi(z+2(x-z))$ for all $x\in B_{\rho/2}(z)$ and

$$\tilde{w}_{B_{\rho/2}(z)}=\left(\int_{B_{\rho/2}(z)}\varphi_1(x)^2dx\right)^{-1}\int_{B_{\rho/2}(z)}w(x,t)\varphi_1(x)^2dx.$$

In fact, since $0\le\varphi\le 1$ and $\varphi=1$ in $B_{\rho/2}(z)$ thus

$$\begin{aligned}&\int_{B_{\rho/2}(z)}(w(s')-\tilde{w}_{B_{\rho/2}(z)}(s'))^2dx\\ &\le 2\int_{B_{\rho/2}(z)}(w(s')-\tilde{w}_{B_\rho(z)}(s'))^2dx+2^{N+1}(\tilde{w}_{B_{\rho/2}(z)}(s')-\tilde{w}_{B_\rho}(z)(s'))^2|B_{\rho/4}(z)|\\ &\le 2\int_{B_\rho(z)}(w(s')-\tilde{w}_{B_\rho(z)}(s'))^2\varphi^2dx+2^{N+2}\int_{B_{\rho/2}(z)}(w(s')-\tilde{w}_{B_{\rho/2}(z)}(s'))^2\varphi_1^2dx\\ &\quad+2^{N+2}\int_{B_{\rho/2}(z)}(w(s')-\tilde{w}_{B_\rho(z)}(s'))^2\varphi_1^2dx.\end{aligned}$$

which yields (7.24) due to the following inequality

$$\int_{B_{\rho/2}(z)}(w(s')-\tilde{w}_{B_{\rho/2}(z)}(s'))^2\varphi_1^2dx\le\int_{B_{\rho/2}(z)}(w(s')-l)^2\varphi_1^2dx\quad\forall l\in\mathbb{R}.$$

Therefore,

$$\sup_{s'\in[s-\rho^2/4,s]}\int_{B_{\rho/2}(z)}(w(s')-\tilde{w}_{B_{\rho/2}(z)}(s'))^2dx \\ +\int_{Q_{\rho/2}(z,s)}|\nabla w|^2dxdt\le\frac{c_{10}}{\rho^2}\int_{Q_\rho(z,s)}|w-\tilde{w}_{B_\rho(z)}|^2dxdt. \quad (7.25)$$

Now we use estimate (7.25) for $\rho=r/2$, we have

$$\begin{aligned}\int_{Q_{r/4}(z,s)}|\nabla w|^2dxdt&\le\frac{c_{10}}{r^2}\int_{Q_{r/2}(z,s)}(w-\tilde{w}_{B_{r/2}(z)})^2dxdt\\ &\le\frac{c_{10}}{r^2}\left(\sup_{s'\in[s-r^2/4,s]}\int_{B_{r/2}(z)}(w(s')-\tilde{w}_{B_{r/2}(z)}(s'))^2dx\right)^{\frac{2}{N+2}}\\ &\qquad\times\int_{s-r^2/4}^{s}\left(\int_{B_{r/2}(z)}(w-\tilde{w}_{B_{r/2}(z)})^2dx\right)^{\frac{N}{N+2}}dt.\end{aligned}$$

After we again use estimate (7.25) for $\rho = r$ we get

$$\int_{Q_{r/4}(z,s)} |\nabla w|^2 dxdt \le \frac{c_{11}}{r^2} \left(\frac{1}{r^2} \int_{Q_r(z,s)} |w - \tilde{w}_{B_r(z)}|^2 dxdt \right)^{\frac{2}{N+2}} \times \int_{s-r^2/4}^{s} \left(\int_{B_{\rho/2}(z)} (w - \tilde{w}_{B_{r/2}(z)})^2 dx \right)^{\frac{N}{N+2}} dt.$$

Thanks to a Sobolev-Poincare inequality, we obtain

$$\int_{Q_{r/4}(z,s)} |\nabla w|^2 dxdt \le \frac{c_{12}}{r^2} \left(\int_{Q_r(z,s)} |\nabla w|^2 dxdt \right)^{\frac{2}{N+2}} \int_{Q_{r/2}(z,s)} |\nabla w|^{\frac{2N}{N+2}} dxdt.$$

Since $p_0 \in (\frac{2N}{N+2}, 2)$, thanks to Holder inequality we get (7.22).
Finally, we consider the case $B_r(z) \cap \Omega \ne \emptyset$. In this case we choose $z_0 \in \partial\Omega$ such that $|z - z_0| = \text{dist}(z, \partial\Omega)$. Then $|z_0 - z| < r$ and thus $\frac{1}{4}r \le \rho_1 \le \frac{1}{2}r$,

$$B_{\frac{1}{4}r}(z) \subset B_{\frac{5}{4}r}(z_0) \subset B_{\rho_1+r}(z_0) \subset B_{\rho_1+\frac{11}{10}r}(z_0) \subset B_{\frac{16}{10}r}(z_0) \subset B_{\frac{26}{10}r}(z) \subset B_{6R}(x_0). \quad (7.26)$$

Let $\varphi \in C_c^\infty(B_{\rho_1+\frac{11}{10}r}(z_0))$ be such that $0 \le \varphi \le 1$, $\varphi = 1$ in $B_{\rho_1+r}(z_0)$ and $|\nabla\varphi| \le C/r$. For $\frac{1}{2}r \le \rho_2 \le r$, let $\eta \in C_c^\infty((s - \rho_2^2, s])$ be such that $0 \le \eta \le 1$, $\eta = 1$ in $[s - \rho_2^2/4, s]$ and $|\eta_t| \le c/r^2$. Using $\phi = w\varphi^2\eta^2$ as a test function for (7.16) we have for any $s' \in (s - \rho_2^2, s)$

$$\int_{(B_{\rho_1+\frac{11}{10}r}(z_0)\cap\Omega)\times(s-\rho_2^2,s')} w_t w \varphi^2\eta^2 dxdt + \int_{(B_{\rho_1+\frac{11}{10}r}(z_0)\cap\Omega)\times(s-\rho_2^2,s')} A(x,t,\nabla w)\nabla\left(w\varphi^2\eta^2\right) dxdt = 0.$$

As above we also get

$$\sup_{s'\in[s-\rho_2^2/4,s]} \int_{B_{\rho_1+r}(z_0)} w^2(s')dx + \int_{B_{\rho_1+r}(z_0)\times(s-\rho_2^2/4,s)} |\nabla w|^2 dxdt \le \frac{c_{13}}{r^2} \int_{B_{\rho_1+\frac{11}{10}r}(z_0)\times(s-\rho_2^2,s)} w^2 dxdt.$$

In particular, for $\rho_1 = \frac{1}{4}r$, $\rho_2 = \frac{1}{2}r$ and using (7.26) yield

$$\int_{Q_{\frac{1}{4}r}(z,s)} |\nabla w|^2 dxdt \le \frac{c_{14}}{r^2} \int_{B_{\frac{29}{20}r}(z_0)\times(s-r^2/4,s)} w^2 dxdt, \quad (7.27)$$

and $\rho_1 = (\frac{1}{4} + \frac{1}{10})r, \rho_2 = r$,

$$\sup_{s'\in[s-r^2/4,s]} \int_{B_{\frac{1}{4}r+\frac{11}{10}r}(z_0)} w^2(s')dx \le \frac{c_{15}}{r^2} \int_{B_{\frac{29}{20}r}(z_0)\times(s-r^2,s)} w^2 dxdt.$$

Set $K_1 = \{w = 0\} \cap \overline{B}_{\frac{29}{20}r}(z_0)$ and $K_2 = \{w = 0\} \cap \overline{B}_{\frac{1}{4}r+\frac{11}{10}r}(z_0)$, Since $\mathbb{R}^N\backslash\Omega$ satisfies an uniformly 2−thick,we have the following estimates

$$\text{Cap}_2(K_1, B_{\frac{29}{10}r}(z_0)) \ge c_{16}r^{N-2} \quad \text{and} \quad \text{Cap}_{p_0}(K_2, B_{\frac{1}{2}r+\frac{11}{5}r}(z_0)) \ge c_{16}r^{N-p_0}.$$

So, by Sobolev-Poincare's inequality we get

$$\fint_{B_{\frac{29}{20}r}(z_0)} w^2 dx \le c_{17}r^2 \fint_{B_{\frac{5}{2}r}(z)} |\nabla w|^2 dx, \quad (7.28)$$

and

$$\fint_{B_{\frac{1}{4}r+\frac{11}{10}r}(z_0)} w^2dxdt \le c_{18}r^2\left(\fint_{B_{\frac{1}{4}r+\frac{11}{10}r}(z_0)}|\nabla w|^{p_0}dx\right)^{\frac{2}{p_0}} \le c_{19}r^2\left(\fint_{B_{\frac{5}{2}r}(z_0)}|\nabla w|^{p_0}dx\right)^{\frac{2}{p_0}}.$$

Leads to

$$\sup_{s'\in[s-r^2/4,s]}\int_{B_{\frac{1}{4}r+\frac{11}{10}r}(z_0)} w^2(s')dx \le c_{20}\int_{Q_{\frac{5}{2}r}(z,s)}|\nabla w|^2dxdt, \tag{7.29}$$

and

$$\int_{B_{\frac{1}{4}r+\frac{11}{10}r}(z_0)} w^2(t)dx \le c_{21}r^{N+2}\left(\fint_{B_{\frac{5}{2}r}(z_0)}|\nabla w|^{p_0}(t)dx\right)^{\frac{2}{p_0}}. \tag{7.30}$$

From (7.27), we have

$$\fint_{Q_{\frac{1}{4}r}(z,s)}|\nabla w|^2dxdt \le \frac{c_{22}}{r^{N+4}}\int_{B_{\frac{1}{4}r+\frac{11}{10}r}(z_0)\times(s-r^2/4,s)} w^2dxdt$$

$$\le \frac{c_{22}}{r^{N+4}}\left(\sup_{s'\in[s-r^2/4,s]}\int_{B_{\frac{1}{4}r+\frac{11}{10}r}(z_0)} w^2(s')dx\right)^{1-\frac{p_0}{2}}\int_{s-r^2/4}^{s}\left(\int_{B_{\frac{1}{4}r+\frac{11}{10}r}(z_0)} w^2(t)dx\right)^{\frac{p_0}{2}}dt.$$

Using (7.30), (7.29) and Holder's inequality we get

$$\fint_{Q_{\frac{1}{4}r}(z,s)}|\nabla w|^2dxdt \le \frac{c_{23}}{r^{N+4}}\left(\int_{Q_{\frac{5}{2}r}(z,s)}|\nabla w|^2dxdt\right)^{1-\frac{p_0}{2}} r^{\frac{N+2}{2}p_0-N}\int_{Q_{\frac{5}{2}r}(z,s)}|\nabla w|^{p_0}dxdt$$

$$= c_{24}\left(\fint_{Q_{\frac{5}{2}r}(z,s)}|\nabla w|^2dxdt\right)^{1-\frac{p_0}{2}}\fint_{Q_{\frac{5}{2}r}(z,s)}|\nabla w|^{p_0}dxdt$$

$$\le \frac{1}{2}\fint_{Q_{\frac{26}{10}r}(z,s)}|\nabla w|^2dxdt + c_{25}\left(\fint_{Q_{\frac{26}{10}r}(z,s)}|\nabla w|^{p_0}dxdt\right)^{\frac{2}{p_0}}.$$

So we proved (7.22).
Therefore, By Gehring's Lemma (see [60]) we get (7.19).
3. Now we prove (7.20). Let $y \in \partial\Omega$, $Q_{\rho_1}(y,s) \subset Q_{\rho_2}(y,s) \subset Q_{6R}$ with $\rho_1 \le \rho_2/4$. First, we will show that there exists a constant $\beta_2 = \beta_2(N,\Lambda_1,\Lambda_2,c_0) \in (0,1/2]$ such that

$$\text{osc}(w,Q_{\rho_1}(y,s)) \le c_{26}\left(\frac{\rho_1}{\rho_2}\right)^{\beta_2}\text{osc}(w,Q_{\rho_2/2}(y,s)), \tag{7.31}$$

where $\text{osc}(w,A) = \sup_A w - \inf_A w$.
Indeed, since

$$\int_0^1 \frac{\text{Cap}_{1,2}(\Omega^c\cap B_r(z),B_{2r}(z))}{r^{N-2}}\frac{dr}{r} = +\infty \quad \forall z \in \partial\Omega.$$

thus by the Wiener criterion (see [83]), we have w is continuous up to $\partial_p\tilde{\Omega}_{6R}$. So, we can choose $\varphi = (V - M_{4\rho_1})\eta^2 \in L^2(-\infty,T;H_0^1(\Omega\cap B_{6R}(x_0)))$ as test function in (7.16), where
a. $\eta \in C^\infty(Q_{4\rho_1}(y,s))$, $0 \le \eta \le 1$ such that $\eta = 1$ in $Q_{\rho_1/2}(y,s-\frac{17}{4}\rho_1^2)$, $\text{supp}(\eta) \subset\subset Q_{\rho_1}(y,s-4\rho_1^2)$ and $|\nabla\eta| \le c_{27}/\rho_1$, $|\eta_t| \le c_{28}/\rho_1^2$.

b. $M_{4\rho_1} = \sup_{Q_{4\rho_1}(y,s)} w$ and $V = \inf\{M_{4\rho_1} - w, M_{4\rho_1}\}$ in $\tilde{\Omega}_{6R}$, $V = M_{4\rho_1}$ outside $\tilde{\Omega}_{6R}$. We have

$$\int_{\tilde{\Omega}_{6R}} w_t \left(V - M_{4\rho_1}\right) \eta^2 dxdt \\ + \int_{\tilde{\Omega}_{6R}} 2\eta A(x,t,\nabla w)\nabla\eta \left(V - M_{4\rho_1}\right) dxdt + \int_{\tilde{\Omega}_{6R}} \eta^2 A(x,t,\nabla w)\nabla V dxdt = 0,$$

which implies

$$\int_{\tilde{\Omega}_{6R}} \eta^2 A(x,t,-\nabla V)(-\nabla V) dxdt = \int_{\tilde{\Omega}_{6R}} 2\eta A(x,t,-\nabla V)\nabla\eta \left(V - M_{4\rho_1}\right) dxdt \\ - \int_{\tilde{\Omega}_{6R}} \left(V - M_{4\rho_1}\right)_t \left(V - M_{4\rho_1}\right) \eta^2 dxdt.$$

Using (1.2) and (1.3) we get

$$\Lambda_2 \int_{\tilde{\Omega}_{6R}} \eta^2 |\nabla V|^2 dxdt \\ \leq 2\Lambda_1 \int_{\tilde{\Omega}_{6R}} \eta |\nabla V||\nabla\eta||V - M_{4\rho_1}| dxdt - 1/2 \int_{\tilde{\Omega}_{6R}} \left(\left(V - M_{4\rho_1}\right)^2 - M_{4\rho_1}^2\right) (\eta^2)_t dxdt \\ \leq 2\Lambda_1 M_{4\rho_1} \int_{\tilde{\Omega}_{6R}} \eta |\nabla V||\nabla\eta| dxdt + 2M_{4\rho_1} \int_{\tilde{\Omega}_{6R}} \eta V |\eta_t| dxdt.$$

Since $\mathrm{supp}(|\nabla V|) \cap \mathrm{supp}(\eta) \subset \tilde{\Omega}_{6R}$, thus

$$\int_{\mathbb{R}^{N+1}} |\nabla(\eta V)|^2 dxdt \leq c_{29} M_{4\rho_1} \left(\int_{\mathbb{R}^{N+1}} \eta |\nabla V||\nabla\eta| dxdt + \int_{\mathbb{R}^{N+1}} V \left(\eta|\eta_t| + |\nabla\eta|^2\right) dxdt\right) \\ \leq c_{30} M_{4\rho_1} \left(\int_{\mathbb{R}^{N+1}} \eta |\nabla V||\nabla\eta| dxdt + \frac{1}{\rho_1^2} \int_{Q_{\rho_1}(y,s-4\rho_1^2)} V dxdt\right). \quad (7.32)$$

By [50, Theorem 6.31, p. 132], for any $\sigma \in (0, 1 + 2/N)$ there holds

$$\left(\fint_{Q_{\rho_1}(y,s-4\rho_1^2)} V^\sigma dxdt\right)^{1/\sigma} \leq c_{31} \inf_{Q_{\rho_1}(y,s)} V = c_{31}(M_{4\rho_1} - \sup_{Q_{\rho_1}(y,s)} w) = c_{31}(M_{4\rho_1} - M_{\rho_1}). \quad (7.33)$$

In particular,

$$\frac{1}{\rho_1^2} \int_{Q_{\rho_1}(y,s-4\rho_1^2)} V dxdt \leq c_{32} \rho_1^N (M_{4\rho_1} - M_{\rho_1}). \quad (7.34)$$

We need to estimate $\int_{\tilde{\Omega}_{6R}} \eta |\nabla V||\nabla\eta| dxdt$. Using Holder inequality and (7.33), for $\varepsilon \in (0, \min\{2/N, 1\})$ we have

$$\int_{\tilde{\Omega}_{6R}} \eta |\nabla V||\nabla\eta| dxdt \leq \left(\int_{\tilde{\Omega}_{6R}} \eta^2 V^{-(1+\varepsilon)} |\nabla V|^2 dxdt\right)^{1/2} \left(\int_{\tilde{\Omega}_{6R}} V^{1+\varepsilon} |\nabla\eta|^2 dxdt\right)^{1/2} \\ \leq c_{28} \left(\int_{\tilde{\Omega}_{6R}} \eta^2 V^{-(1+\varepsilon)} |\nabla V|^2 dxdt\right)^{1/2} \left(\int_{Q_{\rho_1}(y,s-4\rho_1^2)} V^{1+\varepsilon} dxdt\right)^{1/2} \\ \leq c_{33} \left(\int_{\tilde{\Omega}_{6R}} \eta^2 V^{-(1+\varepsilon)} |\nabla V|^2 dxdt\right)^{1/2} \rho_1^{N/2} (M_{4\rho_1} - M_{\rho_1})^{(1+\varepsilon)/2}.$$

To estimate $\left(\int_{\tilde{\Omega}_{6R}} \eta^2 V^{-(1+\varepsilon)}|\nabla V|^2 dxdt\right)^{1/2}$, we can choose $\varphi = ((V+\delta)^{-\varepsilon} - (M_{4\rho_1}+\delta)^{-\varepsilon})\eta^2$, for $\delta > 0$, as test function in (7.16), we will get

$$\int_{\tilde{\Omega}_{6R}} \eta^2 (V+\delta)^{-(1+\varepsilon)}|\nabla V|^2 dxdt \\ \leq c_{34} \int_{\tilde{\Omega}_{6R}} \eta (V+\delta)^{-\varepsilon}|\nabla V||\nabla \eta| dxdt + c_{34} \int_{\tilde{\Omega}_{6R}} \eta (V+\delta)^{1-\varepsilon}|\eta_t| dxdt.$$

Thanks to Holder's inequality, we obtain

$$\int_{\tilde{\Omega}_{6R}} \eta^2 (V+\delta)^{-(1+\varepsilon)}|\nabla V|^2 dxdt \leq c_{35} \int_{\tilde{\Omega}_{6R}} (V+\delta)^{1-\varepsilon}\left(\eta|\eta_t| + |\nabla \eta|^2\right) dxdt \\ \leq c_{36}\rho_1^2 \int_{Q_{\rho_1}(y,s-4\rho_1^2)} (V+\delta)^{1-\varepsilon} dxdt.$$

Letting $\delta \to 0$ and using (7.33), we get

$$\int_{\tilde{\Omega}_{6R}} \eta^2 V^{-(1+\varepsilon)}|\nabla V|^2 dxdt \leq c_{36}\rho_1^2 \int_{Q_{\rho_1}(y,s-4\rho_1^2)} V^{1-\varepsilon} dxdt \\ \leq c_{37}\rho_1^N \left(M_{4\rho_1} - M_{\rho_1}\right)^{1-\varepsilon}.$$

Thus,

$$\int_{\tilde{\Omega}_{6R}} \eta|\nabla V||\nabla \eta| dxdt \leq c_{38}\rho_1^N (M_{4\rho_1} - M_{\rho_1}).$$

Combining this with (7.32) and (7.34),

$$\int_{\mathbb{R}^{N+1}} |\nabla(\eta V)|^2 dxdt \leq c_{39}\rho_1^N M_{4\rho_1}\left(M_{4\rho_1} - M_{\rho_1}\right).$$

Note that $\eta V = M_{4\rho_1}$ in $\left(\Omega^c \cap B_{\rho_1/2}(y)\right) \times (s - \frac{9}{2}\rho_1^2, s - \frac{17}{4}\rho_1^2)$ thus

$$\int_{\mathbb{R}^{N+1}} |\nabla(\eta V)|^2 dxdt \geq \int_{s-\frac{9}{2}\rho_1^2}^{s-\frac{17}{4}\rho_1^2} \int_{\mathbb{R}^{\mathbb{N}}} |\nabla(\eta V)|^2 dxdt \\ \geq \int_{s-\frac{9}{2}\rho_1^2}^{s-\frac{17}{4}\rho_1^2} M_{4\rho_1}^2 \mathrm{Cap}_{1,2}(\Omega^c \cap B_{\rho_1/2}(y), B_{\rho_1}(y)) dt \\ \geq c_{40} M_{4\rho_1}^2 \rho_1^N.$$

Here we used $Cap_{1,2}(\Omega^c \cap B_{\rho_1/2}(y), B_{\rho_1}(y)) \geq c\rho_1^{N-2}$ in the last inequality. It follows

$$M_{4\rho_1} \leq c_{41}(M_{4\rho_1} - M_{\rho_1}).$$

So

$$\sup_{Q_{\rho_1}(y,s)} w \leq \gamma \sup_{Q_{4\rho_1}(y,s)} w \quad \text{where } \gamma = \frac{c_{41}}{c_{41}+1} < 1.$$

Of course, above estimate is also true when we replace w by $-w$. These give,

$$\mathrm{osc}(w, Q_{\rho_1}(y,s)) \leq \gamma \mathrm{osc}(w, Q_{4\rho_1}(y,s)).$$

It follows (7.31).
We come back the proof of (7.20).
Since $w = 0$ outside Ω_T this leads to

$$\left(\fint_{Q_{\rho_1}(y,s)} |w|^2 dxdt\right)^{1/2} \leq c_{42}\mathrm{osc}(w, Q_{\rho_2/2}(y,s)).$$

On the other hand, By [50, Theorem 6.30, p. 132] we have

$$\sup_{Q_{\rho_2/2}(y,s)} w \le c_{43}\left(\fint_{Q_{\rho_2}(y,s)} (w^+)^2dxdt\right)^{1/2} \quad \text{and}$$
$$\sup_{Q_{\rho_2/2}(y,s)} (-w) \le c_{44}\left(\fint_{Q_{\rho_2}(y,s)} (w^-)^2dxdt\right)^{1/2}.$$

Thus, we get (7.20).
Next, we have (7.21) for case $z = y \in \partial\Omega$ since from Caccippoli's inequality,

$$\int_{Q_{\rho_1}(z,s)} |\nabla w|^2dxdt \le \frac{c_{45}}{\rho_1^2}\int_{Q_{2\rho_1}(z,s)} |w|^2dxdt,$$

and using Sobolev-Poincare's inequality as in (7.28),

$$\int_{Q_{\rho_2}(z,s)} |w|^2dxdt \le c_{46}\rho_2^2\int_{Q_{\rho_2}(z,s)} |\nabla w|^2dxdt.$$

We now prove (7.21). Take $Q_{\rho_1}(z,s) \subset Q_{\rho_2}(z,s) \subset Q_{6R}$, it is enough to consider the case $\rho_1 \le \rho_2/20$. Clearly, if $B_{\rho_2/4}(z) \subset \Omega$ then (7.21) follows from (7.7) in Theorem 7.1. We consider $B_{\rho_2/4}(z) \cap \partial\Omega \ne \emptyset$, let $z_0 \in B_{\rho_2/4}(z) \cap \partial\Omega$ such that $|z - z_0| = \text{dist}(z, \partial\Omega) \le \rho_2/4$. Obviously, if $\rho_1 < |z - z_0|/4$ and $z \notin \Omega$, then (7.21) is trivial. If $\rho_1 < |z - z_0|/4$ and $z \in \Omega$, then (7.21) follows from (7.7) in Theorem 7.1.
Now assume $\rho_1 \ge |z - z_0|/4$ then since $Q_{\rho_1}(z,s) \subset Q_{5\rho_1}(z_0,s)$

$$\begin{aligned}\left(\fint_{Q_{\rho_1}(z,s)} |\nabla w|^2dxdt\right)^{1/2} &\le c_{47}\left(\fint_{Q_{5\rho_1}(z_0,s)} |\nabla w|^2dxdt\right)^{1/2}\\ &\le c_{48}\left(\frac{\rho_1}{\rho_2}\right)^{\beta_2-1}\left(\fint_{Q_{\rho_2/4}(z_0,s)} |\nabla w|^2dxdt\right)^{1/2}\\ &\le c_{49}\left(\frac{\rho_1}{\rho_2}\right)^{\beta_2-1}\left(\fint_{Q_{\rho_2/2}(z,s)} |\nabla w|^2dxdt\right)^{1/2},\end{aligned}$$

which implies (7.21). ■

Corollary 7.6 *Suppose that $\mathbb{R}^N\backslash\Omega$ satisfies uniformly $2-$thick with constants c_0, r_0. Let β_2 be the constant in Theorem 7.5. For $2 - \beta_2 < \theta < N + 2$, there exists a constant $C = C(N, \Lambda_1, \Lambda_2, \theta) > 0$ such that for any $B_\rho(y) \cap \partial\Omega \ne \emptyset$, $s \in (-T, T)$, $0 < \rho \le r_0$*

$$\int_{Q_\rho(y,s)} |\nabla u|dxdt \le C\rho^{N+3-\theta}\left(\left(\frac{T_0}{r_0}\right)^{N+3-\theta} + 1\right)||\mathbb{M}_\theta[\mu]||_{L^\infty(\Omega\times(-T,T))}, \tag{7.35}$$

where $T_0 = diam(\Omega) + T^{1/2}$.

Proof. Take $B_{\rho_2/4}(y)\cap\partial\Omega \ne \emptyset$ and $s \in (-T, T)$, $\rho_2 \le 2r_0$. Let $y_0 \in B_{\rho_2/4}(y)\cap\partial\Omega$ such that $|y - y_0| = dist(y, \partial\Omega) \le \rho_2/4$, thus $Q_{\rho_2/4}(y,s) \subset Q_{\rho_2/2}(y_0,s)$ For any $Q_{\rho_1}(y,s) \subset Q_{\rho_2}(y,s)$ with $\rho_1 \le \rho_2/4$, we take w as in Theorem 7.5 with $Q_{6R} = Q_{\rho_2/2}(y_0,s)$. Thus,

$$\int_{Q_{\rho_1}(y,s)} |\nabla w|dxdt \le c_1\left(\frac{\rho_1}{\rho_2}\right)^{N+\beta_1+1}\int_{Q_{\rho_2/4}(y,s)} |\nabla w|dxdt,$$
$$\int_{Q_{\rho_2/2}(y_0,s)} |\nabla u - \nabla w|dxdt \le c_2\rho_2|\mu|(Q_{\rho_2/2}(y_0,s)).$$

As in the proof of Corollary 7.2, we get the result. ■

7.2.2 Reifenberg flat domain

In this subsection, we always assume that A satisfies (2.27). Also, we assume that Ω is a (δ, R_0)- Reifenberg flat domain with $0<\delta<1/2$. Fix $x_0 \in \partial\Omega$ and $0<R<R_0/6$. We have a density estimate

$$|B_t(x)\cap(\mathbb{R}^N\backslash\Omega)| \geq c|B_t(x)| \ \forall x\in\partial\Omega, 0<t<R_0, \tag{7.36}$$

with $c=((1-\delta)/2)^N \geq 4^{-N}$.
In particular, $\mathbb{R}^N\backslash\Omega$ satisfies uniformly 2−thick with constants $c, r_0 = R_0$.
Next we set $\rho = R(1-\delta)$ so that $0<\rho/(1-\delta)<R_0/6$. By the definition of Reifenberg flat domains, there exists a coordinate system $\{y_1, y_2, ..., y_N\}$ with the origin $0\in\Omega$ such that in this coordinate system $x_0 = (0, ..., 0, -\rho\delta/(1-\delta))$ and

$$B_\rho^+(0) \subset \Omega\cap B_\rho(0) \subset B_\rho(0)\cap\{y=(y_1,y_2,....,y_N): y_N > -2\rho\delta/(1-\delta)\}.$$

Since $\delta<1/2$ we have

$$B_\rho^+(0) \subset \Omega\cap B_\rho(0) \subset B_\rho(0)\cap\{y=(y_1,y_2,....,y_N): y_N > -4\rho\delta\},$$

where $B_\rho^+(0) := B_\rho(0)\cap\{y=(y_1,y_2,...,y_N): y_N>0\}$.
Furthermore we consider the unique solution

$$v\in C(t_0-\rho^2, t_0; L^2(\Omega\cap B_\rho(0)))\cap L^2(t_0-\rho^2, t_0; H^1(\Omega\cap B_\rho(0))) \tag{7.37}$$

to the following equation

$$\begin{cases} v_t - div\left(\overline{A}_{B_\rho(0)}(t,\nabla v)\right) = 0 \ in \ \tilde{\Omega}_\rho(0), \\ v = w \quad\quad on \ \ \partial_p\tilde{\Omega}_\rho(0), \end{cases} \tag{7.38}$$

where $\tilde{\Omega}_\rho(0) = (\Omega\cap B_\rho(0))\times(t_0-\rho^2, t_0)$ $(-T<t_0<T)$.
We put $v=w$ outside $\tilde{\Omega}_\rho(0)$. As Lemma 7.3 we have the following Lemma.

Lemma 7.7 *Let θ_2 be the constant in Theorem 7.5. There exists constants $C_1 = C_1(N, \Lambda_1, \Lambda_2)$, $C_2 = C_2(\Lambda_1,\Lambda_2)$ such that*

$$\left(\fint_{Q_\rho(0,t_0)} |\nabla w - \nabla v|^2\right)^{1/2} \leq [A]_{s_2}^R \fint_{Q_\rho(0,t_0)} |\nabla w| dxdt, \tag{7.39}$$

with $s_2 = \frac{2\theta_2}{\theta_2 - 2}$ and

$$C_2^{-1}\int_{Q_\rho(0,t_0)} |\nabla v|^2 dxdt \leq \int_{Q_\rho(0,t_0)} |\nabla w|^2 dxdt \leq C_2 \int_{Q_\rho(0,t_0)} |\nabla v|^2 dxdt. \tag{7.40}$$

We can see that if the boundary of Ω is bad enough, then the L^∞-norm of ∇v up to $\partial\Omega\cap B_\rho(0)\times(t_0-\rho^2, t_0)$ could be unbounded. For our purpose, we will consider another equation:

$$\begin{cases} V_t - div\left(\overline{A}_{B_\rho(0)}(t,\nabla V)\right) = 0 \ \text{ in } \ Q_\rho^+(0,t_0), \\ V = 0 \quad\quad \text{on } \ T_\rho(0,t_0), \end{cases} \tag{7.41}$$

where $Q_\rho^+(0,t_0) = B_\rho^+(0)\times(t_0-\rho^2,t_0)$ and $T_\rho(0,t_0) = Q_\rho(0,t_0)\cap\{x_N=0\}$.
A weak solution V of above problem is understood in the following sense: the zero extension of V to $Q_\rho(0,t_0)$ is in $V\in C(t_0-\rho^2,t_0;L^2(B_\rho(0)))\cap L^2_{\text{loc}}(t_0-\rho^2,t_0;H^1(B_\rho(0)))$ and for every $\varphi\in C_c^1(Q_\rho^+(0,t_0))$ there holds

$$-\int_{Q_\rho^+(0,t_0)} V\varphi_t dxdt + \int_{Q_\rho^+(0,t_0)} \overline{A}_{B_\rho(0)}(t,\nabla V)\nabla\varphi dxdt = 0.$$

We have the following gradient L^∞ estimate up to the boundary for V.

Lemma 7.8 (see [48, 49]) *For any weak solution* $V \in C(t_0-\rho^2, t_0; L^2(B_\rho^+(0))) \cap L^2_{loc}(t_0-\rho^2, t_0; H^1(B_\rho^+(0)))$ *of* (7.41), *we have*

$$||\nabla V||_{L^\infty(Q^+_{\rho'/2}(0,t_0))} \leq C \fint_{Q^+_{\rho'}(0,t_0)} |\nabla V|^2 dxdt \quad \forall\ 0 < \rho' \leq \rho. \tag{7.42}$$

for some constant $C = C(N, \Lambda_1, \Lambda_2) > 0$. *Moreover,* ∇V *is continuous up to* $T_\rho(0, t_0)$.

Lemma 7.9 *If* $V \in C(t_0-\rho^2, t_0; L^2(B_\rho^+(0))) \cap L^2(t_0-\rho^2, t_0; H^1(B_\rho^+(0)))$ *is a weak solution of* (7.41), *then its zero extension from* $Q_\rho^+(0, t_0)$ *to* $Q_\rho(0, t_0)$ *solves*

$$V_t - div\left(\overline{A}_{B_\rho(0)}(t, \nabla V)\right) = \frac{\partial F}{\partial x_N}, \tag{7.43}$$

weakly in $Q_\rho(0, t_0)$, *for* $(x, t) = (x', x_N, t) \in Q_\rho(0, t_0)$,
$\overline{A}_{B_\rho(0)} = (\overline{A}^1_{B_\rho(0)}, \overline{A}^2_{B_\rho(0)}, ..., \overline{A}^N_{B_\rho(0)})$, *and* $F(x, t) = \chi_{x_N<0}\overline{A}^N_{B_\rho(0)}(t, \nabla V(x', 0, t))$.

Proof. Let $g \in C^\infty(\mathbb{R})$ with $g = 0$ on $(-\infty, 1/2)$ and $g = 1$ on $(1, \infty)$. Then, for any $\varphi \in C_c^\infty(Q_\rho(0, t_0))$ and $n \in \mathbb{N}$. We have $\varphi_n(x, t) = \varphi_n(x', x_N, t) = g(nx_N)\varphi(x, t) \in C_c^\infty(Q_\rho^+(0, t_0)$. Thus, we get

$$\int_{Q_\rho^+(0,t_0)} V_t\varphi_n dxdt + \int_{Q_\rho^+(0,t_0} \overline{A}_{B_\rho(0)}(t, \nabla V)\nabla\left(g(nx_N)\varphi(x, t)\right) dxdt = 0,$$

which implies

$$\int_{Q_\rho^+(0,t_0)} V_t\varphi_n dxdt + \int_{Q_\rho^+(0,t_0)} \overline{A}_{B_\rho(0)}(t, \nabla V)\nabla\varphi(x, t)g(nx_N)dxdt$$
$$= -\int_0^\rho G(x_N)g'(nx_N)ndx_N.$$

where

$$G(x_N) = \int_{t_0-\rho^2}^{t_0} \int_{|x'|<\sqrt{\rho^2-x_N^2}} \overline{A}^N_{B_\rho(0)}(t, \nabla V)\varphi(x', x_N, t)dx'dt \in C([0, \infty)).$$

Letting $n \to \infty$ we get

$$\int_{Q_\rho^+(0,t_0)} V_t\varphi dxdt + \int_{Q_\rho^+(0,t_0)} \overline{A}_{B_\rho(0)}(t, \nabla V)\nabla\varphi(x, t)dxdt = -G(0)$$
$$= -\int_{Q_\rho(0,t_0)} F\frac{\partial\varphi}{\partial x_N}dxdt.$$

Since $\nabla V = 0, V = 0$ outside Q_ρ^+, therefore we get the result. ∎

We now consider a scaled version of equation (7.38)

$$\begin{cases} v_t - div\left(\overline{A}_{B_1(0)}(t, \nabla v)\right) = 0 \quad \text{in } \tilde{\Omega}_1(0), \\ v = 0 \quad \text{on } \partial_p\tilde{\Omega}_1(0)\backslash\left(\Omega \times (-T, T)\right), \end{cases} \tag{7.44}$$

under assumption

$$B_1^+(0) \subset \Omega \cap B_1(0) \subset B_1(0) \cap \{x_N > -4\delta\}. \tag{7.45}$$

Lemma 7.10 *For any* $\varepsilon > 0$ *there exists a small* $\delta = \delta(N, \Lambda_1, \Lambda_2, \varepsilon) > 0$ *such that if* $v \in C(t_0-1, t_0; L^2(\Omega \cap B_1(0))) \cap L^2(t_0-1, t_0; H^1(\Omega \cap B_1(0)))$ *is a solution of* (7.44) *and* (7.45) *is satisfied and the bounded*

$$\fint_{Q_1(0,t_0)} |\nabla v|^2 dxdt \leq 1, \tag{7.46}$$

then there exists a weak solution $V \in C(t_0-1,t_0;L^2(B_1^+(0)))\cap L^2(t_0-1,t_0;H^1(B_1^+(0)))$ *of* (7.41) *with* $\rho=1$, *whose zero extension to* $Q_1(0,t_0)$ *satisfies*

$$\fint_{Q_1(0,t_0)} |v-V|^2 dxdt \le \varepsilon^2, \tag{7.47}$$

Proof. We argue by contradiction. Suppose that the conclusion were false. Then, there exist a constant $\varepsilon_0>0$, $t_0\in\mathbb{R}$ and a sequence of nonlinearities $\{A_k\}$ satisfying (1.2) and (2.27), a sequence of domains $\{\Omega^k\}$, and a sequence of functions $\{v_k\}\subset C(t_0-1,t_0;L^2(\Omega^k\cap B_1(0)))\cap L^2(t_0-1,t_0;H^1(\Omega^k\cap B_1(0)))$ such that

$$B_1^+(0)\subset\Omega^k\cap B_1(0)\subset B_1(0)\cap\{x_N>-1/2k\}, \tag{7.48}$$

$$\begin{cases} (v_k)_t - div\left(\overline{A}_{k,B_1(0)}(t,\nabla v_k)\right)=0 \quad \text{in } \tilde{\Omega}_1^k(0), \\ v_k=0 \quad \text{on} \quad (\partial_p\tilde{\Omega}_1^k(0))\backslash(\Omega^k\times(-T,T)), \end{cases} \tag{7.49}$$

and the zero extension of each v_k to $Q_1(0,t_0)$ satisfies

$$\fint_{Q_1(0,t_0)} |\nabla v_k|^2 dxdt \le 1 \text{ but} \tag{7.50}$$

$$\fint_{Q_1(0,t_0)} |v_k-V_k|^2 dxdt \ge \varepsilon_0^2, \tag{7.51}$$

for any weak solution V_k of

$$\begin{cases} (V_k)_t - div\left(\overline{A}_{k,B_1(0)}(t,\nabla V_k)\right)=0, \text{ in } Q_1^+(0,t_0), \\ V_k=0 \text{ on } T_1(0,t_0). \end{cases} \tag{7.52}$$

By (7.48) and (7.50) and Poincare's inequality it following that

$$||v_k||_{L^2(t_0-1,t_0;H^1(B_1(0)))}\le c_1||\nabla v_k||_{L^2(Q_1(0,t_0)}\le c_2,$$

and

$$\begin{aligned} ||(v_k)_t||_{L^2(t_0-1,t_0;H^{-1}(B_1(0)))} &= ||\overline{A}_{k,Q_1(0,t_0)}(\nabla v_k)||_{L^2(t_0-1,t_0;H^{-1}(B_1(0)))} \\ &\le \int_{Q_1(0,t_0)} |\overline{A}_{k,B_1(0)}(t,\nabla v_k)|^2 dxdt \\ &\le c_3\int_{Q_1(0,t_0)} |\nabla v_k|^2 dxdt \\ &\le c_4. \end{aligned}$$

Therefore, using Aubin−Lions Lemma, one can find v_0 and a subsequence, still denoted by $\{v_k\}$ such that

$$v_k\to v_0 \text{ weakly in } L^2(t_0-1,t_0,H^1(B_1(0))) \text{ and strongly in } L^2(t_0-1,t_0,L^2(B_1(0))),$$

and

$$(v_k)_t\to(v_0)_t \quad \text{weakly in } L^2(t_0-1,t_0,H^{-1}(B_1(0))).$$

Moreover, $v_0=0$ in $Q_1^-(0,t_0):=(B_1(0)\cap\{x_N<0\})\times(1-t_0,1)$ since $v_k=0$ on outside $\Omega^k\cap Q_1(0,t_0)$ for all k.
To get a contradiction we take V_k to be the unique solution of $(V_k)_t - div\left(\overline{A}_{k,B_1(0)}(t,\nabla V_k)\right)=0$ in $Q_1^+(0,t_0)$ and $V_k-v_0\in L^2(t_0-1,t_0,H_0^1(B_1^+(0)))$ and $V_k(t_0-1)=v_0(t_0-1)$. As above, one can find V_0 and a subsequence, still denoted by $\{V_k\}$ such that

$$V_k\to V_0 \text{ weakly in } L^2(t_0-1,t_0,H^1(B_1(0))) \text{ and strongly in } L^2(t_0-1,t_0,L^2(B_1(0))),$$

and

$$(V_k)_t \to (V_0)_t \quad \text{weakly in} \quad L^2(t_0-1,t_0,H^{-1}(B_1)),$$

for some $V_0 \in v_0 + L^2(t_0-1,t_0,H_0^1(B_1^+(0))$ and $V_0(t_0-1) = v_0(t_0-1)$.
Thanks to (7.51), the proof would be complete if we could show that $v_0 = V_0$. In fact,
Let $\mathcal{J}_k : X \to L^2(Q_1^+(0,t_0),\mathbb{R}^N)$ determined by

$$\mathcal{J}_k(\phi(x,t)) = \overline{A}_{k,B_1(0)}(t,\nabla\phi(x,t)) \quad \text{for any} \quad \phi \in X,$$

where $X \subset L^2(t_0-1,t_0,H^1(B_1(0)))$ is closures (in the strong topology of $L^2(t_0-1,t_0,H^1(B_1(0))))$ of convex combinations of $\{v_k\}_{k\geq 1} \cup \{V_k\}_{k\geq 1} \cup \{0\}$.
Since v_k, V_k converge weakly to v_0, V_0 in $L^2(t_0-1,t_0,H^1(B_1(0)))$ resp., thus by Mazur Theorem, X is compact subset of $L^2(t_0-1,t_0,H^1(B_1(0)))$ and $v_0, V_0 \in X$.
Thanks to (1.2) and (2.27), we get $\mathcal{J}_k(0) = 0$ and

$$||\mathcal{J}_k(\phi_1) - \mathcal{J}_k(\phi_2)||_{L^2(Q_1^+(0,t_0),\mathbb{R}^N)} \leq \Lambda_1 ||\phi_1 - \phi_2||_{L^2(t_0-1,t_0,H^1(B_1(0)))},$$

for every $\phi_1, \phi_2 \in X$ and $k \in \mathbb{N}$. Thus, by Ascoli Theorem, there exist $\mathcal{J} \in C(X, L^2(Q_1^+(0,t_0),\mathbb{R}^N))$ and a subsequence of $\{\mathcal{J}_k\}$, still denote by it, such that

$$\sup_{\phi\in X} ||\mathcal{J}_k(\phi) - \mathcal{J}(\phi)||_{L^2(Q_1^+(0,t_0),\mathbb{R}^N)} \to 0 \quad \text{as} \quad k \to \infty, \tag{7.53}$$

and also for any $\phi_1, \phi_2 \in X$,

$$\int_{Q_1^+(0,t_0)} (\mathcal{J}(\phi_1) - \mathcal{J}(\phi_2)) \, . \, (\nabla\phi_1 - \nabla\phi_2) \, dxdt \geq \Lambda_2 |||\nabla\phi_1 - \nabla\phi_2|||_{L^2(Q_1^+(0,t_0))}. \tag{7.54}$$

From (7.48), we deduce

$$\int_{Q_1^+(0,t_0)} (v_k - V_k)_t (v_0 - V_0) dxdt + \int_{Q_1^+(0,t_0)} \left(\overline{A}_{k,B_1(0)}(t,\nabla v_k) - \overline{A}_{k,B_1(0)}(t,\nabla V_k)\right) . \nabla(v_0 - V_0) dxdt = 0.$$

We have

$$\int_{Q_1^+(0,t_0)} |\overline{A}_{k,B_1(0)}(\nabla v_k)|^2 dxdt \leq c_9 \int_{Q_1^+(0,t_0)} |\nabla v_k|^2 dxdt \leq c_{10} \quad \text{and}$$
$$\int_{Q_1^+(0,t_0)} |\overline{A}_{k,B_1(0)}(\nabla V_k)|^2 dxdt \leq c_9 \int_{Q_1^+(0,t_0)} |\nabla V_k|^2 dxdt \leq c_{11}.$$

for every k.
Thus there exists a subsequence, still denoted by $\{\overline{A}_{k,B_1(0)}(t,\nabla v_k), \overline{A}_{k,B_1(0)}(t,\nabla V_k)\}$ and a vector field A_1, A_2 belonging to $L^2(Q_1^+(0,t_0),\mathbb{R}^N)$ such that

$$\overline{A}_{k,B_1(0)}(t,\nabla v_k) \to A_1 \quad \text{and} \quad \overline{A}_{k,B_1(0)}(t,\nabla V_k) \to A_2,$$

weakly in $L^2(Q_1^+(0,t_0),\mathbb{R}^N)$. It follows

$$\int_{Q_1^+(0,t_0)} (v_0 - V_0)_t (v_0 - V_0) dxdt + \int_{Q_1^+(0,t_0)} (A_1 - A_2) . \nabla(v_0 - V_0) dxdt = 0.$$

Since

$$\int_{Q_1^+(0,t_0)} (v_0 - V_0)_t (v_0 - V_0) dxdt = \int_{B_1^+(0)} (v_0 - V_0)^2 (t_0) dx \geq 0,$$

we get

$$\int_{Q_1^+(0,t_0)} (A_1 - A_2).\nabla(v_0 - V_0)dxdt \le 0. \tag{7.55}$$

For our purpose, we need to show that

$$\int_{Q_1^+(0,t_0)} (A_1 - \mathcal{J}(v_0)).\nabla(v_0 - V_0)dxdt \ge 0 \text{ and} \tag{7.56}$$

$$\int_{Q_1^+(0,t_0)} (A_2 - \mathcal{J}(V_0)).\nabla(V_0 - v_0)dxdt \ge 0. \tag{7.57}$$

To do this, we fix a function $g \in X$ and any $\varphi \in C_c^1(Q_1^+(0,t_0))$ such that $\varphi \ge 0$. We have

$$\begin{aligned}
0 &\le \int_{Q_1^+(0,t_0)} \varphi\left(\overline{A}_{k,B_1(0)}(t,\nabla v_k) - \overline{A}_{k,B_1(0)}(t,\nabla g)\right)(\nabla v_k - \nabla g)\,dxdt \\
&= \int_{Q_1^+(0,t_0)} \varphi \overline{A}_{k,B_1(0)}(t,\nabla v_k)\nabla v_k dxdt - \int_{Q_1^+(0,t_0)} \varphi \overline{A}_{k,B_1(0)}(t,\nabla v_k)\nabla g dxdt \\
&- \int_{Q_1^+(0,t_0)} \varphi \overline{A}_{k,B_1(0)}(t,\nabla g)(\nabla v_k - \nabla g)\,dxdt \\
&:= B_1 + B_2 + B_3.
\end{aligned}$$

It is easy to see that

$$\lim_{k\to\infty} B_2 = -\int_{Q_1^+(0,t_0)} \varphi A_1 \nabla g dxdt \quad \text{and} \quad \lim_{k\to\infty} B_3 = -\int_{Q_1^+(0,t_0)} \varphi \mathcal{J}(g)(\nabla v_0 - \nabla g)\,dxdt.$$

Moreover, we have

$$\begin{aligned}
B_1 &= -\int_{Q_1^+(0,t_0)} (v_k)_t \varphi v_k dxdt - \int_{Q_1^+(0,t_0)} \overline{A}_{k,Q_1(0,t_0)}(\nabla v_k)\nabla\varphi v_k dxdt \\
&= \frac{1}{2}\int_{Q_1^+(0,t_0)} v_k^2 \varphi_t dxdt - \int_{Q_1^+(0,t_0)} \overline{A}_{k,Q_1(0,t_0)}(\nabla v_k)\nabla\varphi v_k dxdt.
\end{aligned}$$

Thus,

$$\begin{aligned}
\lim_{k\to\infty} B_1 &= \frac{1}{2}\int_{Q_1^+(0,t_0)} v_0^2 \varphi_t dxdt - \int_{Q_1^+(0,t_0)} A_1 \nabla\varphi v_0 dxdt \\
&= -\int_{Q_1^+(0,t_0)} (v_0)_t \varphi v_0 dxdt - \int_{Q_1^+(0,t_0)} A_1 \nabla(\varphi v_0)dxdt + \int_{Q_1^+(0,t_0)} \varphi A_1 \nabla v_0 dxdt \\
&= \int_{Q_1^+(0,t_0)} \varphi A_1 \nabla v_0 dxdt.
\end{aligned}$$

Hence,

$$0 \le \int_{Q_1^+(0,t_0)} \varphi\left(A_1 - \mathcal{J}(g)\right)(\nabla v_0 - \nabla g)\,dxdt$$

holds for all $\varphi \in C_c^1(Q_1^+(0,t_0))$, $\varphi \ge 0$ and $g \in X$. Now we choose $g = v_0 - \xi(v_0 - V_0) = (1-\xi)v_0 + \xi V_0 \in X$ for $\xi \in (0,1)$, so

$$0 \le \int_{Q_1^+(0,t_0)} \varphi\left(A - \mathcal{J}(v_0 - \xi(v_0 - V_0))\right)(\nabla v_0 - \nabla V_0)\,dxdt$$

Letting $\xi \to 0^+$ and $\varphi \to \chi_{Q_1^+(0,t_0)}$, we get (7.56). Similarly, we also obtain (7.57). Thus,

$$\int_{Q_1^+(0,t_0)} (A_1 - A_2)\nabla(v_0 - V_0)dxdt \geq \int_{Q_1^+(0,t_0)} (\mathcal{J}(v_0) - \mathcal{J}(V_0))\nabla(v_0 - V_0)dxdt.$$

Combining this with (7.54), (7.55) and $v_0 - V_0 \in L^2(t_0 - 1, t_0, H_0^1(B_1^+(0)))$, yields $v_0 = V_0$. This completes the proof of Lemma. ■

Lemma 7.11 *For any $\varepsilon > 0$ there exists a small $\delta = \delta(N, \Lambda_1, \Lambda_2, \varepsilon) > 0$ such that if $v \in C(t_0 - 1, t_0; L^2(\Omega \cap B_1(0))) \cap L^2(t_0 - 1, t_0; H^1(\Omega \cap B_1(0)))$ is a solution of* (7.44) *and* (7.45) *is satisfied and the bounded*

$$\fint_{Q_1(0,t_0)} |\nabla v|^2 dxdt \leq 1, \tag{7.58}$$

then there exists a weak solution $V \in C(t_0 - 1, t_0; L^2(B_1^+(0))) \cap L^2(t_0 - 1, t_0; H^1(B_1^+(0)))$ of (7.41) *with $\rho = 1$, whose zero extension to $Q_1(0, t_0)$ satisfies*

$$||\nabla V||_{L^\infty(Q_{1/4}(0,t_0))} \leq C \quad \text{and} \tag{7.59}$$

$$\fint_{Q_{1/8}(0,t_0)} |\nabla v - \nabla V|^2 dxdt \leq \varepsilon^2, \tag{7.60}$$

for some $C = C(N, \Lambda_1, \Lambda_2) > 0$.

Proof. Given $\varepsilon_1 \in (0, 1)$ by applying Lemma 7.10 one finds a small $\delta = \delta(N, \Lambda_1, \Lambda_2, \varepsilon_1) > 0$ and a weak solution $V \in C(t_0 - 1, t_0; L^2(B_1^+(0))) \cap L^2(t_0 - 1, t_0; H^1(B_1^+(0)))$ of (7.41) with $\rho = 1$ such that

$$\fint_{Q_1(0,t_0)} |v - V|^2 dxdt \leq \varepsilon_1^2, \tag{7.61}$$

Using $\phi^2 V$ with $\phi \in C_c^\infty(B_1 \times (t_0 - 1, t_0])$, $0 \leq \phi \leq 1$ and $\phi = 1$ in $Q_{1/2}(0, t_0)$ as test function in (7.41), we can obtain

$$\int_{Q_{1/2}(0,t_0)} |\nabla V|^2 dxdt \leq c_1 \int_{Q_1(0,t_0)} |V|^2 dxdt.$$

This implies

$$\begin{aligned}
\int_{Q_{1/2}(0,t_0)} |\nabla V|^2 dxdt &\leq c_2 \int_{Q_1(0,t_0)} \left(|v - V|^2 + |v|^2\right) dxdt \\
&\leq c_3 \int_{Q_1(0,t_0)} \left(|v - V|^2 + |\nabla v|^2\right) dxdt \\
&\leq c_4,
\end{aligned}$$

since (7.58), (7.61) and Poincare's inequality. Thus, using Lemma 7.8 we get (7.59). Next, we will prove (7.60). By Lemma 7.9, the zero extension of V to $Q_1(0, t_0)$ satisfies

$$V_t - div\left(\overline{A}_{B_1(0)}(t, \nabla V)\right) = \frac{\partial F}{\partial x_N} \quad \text{in weakly} \quad Q_1(0, t_0).$$

where $F(x, t) = \chi_{x_N<0}\overline{A}^N_{B_\rho(0)}(t, \nabla V(x', 0, t))$. Thus, we can write

$$\begin{aligned}
&\int_{\tilde{\Omega}_1(0,t_0)} (V - v)_t \varphi dxdt \\
&\quad + \int_{\tilde{\Omega}_1(0,t_0)} \left(\overline{A}_{B_1(0)}(t, \nabla V) - \overline{A}_{B_1(0)}(t, \nabla v)\right) \nabla\varphi dxdt = -\int_{\tilde{\Omega}_1(0,t_0)} F \frac{\partial \varphi}{\partial x_N} dxdt,
\end{aligned}$$

for any $\varphi \in L^2(t_0-1,t_0,H_0^1(\Omega\cap B_1(0)))$.
We take $\varphi = \phi^2(V-v)$ where $\varphi \in C_c^\infty(B_{1/4}\times(t_0-(1/4)^2,t_0])$, $0\le\phi\le 1$ and $\phi = 1$ on $\overline{Q}_{1/8}(0,t_0)$, so

$$\begin{aligned}
&\int_{\tilde{\Omega}_1(0,t_0)} \phi^2\left(\overline{A}_{B_1(0)}(t,\nabla V)-\overline{A}_{B_1(0)}(t,\nabla v)\right)(\nabla V-\nabla v)\,dxdt\\
&\quad= -2\int_{\tilde{\Omega}_1(0,t_0)} \phi(V-v)\left(\overline{A}_{B_1(0)}(t,\nabla V)-\overline{A}_{B_1(0)}(t,\nabla v)\right)\nabla\phi dxdt\\
&\quad - \int_{\tilde{\Omega}_1(0,t_0)} \phi^2(V-v)_t(V-v)dxdt\\
&\quad - \int_{\tilde{\Omega}_1(0,t_0)} \left(\phi^2 F\frac{\partial(V-v)}{\partial x_N} + 2\phi F(V-v)\frac{\partial\phi}{\partial x_N}\right)dxdt.
\end{aligned}$$

We can rewrite $I_1 = I_2+I_3+I_4$.
We see that

$$I_1 \ge c_5\int_{\tilde{\Omega}_1(0,t_0)} \phi^2|\nabla V-\nabla v|^2dxdt$$

and using Holder's inequality

$$\begin{aligned}
|I_2| &\le c_6\int_{\tilde{\Omega}_1(0,t_0)} \phi|V-v|(|\nabla V|+|\nabla v|)|\nabla\phi|dxdt\\
&\le \varepsilon_2\int_{\tilde{\Omega}_1(0,t_0)} \phi^2(|\nabla V|^2+|\nabla v|^2)dxdt + c_7(\varepsilon_2)\int_{\tilde{\Omega}_1(0,t_0)} |V-v|^2|\nabla\phi|^2dxdt.
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
|I_4| &\le \varepsilon_2\int_{\tilde{\Omega}_1(0,t_0)} \phi^2(|\nabla V|^2+|\nabla v|^2)dxdt + c_8(\varepsilon_2)\int_{\tilde{\Omega}_1(0,t_0)} |V-v|^2|\nabla\phi|^2dxdt\\
&\quad + c_8(\varepsilon_2)\int_{\tilde{\Omega}_1(0,t_0)} |F|^2\phi^2dxdt,
\end{aligned}$$

and

$$I_3 \le \int_{\tilde{\Omega}_1(0,t_0)} \phi_t\phi(V-v)^2dxdt \le c_9\int_{\tilde{\Omega}_{1/4}(0,t_0)} |V-v|^2dxdt.$$

Hence,

$$\begin{aligned}
&\int_{\tilde{\Omega}_{1/8}(0,t_0)} |\nabla V-\nabla v|^2\\
&\le c_{10}\varepsilon_2\int_{\tilde{\Omega}_{1/4}(0,t_0)} (|\nabla V|^2+|\nabla v|^2) + c_{11}(\varepsilon_2)\int_{\tilde{\Omega}_{1/4}(0,t_0)} (|V-v|^2+|F|^2)\\
&\le c_{12}\varepsilon_2 + c_{13}(\varepsilon_2)\left(\varepsilon_1^2 + \int_{\tilde{\Omega}_{1/4}(0,t_0)\cap\{-4\delta<x_N<0\}} |\nabla V(x',0,t)|^2dxdt\right)\\
&\le\ c_{12}\varepsilon_2 + c_{14}(\varepsilon_2)\left(\varepsilon_1^2+\delta\right).
\end{aligned}$$

Finally, for any $\varepsilon > 0$ by choosing $\varepsilon_2, \varepsilon_1$ and δ appropriately we get (7.60). This completes the proof of Lemma. ■

Lemma 7.12 *For any $\varepsilon > 0$ there exists a small $\delta = \delta(N,\Lambda_1,\Lambda_2,\varepsilon) > 0$ such that if $v \in C(t_0-\rho^2,t_0;L^2(\Omega\cap B_\rho(0)))\cap L^2(t_0-\rho^2,t_0;H^1(\Omega\cap B_\rho(0)))$ is a solution of*

$$\begin{cases} v_t - div\left(\overline{A}_{B_\rho(0)}(t,\nabla v)\right) = 0 \ in\ \tilde{\Omega}_\rho(0) \\ v = 0 \quad\quad on\ \ \partial_p\tilde{\Omega}_\rho(0)\backslash(\Omega\times(-T,T)) \end{cases} \tag{7.62}$$

and

$$B_\rho^+(0) \subset \Omega \cap B_\rho(0) \subset B_\rho(0) \cap \{x_N > -4\rho\delta\}. \tag{7.63}$$

then there exists a weak solution $V \in C(t_0-\rho^2, t_0; L^2(B_\rho^+(0))) \cap L^2(t_0-\rho^2, t_0; H^1(B_\rho^+(0)))$ *of* (7.41)*, whose zero extension to* $Q_1(0,t_0)$ *satisfies*

$$||\nabla V||^2_{L^\infty(Q_{\rho/4}(0,t_0))} \le C \fint_{Q_\rho(0,t_0)} |\nabla v|^2 dxdt \quad \text{and} \tag{7.64}$$

$$\fint_{Q_{\rho/8}(0,t_0)} |\nabla v - \nabla V|^2 dxdt \le \varepsilon^2 \fint_{Q_\rho(0,t_0)} |\nabla v|^2 dxdt. \tag{7.65}$$

for some $C = C(N, \Lambda_1, \Lambda_2) > 0$.

Proof. We set

$$\mathcal{A}(x,t,\xi) = A(\rho x, t_0 + \rho^2(t-t_0), \kappa\xi)/\kappa \quad \text{and} \quad \tilde{v}(x,t) = v(\rho x, t_0 + \rho^2(t-t_0))/(\rho\kappa)$$

where $\kappa = \left(\frac{1}{|Q_\rho(0,t_0)|}\int_{Q_\rho(0,t_0)} |\nabla v|^2 dxdt\right)^{1/2}$. Then $\mathcal{A}$ satisfies conditions (1.2) and (2.27) with the same constants Λ_1 and Λ_2. We can see that $\tilde{v}$ is a solution of

$$\begin{cases} \tilde{v}_t - div\left(\overline{\mathcal{A}}_{B_1(0)}(t, \nabla\tilde{v})\right) = 0 & \text{in } \tilde{\Omega}_1^\rho(0) \\ \tilde{v} = 0 & \text{on } ((\partial\Omega^\rho \cap B_1(0)) \times (t_0-1, t_0)) \cup ((\Omega^\rho \cap B_1(0)) \times \{t = t_0 - 1\}) \end{cases} \tag{7.66}$$

where $\Omega^\rho = \{z = x/\rho : x \in \Omega\}$ and satisfies $\fint_{Q_1(0,t_0)} |\nabla\tilde{v}|^2 dxdt = 1$. We also have

$$B_1^+(0) \subset \Omega^\rho \cap B_1(0) \subset B_1(0) \cap \{x_N > -4\delta\}.$$

Therefore, applying Lemma 7.11 for any $\varepsilon > 0$, there exist a constant $\delta = \delta(N, \Lambda_1, \Lambda_2, \varepsilon) > 0$ and $\tilde{V}$ satisfies

$$||\nabla\tilde{V}||_{L^\infty(Q_{1/4}(0,t_0))} \le c_1 \quad \text{and} \fint_{Q_{1/8}(0,t_0)} |\nabla\tilde{v} - \nabla\tilde{V}|^2 dxdt \le \varepsilon^2.$$

We complete the proof by choosing $V(x,t) = k\rho\tilde{V}(x/\rho, t_0 + (t-t_0)/\rho^2)$. ■

Lemma 7.13 *Let* s_2 *be as in Lemma 7.7. For any* $\varepsilon > 0$ *there exists a small* $\delta = \delta(N, \Lambda_1, \Lambda_2, \varepsilon) > 0$ *such that the following holds. If* Ω *is a* (δ, R_0)*-Reifenberg flat domain and* $u \in C(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$ *is a solution to equation* (2.4) *with* $\mu \in L^2(\Omega \times (-T,T))$ *and* $u(-T) = 0$*, for* $x_0 \in \partial\Omega$*,* $-T < t_0 < T$ *and* $0 < R < R_0/6$ *then there is a function* $V \in L^2(t_0 - (R/9)^2, t_0; H^1(B_{R/9}(x_0))) \cap L^\infty(t_0 - (R/9)^2, t_0; W^{1,\infty}(B_{R/9}(x_0)))$ *such that*

$$||\nabla V||_{L^\infty(Q_{R/9}(x_0,t_0))} \le c \fint_{Q_{6R}(x_0,t_0)} |\nabla u| dxdt + c\frac{|\mu|(Q_{6R}(x_0,t_0))}{R^{N+1}} \tag{7.67}$$

and

$$\fint_{Q_{R/9}(x_0,t_0)} |\nabla u - \nabla V| dxdt$$
$$\le c(\varepsilon + [A]_{s_2}^{R_0}) \fint_{Q_{6R}(x_0,t_0)} |\nabla u| dxdt + c(\varepsilon + 1 + [A]_{s_2}^{R_0}) \frac{|\mu|(Q_{6R}(x_0,t_0))}{R^{N+1}}, \tag{7.68}$$

for some $c = c(N, \Lambda_1, \Lambda_2) > 0$.

Proof. Let $x_0 \in \partial\Omega$, $-T < t_0 < T$ and $\rho = R(1-\delta)$, we may assume that $0 \in \Omega$, $x_0 = (0, ..., -\delta\rho/(1-\delta))$ and

$$B_\rho^+(0) \subset \Omega \cap B_\rho(0) \subset B_\rho(0) \cap \{x_N > -4\rho\delta\}. \tag{7.69}$$

We also have

$$Q_{R/9}(x_0,t_0)\subset Q_{\rho/8}(0,t_0)\subset Q_{\rho/4}(0,t_0)\subset Q_{\rho}(0,t_0)\subset Q_{6\rho}(0,t_0)\subset Q_{6R}(x_0,t_0), \qquad (7.70)$$

provided that $0<\delta<1/625$.
Let w and v be in Theorem 7.5 and Lemma 7.7. By Lemma 7.12 for any $\varepsilon>0$ we can find a small positive $\delta=\delta(N,\alpha,\beta,\varepsilon)<1/625$ such that there is a function $V\in L^2(t_0-\rho^2,t_0;H^1(B_\rho(0)))\cap L^\infty(t_0-\rho^2,t_0;W^{1,\infty}(B_\rho(0)))$ satisfying

$$||\nabla V||^2_{L^\infty(Q_{\rho/4}(0,t_0))}\le c_1\fint_{Q_\rho(0,t_0)}|\nabla v|^2dxdt \text{ and}$$

$$\fint_{Q_{\rho/8}(0,t_0)}|\nabla v-\nabla V|^2\le \varepsilon^2\fint_{Q_\rho(0,t_0)}|\nabla v|^2dxdt.$$

Then, by (7.40) in Lemma 7.7 and (7.19) in Theorem 7.5 and (7.70) we get

$$\begin{aligned}||\nabla V||_{L^\infty(Q_{R/9}(x_0,t_0))}&\le c_2\left(\fint_{Q_\rho(0,t_0)}|\nabla w|^2dxdt\right)^{1/2}\\ &\le c_3\fint_{Q_{6R}(x_0,t_0)}|\nabla w|dxdt \qquad (7.71)\end{aligned}$$

and

$$\begin{aligned}\fint_{Q_{\rho/8}(0,t_0)}|\nabla v-\nabla V|dxdt&\le c_4\varepsilon\left(\fint_{Q_\rho(0,t_0)}|\nabla w|^2dxdt\right)^{1/2}\\ &\le c_5\varepsilon\fint_{Q_{6R}(x_0,t_0)}|\nabla w|dxdt. \qquad (7.72)\end{aligned}$$

Therefore, from (7.18) in Theorem 7.5 and (7.71) we get (7.67).
Now we prove (7.68), we have

$$\begin{aligned}\fint_{Q_{R/9}(x_0,t_0)}|\nabla u-\nabla V|dxdt&\le c_6\fint_{Q_{\rho/8}(0,t_0)}|\nabla u-\nabla V|dxdt\\ \le c_6\fint_{Q_{\rho/8}(0,t_0)}&|\nabla u-\nabla w|dxdt+c_6\fint_{Q_{\rho/8}(0,t_0)}|\nabla w-\nabla v|dxdt\\ +c_8\fint_{Q_{\rho/8}(0,t_0)}&|\nabla v-\nabla V|dxdt.\end{aligned}$$

From Lemma 7.7 and Theorem 7.5 and (7.72) it follows that

$$\fint_{Q_{\rho/8}(0,t_0)}|\nabla u-\nabla w|dxdt\le c_7\frac{|\mu|(Q_{6R}(x_0,t_0))}{R^{N+1}},$$

$$\begin{aligned}\fint_{Q_{\rho/8}(0,t_0)}|\nabla v-\nabla w|dxdt&\le c_8[A]_{s_2}^{R_0}\fint_{Q_{6\rho}(0,t_0)}|\nabla w|dxdt\\ &\le c_9[A]_{s_2}^{R_0}\fint_{Q_{6R}(x_0,t_0)}|\nabla w|dxdt\\ &\le c_{10}[A]_{s_2}^{R_0}\left(\fint_{Q_{6R}(x_0,t_0)}|\nabla u|dxdt+\frac{|\mu|(Q_{6R}(x_0,t_0))}{R^{N+1}}\right),\end{aligned}$$

and

$$\begin{aligned}\fint_{Q_{\rho/8}(0,t_0)}|\nabla v-\nabla V|dxdt&\le c_{11}\varepsilon\fint_{Q_{6R}(x_0,t_0)}|\nabla w|dxdt\\ &\le c_{12}\varepsilon\left(\fint_{Q_{6R}(x_0,t_0)}|\nabla u|dxdt+\frac{|\mu|(Q_{6R}(x_0,t_0))}{R^{N+1}}\right).\end{aligned}$$

Hence we get (7.68).

∎

8 Global Integral Gradient Bounds for Parabolic equations

8.1 Global estimates on 2-Capacity uniform thickness domains

We use the Theorem 7.1 and 7.5 to prove the following theorem.

Theorem 8.1 *Suppose that $\mathbb{R}^N\backslash\Omega$ satisfies uniformly $2-$thick with constants c_0, r_0. Let θ_1, θ_2 be in Theorem 7.1 and 7.5. Set $\theta = \min\{\theta_1, \theta_2\}$ and $T_0 = diam(\Omega) + T^{1/2}$. Let $Q = B_{diam(\Omega)}(x_0) \times (0, T)$ that contains Ω_T. Let $B_1 = \tilde{Q}_{R_1}(y_0, s_0)$, $B_2 = 4B_1 := \tilde{Q}_{4R_1}(y_0, s_0)$ for $R_1 > 0$. For $\mu \in \mathfrak{M}_b(\Omega_T)$, $\sigma \in \mathfrak{M}_b(\Omega)$, set $\omega = |\mu| + |\sigma| \otimes \delta_{\{t=0\}}$, there exist a distribution solution u of equation (2.4) with data μ, $u_0 = \sigma$ and constants $C_1 = C_1(N, \Lambda_1, \Lambda_2, c_0, T_0/r_0), c_2 > 0$, $\varepsilon_1 = \varepsilon_1(N, \Lambda_1, \Lambda_2, c_0, T_0/r_0), \varepsilon_2 = \varepsilon_1(N, \Lambda_1, \Lambda_2, c_0) > 0$ such that*

$$|\{\mathbb{M}(|\nabla u|) > \varepsilon^{-1/\theta}\lambda, \mathbb{M}_1[\omega] \le \varepsilon^{1-\frac{1}{\theta}}\lambda\} \cap Q| \le C_1\varepsilon|\{\mathbb{M}(|\nabla u|) > \lambda\} \cap Q|, \tag{8.1}$$

for all $\lambda > 0, \varepsilon \in (0, \varepsilon_1)$ and

$$|\{\mathbb{M}(\chi_{B_2}|\nabla u|) > \varepsilon^{-1/\theta}\lambda, \mathbb{M}_1[\chi_{B_2}\omega] \le \varepsilon^{1-\frac{1}{\theta}}\lambda\} \cap B_1| \le C_1\varepsilon|\{\mathbb{M}(\chi_{B_2}|\nabla u|) > \lambda\} \cap B_1|, \tag{8.2}$$

for all $\lambda > \varepsilon^{-1+\frac{1}{\theta}}||\nabla u||_{L^1(\Omega_T \cap B_2)} R_2^{-N-2}$, $\varepsilon \in (0, \varepsilon_2)$ with $R_2 = \inf\{r_0, R_1\}/16$.
Moreover, if $\sigma \in L^1(\Omega)$ then u is a renormalized solution.

Proof of Theorem 8.1. Let $\{\mu_n\} \subset C_c^\infty(\Omega_T), \{\sigma_n\} \subset C_c^\infty(\Omega)$ be as in the proof of Theorem 2.1. We have $|\mu_n| \le \varphi_n * |\mu|$ and $|\sigma_n| \le \varphi_{1,n} * |\sigma|$ for any $n \in \mathbb{N}$, $\{\varphi_n\}, \{\varphi_{1,n}\}$ are sequences of standard mollifiers in $\mathbb{R}^{N+1}, \mathbb{R}^N$, respectively.
Let u_n be solution of equation

$$\begin{cases} (u_n)_t - \operatorname{div}(A(x, t, \nabla u_n)) = \mu_n & \text{in } \Omega_T, \\ u_n = 0 & \text{on } \partial\Omega \times (0, T), \\ u_n(0) = \sigma_n & \text{in } \Omega. \end{cases} \tag{8.3}$$

By Proposition 3.5 and Theorem 3.6, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$ converging to a distribution solution u of (2.4) with data $\mu \in \mathfrak{M}_b(\Omega_T)$ and $u_0 = \sigma$ such that $u_n \to u$ in $L^s(0, T, W_0^{1,s}(\Omega))$ for any $s \in \left[1, \frac{N+2}{N+1}\right)$ and if $\sigma \in L^1(\Omega)$ then u is a renormalized solution.
By Remark 3.3 and Theorem 3.6, a sequence $\{u_{n,m}\}_m$ of solutions to equations

$$\begin{cases} (u_{n,m})_t - div(A(x, t, \nabla u_{n,m})) = \mu_{n,m} & \text{in } \Omega \times (-T, T), \\ u_{n,m} = 0 \quad \text{on } \partial\Omega \times (-T, T), \\ u_{n,m}(-T) = 0 \quad \text{on } \Omega, \end{cases}$$

converges to $\chi_{\Omega_T} u_n$ in $L^s(-T, T, W_0^{1,s}(\Omega))$ for any $s \in \left[1, \frac{N+2}{N+1}\right)$, where $\mu_{n,m} = (g_{n,m})_t + \chi_{\Omega_T}\mu_n$, $g_{n,m}(x, t) = \sigma_n(x) \int_{-T}^t \varphi_{2,m}(s)ds$ and $\{\varphi_{2,m}\}$ is a sequence of mollifiers in $\mathbb{R}$.
Set

$$E^1_{\lambda,\varepsilon} = \{\mathbb{M}(|\nabla u|) > \varepsilon^{-1/\theta}\lambda, \mathbb{M}_1[\omega] \le \varepsilon^{1-\frac{1}{\theta}}\lambda\} \cap Q, \quad F^1_\lambda = \{\mathbb{M}(|\nabla u|) > \lambda\} \cap Q,$$
$$E^2_{\lambda,\varepsilon} = \{\mathbb{M}(\chi_{B_2}|\nabla u|) > \varepsilon^{-1/\theta}\lambda, \mathbb{M}_1[\chi_{B_2}\omega] \le \varepsilon^{1-\frac{1}{\theta}}\lambda\} \cap B_1, \quad F^2_\lambda = \{\mathbb{M}(\chi_{B_2}|\nabla u|) > \lambda\} \cap B_1,$$

for $\varepsilon \in (0, 1)$ and $\lambda > 0$.
We verify that

$$|E^1_{\lambda,\varepsilon}| \le c_1\varepsilon|\tilde{Q}_{R_3}| \quad \forall\, \lambda > 0, \varepsilon \in (0, 1) \text{ and} \tag{8.4}$$

$$|E^2_{\lambda,\varepsilon}| \le c_2\varepsilon|\tilde{Q}_{R_2}| \quad \forall \quad \lambda > \varepsilon^{-1+\frac{1}{\theta}}||\nabla u||_{L^1(\Omega_T\cap A)}R_2^{-N-2}, \varepsilon\in(0,1) \tag{8.5}$$

for some $c_1 = c_1(T_0/r_0), c_2 > 0$ and $R_3 = \inf\{r_0, T_0\}/16$.
In fact, we can assume that $E^1_{\lambda,\varepsilon} \ne \emptyset$ so $(|\mu|(\Omega_T) + |\sigma|(\Omega)) \le T_0^{N+1}\varepsilon^{1-\frac{1}{\theta}}\lambda$. We have

$$|E^1_{\lambda,\varepsilon}| \le \frac{c_3}{\varepsilon^{-1/\theta}\lambda}\int_{\Omega_T}|\nabla u|dxdt.$$

By Remark 3.2, $\int_{\Omega_T}|\nabla u_n|dxdt \le c_4T_0\left(|\mu_n|(\Omega_T) + |\sigma_n|(\Omega)\right)$ for all n. Letting $n\to\infty$ we get $\int_{\Omega_T}|\nabla u|dxdt \le c_4T_0\left(|\mu|(\Omega_T) + |\sigma|(\Omega)\right)$. Thus,

$$|E^1_{\lambda,\varepsilon}| \le \frac{c_3c_4}{\varepsilon^{-1/\theta}\lambda}T_0\left(|\mu|(\Omega_T) + |\sigma|(\Omega)\right) \le \frac{c_3c_4}{\varepsilon^{-1/\theta}\lambda}T_0^{N+2}\varepsilon^{1-\frac{1}{\theta}}\lambda = c_5\varepsilon|\tilde{Q}_{R_3}|.$$

Hence, (8.4) holds with $c_1 = c_5(T_0/r_0)$.
For any $\lambda > \varepsilon^{-1+\frac{1}{\theta}}||\nabla u||_{L^1(\Omega_T\cap B_2)}R_2^{-N-2}$ we have

$$|E^2_{\lambda,\varepsilon}| \le \frac{c_3}{\varepsilon^{-1/\theta}\lambda}\int_{\Omega_T}\chi_{B_2}|\nabla u|dxdt < c_2\varepsilon|\tilde{Q}_{R_2}|.$$

Hence, (8.5) holds.
Next we verify that for all $(x,t)\in Q$ and $r\in(0,R_3]$ and $\lambda>0, \varepsilon\in(0,1)$ we have $\tilde{Q}_r(x,t)\cap Q\subset F^1_\lambda$ if $|E^1_{\lambda,\varepsilon}\cap\tilde{Q}_r(x,t)| \ge c_6\varepsilon|\tilde{Q}_r(x,t)|$ where the constant c_6 does not depend on λ and ε. Indeed, take $(x,t)\in Q$ and $0<r\le R_3$. Now assume that $\tilde{Q}_r(x,t)\cap Q\cap(F^1_\lambda)^c\ne\emptyset$ and $E^1_{\lambda,\varepsilon}\cap\tilde{Q}_r(x,t)\ne\emptyset$ i.e, there exist $(x_1,t_1),(x_2,t_2)\in\tilde{Q}_r(x,t)\cap Q$ such that $\mathbb{M}(|\nabla u|)(x_1,t_1)\le\lambda$ and $\mathbb{M}_1[\omega](x_2,t_2)\le\varepsilon^{1-\frac{1}{\theta}}\lambda$. We need to prove that

$$|E^1_{\lambda,\varepsilon}\cap\tilde{Q}_r(x,t))| < c_6\varepsilon|\tilde{Q}_r(x,t)| \tag{8.6}$$

Obviously, we have for all $(y,s)\in\tilde{Q}_r(x,t)$ there holds

$$\mathbb{M}(|\nabla u|)(y,s) \le \max\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u|\right)(y,s), 3^{N+2}\lambda\}.$$

Leads to, for all $\lambda>0$ and $\varepsilon\in(0,\varepsilon_0)$ with $\varepsilon_0\le 3^{-(N+2)\theta}$,

$$E^1_{\lambda,\varepsilon}\cap\tilde{Q}_r(x,t) = \{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u|\right) > \varepsilon^{-1/\theta}\lambda, \mathbb{M}_1[\omega]\le\varepsilon^{1-\frac{1}{\theta}}\lambda\}\cap Q\cap\tilde{Q}_r(x,t). \tag{8.7}$$

In particular, $E^1_{\lambda,\varepsilon}\cap\tilde{Q}_r(x,t)=\emptyset$ if $\overline{B}_{4r}(x)\subset\subset\mathbb{R}^N\backslash\Omega$. Thus, it is enough to consider the case $B_{4r}(x)\subset\subset\Omega$ and $B_{4r}(x)\cap\Omega\ne\emptyset$.
We consider the case $B_{4r}(x)\subset\subset\Omega$. Let $w_{n,m}$ be as in Theorem 7.1 with $Q_{2R}=Q_{4r}(x,t_0)$ and $u=u_{n,m}$ where $t_0=\min\{t+2r^2,T\}$. We have

$$\fint_{Q_{4r}(x,t_0)}|\nabla u_{n,m}-\nabla w_{n,m}|dxdt \le c_7\frac{|\mu_{n,m}|(Q_{4r}(x,t_0))}{r^{N+1}} \quad \text{and} \tag{8.8}$$

$$\fint_{Q_{2r}(x,t_0)}|\nabla w_{n,m}|^\theta dxdt \le c_8\left(\fint_{Q_{4r}(x,t_0)}|\nabla w_{n,m}|dxdt\right)^\theta. \tag{8.9}$$

From (8.7), we have

$$
\begin{aligned}
|E^1_{\lambda,\varepsilon} \cap \tilde{Q}_r(x,t)| &\le |\{\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla w_{n,m}|\right) > \varepsilon^{-1/\theta}\lambda/4\} \cap \tilde{Q}_r(x,t)\}| \\
&\quad + |\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u_{n,m} - \nabla w_{n,m}|\right) > \varepsilon^{-1/\theta}\lambda/4\} \cap \tilde{Q}_r(x,t)| \\
&\quad + |\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u_{n,m} - \nabla u_{n}|\right) > \varepsilon^{-1/\theta}\lambda/4\} \cap \tilde{Q}_r(x,t)| \\
&\quad + |\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u_{n} - \nabla u|\right) > \varepsilon^{-1/\theta}\lambda/4\} \cap \tilde{Q}_r(x,t)| \\
&\le c_9\varepsilon\lambda^{-\theta}\int_{\tilde{Q}_{2r}(x,t)} |\nabla w_{n,m}|^\theta dxdt + c_9\varepsilon^{1/\theta}\lambda^{-1}\int_{\tilde{Q}_{2r}(x,t)} |\nabla u_{n,m} - \nabla w_{n,m}| dxdt \\
&\quad + c_9\varepsilon^{1/\theta}\lambda^{-1}\int_{\tilde{Q}_{2r}(x,t)} |\nabla u_{n,m} - \nabla u_n| dxdt + c_9\varepsilon^{1/\theta}\lambda^{-1}\int_{\tilde{Q}_{2r}(x,t)} |\nabla u_n - \nabla u| dxdt.
\end{aligned}
$$

Thanks to (8.8) and (8.9) we can continue

$$
\begin{aligned}
&|E^1_{\lambda,\varepsilon} \cap \tilde{Q}_r(x,t)| \le c_{10}\varepsilon\lambda^{-\theta}|\tilde{Q}_r(x,t)|\left(\fint_{Q_{4r}(x,t_0)} |\nabla u_{n,m}| dxdt\right)^\theta \\
&+ c_{10}\varepsilon\lambda^{-\theta}|\tilde{Q}_r(x,t)|\left(\frac{|\mu_{n,m}|(Q_{4r}(x,t_0))}{r^{N+1}}\right)^\theta + c_{10}\varepsilon^{1/\theta}\lambda^{-1}|\tilde{Q}_r(x,t)|\frac{|\mu_{n,m}|(Q_{4r}(x,t_0))}{r^{N+1}} \\
&+ c_{10}\varepsilon^{1/\theta}\lambda^{-1}\int_{Q_{2r}(x,t_0)} |\nabla u_{n,m} - \nabla u_n| dxdt + c_{10}\varepsilon^{1/\theta}\lambda^{-1}\int_{Q_{2r}(x,t_0)} |\nabla u_n - \nabla u| dxdt.
\end{aligned}
$$

Letting $m \to \infty$ and $n \to \infty$, we get

$$
\begin{aligned}
&|E_{\lambda,\varepsilon} \cap \tilde{Q}_r(x,t)| \le c_{10}\varepsilon\lambda^{-\theta}|\tilde{Q}_r(x,t)|\left(\fint_{Q_{4r}(x,t_0)} |\nabla u| dxdt\right)^\theta \\
&\quad + c_{10}\varepsilon\lambda^{-\theta}|\tilde{Q}_r(x,t)|\left(\frac{\omega(\overline{Q_{4r}(x,t_0)})}{r^{N+1}}\right)^\theta + c_{10}\varepsilon^{1/\theta}\lambda^{-1}|\tilde{Q}_r(x,t)|\frac{\omega(\overline{Q_{4r}(x,t_0)})}{r^{N+1}}.
\end{aligned}
$$

Since, $\mathbb{M}(|\nabla u|)(x_1,t_1) \le \lambda$ and $\mathbb{M}_1[\omega](x_2,t_2) \le \varepsilon^{1-\frac{1}{\theta}}\lambda$ we have

$$
\int_{Q_{4r}(x,t_0)} |\nabla u| dxdt \le \int_{\tilde{Q}_{8r}(x,t)} |\nabla u| dxdt \le \int_{\tilde{Q}_{9r}(x_1,t_1)} |\nabla u| dxdt \le |\tilde{Q}_{9r}(x_1,t_1)|\lambda,
$$

and

$$
\omega(\overline{Q_{4r}(x,t_0)}) \le \omega(\tilde{Q}_{8r}(x,t)) \le \omega(\tilde{Q}_{9r}(x_2,t_2) \le \varepsilon^{1-\frac{1}{\theta}}\lambda(9r)^{N+1}.
$$

Thus

$$
|E_{\lambda,\varepsilon} \cap \tilde{Q}_r(x,t)| \le c_{11}\varepsilon|\tilde{Q}_r(x,t)|.
$$

Next, we consider the case $B_{4r}(x) \cap \Omega \ne \emptyset$. Let $x_3 \in \partial\Omega$ such that $|x_3 - x| = \text{dist}(x, \partial\Omega)$. Let w_n be as in Theorem 7.5 with $\tilde{\Omega}_{6R} = \tilde{\Omega}_{16r}(x_3,t_0)$ and $u = u_{n,m}$ where $t_0 = \min\{t + 2r^2, T\}$. We have $Q_{12r}(x,t_0) \subset Q_{16r}(x_3,t_0)$,

$$
\fint_{Q_{12r}(x,t_0)} |\nabla u_{n,m} - \nabla w_{n,m}| dxdt \le c_{12}\frac{|\mu_{n,m}|(\tilde{\Omega}_{16r}(x_3,t_0))}{r^{N+1}} \quad \text{and}
$$

$$
\left(\fint_{Q_{2r}(x,t_0)} |\nabla w_{n,m}|^\theta dxdt\right)^{\frac{1}{\theta}} \le c_{13}\fint_{Q_{12r}(x,t_0)} |\nabla w_{n,m}| dxdt.
$$

As above we also obtain

$$|E^1_{\lambda,\varepsilon}\cap \tilde{Q}_r(x,t)| \le c_{14}\varepsilon\lambda^{-\theta}|\tilde{Q}_r(x,t)|\left(\fint_{Q_{12r}(x,t_0)}|\nabla u|dxdt\right)^{\theta}$$
$$+c_{14}\varepsilon\lambda^{-\theta}|\tilde{Q}_r(x,t)|\left(\frac{\omega(\overline{Q_{16r}(x_3,t_0)})}{r^{N+1}}\right)^{\theta}+c_{14}\varepsilon^{1/\theta}\lambda^{-1}|\tilde{Q}_r(x,t)|\frac{\omega(\overline{Q_{16r}(x_3,t_0)})}{r^{N+1}}.$$

Since, $\mathbb{M}(|\nabla u|)(x_1,t_1)\le\lambda$ and $\mathbb{M}_1[\omega](x_2,t_2)\le\varepsilon^{1-\frac{1}{\theta}}\lambda$ we have

$$\int_{Q_{12r}(x,t_0)}|\nabla u|dxdt \le \int_{\tilde{Q}_{24r}(x,t)}|\nabla u|dxdt \le \int_{\tilde{Q}_{25r}(x_1,t_1)}|\nabla u|dxdt \le |\tilde{Q}_{25r}(x_1,t_1)|\lambda$$

and

$$\omega(\overline{Q_{16r}(x_3,t_0)})\le\omega(\tilde{Q}_{32r}(x_3,t))\le\omega(\tilde{Q}_{36r}(x,t))\le\omega(\tilde{Q}_{37r}(x_2,t_2))\le\varepsilon^{1-\frac{1}{\theta}}\lambda(37r)^{N+1}.$$

Thus

$$|E^1_{\lambda,\varepsilon}\cap\tilde{Q}_r(x,t)|\le c_{15}\varepsilon|\tilde{Q}_r(x,t)|.$$

Hence, (8.6) holds with $c_6=2\max\{c_{11},c_{15}\}$.
Similarly, we also prove that for all $(x,t)\in B_1$ and $r\in(0,R_2]$ and $\lambda>0,\varepsilon\in(0,1)$ we have $\tilde{Q}_r(x,t)\cap B_1\subset F^2_\lambda$ if $|E^2_{\lambda,\varepsilon}\cap\tilde{Q}_r(x,t)|\ge c_{16}\varepsilon|\tilde{Q}_r(x,t)|$ where a constant c_{26} does not depend on λ and ε. Now, choose $\varepsilon_1=(2\max\{1,c_1,c_6\})^{-1}$ and $\varepsilon_2=(2\max\{1,c_2,c_{16}\}^{-1}$. We apply Lemma 3.21 with $E=E^1_{\lambda,\varepsilon},F=F^1_\lambda$ and ε is replaced by $\max\{c_1,c_6\}\varepsilon$ for any $0<\varepsilon<\varepsilon_1$ and $\lambda>0$ we get (8.1), for $E=E^2_{\lambda,\varepsilon},F=F^2_\lambda$ and ε is replaced by $\max\{c_1,c_{17}\}\varepsilon$ for any $0<\varepsilon<\varepsilon_2$ and $\lambda>\varepsilon^{-1+\frac{1}{\theta}}||\nabla u||_{L^1(\Omega_T\cap B_2)}R_2^{-N-2}$ we get (8.2).
This completes the proof of the Theorem. ■

Proof of Theorem 2.17. By theorem 8.1, there exist constants $c_1>0$, $0<\varepsilon_0<1$ and a renormalized solution u of equation (2.4) with data μ, $u_0=\sigma$ such that for any $\varepsilon\in(0,1)$, $\lambda>0$

$$|\{\mathbb{M}(|\nabla u|)>\varepsilon^{-1/\theta}\lambda,\mathbb{M}_1[\omega]\le\varepsilon^{1-\frac{1}{\theta}}\lambda\}\cap Q|\le c_1\varepsilon|\{\mathbb{M}(|\nabla u|)>\lambda\}\cap Q|.$$

Therefore, if $0<s<\infty$

$$\begin{aligned}||\mathbb{M}(|\nabla u|)||^s_{L^{p,s}(Q)}&=\varepsilon^{-s/\theta}p\int_0^\infty\lambda^s|\{(x,t)\in Q:\mathbb{M}(|\nabla u|)>\varepsilon^{-1/\theta}\lambda\}|^{\frac{s}{p}}\frac{d\lambda}{\lambda}\\&\le c_1^{s/p}\varepsilon^{\frac{s(\theta-p)}{\theta p}}p\int_0^\infty\lambda^s|\{(x,t)\in Q:\mathbb{M}(|\nabla u|)>\lambda\}|^{\frac{s}{p}}\frac{d\lambda}{\lambda}\\&+\varepsilon^{-s/\theta}p\int_0^\infty\lambda^s|\{(x,t)\in Q:\mathbb{M}_1[\omega]>\varepsilon^{1-\frac{1}{\theta}}\lambda\}|^{\frac{s}{p}}\frac{d\lambda}{\lambda}\\&=c_1^{s/p}\varepsilon^{\frac{s(\theta-p)}{\theta p}}||\mathbb{M}(|\nabla u|)||^s_{L^{p,s}(Q)}+\varepsilon^{-s}||\mathbb{M}_1[\omega]||^s_{L^{p,s}(Q)}.\end{aligned}$$

Since $p<\theta$, we can choose $0<\varepsilon<\varepsilon_0$ such that $c_1^{s/p}\varepsilon^{\frac{s(\theta-p)}{\theta p}}\le 1/2$ we get the result for case $0<s<\infty$. Similarly, we also get the result for case $s=\infty$.
Also, we get (2.29) by using (4.16) in Proposition 4.8, (4.28) in Proposition 4.19. This completes the proof. ■

Remark 8.2 *Thanks to Proposition 4.4 we have for any* $s\in\left(\frac{N+2}{N+1},\frac{N+2+\theta}{N+2}\right)$ *if* $\mu\in L^{\frac{(s-1)(N+2)}{s},\infty}(\Omega_T)$ *and* $\sigma\equiv 0$ *then*

$$|||\nabla u|^s||_{L^{\frac{(s-1)(N+2)}{s},\infty}(\Omega_T)}\le c_2||\mu||^s_{L^{\frac{(s-1)(N+2)}{s},\infty}(\Omega_T)},$$

where constant c_2 *depends on* $N,\Lambda_1,\Lambda_2,s,c_0,T_0/r_0$.

As the proof of Theorem 8.1, we also get

Theorem 8.3 *Suppose that $\mathbb{R}^N\backslash\Omega$ satisfies uniformly 2−thick with constants c_0, r_0. Let θ be as in Theorem 8.1. Let $1 \leq p < \theta$, $0 < s \leq \infty$ and $\mu \in \mathfrak{M}_b(\Omega_T)$, $\sigma \in \mathfrak{M}_b(\Omega)$, set $\omega = |\mu| + |\sigma| \otimes \delta_{\{t=0\}}$. There exist $C_1 = C_1(N, \Lambda_1, \Lambda_2, p, s, c_0) > 0$ and a distribution solution u of equation (2.4) with data μ and $u_0 = \sigma$ such that*

$$\begin{aligned}||\mathbb{M}(\chi_{\tilde{Q}_{4R}(y_0,s_0)}|\nabla u|)||_{L^{p,s}(\tilde{Q}_R(y_0,s_0))} \leq C_1 R^{\frac{N+2}{p}} \inf\{r_0, R\}^{-N-2} ||\nabla u||_{L^1(\tilde{Q}_{4R}(y_0,s_0))} \\ + C_1 ||\mathbb{M}_1[\chi_{\tilde{Q}_{4R}(y_0,s_0)}\omega]||_{L^{p,s}(\tilde{Q}_R(y_0,s_0))}, \qquad (8.10)\end{aligned}$$

for any $\tilde{Q}_R(y_0, s_0) \subset \mathbb{R}^{N+1}$ and if $\sigma \in L^1(\Omega)$ then u is a renormalized solution.

Proof of Theorem 2.19. Let $\{u_{n,m}\}$ and $\mu_{n,m}$ be in the proof of Theorem 8.1. From Corollary 7.2 and 7.6 we assert: for $2 - \inf\{\beta_1, \beta_2\} < \gamma < N + 2$, there exists a constant $C = C(N, \Lambda_1, \Lambda_2, c_0, \gamma) > 0$ such that for any $0 < \rho \leq T_0$

$$\int_{Q_\rho(y,s)} |\nabla u_{n,m}| dxdt \leq C(N, \Lambda_1, \Lambda_2, \gamma, c_0, T_0/r_0) \rho^{N+3-\gamma} ||\mathbb{M}_\gamma[|\mu_{n,m}|]||_{L^\infty(\Omega\times(-T,T))},$$

where β_1, β_2 are constants in Theorem 7.1 and Theorem 7.5. It is easy to see that

$$||\mathbb{M}_\gamma[|\mu_{n,m}|]||_{L^\infty(\Omega\times(-T,T))} \leq ||\mathbb{M}_\gamma[\omega]||_{L^\infty(\Omega\times(-T,T))} = ||\mathbb{M}_\gamma[\omega]||_{L^\infty(\Omega_T)},$$

for any n, m large enough.
Letting $m \to \infty, n \to \infty$, yield

$$\int_{Q_\rho(y,s)} |\nabla u| dxdt \leq C(N, \Lambda_1, \Lambda_2, \gamma, c_0, T_0/r_0) \rho^{N+3-\gamma} ||\mathbb{M}_\gamma[\omega]||_{L^\infty(\Omega_T)}$$

By Theorem 8.3 we get

$$\begin{aligned}|||\nabla u|||_{L^{p,s}(\tilde{Q}_R(y_0,s_0)\cap\Omega_T)} \leq c_1(T_0/r_0) R^{\frac{N+2}{p}+1-\gamma} ||\mathbb{M}_\gamma[\omega]||_{L^\infty(\Omega_T)} \\ + c_2 ||\mathbb{M}_1[\chi_{\tilde{Q}_R(y_0,s_0)}\omega]||_{L^{p,s}(\tilde{Q}_R(y_0,s_0))}\end{aligned}$$

for any $\tilde{Q}_R(y_0, s_0) \subset \mathbb{R}^{N+1}$ and $0 < R \leq T_0$. It follows (2.30).

Finally, if $\mu \in L_*^{\frac{(\gamma-1)p}{\gamma}, \frac{(\gamma-1)s}{\gamma}; (\gamma-1)p}(\Omega_T)$ and $\sigma \equiv 0$, then clearly u is a unique renormalized solution. It suffices to show that

$$||\mathbb{M}_\gamma[|\mu|]||_{L^\infty(\Omega_T)} \leq c_3 ||\mu||_{L_*^{\frac{(\gamma-1)p}{\gamma}, \frac{(\gamma-1)s}{\gamma}; (\gamma-1)p}(\Omega_T)} \quad \text{and} \qquad (8.11)$$

$$R^{\frac{p(\gamma-1)-N-2}{p}} ||\mathbb{M}_1[\chi_{\tilde{Q}_R(y,s_0)}|\mu|]||_{L^{p,s}(\tilde{Q}_R(y_0,s_0))} \leq c_3 ||\mu||_{L_*^{\frac{(\gamma-1)p}{\gamma}, \frac{(\gamma-1)s}{\gamma}; (\gamma-1)p}(\Omega_T)} \qquad (8.12)$$

for any $\tilde{Q}_R(y_0, s_0) \subset \mathbb{R}^{N+1}$ and $0 < R \leq T_0$, where $c_3 = c_3(N, \Lambda_1, \Lambda_2, p, s, \gamma, c_0, T_0/r_0)$.
In fact, for $0 < \rho < T_0$ and $(x, t) \in \Omega_T$ we have

$$\begin{aligned}||\mu||_{L_*^{\frac{(\gamma-1)p}{\gamma}, \frac{(\gamma-1)s}{\gamma}; (\gamma-1)p}(\Omega_T)} &\geq ||\mu||_{L_*^{\frac{(\gamma-1)p}{\gamma}, \infty; (\gamma-1)p}(\Omega_T)} \\ &\geq \rho^{\frac{(\gamma-1)p-N-2}{\frac{(\gamma-1)p}{\gamma}}} ||\mu||_{L^{\frac{(\gamma-1)p}{\gamma}, \infty}(\tilde{Q}_\rho(x,t)\cap\Omega_T)} \\ &\geq c_4 \rho^{\frac{(\gamma-1)p-N-2}{\frac{(\gamma-1)p}{\gamma}}} |\tilde{Q}_\rho(x,t)|^{-1+\frac{\gamma}{(\gamma-1)p}} |\mu|(\tilde{Q}_\rho(x,t) \cap \Omega_T) \\ &= c_5 \frac{|\mu|(\tilde{Q}_\rho(x,t) \cap \Omega_T)}{\rho^{N+2-\gamma}},\end{aligned}$$

which obviously implies (8.11).
Next, we note that

$$\mathbb{M}_1[\chi_{\tilde{Q}_R(y_0,s_0)}|\mu|](x,t) \le c_6 \left(\mathbb{M}\left(\chi_{\tilde{Q}_R(y_0,s_0)}|\mu|\right)(x,t)\right)^{1-\frac{1}{\gamma}} ||\mu||^{\frac{1}{\gamma}}_{L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\Omega_T)}.$$

We derive

$$\begin{aligned} &R^{\frac{p(\gamma-1)-N-2}{p}}||\mathbb{M}_1[\chi_{\tilde{Q}_R(y,s_0)}|\mu|]||_{L^{p,s}(\tilde{Q}_R(y_0,s_0))} \\ &\le c_6 R^{\frac{p(\gamma-1)-N-2}{p}}||\mathbb{M}\left(\chi_{\tilde{Q}_R(y_0,s_0)}|\mu|\right)||^{1-\frac{1}{\gamma}}_{L^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma}}(\tilde{Q}_R(y_0,s_0))}||\mu||^{\frac{1}{\gamma}}_{L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\Omega_T)} \\ &\le c_7 R^{\frac{p(\gamma-1)-N-2}{p}}|||\mu|||^{1-\frac{1}{\gamma}}_{L^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma}}(\tilde{Q}_R(y_0,s_0))}||\mu||^{\frac{1}{\gamma}}_{L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\Omega_T)}. \end{aligned}$$

Here we used the boundedness property of $\mathbb{M}$ in $L^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma}}(\mathbb{R}^{N+1})$ for $\frac{(\gamma-1)p}{\gamma} > 1$. Therefore, immediately we get (8.12). This completes the proof of theorem.

■

8.2 Global estimates on Reifenberg flat domains

Now we prove results for Reifenberg flat domain. First, we will use Lemma 7.4, 7.13 and Lemma 3.19 to get the following result.

Theorem 8.4 *Suppose that A satisfies (2.27). Let s_1, s_2 be in Lemma 7.3 and 7.7, set $s_0 = \max\{s_1, s_2\}$. Let $w \in A_\infty$, $\mu \in \mathfrak{M}_b(\Omega_T)$, $\sigma \in \mathfrak{M}_b(\Omega)$, set $\omega = |\mu| + |\sigma| \otimes \delta_{\{t=0\}}$. There exists a distribution solution of (2.4) with data μ and $u_0 = \sigma$ such that following holds. For any $\varepsilon > 0, R_0 > 0$ one finds $\delta_1 = \delta_1(N, \Lambda_1, \Lambda_2, \varepsilon, [w]_{A_\infty}) \in (0,1)$ and $\delta_2 = \delta_2(N, \Lambda_1, \Lambda_2, \varepsilon, [w]_{A_\infty}, T_0/R_0) \in (0,1)$ and $\Lambda = \Lambda(N, \Lambda_1, \Lambda_2) > 0$ such that if Ω is (δ_1, R_0)-Reifenberg flat domain and $[\mathcal{A}]^{R_0}_{s_0} \le \delta_1$ then*

$$w(\{\mathbb{M}(|\nabla u|) > \Lambda\lambda, \mathbb{M}_1[\omega] \le \delta_2\lambda\} \cap \Omega_T) \le B\varepsilon w(\{\mathbb{M}(|\nabla u|) > \lambda\} \cap \Omega_T) \tag{8.13}$$

for all $\lambda > 0$, where the constant B depends only on $N, \Lambda_1, \Lambda_2, T_0/R_0, [w]_{A_\infty}$.
Furthermore, if $\sigma \in L^1(\Omega)$ then u is a renormalized solution.

Proof. Let $\{\mu_n\}, \{\sigma_n\}, \{\mu_{n,m}\}, \{u_n\}, \{u_{n,m}\}, u$ be as in the proof of Theorem 8.1. Let ε be in $(0,1)$. Set $E_{\lambda,\delta_2} = \{\mathbb{M}(|\nabla u|) > \Lambda\lambda, \mathbb{M}_1[\omega] \le \delta_2\lambda\} \cap \Omega_T$ and $F_\lambda = \{\mathbb{M}(|\nabla u|) > \lambda\} \cap \Omega_T$ for $\varepsilon \in (0,1)$ and $\lambda > 0$. Let $\{y_i\}_{i=1}^L \subset \Omega$ and a ball B_0 with radius $2T_0$ such that

$$\Omega \subset \bigcup_{i=1}^L B_{r_0}(y_i) \subset B_0$$

where $r_0 = \min\{R_0/1080, T_0\}$. Let $s_j = T - jr_0^2/2$ for all $j = 0, 1, ..., [\frac{2T}{r_0^2}]$ and $Q_{2T_0} = B_0 \times (T - 4T_0^2, T)$. So,

$$\Omega_T \subset \bigcup_{i,j} Q_{r_0}(y_i, s_j) \subset Q_{2T_0}.$$

We verify that

$$w(E_{\lambda,\delta_2}) \le \varepsilon w(\tilde{Q}_{r_0}(y_i, s_j)) \ \ \forall \ \lambda > 0 \tag{8.14}$$

for some δ_2 small enough, depended on $n, p, \alpha, \beta, \epsilon, [w]_{A_\infty}, T_0/R_0$.
In fact, we can assume that $E_{\lambda,\delta_2} \ne \emptyset$ so $|\mu|(\Omega_T) + |\sigma|(\Omega) \le T_0^{N+1}\delta_2\lambda$. We have

$$|E_{\lambda,\delta_2}| \le \frac{c_1}{\Lambda\lambda}\int_{\Omega_T} |\nabla u| dxdt.$$

We also have
$$\int_{\Omega_T} |\nabla u| dxdt \le c_2 T_0 (|\mu|(\Omega_T) + |\sigma|(\Omega)).$$
Thus,
$$|E_{\lambda,\varepsilon}| \le \frac{c_3}{\Lambda\lambda} T_0(|\mu|(\Omega_T) + |\sigma|(\Omega)) \le \frac{c_3}{\Lambda\lambda} T_0^{N+2}\delta_2\lambda = c_4\delta_2|Q_{2T_0}|.$$
which implies
$$w(E_{\lambda,\delta_2}) \le A\left(\frac{|E_{\lambda,\delta_2}|}{|Q_{2T_0}|}\right)^\nu w(Q_{2T_0}) \le A\,(c_4\delta_2)^\nu\, w(Q_{2T_0})$$
where (A,ν) is a pair of A_∞ constants of w. It is known that (see, e.g [33]) there exist $A_1 = A_1(N,A,\nu)$ and $\nu_1 = \nu_1(N,A,\nu)$ such that
$$\frac{w(\tilde{Q}_{2T_0})}{w(\tilde{Q}_{r_0}(y_i,s_j))} \le A_1\left(\frac{|\tilde{Q}_{2T_0}|}{|\tilde{Q}_{r_0}(y_i,s_j)|}\right)^{\nu_1} \quad \forall i,j.$$
So,
$$w(E_{\lambda,\delta_2}) \le A\,(c_4\delta_2)^\nu A_1\left(\frac{|\tilde{Q}_{T_0}|}{|\tilde{Q}_{r_0}(y_i,s_j)|}\right)^{\nu_1} w(\tilde{Q}_{r_0}(y_i,s_j)) < \varepsilon w(\tilde{Q}_{r_0}(y_i,s_j)) \;\; \forall\; i,j$$
where $\delta_2 \le \left(\frac{\varepsilon}{2c_5(T_0 r_0^{-1})^{(N+2)\nu_1}}\right)^{1/\nu}$. It follows (8.14).
Next we verify that for all $(x,t)\in\Omega_T$ and $r\in(0,2r_0]$ and $\lambda>0$ we have $\tilde{Q}_r(x,t)\cap\Omega_T\subset F_\lambda$ if $w(E_{\lambda,\delta_2}\cap\tilde{Q}_r(x,t)) \ge \varepsilon w(Q_r(x,t))$ for some $\delta_2 \le \left(\frac{\varepsilon}{2c_5(T_0 r_0^{-1})^{(N+2)\nu_1}}\right)^{1/\nu}$.
Indeed, take $(x,t)\in\Omega_T$ and $0<r\le 2r_0$. Now assume that $\tilde{Q}_r(x,t)\cap\Omega_T\cap F_\lambda^c \ne \emptyset$ and $E_{\lambda,\delta_2}\cap \tilde{Q}_r(x,t)\ne\emptyset$ i.e, there exist $(x_1,t_1),(x_2,t_2)\in\tilde{Q}_r(x,t)\cap\Omega_T$ such that $\mathbb{M}(|\nabla u|)(x_1,t_1)\le\lambda$ and $\mathbb{M}_1[\omega](x_2,t_2)\le\delta_2\lambda$. We need to prove that
$$w(E_{\lambda,\delta_2}\cap\tilde{Q}_r(x,t))) < \varepsilon w(\tilde{Q}_r(x,t)). \tag{8.15}$$
Clearly,
$$\mathbb{M}(|\nabla u|)(y,s) \le \max\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u|\right)(y,s), 3^{N+2}\lambda\} \;\; \forall (y,s)\in\tilde{Q}_r(x,t).$$
Therefore, for all $\lambda>0$ and $\Lambda\ge 3^{N+2}$,
$$E_{\lambda,\delta_2}\cap\tilde{Q}_r(x,t) = \{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u|\right) > \Lambda\lambda, \mathbb{M}_1[\omega]\le\delta_2\lambda\}\cap\Omega_T\cap\tilde{Q}_r(x,t). \tag{8.16}$$
In particular, $E_{\lambda,\delta_2}\cap\tilde{Q}_r(x,t)=\emptyset$ if $\overline{B}_{8r}(x)\subset\subset\mathbb{R}^N\backslash\Omega$. Thus, it is enough to consider the case $B_{8r}(x)\subset\subset\Omega$ and $B_{8r}(x)\cap\Omega\ne\emptyset$.
We consider the case $B_{8r}(x)\subset\subset\Omega$. Let $v_{n,m}$ be as in Lemma 7.4 with $Q_{2R}=Q_{8r}(x,t_0)$ and $u=u_{n,m}$ where $t_0=\min\{t+2r^2,T\}$. We have
$$||\nabla v_{n,m}||_{L^\infty(Q_{2r}(x,t_0))} \le c_6 \fint_{Q_{8r}(x,t_0)} |\nabla u_{n,m}| dxdt + c_6\frac{|\mu_{n,m}|(Q_{8r}(x,t_0))}{r^{N+1}}, \tag{8.17}$$
and
$$\fint_{Q_{4r}(x,t_0)} |\nabla u_{n,m}-\nabla v_{n,m}| dxdt \le c_8\frac{|\mu_{n,m}|(Q_{8r}(x,t_0))}{r^{N+1}} + c_8[A]_{s_0}^{R_0}\left(\fint_{Q_{8r}(x,t_0)}|\nabla u_{n,m}|dxdt + \frac{|\mu_{n,m}|(Q_{8r}(x,t_0))}{r^{N+1}}\right).$$

Thanks to $\mathbb{M}(|\nabla u|)(x_1,t_1)\le\lambda$ and $\mathbb{M}_1[\omega](x_2,t_2)\le\delta_2\lambda$ with $(x_1,t_1),(x_2,t_2)\in Q_r(x,t)$, we get

$$\begin{aligned}\limsup_{n\to\infty}\limsup_{m\to\infty}||\nabla v_{n,m}||_{L^\infty(Q_{2r}(x,t))}&\le c_9\fint_{\tilde{Q}_{17r}(x_1,t_1)}|\nabla u|dxdt+c_9\frac{\omega(\overline{\tilde{Q}_{17r}(x_2,t_2)})}{r^{N+1}}\\&\le c_9\lambda+c_9\delta_2\lambda\\&\le c_{10}\lambda,\end{aligned}$$

and

$$\begin{aligned}&\limsup_{n\to\infty}\limsup_{m\to\infty}\fint_{Q_{4r}(x,t_0)}|\nabla u_n-\nabla v_n|dxdt\\&\quad\le c_{11}\frac{\omega(\overline{\tilde{Q}_{17r}(x_2,t_2)})}{r^{N+1}}+c_{11}[A]_{s_0}^{R_0}\left(\fint_{\tilde{Q}_{17r}(x_1,t_1)}|\nabla u|dxdt+\frac{\omega(\overline{\tilde{Q}_{17r}(x_2,t_2)})}{r^{N+1}}\right)\\&\quad\le c_{11}\delta_2\lambda+c_{11}[A]_{s_0}^{R_0}\left(\lambda+\delta_2\lambda\right)\\&\quad\le c_{11}\left(\delta_2+\delta_1(1+\delta_2)\right)\lambda.\end{aligned}$$

Here we used $[A]_{s_0}^{R_0}\le\delta_1$ in the last inequality.
So, we can find n_0 large enough and a sequence $\{k_n\}$ such that

$$||\nabla v_{n,m}||_{L^\infty(\tilde{Q}_{2r}(x,t))}=||\nabla v_{n,m}||_{L^\infty(Q_{2r}(x,t_0))}\le 2c_{10}\lambda\ \text{ and}\tag{8.18}$$

$$\fint_{Q_{4r}(x,t_0)}|\nabla u_{n,m}-\nabla v_{n,m}|dxdt\le 2c_{11}\left(\delta_2+\delta_1(1+\delta_2)\right)\lambda,\tag{8.19}$$

for all $n\ge n_0$ and $m\ge k_n$.
In view of (8.18) we see that for $\Lambda\ge\max\{3^{N+2},8c_{10}\}$ and $n\ge n_0$, $m\ge k_n$,

$$|\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla v_{n,m}|\right)>\Lambda\lambda/4\}\cap\tilde{Q}_r(x,t)|=0.$$

Leads to

$$\begin{aligned}|E_{\lambda,\delta_2}\cap\tilde{Q}_r(x,t)|&\le|\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u_{n,m}-\nabla v_{n,m}|\right)>\Lambda\lambda/4\}\cap\tilde{Q}_r(x,t)|\\&\quad+|\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u_n-\nabla u_{n,m}|\right)>\Lambda\lambda/4\}\cap\tilde{Q}_r(x,t)|\\&\quad+|\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u-\nabla u_n|\right)>\Lambda\lambda/4\}\cap\tilde{Q}_r(x,t)|.\end{aligned}$$

Therefore, by (8.19) and $\tilde{Q}_{2r}(x,t)\subset Q_{4r}(x,t_0)$ we obtain for any $n\ge n_0$ and $m\ge k_n$

$$\begin{aligned}|E_{\lambda,\delta_2}\cap\tilde{Q}_r(x,t)|&\le\frac{c_{12}}{\lambda}\int_{\tilde{Q}_{2r}(x,t)}|\nabla u_{n,m}-\nabla v_{n,m}|dxdt\\&\quad+\frac{c_{12}}{\lambda}\int_{\tilde{Q}_{2r}(x,t)}|\nabla u_n-\nabla u_{n,m}|dxdt+\frac{c_{12}}{\lambda}\int_{\tilde{Q}_{2r}(x,t)}|\nabla u-\nabla u_n|dxdt\\&\le c_{13}\left(\delta_2+\delta_1(1+\delta_2)\right)|Q_r(x,t)|\\&\quad+\frac{c_{12}}{\lambda}\int_{\tilde{Q}_{2r}(x,t)}|\nabla u_n-\nabla u_{n,m}|dxdt+\frac{c_{12}}{\lambda}\int_{\tilde{Q}_{2r}(x,t)}|\nabla u-\nabla u_n|dxdt.\end{aligned}$$

Letting $m\to\infty$ and $n\to\infty$ we get

$$|E_{\lambda,\delta_2}\cap\tilde{Q}_r(x,t)|\le c_{13}\left(\delta_2+\delta_1(1+\delta_2)\right)|\tilde{Q}_r(x,t)|.$$

Thus,

$$\begin{aligned} w(E_{\lambda,\delta_2}\cap\tilde{Q}_r(x,t)) &\le C\left(\frac{|E_{\lambda,\delta_2}\cap\tilde{Q}_r(x,t)|}{|\tilde{Q}_r(x,t)|}\right)^{\nu} w(\tilde{Q}_r(x,t))\\ &\le C\left(c_{13}\left(\delta_2+\delta_1(1+\delta_2)\right)\right)^{\nu} w(\tilde{Q}_r(x,t))\\ &< \varepsilon w(\tilde{Q}_r(x,t)). \end{aligned}$$

where δ_2,δ_1 are appropriately chosen, (C,ν) is a pair of A_∞ constants of w.
Next we consider the case $B_{8r}(x)\cap\Omega\neq\emptyset$. Let $x_3\in\partial\Omega$ such that $|x_3-x|=\text{dist}(x,\partial\Omega)$. Set $t_0=\min\{t+2r^2,T\}$. We have

$$Q_{2r}(x,t_0)\subset Q_{10r}(x_3,t_0)\subset Q_{540r}(x_3,t_0)\subset\tilde{Q}_{1080r}(x_3,t)\subset\tilde{Q}_{1088r}(x,t)\subset\tilde{Q}_{1089r}(x_1,t_1) \tag{8.20}$$

and

$$Q_{540r}(x_3,t_0)\subset\tilde{Q}_{1080r}(x_3,t)\subset\tilde{Q}_{1088r}(x,t)\subset\tilde{Q}_{1089r}(x_2,t_2) \tag{8.21}$$

Let $V_{n,m}$ be as in Lemma 7.13 with $Q_{6R}=Q_{540r}(x_3,t_0)$, $u=u_{n,m}$ and $\varepsilon=\delta_3\in(0,1)$. We have

$$||\nabla V_{n,m}||_{L^\infty(Q_{10r}(x_3,t_0))}\le c_{14}\fint_{Q_{540r}(x_3,t_0)}|\nabla u_{n,m}|dxdt+c_{14}\frac{|\mu_{n,m}|(Q_{540r}(x_3,t_0))}{R^{N+1}}$$

and

$$\begin{aligned} &\fint_{Q_{10r}(x_3,t_0)}|\nabla u_{n,m}-\nabla V_{n,m}|dxdt\\ &\quad\le c_{15}(\delta_3+[A]_{s_0}^{R_0})\fint_{Q_{540r}(x_3,t_0)}|\nabla u_{n,m}|dxdt+c_{15}(\delta_3+1+[A]_{s_0}^{R_0})\frac{|\mu_{n,m}|(Q_{540r}(x_3,t_0))}{R^{N+1}}. \end{aligned}$$

Since $\mathbb{M}(|\nabla u|)(x_1,t_1)\le\lambda$, $\mathbb{M}_1[\omega](x_2,t_2)\le\delta_2\lambda$ and (8.20), (8.21) we get

$$\begin{aligned} \limsup_{n\to\infty}\limsup_{m\to\infty}||\nabla V_{n,m}||_{L^\infty(Q_{2r}(x,t_0))} &\le \limsup_{n\to\infty}\limsup_{m\to\infty}||\nabla V_{n,m}||_{L^\infty(Q_{10r}(x_3,t_0))}\\ &\le c_{14}\fint_{Q_{540r}(x_3,t_0)}|\nabla u|dxdt+c_{14}\frac{\omega(\overline{Q_{540r}(x_3,t_0)})}{R^{N+1}}\\ &\le c_{15}\fint_{\tilde{Q}_{1089r}(x_1,t_1)}|\nabla u|dxdt+c_{15}\frac{\omega(\tilde{Q}_{1089r}(x_2,t_2))}{R^{N+1}}\\ &\le c_{16}\lambda+c_{16}\delta_2\lambda\\ &\le c_{17}\lambda \end{aligned}$$

and

$$\begin{aligned} &\limsup_{n\to\infty}\limsup_{m\to\infty}\fint_{Q_{2r}(x,t_0)}|\nabla u_{n,m}-\nabla V_{n,m}|dxdt\\ &\quad\le c_{18}(\delta_3+[A]_{s_0}^{R_0})\fint_{Q_{540r}(x_3,t_0)}|\nabla u|dxdt+c_{18}(\delta_3+1+[A]_{s_0}^{R_0})\frac{\omega(\overline{Q_{540r}(x_3,t_0)})}{r^{N+1}}\\ &\quad\le c_{19}(\delta_3+[A]_{s_0}^{R_0})\fint_{\tilde{Q}_{1089r}(x_1,t_1)}|\nabla u|dxdt+c_{19}(\delta_3+1+[A]_{s_0}^{R_0})\frac{\omega(\tilde{Q}_{1089}(x_2,t_2))}{r^{N+1}}\\ &\quad\le c_{20}(\delta_3+[A]_{s_0}^{R_0})\lambda+c_{21}(\delta_3+1+[A]_{s_0}^{R_0})\delta_2\lambda\\ &\quad\le c_{20}\left((\delta_3+\delta_1)+(\delta_3+1+\delta_1)\delta_2\right)\lambda. \end{aligned}$$

Here we used $[A]_s^{R_0}\le\delta_1$ in the last inequality.
So, we can find n_0 large enough and a sequence $\{k_n\}$ such that

$$||\nabla V_{n,m}||_{L^\infty(\tilde{Q}_{2r}(x,t))}=||\nabla V_{n,m}||_{L^\infty(Q_{2r}(x,t_0))}\le 2c_{17}\lambda\quad\text{and} \tag{8.22}$$

$$\fint_{Q_{2r}(x,t_0)} |\nabla u_{n,m} - \nabla V_{n,m}| dxdt \le 2c_{21}\left((\delta_3+\delta_1)+(\delta_3+1+\delta_1)\delta_2\right)\lambda, \tag{8.23}$$

for all $n \ge n_0$ and $m \ge k_n$.
Now set $\Lambda = \max\{3^{N+2}, 8c_{10}, 8c_{17}\}$. As above we also have for $n \ge n_0$, $m \ge k_n$

$$\begin{aligned}|E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t)| \le\, & |\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u_{n,m} - \nabla V_{n,m}|\right) > \Lambda\lambda/4\} \cap \tilde{Q}_r(x,t)| \\ & + |\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u_n - \nabla u_{n,m}|\right) > \Lambda\lambda/4\} \cap \tilde{Q}_r(x,t)| \\ & + |\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u - \nabla u_n|\right) > \Lambda\lambda/4\} \cap \tilde{Q}_r(x,t)|.\end{aligned}$$

Therefore from (8.23) we obtain

$$\begin{aligned}|E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t)| \le\, & \frac{c_{22}}{\lambda}\int_{\tilde{Q}_{2r}(x,t)} |\nabla u_{n,m} - \nabla V_{n,m}| dxdt \\ & + \frac{c_{22}}{\lambda}\int_{\tilde{Q}_{2r}(x,t)} |\nabla u_n - \nabla u_{n,m}| dxdt + \frac{c_{22}}{\lambda}\int_{\tilde{Q}_{2r}(x,t)} |\nabla u - \nabla u_n| dxdt \\ \le\, & c_{23}\left((\delta_3+\delta_1)+(\delta_3+1+\delta_1)\delta_2\right)|\tilde{Q}_r(x,t)| \\ & + \frac{c_{22}}{\lambda}\int_{\tilde{Q}_{2r}(x,t)} |\nabla u_n - \nabla u_{n,m}| dxdt + \frac{c_{22}}{\lambda}\int_{\tilde{Q}_{2r}(x,t)} |\nabla u - \nabla u_n| dxdt.\end{aligned}$$

Letting $m \to \infty$ and $n \to \infty$ we get

$$|E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t)| \le c_{22}\left((\delta_3+\delta_1)+(\delta_3+1+\delta_1)\delta_2\right)|\tilde{Q}_r(x,t)|.$$

Thus

$$\begin{aligned}w(E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t)) \le\, & C\left(\frac{|E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t)|}{|\tilde{Q}_r(x,t)|}\right)^\nu w(\tilde{Q}_r(x,t)) \\ \le\, & C\left(c_{22}\left((\delta_3+\delta_1)+(\delta_3+1+\delta_1)\delta_2\right)\right)^\nu w(\tilde{Q}_r(x,t)) \\ <\, & \varepsilon w(\tilde{Q}_r(x,t)),\end{aligned}$$

where $\delta_3, \delta_1, \delta_2$ are appropriately chosen, (C,ν) is a pair of A_∞ constants of w.
Therefore, for all $(x,t) \in \Omega_T$ and $r \in (0, 2r_0]$ and $\lambda > 0$ if $w(E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t)) \ge \varepsilon w(\tilde{Q}_r(x,t))$ then $\tilde{Q}_r(x,t) \cap \Omega_T \subset F_\lambda$ where $\delta_1 = \delta_1(N, \Lambda_1, \Lambda_2, \varepsilon, [w]_{A_\infty}) \in (0,1)$ and $\delta_2 = \delta_2(N, \Lambda_1, \Lambda_2, \varepsilon, [w]_{A_\infty}, T_0/R_0) \in (0,1)$. Applying Lemma 3.19 we get the result. ■

Proof of Theorem 2.20. As in the proof of Theorem 2.17, we can prove (2.32) by using estimate (8.13) in Theorem 8.4. In particular, thanks to Proposition 4.4 for $q > \frac{N+2}{N+1}$, $\mu \in L^{\frac{(N+2)(q-1)}{q},\infty}(\Omega_T)$ and $\sigma \equiv 0$,

$$|||\nabla u|^q||_{L^{\frac{(N+2)(q-1)}{q},\infty}(\Omega_T)} \le c||\mu||^q_{L^{\frac{(N+2)(q-1)}{q},\infty}(\Omega_T)}, \tag{8.24}$$

where the constant c depends only on $N, \Lambda_1, \Lambda_2, q$ and T_0/R_0. ■

Proof of Theorem 2.22. By Theorem 2.20, there exists a renormalized solution of (2.4) with data μ, $u(0) = \sigma$ satisfied

$$\int_{\Omega_T} |\nabla u|^q dw \le c_1 \int_{\Omega_T} (\mathbb{M}_1[\omega])^q dw \tag{8.25}$$

for any $w \in A_\infty$, where $c_1 = c_1(N, \Lambda_1, \Lambda_2, q, T_0/R_0, [w]_{A_\infty})$.
For $0 < \delta < 1$ we have $\mathbb{M}_1[\omega] \le c_2 \mathbb{I}_1^{2T_0,\delta}[\omega]$ in Ω_T. Thus, (8.25) can be rewritten

$$\int_{\Omega_T} |\nabla u|^q dw \le c_1 c_2^q \int_{\Omega_T} \left(\mathbb{I}_1^{2T_0,\delta}[\omega]\right)^q dw. \tag{8.26}$$

Thanks to Proposition 4.23 and Corollary 4.39 and 4.38 we obtain the result. ■

In follow that we usually employ the the Minkowski inequality, for convenience we recall it, for any $0 < q_1 \leq q_2 < \infty$ there holds

$$\left(\int_X \left(\int_Y |f(x,y)|^{q_1} d\mu_2(y)\right)^{\frac{q_2}{q_1}} d\mu_1(x)\right)^{\frac{1}{q_2}} \leq \left(\int_Y \left(\int_X |f(x,y)|^{q_2} d\mu_1(x)\right)^{\frac{q_1}{q_2}} d\mu_2(y)\right)^{\frac{1}{q_1}}$$

for any measure function f in $X \times Y$, where μ_1, μ_2 are nonnegative measure in X and Y respectively.

Proof of Theorem 2.21. We will consider only the case $s \neq \infty$ and leave the case $s = \infty$ to the readers. Take $\kappa_1 \in (0, \kappa)$. It is easy to see that for $(x_0, t_0) \in \Omega_T$ and $0 < \rho < \text{diam}(\Omega) + T^{1/2}$

$$w(x,t) = \min\{\rho^{-N-2+\kappa-\kappa_1}, \max\{|x - x_0|, \sqrt{2|t - t_0|}\}^{-N-2+\kappa-\kappa_1}\} \in A_\infty$$

where $[w]_{A_\infty}$ is independent of (x_0, t_0) and ρ. Thus, from (2.32) in Theorem 2.20 we have

$$\begin{aligned}
||\mathbb{M}(|\nabla u|)||^s_{L^{q,s}(\tilde{Q}_\rho(x_0,t_0)\cap\Omega_T)} &= \rho^{\frac{(N+2-\kappa+\kappa_1)s}{q}} ||\mathbb{M}(|\nabla u|)||^s_{L^{q,s}(\tilde{Q}_\rho(x_0,t_0)\cap\Omega_T,dw)} \\
&\leq c_1 \rho^{\frac{(N+2-\kappa+\kappa_1)s}{q}} ||\mathbb{M}_1[\omega]||^s_{L^{q,s}(\Omega_T,dw)} \\
&= q c_1 \rho^{\frac{(N+2-\kappa+\kappa_1)s}{q}} \int_0^\infty (\lambda^q w(\{\mathbb{M}_1[\omega] > \lambda\} \cap \Omega_T))^{\frac{s}{q}} \frac{d\lambda}{\lambda} \\
&= q c_1 \rho^{\frac{(N+2-\kappa+\kappa_1)s}{q}} \int_0^\infty \left(\lambda^q \int_0^\infty |\{\mathbb{M}_1[\omega] > \lambda, w > \tau\} \cap \Omega_T| d\tau\right)^{\frac{s}{q}} \frac{d\lambda}{\lambda} \\
&=: c_1 \rho^{\frac{(N+2-\kappa+\kappa_1)s}{q}} A. \qquad (8.27)
\end{aligned}$$

Since $w \leq \rho^{-N-2+\kappa-\kappa_1}$ and $\{\mathbb{M}_1[\omega] > \lambda, w > \tau\} \subset \{\mathbb{M}_1[\omega] > \lambda\} \cap \tilde{Q}_{\tau^{\frac{1}{-N-2+\kappa-\kappa_1}}}(x_0, t_0)$,

$$A \leq q \int_0^\infty \left(\lambda^q \int_0^{\rho^{-N-2+\kappa-\kappa_1}} |\{\mathbb{M}_1[\omega] > \lambda\} \cap \tilde{Q}_{\tau^{\frac{1}{-N-2+\kappa-\kappa_1}}}(x_0, t_0) \cap \Omega_T| d\tau\right)^{\frac{s}{q}} \frac{d\lambda}{\lambda}.$$

We divide to two cases.

Case 1: $0 < s \leq q$. We can verify that for any nonincreasing function F in $(0, \infty)$ and $0 < a \leq 1$ we have

$$\left(\int_0^\infty F(\tau) d\tau\right)^a \leq 4 \int_0^\infty (\tau F(\tau))^a \frac{d\tau}{\tau}.$$

Hence,

$$\begin{aligned}
A &\leq 4q \int_0^\infty \int_0^{\rho^{-N-2+\kappa-\kappa_1}} \left(\lambda^q \tau |\{\mathbb{M}_1[\omega] > \lambda\} \cap \tilde{Q}_{\tau^{\frac{1}{-N-2+\kappa-\kappa_1}}}(x_0, t_0) \cap \Omega_T|\right)^{\frac{s}{q}} \frac{d\tau}{\tau} \frac{d\lambda}{\lambda} \\
&= 4q \int_0^{\rho^{-N-2+\kappa-\kappa_1}} \int_0^\infty \left(\lambda^q |\{\mathbb{M}_1[\omega] > \lambda\} \cap \tilde{Q}_{\tau^{\frac{1}{-N-2+\kappa-\kappa_1}}}(x_0, t_0) \cap \Omega_T|\right)^{\frac{s}{q}} \frac{d\lambda}{\lambda} \tau^{\frac{s}{q}} \frac{d\tau}{\tau} \\
&= 4 \int_0^{\rho^{-N-2+\kappa-\kappa_1}} ||\mathbb{M}_1[\omega]||^s_{L^{q,s}(\tilde{Q}_{\tau^{\frac{1}{-N-2+\kappa-\kappa_1}}}(x_0,t_0)\cap\Omega_T)} \tau^{\frac{s}{q}} \frac{d\tau}{\tau} \\
&\leq 4 \int_0^{\rho^{-N-2+\kappa-\kappa_1}} ||\mathbb{M}_1[\omega]||^s_{L^{q,s;\kappa}(\Omega_T)} \tau^{\frac{(N+2-\kappa)s}{(-N-2+\kappa-\kappa_1)q}} \tau^{\frac{s}{q}} \frac{d\tau}{\tau} \\
&= c_2 ||\mathbb{M}_1[\omega]||^s_{L^{q,s;\kappa}(\Omega_T)} \rho^{-\frac{s\kappa_1}{q}}.
\end{aligned}$$

Case 2: $s > q$. Using the Minkowski inequality, yields

$$\begin{aligned}
A &\leq c_3 \left(\int_0^{\rho^{-N-2+\kappa-\kappa_1}} \left(\int_0^\infty \left(\lambda^q |\{\mathbb{M}_1[\omega] > \lambda\} \cap \tilde{Q}_{\tau^{\frac{1}{-N-2+\kappa-\kappa_1}}}(x_0,t_0) \cap \Omega_T| \right)^{\frac{s}{q}} \frac{d\lambda}{\lambda} \right)^{\frac{q}{s}} d\tau \right)^{\frac{s}{q}} \\
&\leq c_4 \left(\int_0^{\rho^{-N-2+\kappa-\kappa_1}} \left(||\mathbb{M}_1[\omega]||^s_{L^{q,s;\kappa}(\Omega_T)} \tau^{\frac{(N+2-\kappa)s}{(-N-2+\kappa-\kappa_1)q}} \right)^{\frac{q}{s}} d\tau \right)^{\frac{s}{q}} \\
&= c_5 ||\mathbb{M}_1[\omega]||^s_{L^{q,s;\kappa}(\Omega_T)} \rho^{-\frac{s\kappa_1}{q}}.
\end{aligned}$$

Therefore, we always have

$$A \leq c_6 ||\mathbb{M}_1[\omega]||^s_{L^{q,s;\kappa}(\Omega_T)} \rho^{-\frac{s\kappa_1}{q}}.$$

which implies (2.33) from (8.27).
Similarly, we obtain estimate (2.46) by adapting

$$w(x,t) = \min\{\rho^{-N+\vartheta-\vartheta_1}, |x-x_0|^{-N+\vartheta-\vartheta_1}\} \in A_\infty$$

in above argument, where $0 < \vartheta_1 < \vartheta$, $x_0 \in \Omega$ and $0 < \rho < \text{diam}(\Omega)$ and $[w]_{A_\infty}$ is independent of x_0 and ρ.
Next, to archive (2.35) we need to show that for any ball $B_\rho \subset \mathbb{R}^N$

$$\left(\int_0^T |\text{osc}_{B_\rho \cap \overline{\Omega}} u(t)|^q dt \right)^{\frac{1}{q}} \leq c_7 \rho^{1-\frac{\vartheta}{q}} |||\nabla u|||_{L^{q;\vartheta}_{**}(\Omega_T)} \tag{8.28}$$

Since the extension of u over $(\Omega_T)^c$ is zero and $u \in L^1(0,T,W_0^{1,1}(\Omega))$ thus we have for a.e $t \in (0,T)$, $u(.,t) \in W^{1,1}(\mathbb{R}^N)$. Applying [32, Lemma 7.16] to a ball $B_\rho \subset \mathbb{R}^N$, we get for a.e $t \in (0,T)$ and $x \in B_\rho$

$$\begin{aligned}
|u(x,t) - u_{B\rho}(t)| &\leq \frac{2^N}{N|B_1(0)|} \int_{B_\rho} \frac{|\nabla u(y,t)|}{|x-y|^{N-1}} dy \\
&\leq \frac{2^N}{N|B_1(0)|} \int_{B_{2\rho}(x)} \frac{|\nabla u(y,t)|}{|x-y|^{N-1}} dy \\
&\leq c_8 \int_0^{3\rho} \frac{\int_{B_r(x)} |\nabla u(y,t)| dy}{r^{N-1}} \frac{dr}{r},
\end{aligned}$$

here $u_{B\rho}(t)$ is the average of $u(.,t)$ over B_ρ, i.e $u_{B\rho}(t) = \frac{1}{|B_\rho|} \int_{B_\rho} u(x,t) dx$.
Using the Minkowski and the Holder inequality, we discover that for a.e $x \in B_\rho$

$$\begin{aligned}
\left(\int_0^T |u(x,t) - u_{B\rho}(t)|^q dt \right)^{\frac{1}{q}} &\leq c_8 \left(\int_0^T \left(\int_0^{3\rho} \frac{\int_{B_r(x)} |\nabla u(y,t)| dy}{r^{N-1}} \frac{dr}{r} \right)^q dt \right)^{\frac{1}{q}} \\
&\leq c_8 \int_0^{3\rho} \int_{B_r(x)} \left(\int_0^T |\nabla u(y,t)|^q dt \right)^{\frac{1}{q}} dy \frac{dr}{r^N} \\
&\leq c_8 \int_0^{3\rho} \left(\int_{B_r(x)} \int_0^T |\nabla u(y,t)|^q dt dy \right)^{\frac{1}{q}} |B_r(x)|^{\frac{q-1}{q}} \frac{dr}{r^N} \\
&\leq c_8 |B_1(x)|^{\frac{q-1}{q}} \int_0^{3\rho} r^{\frac{N-\vartheta}{q}} r^{\frac{N(q-1)}{q}} \frac{dr}{r^N} |||\nabla u|||_{L^{q;\vartheta}_{**}(\Omega_T)} \\
&= c_9 \rho^{1-\frac{\vartheta}{q}} |||\nabla u|||_{L^{q;\vartheta}_{**}(\Omega_T)}.
\end{aligned}$$

Therefore, we find (8.28) with $c_7 = 2c_9$. ∎

Proof of Proposition 2.28. Clearly, estimate (2.46) is followed by (4.12) in Proposition 4.7. We want to emphasize that almost every estimates in this proof will be used the Minkowski inequality. For a ball $B_\rho \subset \mathbb{R}^N$, we have for a.e $x \in \mathbb{R}^N$

$$\begin{aligned} ||\mathbb{I}_1[\mu](x,.)||_{L^q(\mathbb{R})} &= \left(\int_{-\infty}^{+\infty}\left(\int_0^\infty \frac{\mu(\tilde{Q}_r(x,t))}{r^{N+1}}\frac{dr}{r}\right)^q dt\right)^{\frac{1}{q}} \\ &\le \int_0^\infty \left(\int_{-\infty}^{+\infty}(\mu(\tilde{Q}_r(x,t)))^q dt\right)^{\frac{1}{q}}\frac{dr}{r^{N+2}}. \end{aligned} \tag{8.29}$$

Now, we need to estimate $\left(\int_{-\infty}^{+\infty}(\mu(\tilde{Q}_r(x,t)))^q dt\right)^{\frac{1}{q}}$.

b. We have

$$\begin{aligned} \left(\int_{-\infty}^{+\infty}(\mu(\tilde{Q}_r(x,t)))^q dt\right)^{\frac{1}{q}} &= \left(\int_{-\infty}^{+\infty}\left(\int_{\mathbb{R}^{N+1}}\chi_{\tilde{Q}_r(x,t)}(x_1,t_1)d\mu(x_1,t_1)\right)^q dt\right)^{\frac{1}{q}} \\ &\le \int_{\mathbb{R}^{N+1}}\left(\int_{-\infty}^{+\infty}\chi_{\tilde{Q}_r(x,t)}(x_1,t_1)dt\right)^{\frac{1}{q}} d\mu(x_1,t_1) \\ &= r^{\frac{2}{q}}\mu_1(B_r(x)) \end{aligned}$$

Combining this with (8.29) we obtain (2.47) and (2.49).
Thus, we also assert (2.49) from [1, Theorem 3.1].
c. Set $d\mu_2(x) = ||\mu(x,.)||_{L^{q_1}(\mathbb{R})}dx$. Using Holder's inequality, yields

$$\mu(\tilde{Q}_r(x,t)) \le r^{\frac{2(q_1-1)}{q_1}}\int_{B_r(x)}\left(\int_{t-\frac{\rho^2}{2}}^{t+\frac{\rho^2}{2}}(w(x_1,t_1))^{q_1}dt_1\right)^{\frac{1}{q_1}}dx_1.$$

Leads to

$$\left(\int_{-\infty}^{+\infty}(\mu(\tilde{Q}_r(x,t)))^q dt\right)^{\frac{1}{q}} \le r^{\frac{2(q_1-1)}{q_1}}\int_{B_r(x)}\left(\int_{-\infty}^{+\infty}\left(\int_{t-\frac{\rho^2}{2}}^{t+\frac{\rho^2}{2}}(w(x_1,t_1))^{q_1}dt_1\right)^{\frac{q}{q_1}}dt\right)^{\frac{1}{q}}dx_1.$$

Note that

$$\begin{aligned} &\left(\int_{-\infty}^{+\infty}\left(\int_{t-\frac{\rho^2}{2}}^{t+\frac{\rho^2}{2}}(w(x_1,t_1))^{q_1}dt_1\right)^{\frac{q}{q_1}}dt\right)^{\frac{q_1}{q}} \\ &\quad= \left(\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}\chi_{\left(t-\frac{\rho^2}{2},t+\frac{\rho^2}{2}\right)}(t_1)(w(x_1,t_1))^{q_1}dt_1\right)^{\frac{q}{q_1}}dt\right)^{\frac{q_1}{q}} \\ &\quad\le \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}\chi_{\left(t-\frac{\rho^2}{2},t+\frac{\rho^2}{2}\right)}(t_1)dt\right)^{\frac{q_1}{q}}(w(x_1,t_1))^{q_1}dt_1 \\ &\quad= \rho^{\frac{2q_1}{q}}\int_{-\infty}^{+\infty}(w(x_1,t_1))^{q_1}dt_1. \end{aligned}$$

Hence

$$\begin{aligned} \left(\int_{-\infty}^{+\infty}(\mu(\tilde{Q}_r(x,t)))^q dt\right)^{\frac{1}{q}} &\le r^{\frac{2(q_1-1)}{q_1}+\frac{2}{q}}\int_{B_r(x)}||\mu(x_1,.)||_{L^{q_1}(\mathbb{R})}dx_1 \\ &= r^{\frac{2(q_1-1)}{q_1}+\frac{2}{q}}\mu_2(B_r(x)). \end{aligned}$$

Consequently, since (8.29) we derive (2.50) and (2.51).
We also obtain (2.52) from [1, Theorem 3.1]. ■

8.3 Global estimates in $\mathbb{R}^N\times(0,\infty)$ and $\mathbb{R}^{N+1}$

Now, we present the proofs of Theorem 2.25 and 2.27.
Proof of Theorem 2.25 and Theorem 2.27. For any $n\geq 1$, it is easy to see that

i. $\mathbb{R}^N\backslash B_n(0)$ satisfies uniformly 2−thick with constants $c_0=\frac{\text{Cap}_p(B_{1/4}(z_0),B_2(0))}{\text{Cap}_p(B_1(0),B_2(0))}$, $z_0=(1/2,0,...,0)\in\mathbb{R}^N$ and $r_0=n$.

ii. for any $\delta\in(0,1)$, $B_n(0)$ is a $(\delta,2n\delta)-$ Reifenberg flat domain.

iii. $[\mathcal{A}]_{s_0}^n\leq[\mathcal{A}]_{s_0}^\infty$.

Assume that $||\mathbb{M}_1[|\omega|]||_{L^{p,s}(\mathbb{R}^{N+1})}<\infty$. Thus by Remark 2.26 we have

$$\mathbb{I}_2[|\omega|](x,t)<\infty \text{ for a.e } (x,t)\in\mathbb{R}^{N+1}. \tag{8.30}$$

In view of the proof of the Theorem 2.5 and applying Theorem 2.17 to $B_n(0)\times(-n^2,n^2)$ and with data $\chi_{B_{n-1}(0)\times(-(n-1)^2,(n-1)^2)}\omega$ for any $n\geq 2$, there exists a sequence renormalized solution $\{u_n\}$ (we will take its subsequence if need) of

$$\begin{cases}(u_n)_t-div(A(x,t,\nabla u_n))=\chi_{B_{n-1}(0)\times(-(n-1)^2,(n-1)^2)}\omega \text{ in } B_n(0)\times(-n^2,n^2),\\ u_n=0 \quad\text{on } \partial B_n(0)\times(-n^2,n^2),\\ u_n(-n^2)=0 \text{ in } B_n(0),\end{cases}$$

converging to a distribution solution u in $L^1_{\text{loc}}(\mathbb{R};W^{1,1}_{\text{loc}}(\mathbb{R}^N))$ of 2.6 with data $\mu=\omega$ such that

$$\begin{aligned}|||\nabla u_n|||_{L^{p,s}(B_n(0)\times(-n^2,n^2))}&\leq c_1||\mathbb{M}_1[\chi_{B_{n-1}(0)\times(-(n-1)^2,(n-1)^2)}|\omega|]||_{L^{p,s}(B_{2n}(0)\times(-n^2,n^2))}\\&\leq c_1||\mathbb{M}_1[|\omega|]||_{L^{p,s}(\mathbb{R}^{N+1})}.\end{aligned}$$

Here $c_1=c_1(N,\Lambda_1,\Lambda_2,p,s)$ is not depending on n since $\frac{T_0}{r_0}=\frac{2n+(1+n^2)^{1/2}}{n}\approx 1$.
Using Fatou Lemma, we get estimate (2.38).
As above, we also obtain (2.39).
And similarly, we can prove Theorem 2.27 by this way.
This completes the proof of Theorem. ■

Remark 8.5 (sharpness) *The inequality* (2.41) *is in a sense optimal as follows:*

$$C^{-1}||\mathbb{M}_1[|\omega|]||_{L^q(\mathbb{R}^{N+1})}\leq|||\nabla\mathcal{H}_2|*|\omega|||_{L^q(\mathbb{R}^N\times(0,\infty))}\leq C||\mathbb{M}_1[|\omega|]||_{L^q(\mathbb{R}^{N+1})} \tag{8.31}$$

for every $q>1$ where $C=C(N,q)$. Indeed, we have

$$\nabla\mathcal{H}_2(x,t)=-\frac{C_\alpha}{2}\frac{\chi_{(0,\infty)}(t)}{t^{(N+1)/2}}\exp(-\frac{|x|^2}{4t})\frac{x}{\sqrt{t}},$$

leads to

$$\frac{c_1^{-1}}{t^{\frac{N+1}{2}}}\chi_{t>0}\chi_{\frac{1}{2}\sqrt{t}\leq|x|\leq 2\sqrt{t}}\leq|\nabla\mathcal{H}_\alpha(x,t)|\leq\frac{c_1}{\max\{|x|,\sqrt{2|t|}\}^{N+1}}.$$

Immediately, we get

$$c_2^{-1}\int_0^\infty\frac{\omega\left((B_r(x)\backslash B_{r/2}(x))\times(t-r^2,t-r^2/4)\right)}{r^{N+1}}\frac{dr}{r}\leq|\nabla\mathcal{H}_2|*|\omega|(x,t)\leq c_2\mathbb{I}_1[\omega](x,t).$$

By Theorem 4.2, gives the right-hand side inequality of (8.31). *So, it is enough to show that*

$$A := \int_{\mathbb{R}^{N+1}} \left(\int_0^\infty \frac{\omega\left((B_r(x)\backslash B_{r/2}(x)) \times (t-r^2, t-r^2/4)\right)}{r^{N+1}} \frac{dr}{r} \right)^q dxdt \geq c_3 ||\mathbb{M}_1[\omega]||^q_{L^q(\mathbb{R}^{N+1})} \tag{8.32}$$

To do this, we take $r_k = (3/2)^k$ *for* $k \in \mathbb{Z}$,

$$\left(\int_0^\infty \frac{\omega\left((B_r(x)\backslash B_{r/2}(x)) \times (t-r^2, t-r^2/4)\right)}{r^{N+1}} \frac{dr}{r} \right)^q$$
$$\geq c_4 \sum_{k=-\infty}^{\infty} \left(\frac{\omega\left((B_{r_k}(x)\backslash B_{3r_k/4}(x)) \times (t-r_k^2, t-9r_k^2/16)\right)}{r_k^{N+1}} \right)^q .$$

We deduce that

$$A \geq c_4 \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}^{N+1}} \left(\frac{\omega\left((B_{r_k}(x)\backslash B_{3r_k/4}(x)) \times (t-r_k^2, t-9r_k^2/16)\right)}{r_k^{N+1}} \right)^q dxdt.$$

For any k, *put* $y = x + \frac{7}{8}r_k$ *and* $s = t - \frac{25}{32}r_k^2$, *so* $B_{r_k}(x)\backslash B_{3r_k/4}(x) \supset B_{r_k/8}(y)$ *and*

$$\int_{\mathbb{R}^{N+1}} \left(\frac{\omega\left((B_{r_k}(x)\backslash B_{3r_k/4}(x)) \times (t-r_k^2, t-9r_k^2/16)\right)}{r_k^{N+1}} \right)^q dxdt$$
$$\geq \int_{\mathbb{R}^{N+1}} \left(\frac{\omega\left(B_{r_k/8}(y) \times (s-7r_k^2/32, t+7r_k^2/32)\right)}{r_k^{N+1}} \right)^q dyds.$$

Consequently,

$$A \geq c_4 \int_{\mathbb{R}^{N+1}} \sum_{k=-\infty}^{\infty} \left(\frac{\omega\left(B_{r_k/8}(y) \times (s-7r_k^2/32, t+7r_k^2/32)\right)}{r_k^{N+1}} \right)^q dyds.$$

It follows (8.32).

9 Quasilinear Riccati Type Parabolic Equations

9.1 Quasilinear Riccati Type Parabolic Equation in Ω_T

We provide below only the proof of Theorem 2.30, 2.32 and 2.33. The proof of Theorem 2.31 can be proceeded by a similar argument.

Proof of Theorem 2.30. Let $\{\mu_n\} \subset C_c^\infty(\Omega_T)$ be as in the proof of Theorem 2.1. We have $|\mu_n|(\Omega_T) \leq |\mu|(\Omega_T)$ for any $n \in \mathbb{N}$. Let $\sigma_n \in C_c^\infty(\Omega)$ be converging to σ in the narrow topology of measures and in $L^1(\Omega)$ if $\sigma \in L^1(\Omega)$ such that $||\sigma_n||_{L^1(\Omega)} \leq |\sigma|(\Omega)$. For $n_0 \in \mathbb{N}$, we prove that the problem (2.53) has a solution with data $\mu = \mu_{n_0}$ and $\sigma = \sigma_{n_0}$. Now we put

$$\mathbf{E}_\Lambda = \{u \in L^1(0,T,W_0^{1,1}(\Omega)) : |||\nabla u|||_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} \leq \Lambda\},$$

where $L^{\frac{N+2}{N+1},\infty}(\Omega_T)$ is Lorent space with norm

$$||f||_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} := \sup_{0<|D|<\infty} \left(|D|^{-\frac{1}{N+2}} \int_{D\cap\Omega_T} |f| \right).$$

By Fatou's lemma, $\mathbf{E}_\Lambda$ is closed under the strong topology of $L^1(0,T,W_0^{1,1}(\Omega))$ and convex. We consider a map $S:\mathbf{E}_\Lambda\to\mathbf{E}_\Lambda$ defined for each $v\in\mathbf{E}_\Lambda$ by $S(v)=u$, where $u\in L^1(0,T,W_0^{1,1}(\Omega))$ is the unique solution of

$$\begin{cases} u_t - div\,(A(x,t,\nabla u)) = |\nabla v|^q + \mu_{n_0} \quad \text{in } \Omega_T, \\ u=0 \quad \text{on} \quad \partial\Omega\times(0,T) \\ u(0)=\sigma_{n_0}. \end{cases} \tag{9.1}$$

By Remark 3.2, we have

$$|||\nabla u|||_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} \le c_1\left(|||\nabla v|^q||_{L^1(\Omega_T)} + |\mu_{n_0}|(\Omega_T) + ||\sigma_{n_0}||_{L^1(\Omega)}\right),$$

for some $c_1=c_1(N,\Lambda_1,\Lambda_2)$. It leads to

$$\begin{aligned} |||\nabla u|||_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} &\le c_1\left(c_2|\Omega_T|^{1-\frac{q(N+1)}{N+2}}|||\nabla v|||^q_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} + |\mu|(\Omega_T)+|\sigma|(\Omega)\right) \\ &\le c_1\left(c_2|\Omega_T|^{1-\frac{q(N+1)}{N+2}}\Lambda^q + |\mu|(\Omega_T)+|\sigma|(\Omega)\right), \end{aligned}$$

for some $c_2=c_2(N,q)>0$. Thus, we now suppose that

$$|\Omega_T|^{-1+\frac{q'}{N+2}}\left(|\mu|(\Omega_T)+|\sigma|(\Omega)\right) \le (2c_1)^{-q'}c_2^{-\frac{1}{q-1}},$$

then

$$|||\nabla u|||_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} \le \Lambda := 2c_1(|\mu|(\Omega)+|\sigma|(\Omega)),$$

which implies that S is well defined.
Now we show that S **is continuous**. Let $\{v_n\}$ be a sequence in $\mathbf{E}_\Lambda$ such that v_n converges strongly in $L^1(0,T,W_0^{1,1}(\Omega))$ to a function $v\in\mathbf{E}_\Lambda$. Set $u_n=S(v_n)$. We need to show that $u_n\to S(v)$ in $L^1(0,T,W_0^{1,1}(\Omega))$. We have

$$\begin{cases} (u_n)_t - \operatorname{div}(A(x,t,\nabla u_n)) = |\nabla v_n|^q + \mu_{n_0} \quad \text{in} \quad \Omega_T, \\ u_n=0 \quad \text{on} \quad \partial\Omega\times(0,T), \\ u_n(0)=\sigma_{n_0} \quad \text{in} \quad \Omega, \end{cases} \tag{9.2}$$

satisfied

$$|||\nabla u_n|||_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} \le \Lambda, \quad |||\nabla v_n|||_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} \le \Lambda.$$

Thus, $|\nabla v_n|^q\to|\nabla v|^q$ in $L^1(\Omega_T)$. Therefore, it is easy to see that we get $u_n\to S(v)$ in $L^1(0,T,W_0^{1,1}(\Omega))$ by Theorem 3.6.
Next we show that S **is pre-compact**. Indeed if $\{u_n\}=\{S(v_n)\}$ is a sequence in $S(\mathbf{E}_\Lambda)$. By Proposition 3.5, there exists a subsequence of $\{u_n\}$ converging to some u in $L^1(0,T,W_0^{1,1}(\Omega))$. Consequently, by Schauder Fixed Point Theorem, S has a fixed point on $\mathbf{E}_\Lambda$ this means: the problem (2.53) has a solution with data μ_{n_0},σ_{n_0}.
Therefore, for any $n\in\mathbb{N}$, there exists a renormalized solution u_n of

$$\begin{cases} (u_n)_t - div\,(A(x,t,\nabla u_n)) = |\nabla u_n|^q + \mu_n \quad \text{in} \quad \Omega_T, \\ u=0 \quad \text{on} \quad \partial\Omega\times(0,T), \\ u_n(0)=\sigma_n, \end{cases} \tag{9.3}$$

which satisfies

$$|||\nabla u_n|||_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} \le 2c_1(|\mu|(\Omega)+|\sigma|(\Omega)).$$

Thanks to Proposition 3.5, there exists a subsequence of $\{u_n\}$ converging to u in $L^1(0,T,W_0^{1,1}(\Omega))$. So, $|||\nabla u|||_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} \le 2c_1(|\mu|(\Omega)+|\sigma|(\Omega))$ and $|\nabla u_n|^q\to|\nabla u|^q$ in $L^1(\Omega)$ since $\{|\nabla u_n|^q\}$ is equi-integrable. It follows the results by Proposition 3.5 and Theorem 3.6. ∎

Proof of Theorem 2.32. **Case a.** A is linear operator. By Theorem 2.22, there exist $\delta = \delta(N, \Lambda_1, \Lambda_2, q) \in (0,1)$ and $s_0 = s_0(N, \Lambda_1, \Lambda_2) > 0$ such that Ω is (δ, R_0)- Reifenberg flat domain and $[\mathcal{A}]_{s_0}^{R_0} \leq \delta$ for some R_0 and a sequence $\{u_n\}_n$ as distribution solutions of

$$\begin{cases} (u_1)_t - \operatorname{div}(A(x,t,\nabla u_1)) = \mu \text{ in } \Omega_T, \\ u_1 = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u_1(0) = \sigma \quad \text{in } \Omega, \end{cases}$$

and

$$\begin{cases} (u_{n+1})_t - \operatorname{div}(A(x,t,\nabla u_{n+1})) = |\nabla u_n|^q + \mu \text{ in } \Omega_T, \\ u_{n+1} = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u_{n+1}(0) = \sigma \quad \text{in } \Omega, \end{cases}$$

which satisfy

$$[|\nabla u_{n+1}|^q]_{\mathfrak{M}^{\mathcal{G}_1,q'}} \leq c_1 [|\nabla u_n|^q + \omega]^q_{\mathfrak{M}^{\mathcal{G}_1,q'}} \quad \forall n \geq 0 \tag{9.4}$$

where $u_0 \equiv 0$ and constant c_1 depends only on $N, \Lambda_1, \Lambda_2, q$ and $T_0/R_0, T_0$. Moreover, if $\sigma \in L^1(\Omega)$ then $\{u_n\}$ is the sequence of renormalized solutions.

i. Suppose

$$[\omega]_{\mathfrak{M}^{\mathcal{G}_1,q'}} \leq (q-1)^{\frac{1}{q}} (qc_1 2^{q-1})^{-\frac{1}{q-1}}, \tag{9.5}$$

we prove that

$$[|\nabla u_n|^q]_{\mathfrak{M}^{\mathcal{G}_1,q'}} \leq \frac{qc_1 2^{q-1}}{q-1} [\omega]^q_{\mathfrak{M}^{\mathcal{G}_1,q'}} \quad \forall n \geq 1. \tag{9.6}$$

Indeed, clearly, we have (9.6) when $n = 1$. Now assume that (9.6) is true with $n = m$, that is,

$$[|\nabla u_m|^q]_{\mathfrak{M}^{\mathcal{G}_1,q'}} \leq \frac{qc_1 2^{q-1}}{q-1} [\omega]^q_{\mathfrak{M}^{\mathcal{G}_1,q'}}.$$

From (9.4) we obtain

$$\begin{aligned} [|\nabla u_{m+1}|^q]_{\mathfrak{M}^{\mathcal{G}_1,q'}} &\leq c_1 [|\nabla u_m|^q + \omega]^q_{\mathfrak{M}^{\mathcal{G}_1,q'}} \\ &\leq c_1 2^{q-1} \left([|\nabla u_m|^q]^q_{\mathfrak{M}^{\mathcal{G}_1,q'}} + [\omega]^q_{\mathfrak{M}^{\mathcal{G}_1,q'}} \right) \\ &\leq c_1 2^{q-1} \left(\left(\frac{qc_1 2^{q-1}}{q-1} \right)^q [\omega]^{q(q-1)}_{\mathfrak{M}^{\mathcal{G}_1,q'}} + 1 \right) [\omega]^q_{\mathfrak{M}^{\mathcal{G}_1,q'}} \\ &\leq \frac{qc_1 2^{q-1}}{q-1} [\omega]^q_{\mathfrak{M}^{\mathcal{G}_1,q'}}. \end{aligned}$$

Here, the last inequality is obtained by using (9.5). So, (9.6) is also true with $n = m+1$. Thus, (9.6) is true for all $n \geq 1$.

ii. Clearly, $u_{n+1} - u_n$ is the unique renormalized solution of

$$\begin{cases} u_t - div\,(A(x,t,\nabla u)) = |\nabla u_n|^q - |\nabla u_{n-1}|^q \text{ in } \Omega_T, \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(0) = 0 \quad \text{in } \Omega. \end{cases} \tag{9.7}$$

So, we have

$$[|\nabla u_{n+1} - \nabla u_n|^q]_{\mathfrak{M}^{\mathcal{G}_1,q'}} \leq c_1 [|\nabla u_n|^q - |\nabla u_{n-1}|^q]^q_{\mathfrak{M}^{\mathcal{G}_1,q'}} \quad \forall n \geq 1.$$

Since, $|s_1^q - s_2^q| \le q|s_1 - s_2|(\max\{s_1, s_2\})^{q-1}$ for any $s_1, s_2 \ge 0$ and using Holder inequality, we get

$$\begin{aligned}[|\nabla u_{n+1} - \nabla u_n|^q]_{\mathfrak{M}^{\mathcal{G}_1,q'}} &\le c_1 q^q \left[|\nabla u_n - \nabla u_{n-1}|^q\right]_{\mathfrak{M}^{\mathcal{G}_1,q'}} \left[(\max\{|\nabla u_n|, |\nabla u_{n-1}|\})^q\right]^{q-1}_{\mathfrak{M}^{\mathcal{G}_1,q'}} \\ &\le c_1 q^q \left[|\nabla u_n - \nabla u_{n-1}|^q\right]_{\mathfrak{M}^{\mathcal{G}_1,q'}} \left([|\nabla u_n|^q]_{\mathfrak{M}^{\mathcal{G}_1,q'}} + [|\nabla u_{n-1}|^q]_{\mathfrak{M}^{\mathcal{G}_1,q'}}\right)^{q-1}\end{aligned}$$

which follows from (9.6),

$$[|\nabla u_{n+1} - \nabla u_n|^q]_{\mathfrak{M}^{\mathcal{G}_1,q'}} \le C\,[|\nabla u_n - \nabla u_{n-1}|^q]_{\mathfrak{M}^{\mathcal{G}_1,q'}} \quad \forall\, n \ge 1$$

where

$$C = c_1 q^q \left(\frac{qc_1 2^q}{q-1}\right)^{q-1} [\omega]^{q(q-1)}_{\mathfrak{M}^{\mathcal{G}_1,q'}}.$$

Hence, if $C < 1$ then u_n converges to $u = u_1 + \sum_{n=1}^{\infty}(u_{n+1} - u_n)$ in $L^q(0, T, W_0^{1,q}(\Omega))$ and satisfied

$$[|\nabla u|^q]_{\mathfrak{M}^{\mathcal{G}_1,q'}} \le \frac{qc_1 2^{q-1}}{q-1}[\omega]^q_{\mathfrak{M}^{\mathcal{G}_1,q'}}.$$

Note that $C < 1$ is equivalent to

$$[\omega]_{\mathfrak{M}^{\mathcal{G}_1,q'}} \le (c_1 q^q)^{-\frac{1}{q(q-1)}} \left(\frac{qc_1 2^q}{q-1}\right)^{-\frac{1}{q}}$$

Combining this with (9.5) and using Theorem 3.6, we conclude that the problem (2.53) has a distribution solution u (a renormalized if $\sigma \in L^1(\Omega)$) , if

$$[\omega]_{\mathfrak{M}^{\mathcal{G}_1,q'}} \le \min\left\{(q-1)^{\frac{1}{q}}(qc_1 2^{q-1})^{-\frac{1}{q-1}}, (c_1 q^q)^{-\frac{1}{q(q-1)}} \left(\frac{qc_1 2^q}{q-1}\right)^{-\frac{1}{q}}\right\}.$$

Next, we will prove **Case b.** and **Case c.**.
Let $\{\mu_n\} \subset C_c^\infty(\Omega_T), \sigma_n \in C_c^\infty(\Omega)$ be as in the proof of Theorem 2.1. We have $|\mu_n| \le \varphi_n * |\mu|, |\sigma_n| \le \varphi_{1,n} * |\sigma|$ for any $n \in \mathbb{N}$, $\{\varphi_n\}$, $\{\varphi_{1,n}\}$ are sequences of standard mollifiers in $\mathbb{R}^{N+1}, \mathbb{R}^N$ respectively. Set $\omega_n = |\mu_n| + |\sigma_n| \otimes \delta_{\{t=0\}}$ and $\omega = |\mu| + |\sigma| \otimes \delta_{\{t=0\}}$.
Case b. For $n_0 \in \mathbb{N}$, $\varepsilon > 0$, we prove that the problem (2.53) has a solution with data $\mu = \mu_{n_0}, \sigma = \sigma_{n_0}$. Now we put

$$\mathbf{E}_\Lambda = \{u \in L^1(0, T, W_0^{1,1}(\Omega)) : [|\nabla u|^{q+\varepsilon}]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}(\Omega_T)} \le \Lambda\}.$$

By Fatou's lemma, $\mathbf{E}_\Lambda$ is closed under the strong topology of $L^1(0, T, W_0^{1,1}(\Omega))$ and convex. We consider a map $S : \mathbf{E}_\Lambda \to \mathbf{E}_\Lambda$ defined for each $v \in \mathbf{E}_\Lambda$ by $S(v) = u$, where $u \in L^1(0, T, W_0^{1,1}(\Omega))$ is the unique solution of problem (9.1). By Theorem 2.22, there exist $\delta = \delta(N, \Lambda_1, \Lambda_2, q+\varepsilon) \in (0, 1)$ and $s_0 = s_0(N, \Lambda_1, \Lambda_2) > 0$ such that Ω is (δ, R_0)- Reifenberg flat domain and $[\mathcal{A}]_{s_0}^{R_0} \le \delta$ for some R_0 we have

$$[|\nabla u|^{q+\varepsilon}]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}} \le c_2[|\nabla v|^q + \omega_{n_0}]^{q+\varepsilon}_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}},$$

where $c_2 = c_2(N, \Lambda_1, \Lambda_2, q+\varepsilon, T_0/R_0, T_0)$. By Remark 4.33, we deduce that

$$[|\nabla v|^q]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}} \le c_3[|\nabla v|^{q+\varepsilon}]^{\frac{q}{q+\varepsilon}}_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}},$$

where a constant c_3 depends on $N, q+\varepsilon$.
Thus,

$$\begin{aligned}[|\nabla u|^{q+\varepsilon}]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}} &\le c_2\left([|\nabla v|^q]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}} + [\omega_{n_0}]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}}\right)^{q+\varepsilon} \\ &\le c_2\left(c_3[|\nabla u|^{q+\varepsilon}]^{\frac{q}{q+\varepsilon}}_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}} + [\omega_{n_0}]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}}\right)^{q+\varepsilon} \\ &\le c_2\left(c_3\Lambda^{\frac{q}{q+\varepsilon}} + [\omega_{n_0}]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}}\right)^{q+\varepsilon} \\ &\le \Lambda,\end{aligned}$$

provided that $[\omega_{n_0}]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}} \le c_4 := 2^{-q'} c_2^{-\frac{q'}{q+\varepsilon}} c_3^{-\frac{1}{q-1}}$ and $\Lambda := 2^{q+\varepsilon} c_2 [\omega_{n_0}]^{q+\varepsilon}_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}}$.
which implies that S is well defined with $[\omega_{n_0}]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}} \le c_4$.
Now we show that S **is continuous**. Let $\{v_n\}$ be a sequence in $\mathbf{E}_\Lambda$ such that v_n converges strongly in $L^1(0,T,W_0^{1,1}(\Omega))$ to a function $v \in \mathbf{E}_\Lambda$. Set $u_n = S(v_n)$. We need to show that $u_n \to S(v)$ in $L^1(0,T,W_0^{1,1}(\Omega))$. We have u_n satisfied (9.2) and

$$[|\nabla u_n|^{q+\varepsilon}]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}} \le \Lambda, \quad [|\nabla v_n|^{q+\varepsilon}]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}} \le \Lambda.$$

In particular, $||\nabla v_n||_{L^{q+\varepsilon}(\Omega_T)} \le \Lambda \mathrm{Cap}_{\mathcal{G}_1,(q+\varepsilon)'}(\overline{\Omega}_T)$ for all n. Thus, $|\nabla v_n|^q \to |\nabla v|^q$ in $L^1(\Omega_T)$. Therefore, it is easy to see that we get $u_n \to S(v)$ in $L^1(0,T,W_0^{1,1}(\Omega))$ by Theorem 3.6. On the other hand, S **is pre-compact**. Therefore, by Schauder Fixed Point Theorem, S has a fixed point on $\mathbf{E}_\Lambda$. Hence the problem (2.53) has a solution with data $\mu = \mu_{n_0}, \sigma = \sigma_{n_0}$.
Thanks to Corollary 4.39 and Remark 4.40 we get

$$[\omega_n]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}} \le c_5 [\omega]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}} \quad \forall \quad n \in \mathbb{N}, \tag{9.8}$$

where $c_5 = c_5(N, q+\varepsilon, T_0)$.
Assume that $[\omega]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}} \le c_4 c_5^{-1}$. So $[\omega_n]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}} \le c_4$ for all n.
Therefore, for any $n \in \mathbb{N}$, there exists a renormalized solution u_n of problem (9.3) which satisfies

$$[|\nabla u_n|^{q+\varepsilon}]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}} \le 2^{q+\varepsilon} c_2 [\omega_n]^{q+\varepsilon}_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}} \le 2^{q+\varepsilon} c_2 c_5^{q+\varepsilon} [\omega]^{q+\varepsilon}_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}}.$$

By Proposition 3.5, there exists a subsequence of $\{u_n\}$ converging to u in $L^1(0,T,W_0^{1,1}(\Omega))$. So, $[|\nabla u|^{q+\varepsilon}]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}(\Omega_T)} \le 2^{q+\varepsilon} c_2 c_5^{q+\varepsilon} [\omega]^{q+\varepsilon}_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}(\Omega_T)}$ and $|\nabla u_n|^q \to |\nabla u|^q$ in $L^1(\Omega)$ since $\{|\nabla u_n|^q\}$ is equi-integrable. It follows the result by Proposition 3.5 and Theorem 3.6.

Case c. For $n_0 \in \mathbb{N}$. We prove that the problem (2.53) has a solution with data $\mu = \mu_{n_0}, \sigma = \sigma_{n_0}$. Now we put

$$\mathbf{E}_\Lambda = \{u \in L^1(0,T,W_0^{1,1}(\Omega)) : |||\nabla u|||_{L^{(N+2)(q-1),\infty}(\Omega_T)} \le \Lambda\},$$

where $L^{(N+2)(q-1),\infty}(\Omega_T)$ is Lorent space with norm

$$||f||_{L^{(N+2)(q-1),\infty}(\Omega_T)} := \sup_{0<|D|<\infty} \left(|D|^{-1+\frac{1}{(N+2)(q-1)}} \int_{D\cap\Omega_T} |f| dxdt \right).$$

By Fatou's lemma, $\mathbf{E}_\Lambda$ is closed under the strong topology of $L^1(0,T,W_0^{1,1}(\Omega))$ and convex. We consider a map $S : \mathbf{E}_\Lambda \to \mathbf{E}_\Lambda$ defined for each $v \in \mathbf{E}_\Lambda$ by $S(v) = u$, where $u \in L^1(0,T,W_0^{1,1}(\Omega))$ is the unique solution of problem (9.1). By Theorem 2.20, there exist $\delta = \delta(N,\Lambda_1,\Lambda_2,q) \in (0,1)$ and $s_0 = s_0(N,\Lambda_1,\Lambda_2) > 0$ such that Ω is (δ, R_0)- Reifenberg flat domain and $[\mathcal{A}]_{s_0}^{R_0} \le \delta$ for some R_0 we have

$$\begin{aligned} |||\nabla u|||_{L^{(N+2)(q-1),\infty}(\Omega_T)} &\le c_6 ||\mathbb{M}_1[|\nabla v|^q + \omega_{n_0}]||_{L^{(N+2)(q-1),\infty}(\Omega_T)} \\ &\le c_6 \left(||\mathbb{M}_1[|\nabla v|^q]||_{L^{(N+2)(q-1),\infty}(\Omega_T)} + ||\mathbb{M}_1[\omega_{n_0}]||_{L^{(N+2)(q-1),\infty}(\Omega_T)} \right), \end{aligned}$$

where $c_6 = c_6(N,\Lambda_1,\Lambda_2,q,T_0/R_0)$ and $T_0 = \mathrm{diam}(\Omega) + T^{1/2}$.
By Proposition 4.4 we have

$$\begin{aligned} ||\mathbb{M}_1[|f|^q]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{n+1})} &\le c_7 ||\mathbb{I}_1[|f|^q]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{n+1})} \\ &\le c_8 ||f||^q_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{n+1})} \quad \forall f \in L^{(N+2)(q-1),\infty}(\mathbb{R}^{n+1}), \end{aligned}$$

where a constant c_8 only depends on N, q. Thus,

$$\begin{aligned} |||\nabla u|||_{L^{(N+2)(q-1),\infty}(\Omega_T)} &\le c_6 \left(c_8 |||\nabla v|||^q_{L^{(N+2)(q-1),\infty}(\Omega_T)} + ||\mathbb{M}_1[\omega_{n_0}]||_{L^{(N+2)(q-1),\infty}(\Omega_T)} \right) \\ &\le c_6 \left(c_8 \Lambda^q + ||\mathbb{M}_1[\omega_{n_0}]||_{L^{(N+2)(q-1),\infty}(\Omega_T)} \right), \end{aligned}$$

which implies that S is well defined with $||\mathbb{M}_1[\omega_{n_0}]||_{L^{(N+2)(q-1),\infty}(\Omega_T)} \leq c_9 := (2c_6)^{-q'} c_8^{-\frac{1}{q-1}}$ and $\Lambda := 2c_6||\mathbb{M}_1[\omega_{n_0}]||_{L^{(N+2)(q-1),\infty}(\Omega_T)}$.
As in **Case b** we can show that $S : \mathbf{E}_\Lambda \to \mathbf{E}_\Lambda$ is continuous and $S(\mathbf{E}_\Lambda)$ is pre-compact, thus by Schauder Fixed Point Theorem, S has a fixed point on $\mathbf{E}_\Lambda$. Hence the problem (2.53) has a solution with data $\mu = \mu_{n_0}, \sigma = \sigma_{n_0}$.
To continue, we need to show that

$$||\mathbb{M}_1[\omega_n]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})} \leq c_{10}||\mathbb{I}_1[|\mu|]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})} + c_{10}||\mathbf{I}_{\frac{2}{(N+2)(q-1)}-1}[|\sigma|]||_{L^{(N+2)(q-1)}(\mathbb{R}^N)}, \quad (9.9)$$

for every $n \geq k_0$. Where k_0 is a constant large enough and $c_{10} = c_{10}(N, q)$ Indeed, we have $\mathbb{M}_1[\omega_n] \leq c_{11}\mathbb{I}_1[\varphi_n * |\mu|] + c_{11}\mathbb{I}_1[(\varphi_{1,n} * |\sigma|) \otimes \delta_{\{t=0\}}]$. Thus, by Proposition 4.19 we deduce

$$\begin{aligned}
&||\mathbb{M}_1[\omega_n]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})} \\
&\leq c_{11}||\mathbb{I}_1[\varphi_n * |\mu|]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})} + c_{12}||\mathbf{I}_{\frac{2}{(N+2)(q-1)}-1}[\varphi_{1,n} * |\sigma|]||_{L^{(N+2)(q-1)}(\mathbb{R}^N)} \\
&= c_{11}||\varphi_n * \mathbb{I}_1[|\mu|]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})} + c_{12}||\varphi_{1,n} * \mathbf{I}_{\frac{2}{(N+2)(q-1)}-1}[|\sigma|]||_{L^{(N+2)(q-1)}(\mathbb{R}^N)} \\
&\to c_{11}||\mathbb{I}_1[|\mu|]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})} + c_{12}||\mathbf{I}_{\frac{2}{(N+2)(q-1)}-1}[|\sigma|]||_{L^{(N+2)(q-1)}(\mathbb{R}^N)} \quad \text{as } n \to \infty.
\end{aligned}$$

It implies (9.9).
Now we assume that

$$||\mathbb{I}_1[|\mu|]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})}, ||\mathbf{I}_{\frac{2}{(N+2)(q-1)}-1}[|\sigma|]||_{L^{(N+2)(q-1)}(\mathbb{R}^N)} \leq c_9(2c_{10})^{-1},$$

then $||\mathbb{M}_1[\omega_n]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})} \leq c_9$ for all $n \geq k_0$. Consequently, there exists a renormalized solution u_n of problem (9.3) satisfied

$$\begin{aligned}
&|||\nabla u_n|||_{L^{(N+2)(q-1),\infty}(\Omega_T)} \leq 2c_6||\mathbb{M}_1[\omega_n]||_{L^{(N+2)(q-1),\infty}(\Omega_T)} \\
&\leq 2c_6c_{10}||\mathbb{I}_1[|\mu|]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})} + 2c_6c_{10}||\mathbf{I}_{\frac{2}{(N+2)(q-1)}-1}[|\sigma|]||_{L^{(N+2)(q-1)}(\mathbb{R}^N)} =: C
\end{aligned}$$

for any $n \geq k_0$. Thanks to Proposition 3.5, there exists a subsequence of $\{u_n\}$ converging to u in $L^1(0, T, W_0^{1,1}(\Omega))$. So, $|||\nabla u|||_{L^{(N+2)(q-1),\infty}(\Omega_T)} \leq C$ and $|\nabla u_n|^q \to |\nabla u|^q$ in $L^1(\Omega)$ since $\{|\nabla u_n|^q\}$ is equi-integrable.
It follows the result by Proposition 3.5 and Theorem 3.6. This completes the proof. ∎

Proof of Theorem 2.33. Let $\{\mu_n\} \subset C_c^\infty(\Omega_T), \sigma_n \in C_c^\infty(\Omega)$ be as in the proof of Theorem 2.1. We have $|\mu_n| \leq \varphi_n * |\mu|, |\sigma_n| \leq \varphi_{1,n} * |\sigma|$ for any $n \in \mathbb{N}$, $\{\varphi_n\}$, $\{\varphi_{1,n}\}$ are sequences of standard mollifiers in $\mathbb{R}^{N+1}, \mathbb{R}^N$ respectively. We can assume that $\text{supp}(\mu_n) \subset (\Omega' + B_{d/4}(0)) \times [0, T]$ and $\text{supp}(\sigma_n) \subset \Omega' + B_{d/4}(0)$ for any $n \in \mathbb{N}$. Set $\omega_n = |\mu_n| + |\sigma_n| \otimes \delta_{\{t=0\}}$ and $\omega = |\mu| + |\sigma| \otimes \delta_{\{t=0\}}$.
First, we prove that the problem (2.53) has a solution with data $\mu = \mu_{n_0}, \sigma = \sigma_{n_0}$ for $n_0 \in \mathbb{N}$. By Corollary 4.39 and Remark 4.40, we have

$$[\omega_n]_{\mathfrak{M}^{\mathcal{G}_1, q'}} \leq c_1\varepsilon_0 \ \forall n \in \mathbb{N}, \quad (9.10)$$

where $c_1 = c_1(N, q, T_0)$ and $\varepsilon_0 = [\omega]_{\mathfrak{M}^{\mathcal{G}_1, q'}}$. By Proposition 4.36 and Remark 4.37, we have

$$\mathbb{I}_1^{2T_0,\delta}\left[\left(\mathbb{I}_1^{2T_0,\delta}[\omega_n]\right)^q\right] \leq c_2\varepsilon_0^{q-1}\mathbb{I}_1^{2T_0,\delta}[\omega_n] \quad \text{a.e in } \mathbb{R}^{N+1} \quad \text{and} \quad (9.11)$$

$$\mathbb{I}_2\left[\left(\mathbb{I}_1^{2T_0,\delta}[\omega_n]\right)^q\right] \leq c_2\varepsilon_0^{q-1}\mathbb{I}_2[\omega_n] \quad \text{a.e in } \mathbb{R}^{N+1}, \quad (9.12)$$

for any $n \in \mathbb{N}$, where $c_2 = c_2(N, \delta, q, T_0)$ and $0 < \delta < 1$. We set

$$\mathbf{E}_\Lambda = \{u \in L^1(0, T, W_0^{1,1}(\Omega)) : |\nabla u| \leq \Lambda\mathbb{I}_1^{2T_0,\delta}[\omega_{n_0}]\}.$$

Clearly, $\mathbf{E}_\Lambda$ is closed under the strong topology of $L^1(0,T,W_0^{1,1}(\Omega))$ and convex.
We consider a map $S:\mathbf{E}_\Lambda\to L^1(0,T,W_0^{1,1}(\Omega))$ defined for each $v\in\mathbf{E}_\Lambda$ by $S(v)=u$, where $u\in L^1(0,T,W_0^{1,1}(\Omega))$ is the unique renormalized solution of problem (9.1). We will show that $S(\mathbf{E}_\Lambda)$ is subset of $\mathbf{E}_\Lambda$ for some $\Lambda>0$ and ε_0 small enough.
We have

$$|\nabla v|\le \Lambda\mathbb{I}_1[\omega_{n_0}]. \tag{9.13}$$

In particular, $|||\nabla v|||_{L^\infty(\Omega_{d/2}\times(0,T))}\le \Lambda(N+1)^{-1}(d/2)^{-N-1}\omega_{n_0}(\overline{\Omega_T})$, where $\Omega_{d/2}=\{x\in\Omega:d(x,\partial\Omega)\le d/2\}$.
From (9.11) and (9.12) lead to

$$\mathbb{I}_1^{2T_0,\delta}[|\nabla v|^q]\le\Lambda^q\mathbb{I}_1^{2T_0,\delta}\left[\left(\mathbb{I}_1^{2T_0,\delta}[\omega_{n_0}]\right)^q\right]\le c_2\Lambda^q\varepsilon_0^{q-1}\mathbb{I}_1^{2T_0,\delta}[\omega_{n_0}]\ \text{ and}$$
$$\mathbb{I}_2[|\nabla v|^q]\le\Lambda^q\mathbb{I}_2\left[\left(\mathbb{I}_1^{2T_0,\delta}[\omega_{n_0}]\right)^q\right]\le c_2\Lambda^q\varepsilon_0^{q-1}\mathbb{I}_2[\omega_{n_0}].$$

Clearly, from [27, Theorem 1.2], we have for any $Q_r(x,t)\subset\subset\Omega\times(-\infty,T)$ with $r\le r_0$

$$\begin{aligned}|\nabla u(x,t)|&\le c_3\fint_{Q_r(x,t)}|\nabla u|dyds+c_3\mathbb{I}_1^{2T_0,\delta}[|\nabla v|^q+\omega_{n_0}](x,t)\\&\le c_3\fint_{Q_r(x,t)}|\nabla u|dyds+c_3\mathbb{I}_1^{2T_0,\delta}[|\nabla v|^q](x,t)+c_3\mathbb{I}_1^{2T_0,\delta}[\omega_{n_0}](x,t)\\&\le c_3\fint_{Q_r(x,t)}|\nabla u|dyds+c_3\left(c_2\Lambda^q\varepsilon_0^{q-1}+1\right)\mathbb{I}_1^{2T_0,\delta}[\omega_{n_0}](x,t),\end{aligned}\tag{9.14}$$

where $c_3=c_3(N,\Lambda_1)$ and $r_0=r_0(N,\Lambda_1,\Lambda_2,\Lambda_3,\beta)>0$.
Since $|||\nabla u|||_{L^1(\Omega_T)}\le c_4T_0\left(|||\nabla v|||^q_{L^q(\Omega_T)}+\omega_{n_0}(\overline{\Omega_T})\right)$, for any $(x,t)\in(\Omega\backslash\Omega_{d/4})\times(-\infty,T)$ where $\Omega_{d/4}=\{x\in\Omega:d(x,\partial\Omega)\le d/4\}$,

$$\begin{aligned}\frac{1}{|Q_{d_0}(x,t)|}\int_{Q_{d_0}(x,t)}|\nabla u|dyds&\le c_5d_0^{-N-2}T_0\left(|||\nabla v|||^q_{L^q(\Omega_T)}+\omega_{n_0}(\overline{\Omega_T})\right)\\&\le c_6\mathbb{I}_1^{2T_0,\delta}[|\nabla v|^q+\omega_{n_0}](x,t)\\&\le c_6\left(c_2\Lambda^q\varepsilon_0^{q-1}+1\right)\mathbb{I}_1^{2T_0,\delta}|[\omega_{n_0}](x,t),\end{aligned}\tag{9.15}$$

where $d_0=\min\{d/8,r_0\}$ and $c_6=c_6(N,p,\Lambda_1,\Lambda_2,T_0/d_0)$.
By regularity theory, we have

$$||\nabla u||_{L^\infty(\Omega_{d/4}\times(0,T))}\le c_7(||u||_{L^\infty(\Omega_{d/2}\times(0,T))}+|||\nabla v|^q||_{L^\infty(\Omega_{d/2}\times(0,T))}),$$

where $c_7=c_7(N,\Lambda_1,\Lambda_2,\Lambda_3,\Omega,T)$.
a. Estimate $|||\nabla v|^q||_{L^\infty(\Omega_{d/2}\times(0,T))}$. Thanks to (9.13),

$$|||\nabla v|^q||_{L^\infty(\Omega_{d/2}\times(0,T))}\le\left(\Lambda(d/2)^{-N-1}(\omega_{n_0}(\overline{\Omega_T}))\right)^q.$$

Since $\omega_{n_0}(\overline{\Omega_T})\le c_1\varepsilon_0\text{Cap}_{\mathcal{G}_1,q'}(\tilde{Q}_{T_0}(x_0,t_0))=c_8(N,q,p,T_0)\varepsilon_0$ with $(x_0,t_0)\in\Omega_T$, thus

$$|||\nabla v|^q||_{L^\infty(\Omega_{d/2}\times(0,T))}\le c_9\Lambda^q\varepsilon_0^{q-1}\mathbb{I}_1^{2T_0,\delta}[\omega_{n_0}](x,t)\ \ \forall(x,t)\in\Omega_T,$$

where $c_9=c_9(N,\Lambda_1,\Lambda_2,\Lambda_3,q,d,\Omega,T)$.
b. Estimate $||u||_{L^\infty(\Omega_{d/2})}$. By Theorem 2.1 we have

$$|u(x,t)|\le c_{10}\mathbb{I}_2[|\nabla v|^q+\omega_{n_0}](x,t)\ \ \forall(x,t)\in\Omega_T,$$

where $c_{10}=c_{10}(N,\Lambda_1,\Lambda_2)$. Thus,

$$\begin{aligned}|u(x,t)|&\le c_{10}\mathbb{I}_2[|\nabla v|^q](x,t)+c_{10}\mathbb{I}_2[\omega_{n_0}](x,t)\\&\le c_{10}\left(c_2\Lambda^q\varepsilon_0^{q-1}+1\right)\mathbb{I}_2[\omega_{n_0}](x,t),\end{aligned}$$

which implies

$$||u||_{L^\infty(\Omega_{d/2}\times(0,T))} \le c_{11}\left(c_2\Lambda^q\varepsilon_0^{q-1}+1\right)d^{-N}\omega_{n_0}(\overline{\Omega_T})$$
$$\le c_{12}\left(c_2\Lambda^q\varepsilon_0^{q-1}+1\right)\mathbb{I}_1^{2T_0,\delta}[\omega_{n_0}](x,t)\quad \forall (x,t)\in\Omega_T,$$

where $c_{12}=c_{12}(N,\Lambda_1,\Lambda_2,\Lambda_3,q,T_0/d)$. Therefore,

$$||\nabla u||_{L^\infty(\Omega_{d/4}\times(0,T))} \le c_{13}\left(c_{14}\Lambda^q\varepsilon_0^{q-1}+1\right)\inf_{(x,t)\in\Omega_T}\mathbb{I}_1^{2T_0,\delta}[\omega_{n_0}](x,t). \tag{9.16}$$

where $c_{13}=c_{13}(N,\Lambda_1,\Lambda_2,\Lambda_3,q,d,\Omega,T)$.
Finally from (9.15) (9.16) and (9.14) we get for all $(x,t)\in\Omega_T$

$$|\nabla u(x,t)| \le c_{14}\left(c_{15}\Lambda^q\varepsilon_0^{q-1}+1\right)\mathbb{I}_1^{2T_0,\delta}[\omega_{n_0}](x,t).$$

where $c_{14}=c_{14}(N,\Lambda_1,\Lambda_2,\Lambda_3,q,d,\Omega,T)$ and $c_{15}=c_{15}(N,\delta,q)$.
So, we suppose that $\Lambda=2c_{14}$ and $\varepsilon_0\le c_{15}^{-\frac{1}{q-1}}(2c_{14})^{-\frac{q}{q-1}}$, it is equivalent to (2.61), (2.62) holding for some $C>0$. Then for any $(x,t)\in\Omega_T$

$$|\nabla u(x,t)| \le \Lambda\mathbb{I}_1^{2T_0,\delta}[\omega_{n_0}](x,t),$$

and S is well defined.
On the other hand, we can see that $S:\mathbf{E}_\Lambda\to\mathbf{E}_\Lambda$ is continuous and $S(E)$ is pre-compact under the strong topology of $L^1(0,T,W_0^{1,1}(\Omega))$.
Thus, by Schauder Fixed Point Theorem, S has a fixed point on $\mathbf{E}_\Lambda$. This means: the problem (2.53) has a solution with data $\mu=\mu_{n_0},\sigma=\sigma_{n_0}$.
Therefore, for any $n\in\mathbb{N}$, there exists a renormalized solution u_n of problem (9.3) which satisfies

$$|\nabla u_n(x,t)| \le \Lambda\mathbb{I}_1^{2T_0,\delta}[\omega_n](x,t)\ \forall\ (x,t)\in\Omega_T.$$

Since $\mathbb{I}_1^{2T_0,\delta}[\omega_n](x,t)\le\varphi_n*\mathbb{I}_1^{2T_0,\delta}[|\mu|](x,t)+\varphi_{1,n}*(\mathbb{I}_1^{2T_0,\delta}[|\sigma|\otimes\delta_{\{t=0\}}](.,t))(x)=:A_n(x,t)$ and A_n converges to $\mathbb{I}_1^{2T_0,\delta}[|\mu|]+\mathbb{I}_1^{2T_0,\delta}[|\sigma|\otimes\delta_{\{t=0\}}]$ in $L^q(\mathbb{R}^{N+1})$, thus $|\nabla u_n|^q$ is equi-integrable. As in the proof of Theorem 2.32, we get the result by using Proposition 3.5 and Theorem 3.6. This completes the proof. ■

9.2 Quasilinear Riccati Type Parabolic Equation in $\mathbb{R}^N\times(0,\infty)$ and $\mathbb{R}^{N+1}$

In this subsection, we provide the proofs of Theorem 2.37 and 2.38. In the same way, we can prove Theorem 2.36.
Proof of Theorem 2.37. As in the proof of Theorem 2.25 and Theorem 2.27, we can apply Theorem 2.32 to obtain: there exists a constant $c_1=c_1(N,\Lambda_1,\Lambda_2,q)$ that if $[A]_{s_0}^\infty\le\delta$ and (2.64) holds with constant c_1 then we can find a sequence of renormalized solutions $\{u_{n_k}\}$ of

$$\begin{cases} (u_{n_k})_t - div(A(x,t,\nabla u_{n_k})) = |\nabla u_{n_k}|^q + \chi_{D_{n_k-1}}\omega \text{ in } D_{n_k},\\ u_{n_k}=0 \quad \text{on } \partial B_{n_k}(0)\times(-n_k^2,n_k^2),\\ u_{n_k}(-n_k^2)=0 \quad \text{on } B_{n_k}(0).\end{cases}$$

converging to some u in $L^1_{\text{loc}}(\mathbb{R};W^{1,1}_{\text{loc}}(\mathbb{R}^N))$ and satisfying

$$|||\nabla u_{n_k}|||_{L^{(q-1)(N+2),\infty}(D_{n_k})} \le c_2||\mathbb{I}_1[|\omega|]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})},$$

for some $c_2 = c_2(N,\Lambda_1,\Lambda_2,q)$, where $D_n = B_n(0)\times(-n^2,n^2)$. It follows $|\nabla u_{n_k}|^q \to |\nabla u|^q$ in $L^1_{\text{loc}}(\mathbb{R}^{N+1})$. Thus, u is a distribution solution of (2.55) which satisfies (2.63).
Furthermore, if $\omega = \mu + \sigma\otimes\delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}(\mathbb{R}^N\times(0,\infty))$ and $\sigma \in \mathfrak{M}(\mathbb{R}^N)$, then $u_{n_k} = 0$ in $B_{n_k}(0)\times(-n_k^2,0)$. So, $u=0$ in $\mathbb{R}^N\times(-\infty,0)$. Therefore, clearly $u|_{\mathbb{R}^N\times[0,\infty)}$ is a distribution solution to (2.54).

■

Proof of Theorem 2.38. Let $\omega_n = \varphi_n * (\chi_{D_{n-1}}\omega)$ for any $n\geq 2$. We have $\mu_n \in C_c^\infty(\mathbb{R}^{N+1})$ with $\text{supp}(\omega_n)\subset D_n$ and $\omega_n \to \omega$ weakly in $\mathfrak{M}(\mathbb{R}^{N+1})$.
According to Corollary 4.39 and Remark 4.40, we have

$$[\omega_n]_{\mathfrak{M}^{\mathcal{H}_1,q'}} \leq c_1\varepsilon_0 \quad \forall n\in\mathbb{N}$$

where $c_1 = c_1(N,q)$ and $[\omega]_{\mathfrak{M}^{\mathcal{H}_1,q'}} \leq \varepsilon_0$. Thus, thanks to Theorem 1.3 we get

$$\mathbb{I}_1\left[(\mathbb{I}_1[\omega_n])^q\right] \leq c_2\varepsilon_0^{q-1}\mathbb{I}_1[\omega_n] \quad \text{and} \tag{9.17}$$

$$\mathbb{I}_2\left[(\mathbb{I}_1[\omega_n])^q\right] \leq c_2\varepsilon_0^{q-1}\mathbb{I}_2[\omega_n] \quad \forall n\in\mathbb{N}, \tag{9.18}$$

where $c_2 = c_2(N,q,c_1)$.
We fix $n_0\in\mathbb{N}$, put:

$$\mathbf{E}_\Lambda = \left\{u\in L^1(-n_0^2,n_0^2,W_0^{1,1}(B_{n_0}(0))) : |\nabla u| \leq \Lambda\mathbb{I}_1[\omega_{n_0}] \ \text{ in } \ B_{n_0/4}(0)\times(-n_0^2,n_0^2)\right\}.$$

By using estimate (5.8) in Remark 5.3, we can apply the argument of the proof of Theorem 2.9, with problem (6.9) replaced by

$$\begin{cases} u_t - div\left(A(t,\nabla u)\right) = \chi_{B_{n_0/4}(0)\times(-n_0^2,n_0^2)}|\nabla v|^q + \omega_{n_0} \ \text{in} \ \ D_{n_0}, \\ u = 0 \quad \text{ on } \quad \partial B_{n_0}(0)\times(-n_0^2,n_0^2), \\ u(-n_0^2) = 0 \quad \text{in} \quad B_{n_0}(0), \end{cases}$$

to obtain: the operator S (in the proof of Theorem 2.9) has a fixed point on $\mathbf{E}_\Lambda$ for some $\Lambda = \Lambda(N,\Lambda_1,\Lambda_2,q) > 0$ and $\varepsilon_0 = \varepsilon_0(N,\Lambda_1,\Lambda_2,q) > 0$. Therefore, for any $n\in\mathbb{N}$ there exists a solution u_n of problem

$$\begin{cases} (u_n)_t - div\left(A(t,\nabla u_n)\right) = \chi_{B_{n/4}(0)\times(-n^2,n^2)}|\nabla u_n|^q + \omega_n \ \text{in} \ \ D_n, \\ u_n = 0 \quad \text{ on } \quad \partial B_n(0)\times(-n^2,n^2), \\ u_n(-n^2) = 0 \quad \text{in} \quad B_n(0), \end{cases}$$

which satisfies

$$|\nabla u_n(x,t)| \leq \Lambda\mathbb{I}_1[\omega_n](x,t) \ \ \forall (x,t)\in B_{n/4}(0)\times(-n^2,n^2).$$

Moreover, combining this with (9.18) and Theorem 2.1 we also obtain

$$\begin{aligned} |u_n(x,t)| &\leq K\mathbb{I}_2\left[\chi_{B_{n/4}(0)\times(-n^2,n^2)}|\nabla u_n|^q + |\omega_n|\right](x,t) \\ &\leq K\Lambda^q\mathbb{I}_2\left[(\mathbb{I}_1[|\omega_n|])^q\right] + K\mathbb{I}_2\left[|\omega_n|\right](x,t) \\ &\leq c_3\mathbb{I}_2\left[|\omega_n|\right](x,t) \\ &\leq c_3\varphi_n * \mathbb{I}_2\left[|\chi_{D_{n-1}}\omega|\right](x,t), \end{aligned}$$

for any $(x,t)\in B_n(0)\times(-n^2,n^2)$.
Since $\mathbb{I}_2[\omega](x_0,t_0) < \infty$ for some $(x_0,t_0)\in\mathbb{R}^{N+1}$, thus $\sup_n\int_{D_m}\chi_{D_n}|u_n|^{q_0}dxdt < \infty$ for all $m\in\mathbb{N}$, $1<q_0<\frac{N+2}{N}$.
In addition, since $\mathbb{I}_1[\omega]\in L^q_{\text{loc}}(\mathbb{R}^{N+1})$, thus $\varphi_n * \mathbb{I}_1\left[|\chi_{D_{n-1}}\omega|\right] \to \mathbb{I}_1[\omega]$ in $L^q_{\text{loc}}(\mathbb{R}^{N+1})$ and $\{\chi_{B_{n/4}(0)\times(-n^2,n^2)}|\nabla u_n|^q\}$ is equi local integrable in $\mathbb{R}^{N+1}$.
Therefore, we can apply Corollary 3.18 to obtain: $u_n \to u$ in $L^1_{\text{loc}}(\mathbb{R};W^{1,1}_{\text{loc}}(\mathbb{R}^N))$ (we will

take its subsequence if need) and u satisfies (2.66). Also, $|\nabla u_n|^q \to |\nabla u|^q$ in $L^1_{loc}(\mathbb{R}^{N+1})$. Finally, we can conclude that u is a distribution solution of problem (2.65). Note that the assumption $[\omega]_{\mathfrak{M}^{\mathcal{H}_{1,q'}}} \leq \varepsilon_0$ is equivalent to (2.67) holding with $C = \varepsilon_0$.
Furthermore, if $\omega = \mu + \sigma \otimes \delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}(\mathbb{R}^N \times (0,\infty))$ and $\sigma \in \mathfrak{M}(\mathbb{R}^N)$, then $u_n = 0$ in $B_n(0) \times (-n^2, a_n)$ where $\operatorname{supp}(\omega_n) \subset \mathbb{R}^N \times (a_n, \infty)$ and $a_n \to 0^-$ as $n \to \infty$. So, $u = 0$ in $\mathbb{R}^N \times (-\infty, 0)$. Therefore, clearly $u|_{\mathbb{R}^N \times [0,\infty)}$ is a distribution solution to (2.68).
This completes the proof of the Theorem. ■

10 Appendix

Proof of the Remark 2.7. For $\omega \in \mathfrak{M}^+(\mathbb{R}^{N+1})$, $0 < \alpha < N+2$ if $\mathbb{I}_\alpha[\omega](x_0,t_0) < \infty$ for some $(x_0,t_0) \in \mathbb{R}^{N+1}$ then for any $0 < \beta \leq \alpha$, $\mathbb{I}_\beta[\omega] \in L^s_{\text{loc}}(\mathbb{R}^{N+1})$ for any $0 < s < \frac{N+2}{N+2-\beta}$.
Indeed, by Remark 4.28 we have $\mathbb{I}_\alpha[\omega] \in L^s_{\text{loc}}(\mathbb{R}^{N+1})$ for any $0 < s < \frac{N+2}{N+2-\beta}$.
Take $0 < \beta \leq \alpha$ and $0 < s < \frac{N+2}{N+2-\beta}$. For $R > 0$, by Proposition 4.4 we have $\mathbb{I}_\beta[\chi_{\tilde{Q}_{2R}(0,0)}\omega] \in L^s_{\text{loc}}(\mathbb{R}^{N+1})$. Thus,

$$\begin{aligned}
&\int_{\tilde{Q}_R(0,0)} (\mathbb{I}_\beta[\omega](x,t))^s \, dxdt \\
&\leq c_1 \int_{\tilde{Q}_R(0,0)} \left(\mathbb{I}_\beta[\chi_{\tilde{Q}_{2R}(0,0)}\omega](x,t)\right)^s dxdt + c_1 \int_{\tilde{Q}_R(0,0)} \left(\mathbb{I}_\beta[\chi_{\tilde{Q}_{2R}(0,0)^c}\omega](x,t)\right)^s dxdt \\
&\leq c_1 \int_{\tilde{Q}_R(0,0)} \left(\mathbb{I}_\beta[\chi_{\tilde{Q}_{2R}(0,0)}\omega](x,t)\right)^s dxdt + c_1 R^{-s(\alpha-\beta)} \int_{\tilde{Q}_R(0,0)} (\mathbb{I}_\alpha[\omega](x,t))^s \, dxdt \\
&< \infty.
\end{aligned}$$

For $0 < \beta < \alpha < N+2$, we consider

$$\omega(x,t) = \sum_{k=4}^{\infty} \frac{a_k}{|\tilde{Q}_{k+1}(0,0) \backslash \tilde{Q}_k(0,0)|} \chi_{\tilde{Q}_{k+1}(0,0)\backslash \tilde{Q}_k(0,0)}(x,t),$$

where $a_k = 2^{n(N+2-\theta)}$ if $k = 2^n$ and $a_k = 0$ otherwise with $\theta \in (\beta, \alpha]$.
It is easy to see that $\mathbb{I}_\alpha[\omega] \equiv \infty$ and $\mathbb{I}_\beta[\omega] < \infty$ in $\mathbb{R}^{N+1}$. ■
Proof of the Remark 2.26. For $\omega \in \mathfrak{M}^+(\mathbb{R}^{N+1})$, since $\mathbb{I}_2[\omega] \leq c_1 I_1[I_1[\omega]]$ thus:
If $\mathbb{I}_1[\omega] \in L^{s,\infty}(\mathbb{R}^{N+1})$ with $1 < s < N+2$, then by Proposition 4.4 in next section

$$||\mathbb{I}_2[\omega]||_{L^{\frac{s(N+1)}{N+2-s},\infty}(\mathbb{R}^{N+1})} \leq c_1 ||\mathbb{I}_1[\omega]||_{L^{s,\infty}(\mathbb{R}^{N+1})} < \infty$$

If $\mathbb{I}_1[\omega] \in L^{N+2,\infty}(\mathbb{R}^{N+1})$, then by Theorem 4.3,

$$\mathbb{I}_2[\omega] \in L^{s_0}_{\text{loc}}(\mathbb{R}^{N+1}) \ \ \forall \ s_0 > 1$$

So, $\mathbb{I}_2[\omega] < \infty$ a.e in $\mathbb{R}^{N+1}$ if $\mathbb{I}_1[\omega] \in L^{s,\infty}(\mathbb{R}^{N+1})$ with $1 < s \leq N+2$.
For $s > N+2$, there exists $\omega \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ such that $\mathbb{I}_2[\omega] \equiv \infty$ in $\mathbb{R}^{N+1}$ and $\mathbb{I}_1[\omega] \in L^s(\mathbb{R}^{N+1})$. Indeed, consider

$$\omega(x,t) = \sum_{k=1}^{\infty} \frac{k^{N-1}}{|\tilde{Q}_{k+1}(0,0) \backslash \tilde{Q}_k(0,0)|} \chi_{\tilde{Q}_{k+1}(0,0)\backslash \tilde{Q}_k(0,0)}(x,t).$$

We have for $(x,t)\in\mathbb{R}^{N+1}$ and $n_0\in\mathbb{N}$ with $n_0>\log_2(\max\{|x|,\sqrt{2|t|}\})$

$$\begin{aligned}\mathbb{I}_2[\omega](x,t) &\geq c_2\sum_{n_0}^{\infty}\frac{\omega(\tilde{Q}_{2^n}(x,t))}{2^{nN}} \geq c_2\sum_{n_0}^{\infty}\frac{\omega(\tilde{Q}_{2^{n-1}}(0,0))}{2^{nN}}\\ &\geq c_2\sum_{n_0}^{\infty}\frac{\sum_{k=1}^{2^{n-1}-1}k^{N-1}}{2^{nN}} = c_2\sum_{k=1}^{\infty}\left(\sum_{n_0}^{\infty}\chi_{k\leq 2^{n-1}-1}\frac{1}{2^{nN}}\right)k^{N-1}\\ &\geq c_4\sum_{k=n_0}^{\infty}k^{-1}=\infty.\end{aligned}$$

On the other hand, for $s_1>\frac{N+2}{2}$

$$\int_{\mathbb{R}^{N+1}}\omega^{s_1}dxdt = c_5\sum_{k=1}^{\infty}\frac{k^{s(N-1)}}{((k+1)^{N+2}-k^{N+2})^{s_1-1}} \leq c_6\sum_{k=1}^{\infty}\frac{k^{s_1(N-1)}}{k^{(s_1-1)(N+1)}}<\infty,$$

since $(s_1-1)(N+1)-s_1(N-1)>1$. Thus,

$$||\mathbb{I}_1[\omega]||_{L^s(\mathbb{R}^{N+1})}\leq c_7||\omega||_{L^{\frac{s(N+2)}{N+2+s}}(\mathbb{R}^{N+1})}<\infty.$$

■

Proof of the Proposition 3.16. We will use an idea in [9, 10] to prove 3.14. For $S'\in W^{1,\infty}(\mathbb{R})$ with $S(0)=0$, $S''\geq 0$, $S'(\tau)\tau\geq 0$ for all $\tau\in\mathbb{R}$ and $||S'||_{L^\infty(\mathbb{R})}\leq 1$ we have

$$\begin{aligned}&-\int_D \eta_t S(u)dxdt+\int_D S'(u)A(x,t,\nabla u)\nabla\eta dxdt\\ &\quad+\int_D S''(u)\eta A(x,t,\nabla u)\nabla u dxdt+\int_D S'(u)\eta L(u)dxdt=\int_D S'(u)\eta d\mu.\end{aligned}$$

Thus,

$$\begin{aligned}&\Lambda_2\int_D S''(u)\eta|\nabla u|^2dxdt\\ &\quad+\int_D S'(u)\eta L(u)dxdt\leq\Lambda_1\int_D|\nabla u||\nabla\eta|dxdt+\int_D\eta d|\mu|+\int_D|\eta_t||u|dxdt.\end{aligned}$$

a. We choose $S'\equiv\varepsilon^{-1}T_\varepsilon$ for $\varepsilon>0$ and let $\varepsilon\to 0$ we will obtain

$$\int_D\eta|L(u)|dxdt\leq\Lambda_1\int_D|\nabla u||\nabla\eta|dxdt+\int_D\eta d|\mu|+\int_D|\eta_t||u|dxdt. \tag{10.1}$$

b. for $S'(u)=(1-(|u|+1)^{-\alpha})\mathrm{sign}(u)$ for $\alpha>0$ then

$$\int_D\frac{|\nabla u|^2}{(|u|+1)^{\alpha+1}}\eta dxdt\leq c_1\left(\int_D|\nabla u||\nabla\eta|dxdt+\int_D\eta d|\mu|+\int_D|\eta_t||u|dxdt\right),$$

Using Holder's inequality, we have

$$\int_D|\nabla u||\nabla\eta|dxdt\leq\frac{1}{2c_1}\int_D\frac{|\nabla u|^2}{(|u|+1)^{\alpha+1}}\eta dxdt+c_2\int_D(|u|+1)^{q_0}\eta dxdt+c_2\int_D|\nabla\eta^{1/q_1}|^{q_1}dxdt.$$

Hence,

$$\int_D|\nabla u||\nabla\eta|dxdt+\int_D\frac{|\nabla u|^2}{(|u|+1)^{\alpha+1}}\eta dxdt\leq c_3B. \tag{10.2}$$

c. for $S'(u)=\frac{-k+\delta+|u|}{2\delta}\text{sign}(u)\chi_{k-\delta<|u|<k+\delta}+\text{sign}(u)\chi_{|u|\geq k+\delta}$, $0<\delta\leq k$ then

$$\frac{1}{2\delta}\int_{k-\delta<|u|<k+\delta}|\nabla u|^2\eta dxdt\leq c_4\left(\int_D|\nabla u||\nabla\eta|dxdt+\int_D\eta d|\mu|+\int_D|\eta_t||u|dxdt\right). \quad (10.3)$$

In particular,

$$\frac{1}{k}\int_D|\nabla T_k(u)|^2\eta dxdt\leq c_5\left(\int_D|\nabla u||\nabla\eta|dxdt+\int_D\eta d|\mu|+\int_D|\eta_t||u|dxdt\right)\quad\forall k>0. \quad (10.4)$$

Consequently, we deduce (3.14) from (10.1)-(10.4).
Next, take $\varphi\in C_c^\infty(D)$ and $S'(u)=\chi_{|u|\leq k-\delta}+\frac{k+\delta-|u|}{2\delta}\chi_{k-\delta<|u|<k+\delta}$, $S(0)=0$ we have

$$\begin{aligned}-\int_D\varphi_t\eta S(u)dxdt&+\int_D S'(u)\eta A(x,t,\nabla u)\nabla\varphi dxdt+\int_D S'(u)\varphi A(x,t,\nabla u)\nabla\eta dxdt\\&-\frac{1}{2\delta}\int_{k-\delta<|u|<k+\delta}\text{sign}(u)\varphi\eta A(x,t,\nabla u)\nabla u dxdt+\int_D S'(u)\varphi\eta L(u)dxdt\\&=\int_D S'(u)\varphi\eta d\mu+\int_D\varphi\eta_t S(u)dxdt.\end{aligned}$$

Combining with (10.1), (10.2) and (10.3), we get

$$-\int_D\varphi_t\eta S(u)dxdt+\int_D S'(u)\eta A(x,t,\nabla u)\nabla\varphi dxdt\leq c_5||\varphi||_{L^\infty(D)}B.$$

Letting $\delta\to 0$, we get

$$-\int_D\varphi_t\eta T_k(u)dxdt+\int_D\eta A(x,t,\nabla T_k(u))\nabla\varphi dxdt\leq c_5||\varphi||_{L^\infty(D)}B.$$

By density, we can take $\varphi=T_\varepsilon(T_k(u)-\langle T_k(w)\rangle_\nu)$,

$$\begin{aligned}-\int_D\frac{\partial}{\partial t}\left(T_\varepsilon(T_k(u)-\langle T_k(w)\rangle_\nu)\right)\eta T_k(u)dxdt&\\+\int_D\eta A(x,t,\nabla T_k(u))\nabla T_\varepsilon(T_k(u)-\langle T_k(w)\rangle_\nu)dxdt&\leq c_5\varepsilon B.\end{aligned}$$

Using integration by part, we have

$$\begin{aligned}-\int_D\frac{\partial}{\partial t}&\left(T_\varepsilon(T_k(u)-\langle T_k(w)\rangle_\nu)\right)\eta T_k(u)dxdt\\&=\frac{1}{2}\int_D(T_\varepsilon(T_k(u)-\langle T_k(w)\rangle_\nu))^2\eta_t dxdt\\&\quad+\int_D T_\varepsilon(T_k(u)-\langle T_k(w)\rangle_\nu)\langle T_k(w)\rangle_\nu\eta_t dxdt\\&\quad+\nu\int_D\eta(T_k(w)-\langle T_k(w)\rangle_\nu)T_\varepsilon(T_k(u)-\langle T_k(w)\rangle_\nu)dxdt.\end{aligned}$$

Thus,

$$\begin{aligned}-\int_D\frac{\partial}{\partial t}&\left(T_\varepsilon(T_k(u)-\langle T_k(w)\rangle_\nu)\right)\eta T_k(u)dxdt\\&\geq-\varepsilon(1+k)||\eta_t||_{L^1(D)}+\nu\int_D\eta\left(T_k(w)-\langle T_k(w)\rangle_\nu\right)T_\varepsilon(T_k(u)-\langle T_k(w)\rangle_\nu)dxdt,\end{aligned}$$

which follows (3.15). ■

Proof of the proposition 3.17. Let $S_k \in W^{2,\infty}(\mathbb{R})$ such that $S_k(z) = z$ if $|z| \le k$ and $S_k(z) = sign(z)2k$ if $|z| > 2k$. For $m \in \mathbb{N}$, let η_m be the cut off function on D_m with respect to D_{m+1}. It is easy to see that from the assumption and Remark 3.4, Proposition 3.15 we get $U_{m,n} = \eta_m S_k(v_n)$, $v_n = u_n - h_n$

$$\sup_{n\ge m+1}\Big(||\left(U_{m,n}\right)_t||_{L^2(-m^2,m^2,H^{-1}(B_m(0)))+L^1(D_m)} + ||U_{m,n}||_{L^2(-m^2,m^2,H_0^1(B_m(0)))} \\ +||u_n||_{L^1(D_m)} + ||v_n||_{L^1(D_m)}\Big) \le M_m < \infty.$$

Thus, $\{U_{m,n}\}_{n\ge m+1}$ is relatively compact in $L^1(D_m)$. On the other hand, for any $n_1, n_2 \ge m+1$

$$|\{|v_{n_1} - v_{n_2}| > \lambda\} \cap D_m| = |\{|\eta_m v_{n_1} - \eta_m v_{n_2}| > \lambda\} \cap D_m| \\ \le \frac{1}{k}\left(||v_{n_1}||_{L^1(D_m)} + ||v_{n_2}||_{L^1(D_m)}\right) + \frac{1}{\lambda}||\eta_m S_k(v_{n_1}) - \eta_m S_k(v_{n_2})||_{L^1(D_m)} \\ \le \frac{2M_m}{k} + \frac{1}{\lambda}||U_{m,n_1} - U_{m,n_2}||_{L^1(D_m)},$$

and h_n is convergent in $L^1_{\text{loc}}(\mathbb{R}^{N+1})$. So, for any $m \in \mathbb{N}$ there is a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$ such that $\{u_n\}$ is a Cauchy sequence (in measure) in D_m. Therefore, there is a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$ such that $\{u_n\}$ converges to u a.e in $\mathbb{R}^{N+1}$ for some u. Clearly, $u \in L^1_{\text{loc}}(\mathbb{R}; W^{1,1}_{loc}(\mathbb{R}^N))$. Now, we prove that $\nabla u_n \to \nabla u$ a.e in $\mathbb{R}^{N+1}$. From (3.15) with $D = D_{m+2}$, $\eta = \eta_m$ and $T_k(w) = T_k(\eta_{m+1}u)$ we have

$$\nu \int_{D_{m+2}} \eta_m \left(T_k(\eta_{m+1}u) - \langle T_k(\eta_{m+1}u)\rangle_\nu\right) T_\varepsilon(T_k(u_n) - \langle T_k(\eta_{m+1}u)\rangle_\nu)dxdt \\ + \int_{D_{m+2}} \eta_m A(x,t,\nabla T_k(u_n))\nabla T_\varepsilon(T_k(u_n) - \langle T_k(\eta_{m+1}u)\rangle_\nu)dxdt \\ \le c_1\varepsilon(1+k)B(n,m) \quad \forall\ n \ge m+2, \tag{10.5}$$

where

$$B(n,m) = ||(\eta_m)_t(|u_n|+1)||_{L^1(D_{m+2})} \\ + \int_{D_{m+2}} (|u_n|+1)^{q_0}\eta dxdt + \int_{D_{m+2}} |\nabla \eta_m^{1/q_1}|^{q_1} dxdt + \int_{D_{m+2}} \eta_m d|\mu_n|,$$

with $q_1 < \frac{q_0-1}{2q_0}$. By the assumption, we verify that the right hand side of (10.5) is bounded by $c_2\varepsilon$, where c_2 does not depend on n.

Since $\{\eta_m T_k(u_n)\}_{n\ge m+2}$ is bounded in $L^2(-(m+2)^2,(m+2)^2; H_0^1(B_{m+2}(0)))$, thus there is a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$ such that

$$\lim_{n\to\infty} \int_{|T_k(u_n) - \langle T_k(\eta_{m+1}u)\rangle_\nu| \le \varepsilon} \eta_m A(x,t,\nabla T_k(u))\nabla\left(T_k(u_n) - T_k(u)\right)dxdt = 0.$$

Therefore, thanks to $u_n \to u$ a.e in D_{m+2} and $\langle T_k(\eta_{m+1}u)\rangle_\nu \to T_k(\eta_{m+1}u)$ in $L^2(-(m+2)^2,(m+2)^2; H_0^1(B_{m+2}(0)))$, we get

$$\limsup_{\nu\to\infty}\limsup_{n\to\infty} \int_{|T_k(u_n) - \langle T_k(\eta_{m+1}u)\rangle_\nu| \le \varepsilon} \eta_{1,m}\Phi_{n,k} dxdt \le c_2\varepsilon \ \forall\ \varepsilon \in (0,1),$$

where $\Phi_{n,k} = (A(x,t,T_k(u_n)) - A(x,t,T_k(u)))\nabla(T_k(u_n) - T_k(u))$. Using Holder inequality,

$$\begin{aligned}
\int_{D_{m+2}} \eta_m \Phi_{k,n}^{1/2} dxdt &= \int_{D_{m+2}} \eta_m \Phi_{k,n}^{1/2} \chi_{|T_k(u_n) - \langle T_k(\eta_{m+1}u)\rangle_\nu| \le \varepsilon} dxdt \\
&+ \int_{D_{m+2}} \eta_m \Phi_{k,n}^{1/2} \chi_{|T_k(u_n) - \langle T_k(\eta_{m+1}u)\rangle_\nu| > \varepsilon} dxdt \\
&\le ||\eta_{1,m}||_{L^1(D_{m+2})}^{1/2} \left(\int_{|T_k(u_n) - \langle T_k(\eta_{m+1}u)\rangle_\nu| \le \varepsilon} \eta_m \Phi_{n,k} dxdt \right)^{1/2} \\
&+ |\{|T_k(u_n) - \langle T_k(\eta_{m+1}u)\rangle_\nu| > \varepsilon\} \cap D_{m+1}|^{1/2} \left(\int_{D_{m+2}} \eta_m^2 \Phi_{k,n} dxdt \right)^{1/2} \\
&= A_{n,\nu,\varepsilon}.
\end{aligned}$$

Clearly, $\limsup_{\varepsilon\to 0} \limsup_{\nu\to\infty} \limsup_{n\to\infty} A_{n,\nu,\varepsilon} = 0$. It follows

$$\limsup_{n\to\infty} \int_{D_{m+2}} \eta_m \Phi_{k,n}^{1/2} dxdt = 0.$$

Since $\Phi_{n,k} \ge \Lambda_2 |\nabla T_k(u_n) - \nabla T_k(u)|^2$, thus $\nabla T_k(u_n) \to \nabla T_k(u)$ in $L^1(D_m)$.
Note that

$$\begin{aligned}
|\{|\nabla u_{n_1} - \nabla u_{n_2}| > \lambda\} \cap D_m| &\le \frac{1}{k}\left(||u_{n_1}||_{L^1(D_m)} + ||u_{n_2}||_{L^1(D_m)}\right) \\
&+ \frac{1}{\lambda}|||\nabla T_k(u_{n_1}) - \nabla T_k(u_{n_2})||_{L^1(D_m)} \\
&\le \frac{2M_m}{k} + \frac{1}{\lambda}|||\nabla T_k(u_{n_1}) - \nabla T_k(u_{n_2})|||_{L^1(D_m)}.
\end{aligned}$$

Thus, we can show that there is a subsequence of $\{\nabla u_n\}$ still denoted by $\{\nabla u_n\}$ converging ∇u a.e in $\mathbb{R}^{N+1}$. ■

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