# Wiener criteria for existence of large solutions of quasilinear elliptic equations with absorption

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# Wiener criteria for existence of large solutions of quasilinear elliptic equations with absorption

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**Abstract**

We obtain sufficient conditions, expressed in terms of Wiener type tests involving Hausdorff or Bessel capacities, for the existence of large solutions to equations (1) $-\Delta_p u + e^u - 1 = 0$ or (2) $-\Delta_p u + u^q = 0$ in a bounded domain $\Omega$ when $q > p-1 > 0$. We apply our results to equations (3) $-\Delta_p u + a|\nabla u|^q + bu^s = 0$, (4) $\Delta_p u + u^{-\gamma} = 0$ with $1 < p \leq 2$, $1 \leq q \leq p$, $a > 0, b > 0$ and $q > p-1$, $s \geq p-1$, $\gamma > 0$.

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*Key words:* quasilinear elliptic equations, Wolff potential, maximal functions, Hausdorff capacities, Bessel capacities.

# 1 Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) and $1 < p \leq N$. We denote $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, $\rho(x) = \mathrm{dist}(x, \partial\Omega)$. In this paper we study some questions relative to the existence of solutions to the problem

$$\begin{array}{ll} -\Delta_p u + g(u) = 0 & \text{in } \Omega \\ \displaystyle\lim_{\rho(x)\to 0} u(x) = \infty & \end{array} \tag{1.1}$$

where $g$ is a continuous nondecreasing function vanishing at 0, and most often $g(u)$ is either $\mathrm{sign}(u)(e^{|u|} - 1)$ or $|u|^{q-1}u$ with $q > p-1$. A solution to problem (1.1) is called a *large solution.* When the domain is regular in the sense that the Dirichlet problem with continuous boundary data $\phi$

$$\begin{array}{ll} -\Delta_p u + g(u) = 0 & \text{in } \Omega, \\ u - \phi \in W_0^{1,p}(\Omega), u \in W_{\mathrm{loc}}^{1,p}(\Omega) \cap L^\infty(\Omega), & \end{array} \tag{1.2}$$

admits a solution $u \in C(\overline{\Omega})$, it is clear that problem (1.1) admits a solution provided problem $-\Delta_p u + g(u) = 0$ in $\Omega$ having a maximal solution, see [14, Chapter 5]. It is known that a

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necessary and sufficient condition for the solvability of problem (1.2) in case $g(u) \equiv 0$ is the *Wiener criterion*, due to Wiener [22] when $p = 2$ and Maz'ya [15], Kilpelainen and Malý [7] when $p \neq 2$, in general case is proved by Malý and Ziemer [12]. This condition is

$$\int_0^1 \left( \frac{C_{1,p}(B_t(x) \cap \Omega^c)}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} = \infty \qquad \forall x \in \partial\Omega, \tag{1.3}$$

where $C_{1,p}$ denotes the capacity associated to the space $W^{1,p}(\mathbb{R}^N)$. The existence of a maximal solution is guaranteed for a large class of nondecreasing nonlinearities $g$ satisfying the Vazquez condition [19]

$$\int_a^\infty \frac{dt}{\sqrt[p]{G(t)}} < \infty \quad \text{where } \ G(t) = \int_0^t g(s)ds \tag{1.4}$$

for some $a > 0$. This is an extension of the Keller-Osserman condition [8], [16], which is the above relation when $p = 2$. If for $R > diam(\Omega)$ there exists a function $v$ which satisfies

$$\begin{aligned} -\Delta_p v + g(v) = 0 \qquad & \text{in } B_R \setminus \{0\}, \\ v = 0 \qquad & \text{on } \partial B_R, \\ \lim_{x \to 0} v(x) = \infty, & \end{aligned} \tag{1.5}$$

then it is easy to see that the maximal solution $u$ of

$$-\Delta_p u + g(u) = 0 \qquad \text{in } \Omega \tag{1.6}$$

is a large solution, without any assumption on the regularity of $\partial\Omega$. Indeed, $x \mapsto v(x - y)$ is a solution of (1.6) in $\Omega$ for all $y \in \partial\Omega$, thus $u(x) \geq v(x - y)$ for any $x \in \Omega, y \in \partial\Omega$. It follows $\lim_{\rho(x)\to 0} u(x) = \infty$ since $\lim_{z\to 0} v(z) = \infty$.

Remark that the existence of a (radial) solution to problem (1.5) needs the fact that equation (1.6) admits solutions with isolated singularities, which is usually not true if the growth of $g$ is too strong since Vazquez and Véron prove in [20] that if

$$\liminf_{|r|\to\infty} |r|^{-\frac{N(p-1)}{N-p}} sign(r) g(r) > 0 \quad \text{with } \ p < N, \tag{1.7}$$

isolated singularities of solutions of (1.6) are removable. Conversely, if $p - 1 < q < \frac{N(p-1)}{N-p}$ with $p < N$, Friedman and Véron [5] characterize the behavior of positive singular solutions to

$$-\Delta_p u + u^q = 0 \tag{1.8}$$

with an isolated singularities. In 2003, Labutin [9] show that a necessary and sufficient condition in order the following problem be solvable

$$\begin{aligned} -\Delta u + |u|^{q-1} u = 0 \qquad & \text{in } \Omega, \\ \lim_{\rho(x)\to 0} u(x) = \infty, & \end{aligned}$$

is that

$$\int_0^1 \frac{C_{2,q'}(B_t(x) \cap \Omega^c)}{t^{N-2}} \frac{dt}{t} = \infty \qquad \forall x \in \partial\Omega,$$

where $C_{2,q'}$ is the capacity associated to the Sobolev space $W^{2,q'}(\mathbb{R}^N)$ and $q' = q/(q-1)$, $N \geq 3$. Notice that this condition is always satisfied if $q$ is subcritical, i.e. $q < N/(N-2)$. We refer to [14] for other related results. Concerning the exponential case of problem (1.1) nothing is known, even in the case $p = 2$, besides the simple cases already mentioned.

In this article we give sufficient conditions, expressed in terms of Wiener tests, in order problem (1.1) be solvable in the two cases $g(u) = \text{sign}(u)(e^{|u|} - 1)$ and $g(u) = |u|^{q-1} u$, $q > p-1$. For $1 < p \leq N$, we denote by $\mathcal{H}_1^{N-p}(E)$ the Hausdorff capacity of a set $E$ defined by

$$\mathcal{H}_1^{N-p}(E) = \inf \left\{ \sum_j h^{N-p}(B_j) : E \subset \bigcup B_j,\ diam(B_j) \leq 1 \right\}$$

where the $B_j$ are balls and $h^{N-p}(B_r) = r^{N-p}$. Our main result concerning the exponential case is the following

**Theorem 1**. *Let $N \geq 2$ and $1 < p \leq N$. If*

$$\int_0^1 \left( \frac{\mathcal{H}_1^{N-p}(\Omega^c \cap B_r(x))}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} = +\infty \quad \forall x \in \partial\Omega, \tag{1.9}$$

*then there exists $u \in C^1(\Omega)$ satisfying*

$$\begin{array}{ll} -\Delta_p u + e^u - 1 = 0 & \text{in } \Omega, \\ \lim\limits_{\rho(x)\to 0} u(x) = \infty. & \end{array} \tag{1.10}$$

Clearly, when $p = N$, we have $\mathcal{H}_1^{N-p}(\{x_0\}) = 1$ for all $x_0 \in \mathbb{R}^N$ thus, (1.9) is true for any open domain $\Omega$.

We also obtain a sufficient condition for the existence of a large solution in the power case expressed in terms of some $C_{\alpha,s}$ Bessel capacity in $\mathbb{R}^N$ associated to the Besov space $B^{\alpha,s}(\mathbb{R}^N)$.

**Theorem 2**. *Let $N \geq 2$, $1 < p < N$ and $q_1 > \frac{N(p-1)}{N-p}$. If*

$$\int_0^1 \left( \frac{C_{p,\frac{q_1}{q_1-p+1}}(\Omega^c \cap B_r(x))}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} = +\infty \quad \forall x \in \partial\Omega, \tag{1.11}$$

*then, for any $p-1 < q < \frac{pq_1}{N}$ there exists $u \in C^1(\Omega)$ satisfying*

$$\begin{array}{ll} -\Delta_p u + u^q = 0 & \text{in } \Omega, \\ \lim\limits_{\rho(x)\to 0} u(x) = \infty. & \end{array} \tag{1.12}$$

We can see that condition (1.9) implies (1.11). In view of Labutin's theorem this previous result is not optimal in the case $p = 2$, since the involved capacity is $C_{2,q_1'}$ with $q_1'$ and thus there exists a solution to

$$\begin{array}{ll} -\Delta_p u + u^{q_1} = 0 & \text{in } \Omega \\ \lim\limits_{\rho(x)\to 0} u(x) = \infty & \end{array}$$

with $q_1 > q$.

At end we apply the previous theorem to quasilinear viscous Hamilton-Jacobi equations:

$$\begin{aligned} -\Delta_p u + a\,|\nabla u|^q + b|u|^{s-1}u = 0 \qquad &\text{in } \Omega, \\ u \in C^1(\Omega),\ \lim_{\rho(x)\to 0} u(x) = \infty. & \end{aligned} \tag{1.13}$$

For $q_1 > p-1$ and $1 < p \le 2$, if equation (1.12) admits a solution with $q = q_1$, then for any $a > 0, b > 0$ and $q \in (p-1, \frac{pq_1}{q_1+1})$, $s \in [p-1, q_1)$ there exists a positive solution to (1.13). Conversely, if for some $a, b > 0$, $s > p-1$ there exists a solution to equation (1.13) with $1 < q = p \le 2$, then for any $q_1 > p-1$, $1 \le q_1 \le p$, $s_1 \ge p-1$, $a_1, b_1 > 0$ there exists a positive solution to equation (1.13) with parameters $q_1, s_1, a_1, b_1$ replacing $q, s, a, b$. Moreover, we also prove that the previous statement holds if for some $\gamma > 0$ there exists $u \in C(\overline{\Omega}) \cap C^1(\Omega)$, $u > 0$ in $\Omega$ satisfying

$$\begin{aligned} -\Delta_p u + u^{-\gamma} = 0 \ \ &\text{in } \Omega, \\ u = 0 \ \ &\text{on } \partial\Omega. \end{aligned}$$

We would like to remark that the case $p = 2$ was studied in [10]. In particular, if the boundary of $\Omega$ is smooth then (1.13) has a solution with $s = 1$ and $1 < q \le 2, a > 0, b > 0$.

## 2 Morrey classes and Wolff potential estimates

In this section we assume that $\Omega$ is a bounded open subset of $\mathbb{R}^N$ and $1 < p < N$. We also denote by $B_r(x)$ the open ball of center $x$ and radius $r$ and $B_r = B_r(0)$. We also recall that a solution of (1.1) belongs to $C^{1,\alpha}_{loc}(\Omega)$ for some $\alpha \in (0,1)$, and is more regular (depending on $g$) on the set $\{x \in \Omega : |\nabla u(x)| \neq 0\}$.

**Definition 2.1** *A function $f \in L^1(\Omega)$ belongs to the Morrey space $\mathcal{M}^s(\Omega)$, $1 \le s \le \infty$, if there is a constant $K$ such that*

$$\int_{\Omega\cap B_r(x)} |f|dy \le K r^{\frac{N}{s'}} \quad \forall r > 0, \forall x \in \mathbb{R}^N.$$

*The norm is defined as the smallest constant $K$ that satisfies this inequality; it is denoted by $||f||_{\mathcal{M}^s(\Omega)}$. Clearly $L^s(\Omega) \subset \mathcal{M}^s(\Omega)$.*

**Definition 2.2** *Let $R \in (0, \infty]$ and $\mu \in \mathfrak{M}^b_+(\Omega)$, the set of nonnegative and bounded Radon measures in $\Omega$. We define the (R-truncated) Wolff potential of $\mu$ by*

$$\mathbf{W}^R_{1,p}[\mu](x) = \int_0^R \left(\frac{\mu(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} \qquad \forall x \in \mathbb{R}^N,$$

*and the (R-truncated) fractional maximal potential of $\mu$ by*

$$\mathbf{M}_{p,R}[\mu](x) = \sup_{0<t<R} \frac{\mu(B_t(x))}{t^{N-p}} \qquad \forall x \in \mathbb{R}^N,$$

*where the measure is extended by 0 in $\Omega^c$.*

We recall a result proved in [6] (see also [2, Theorem 2.4]).

**Theorem 2.3** *Let $\mu$ be a nonnegative Radon measure in $\mathbb{R}^N$. There exist positive constants $C_1, C_2$ depending on $N, p$ such that*

$$\int_{2B} \exp(C_1 \mathbf{W}_{1,p}^R[\chi_B \mu])dx \le C_2 r^N,$$

*for all $B = B_r(x_0) \subset \mathbb{R}^N$, $2B = B_{2r}(x_0)$, $R > 0$ such that $||\mathbf{M}_{p,R}[\mu]||_{L^\infty(\mathbb{R}^N)} \le 1$.*

For $k \ge 0$, we set $T_k(u) = sign(u)\min\{k, |u|\}$.

**Definition 2.4** *Assume $f \in L^1_{loc}(\Omega)$. We say that a measurable function $u$ defined in $\Omega$ is a renormalized supersolution of*

$$-\Delta_p u + f = 0 \qquad \text{in } \Omega \tag{2.1}$$

*if, for any $k > 0$, $T_k(u) \in W^{1,p}_{loc}(\Omega)$, $|\nabla u|^{p-1} \in L^1_{loc}(\Omega)$ and there holds*

$$\int_\Omega (|\nabla T_k(u)|^{p-2} \nabla T_k(u) \nabla \varphi + f\varphi)dx \ge 0$$

*for all $\varphi \in W^{1,p}(\Omega)$ with compact support in $\Omega$ and such that $0 \le \varphi \le k - T_k(u)$, and if $-\Delta_p u + f$ is a positive distribution in $\Omega$.*

The following result is proved in [12, Theorem 4.35].

**Theorem 2.5** *If $f \in \mathcal{M}^{\frac{N}{p-\epsilon}}(\Omega)$ for some $\epsilon \in (0, p)$, $u$ is a nonnegative renormalized supersolution of (2.1) and set $\mu := -\Delta_p u + f$. Then there holds*

$$u(x) + ||f||^{\frac{1}{p-1}}_{\mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)} \ge C \mathbf{W}_{1,p}^{\frac{r}{4}}[\mu](x) \qquad \forall x \in \Omega \ \ s.t. \ B_r(x) \subset \Omega,$$

*for some $C$ depending only on $N, p, \varepsilon, diam(\Omega)$.*

Concerning renormalized solutions (see [3] for the definition) of

$$-\Delta_p u + f = \mu \qquad \text{in } \Omega, \tag{2.2}$$

where $f \in L^1(\Omega)$ and $\mu \in \mathfrak{M}^b_+(\Omega)$, we have

**Corollary 2.6** *Let $f \in \mathcal{M}^{\frac{N}{p-\epsilon}}(\Omega)$ and $\mu \in \mathfrak{M}^b_+(\Omega)$. If $u$ is a renormalized solution to (2.2) and $\inf_\Omega u > -\infty$ then there exists a positive constant $C$ depending only on $N, p, \varepsilon, diam(\Omega)$ such that*

$$u(x) + ||f||^{\frac{1}{p-1}}_{\mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)} \ge \inf_\Omega u + C \mathbf{W}_{1,p}^{\frac{d(x,\partial\Omega)}{4}}[\mu](x) \ \ \forall x \in \Omega.$$

The next result, proved in [2, Theorem 1.1, 1.2], is an important tool for the proof of Theorems **1** and **2**. Before presenting we introduce the notation.

**Definition 2.7** *Let $s > 1$ and $\alpha > 0$. We denote by $C_{\alpha,s}(E)$ the Bessel capacity of Borel set $E \subset \mathbb{R}^N$,*

$$C_{\alpha,s}(E) = \inf\{||\phi||^s_{L^s(\mathbb{R}^N)} : \phi \in L^s_+(\mathbb{R}^N), \quad G_\alpha * \phi \ge \chi_E\}$$

*where $\chi_E$ is the characteristic function of $E$ and $G_\alpha$ the Bessel kernel of order $\alpha$.*
*We say that a measure $\mu$ in $\Omega$ is absolutely continuous with respect to the capacity $C_{\alpha,s}$ in $\Omega$ if*

$$\text{for all } \ E \subset \Omega, E \text{ Borel}, C_{\alpha,s}(E) = 0 \Rightarrow |\mu|(E) = 0.$$

**Theorem 2.8** *Let $\mu \in \mathfrak{M}_+^b(\Omega)$ and $q > p-1$.*

**a.** *If $\mu$ is absolutely continuous with respect to the capacity $C_{p,\frac{q}{q+1-p}}$ in $\Omega$, then there exists a nonnegative renormalized solution $u$ to equation*

$$\begin{aligned} -\Delta_p u + u^q &= \mu \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

*which satisfies*

$$u(x) \le C\mathbf{W}_{1,p}^{2diam(\Omega)}[\mu](x) \quad \forall x \in \Omega \tag{2.3}$$

*where $C$ is a positive constant depending on $p$ and $N$.*

**b.** *If $\exp(C\mathbf{W}_{1,p}^{2diam(\Omega)}[\mu]) \in L^1(\Omega)$ where $C$ is the previous constant, then there exists a nonnegative renormalized solution $u$ to equation*

$$\begin{aligned} -\Delta_p u + e^u - 1 &= \mu \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

*which satisfies* (2.3).

# 3 Estimates from below

If $G$ is any domain in $\mathbb{R}^N$ with a compact boundary and $g$ is nondecreasing, $g(0) = g^{-1}(0) = 0$ and satisfies (1.7)) there always exists a maximal solution to (1.6) in $G$. It is constructed as the limit, when $n \to \infty$, of the solutions of

$$\begin{aligned} -\Delta_p u_n + g(u_n) &= 0 \qquad \text{in } G_n \\ \lim_{\rho_n(x)\to 0} u_n(x) &= \infty \\ \lim_{|x|\to\infty} u_n(x) &= 0 \qquad \text{if } G_n \text{ is unbounded,} \end{aligned} \tag{3.1}$$

where $\{G_n\}_n$ is a sequence of smooth domains such that $G_n \subset \overline{G}_n \subset G_{n+1}$ for all $n$, $\{\partial G_n\}_n$ is a bounded and $\bigcup_{n=1}^{\infty} G_n = G$ and $\rho_n(x) := \mathrm{dist}(x, \partial G_n)$. Our main estimates are the following.

**Theorem 3.1** *Let $K \subset B_{1/4}\backslash\{0\}$ be a compact set and let $U_j \in C^1(K^c)$, $j = 1,2$, be the maximal solutions of*

$$-\Delta_p u + e^u - 1 = 0 \qquad \text{in } K^c \tag{3.2}$$

*for $U_1$ and*

$$-\Delta_p u + u^q = 0 \qquad \text{in } K^c \tag{3.3}$$

*for $U_2$, where $p - 1 < q < \frac{pq_1}{N}$. Then there exist constants $C_k$, $k = 1,2,3,4$, depending on $N$, $p$ and $q$ such that*

$$U_1(0) \ge -C_1 + C_2 \int_0^1 \left( \frac{\mathcal{H}_1^{N-p}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}, \tag{3.4}$$

*and*

$$U_2(0) \ge -C_3 + C_4 \int_0^1 \left( \frac{C_{p,\frac{q_1}{q_1-p+1}}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}. \tag{3.5}$$

**Proof.** *1.* For $j \in \mathbb{Z}$ define $r_j = 2^{-j}$ and $S_j = \{x : r_j \le |x| \le r_{j-1}\}$, $B_j = B_{r_j}$. Fix a positive integer $J$ such that $K \subset \{x : r_J \le |x| < 1/8\}$. Consider the sets $K \cap S_j$ for $j = 3, ..., J$. By [18, Theorem 3.4.27], there exists $\mu_j \in \mathfrak{M}^+(\mathbb{R}^N)$ such that $\operatorname{supp}(\mu_j) \subset K \cap S_j$, $\|\mathbf{M}_{p,1}[\mu_j]\|_{L^\infty(\mathbb{R}^N)} \le 1$ and

$$c_1^{-1}\mathcal{H}_1^{N-p}(K \cap S_j) \le \mu_j(\mathbb{R}^N) \le c_1 \mathcal{H}_1^{N-p}(K \cap S_j) \quad \forall j,$$

for some $c_1 = c_1(N,p)$.
Now, we will show that for $\varepsilon = \varepsilon(N,p) > 0$ small enough, there holds,

$$A := \int_{B_1} \exp\left(\varepsilon \mathbf{W}_{1,p}^1\left[\sum_{k=3}^{J} \mu_k\right](x)\right) dx \le c_2, \tag{3.6}$$

where $c_2$ does not depend on $J$.
Indeed, define $\mu_j \equiv 0$ for all $j \ge J+1$ and $j \le 2$. We have

$$A = \sum_{j=1}^{\infty} \int_{S_j} \exp\left(\varepsilon \mathbf{W}_{1,p}^1\left[\sum_{k=3}^{J} \mu_k\right](x)\right) dx.$$

Since for any $j$

$$\mathbf{W}_{1,p}^1\left[\sum_{k=3}^{J} \mu_k\right] \le c(p)\mathbf{W}_{1,p}^1\left[\sum_{k \ge j+2} \mu_k\right] + c(p)\mathbf{W}_{1,p}^1\left[\sum_{k \le j-2} \mu_k\right] + c(p)\sum_{k=\max\{j-1,3\}}^{j+1} \mathbf{W}_{1,p}^1[\mu_k],$$

with $c(p) = \max\{1, 5^{\frac{2-p}{p-1}}\}$ and $\exp(\sum_{i=1}^{5} a_i) \le \sum_{i=1}^{5} \exp(5a_i)$ for all $a_i$. Thus,

$$\begin{aligned} A \le & \sum_{j=1}^{\infty} \int_{S_j} \exp\left(c_3\varepsilon \mathbf{W}_{1,p}^1\left[\sum_{k \ge j+2} \mu_k\right](x)\right) dx + \sum_{j=1}^{\infty} \int_{S_j} \exp\left(c_3\varepsilon \mathbf{W}_{1,p}^1\left[\sum_{k \le j-2} \mu_k\right](x)\right) dx \\ & + \sum_{j=1}^{\infty} \sum_{k=\max(j-1,3)}^{j+1} \int_{S_j} \exp\left(c_3\varepsilon \mathbf{W}_{1,p}^1[\mu_k](x)\right) dx := A_1 + A_2 + A_3, \text{ with } c_3 = 5c(p). \end{aligned}$$

*Estimate of* $A_3$: We apply Theorem 2.3 for $\mu = \mu_k$ and $B = B_{k-1}$,

$$\int_{2B_{k-1}} \exp\left(c_3\varepsilon \mathbf{W}_{1,p}^1[\mu_k](x)\right) dx \le c_4 r_{k-1}^N$$

with $c_3\varepsilon \in (0, C_1]$, the constant $C_1$ is in Theorem 2.3. In particular,

$$\int_{S_j} \exp\left(c_3\varepsilon \mathbf{W}_{1,p}^1[\mu_k](x)\right) dx \le c_4 r_{k-1}^N \quad \text{for } k = j-1, j, j+1,$$

which implies

$$A_3 \le c_5 \sum_{j=1}^{+\infty} r_j^N = c_5 < \infty. \tag{3.7}$$

*Estimate of* $A_1$: Since $\sum\limits_{k\geq j+2}\mu_k\left(B_t(x)\right)=0$ for all $x\in S_j, t\in(0,r_{j+1})$. Thus,

$$A_1=\sum_{j=1}^{\infty}\int_{S_j}\exp\left(c_3\varepsilon\int\limits_{r_{j+1}}^{1}\left(\frac{\sum\limits_{k\geq j+2}\mu_k(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}}\frac{dt}{t}\right)dx$$

$$\leq\sum_{j=1}^{\infty}\exp\left(c_3\varepsilon\frac{p-1}{N-p}\left(\sum_{k\geq j+2}\mu_k(S_k)\right)^{\frac{1}{p-1}}r_{j+1}^{-\frac{N-p}{p-1}}\right)|S_j|.$$

Note that $\mu_k(S_k)\leq\mu_k(B_{r_{k-1}}(0))\leq r_{k-1}^{N-p}$, which leads to

$$\left(\sum_{k\geq j+2}\mu_k(S_k)\right)^{\frac{1}{p-1}}r_{j+1}^{-\frac{N-p}{p-1}}\leq\left(\sum_{k\geq j+2}r_{k-1}^{N-p}\right)^{\frac{1}{p-1}}r_{j+1}^{-\frac{N-p}{p-1}}=\left(\sum_{k\geq 0}r_k^{N-p}\right)^{\frac{1}{p-1}}=\left(\frac{1}{1-2^{-(N-p)}}\right)^{\frac{1}{p-1}}.$$

Therefore

$$A_1\leq\exp\left(c_3\varepsilon\frac{p-1}{N-p}\left(\frac{1}{1-2^{-(N-p)}}\right)^{\frac{1}{p-1}}\right)|B_1|=c_6. \tag{3.8}$$

*Estimate of* $A_2$: for $x\in S_j$,

$$\mathbf{W}_{1,p}^{1}\left[\sum_{k\leq j-2}\mu_k\right](x)=\int\limits_{r_{j-1}}^{1}\left(\frac{\sum\limits_{k\leq j-2}\mu_k(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}}\frac{dt}{t}=\sum_{i=1}^{j-1}\int\limits_{r_i}^{r_{i-1}}\left(\frac{\sum\limits_{k\leq j-2}\mu_k(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}}\frac{dt}{t}.$$

Since $r_i<t<r_{i-1}$, $\sum\limits_{k\leq i-2}\mu_k(B_t(x))=0, \forall i=1,...,j-1$, thus

$$\mathbf{W}_{1,p}^{1}\left[\sum_{k\leq j-2}\mu_k\right](x)=\sum_{i=1}^{j-1}\int\limits_{r_i}^{r_{i-1}}\left(\frac{\sum\limits_{k=i-1}^{j-2}\mu_k(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}}\frac{dt}{t}\leq\sum_{i=1}^{j-1}\int\limits_{r_i}^{r_{i-1}}\left(\frac{\sum\limits_{k=i-1}^{j-2}\mu_k(S_k)}{t^{N-p}}\right)^{\frac{1}{p-1}}\frac{dt}{t}$$

$$\leq\sum_{i=1}^{j-1}\left(\sum_{k=i-1}^{j-2}r_{k-1}^{N-p}\right)^{\frac{1}{p-1}}r_i^{-\frac{N-p}{p-1}}\leq c_7 j,\ \text{with } c_7=\left(\frac{4^{N-p}}{1-2^{-(N-p)}}\right)^{\frac{1}{p-1}}.$$

Therefore,

$$A_2\leq\sum_{j=1}^{\infty}\int_{S_j}\exp\left(c_3c_7\varepsilon j\right)dx=\sum_{j=1}^{\infty}r_j^N\exp\left(c_3c_7\varepsilon j\right)|S_1|$$

$$=\sum_{j=1}^{\infty}\exp\left(\left(c_3c_7\varepsilon-N\log(2)\right)j\right)|S_1|\leq c_8\quad\text{for}\quad\varepsilon\leq N\log(2)/(2c_3c_7). \tag{3.9}$$

Consequently, from (3.8), (3.9) and (3.7), we obtain $A \le c_2 := c_6 + c_8 + c_5$ for $\varepsilon = \varepsilon(N,p)$ small enough. This implies

$$\left\| \exp\left(\frac{p}{2N}\varepsilon \mathbf{W}^1_{1,p}\left[\sum_{k=3}^{J}\mu_k\right]\right)\right\|_{\mathcal{M}^{\frac{2N}{p}}(B_1)} \le c_9\left(\int_{B_1}\exp\left(\varepsilon\mathbf{W}^1_{1,p}\left[\sum_{k=3}^{J}\mu_k\right](x)\right)dx\right)^{\frac{p}{2N}} \le c_{10}, \tag{3.10}$$

where the constant $c_{10}$ does not depend on $J$. Set $B = B_{\frac{1}{4}}$. For $\varepsilon_0 = (\frac{p\varepsilon}{2NC})^{1/(p-1)}$, where $C$ is the constant in (2.3), by Theorem 2.8 and estimate (3.10), there exists a nonnegative renormalized solution $u$ to equation

$$\begin{array}{ll} -\Delta_p u + e^u - 1 = \varepsilon_0 \sum_{j=3}^{J}\mu_j & \text{in } B, \\ u = 0 & \text{in } \partial B, \end{array}$$

satisfying (2.3) with $\mu = \varepsilon_0\sum_{j=3}^{J}\mu_j$. Thus, from Corollary 2.6 and estimate (3.10), we have

$$u(0) \ge -c_{11} + c_{12}\mathbf{W}^{\frac{1}{4}}_{1,p}\left[\sum_{j=3}^{J}\mu_j\right](0).$$

Therefore

$$\begin{aligned} u(0) &\ge -c_{11} + c_{12}\sum_{i=2}^{\infty}\int_{r_{i+1}}^{r_i}\left(\frac{\sum_{j=3}^{J}\mu_j(B_t(0))}{t^{N-p}}\right)^{\frac{1}{p-1}}\frac{dt}{t} \ge -c_{11} + c_{12}\sum_{i=2}^{J-2}\int_{r_{i+1}}^{r_i}\left(\frac{\mu_{i+2}(B_t(0))}{t^{N-p}}\right)^{\frac{1}{p-1}}\frac{dt}{t} \\ &= -c_{11} + c_{12}\sum_{i=2}^{J-2}\int_{r_{i+1}}^{r_i}\left(\frac{\mu_{i+2}(S_{i+2})}{t^{N-p}}\right)^{\frac{1}{p-1}}\frac{dt}{t} \ge -c_{11} + c_{13}\sum_{i=2}^{J-2}\left(\mathcal{H}_1^{N-p}(K\cap S_{i+2})\right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \\ &= -c_{11} + c_{13}\sum_{i=4}^{\infty}\left(\mathcal{H}_1^{N-p}(K\cap S_i)\right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}}. \end{aligned}$$

From the inequality

$$\left(\mathcal{H}_1^{N-p}(K\cap S_i)\right)^{\frac{1}{p-1}} \ge \frac{1}{\max(1,2^{\frac{2-p}{p-1}})}\left(\mathcal{H}_1^{N-p}(K\cap B_{i-1})\right)^{\frac{1}{p-1}} - \left(\mathcal{H}_1^{N-p}(K\cap B_i)\right)^{\frac{1}{p-1}} \quad \forall i,$$

we deduce that

$$\begin{aligned} u(0) &\ge -c_{11} + c_{13}\sum_{i=4}^{\infty}\left(\frac{1}{\max(1,2^{\frac{2-p}{p-1}})}\left(\mathcal{H}_1^{N-p}(K\cap B_{i-1})\right)^{\frac{1}{p-1}} - \left(\mathcal{H}_1^{N-p}(K\cap B_i)\right)^{\frac{1}{p-1}}\right) r_i^{-\frac{N-p}{p-1}} \\ &\ge -c_{11} + c_{13}\left(\frac{2^{\frac{N-p}{p-1}}}{\max(1,2^{\frac{2-p}{p-1}})} - 1\right)\sum_{i=4}^{\infty}\left(\mathcal{H}_1^{N-p}(K\cap B_i)\right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \\ &\ge -c_{14} + c_{15}\int_0^1\left(\frac{\mathcal{H}_1^{N-p}(K\cap B_t)}{t^{N-p}}\right)^{\frac{1}{p-1}}\frac{dt}{t}. \end{aligned}$$

Since $U_1$ is the maximal solution in $K^c$, $u$ satisfies the same equation in $B\backslash K$ and $U_1 \geq u = 0$ on $\partial B$, it follows that $U_1$ dominates $u$ in $B\backslash K$. Then $U_1(0) \geq u(0)$ and we obtain (3.4).
*2.* By [1, Theorem 2.5.3], there exists $\mu_j \in \mathfrak{M}^+(\mathbb{R}^N)$ such that $\mathrm{supp}(\mu_j) \subset K \cap S_j$ and

$$\mu_j(K \cap S_j) = \int\limits_{\mathbb{R}^N} (G_p[\mu_j](x))^{\frac{q_1}{p-1}} dx = C_{p,\frac{q_1}{q_1-p+1}}(K \cap S_j).$$

By Jensen's inequality, we have for any $a_k \geq 0$,

$$\left(\sum_{k=0}^{\infty} a_k\right)^s \leq \sum_{k=0}^{\infty} \theta_{k,s} a_k^s$$

where $\theta_{k,r}$ has the following expression with $\theta > 0$,

$$\theta_{k,s} = \begin{cases} 1 & \text{if } s \in (0,1], \\ \left(\frac{\theta+1}{\theta}\right)^{s-1} (\theta+1)^{k(s-1)} & \text{if } s > 1. \end{cases}$$

Thus,

$$\begin{aligned}
\int\limits_{B_1} \left(\mathbf{W}^1_{1,p}\left[\sum_{k=3}^{J} \mu_k\right](x)\right)^{q_1} dx &\leq \int_{B_1} \left(\sum_{k=3}^{J} \theta_{k,\frac{1}{p-1}} \mathbf{W}^1_{1,p}[\mu_k](x)\right)^{q_1} dx \\
&\leq \sum_{k=3}^{J} \theta^{q_1}_{k,\frac{1}{p-1}} \theta_{k,q_1} \int\limits_{B_1} \left(\mathbf{W}^1_{1,p}[\mu_k](x)\right)^{q_1} dx \\
&\leq c_{16} \sum_{k=3}^{J} \theta^{q_1}_{k,\frac{1}{p-1}} \theta_{k,q_1} \int\limits_{\mathbb{R}^N} (G_p * \mu_k(x))^{\frac{q_1}{p-1}} dx \\
&= c_{16} \sum_{k=3}^{J} \theta^{q_1}_{k,\frac{1}{p-1}} \theta_{k,q_1} C_{p,\frac{q_1}{q_1-p+1}}(K \cap S_k) \\
&\leq c_{17} \sum_{k=3}^{J} \theta^{q_1}_{k,\frac{1}{p-1}} \theta_{k,q_1} 2^{-k\left(N - \frac{pq_1}{q_1-p+1}\right)} \\
&\leq c_{18},
\end{aligned}$$

for $\theta$ small enough. Here the third inequality follows from [2, Theorem 2.3] and the constant $c_{18}$ does not depend on $J$. Hence,

$$\left\|\left(\mathbf{W}^1_{1,p}\left[\sum_{k=3}^{J} \mu_k\right]\right)^q\right\|_{\mathcal{M}^{\frac{q_1}{q}}(B_1)} \leq c_{19} \left\|\mathbf{W}^1_{1,p}\left[\sum_{k=3}^{J} \mu_k\right]\right\|^q_{L^{q_1}(B_1)} \leq c_{20}, \tag{3.11}$$

where $c_{20}$ is independent of $J$. Take $B = B_{\frac{1}{4}}$. Since $\sum_{j=3}^{J} \mu_j$ is absolutely continuous with respect to the capacity $C_{p,\frac{q}{q+1-p}}$ in $B$, thus by Theorem 2.8, there exists a nonnegative renormalized solution $u$ to equation

$$\begin{aligned} -\Delta_p u + u^q = \textstyle\sum_{j=3}^{J} \mu_j \qquad & \text{in } B, \\ u = 0 \qquad & \text{on } \partial B. \end{aligned}$$

satisfying (2.3) with $\mu = \sum_{j=3}^{J} \mu_j$. Thus, from Corollary 2.6 and estimate (3.11), we have

$$u(0) \geq -c_{21} + c_{22} \mathbf{W}_{1,p}^{\frac{1}{4}} \left[ \sum_{j=3}^{J} \mu_j \right] (0).$$

As above, we also get that

$$u(0) \geq -c_{23} + c_{24} \int_0^1 \left( \frac{C_{p,\frac{q_1}{q_1-p+1}}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}.$$

After we also have $U_2(0) \geq u(0)$. Therefore, we obtain(3.5). ■

# 4 Proof of the main results

First, we prove theorem **1** in the case case $p = N$. To do this we consider the function

$$x \mapsto U(x) = U(|x|) = \log \left( \frac{N-1}{2^{N+1}} \frac{1}{R^N} \left( \frac{R}{|x|} + 1 \right) \right) \quad \text{in} \quad B_R(0) \backslash \{0\}.$$

One has

$$U^{'}(|x|) = \frac{1}{R+|x|} - \frac{1}{|x|} \quad \text{and} \quad U^{''}(|x|) = -\frac{1}{(R+|x|)^2} + \frac{1}{|x|^2},$$

thus, for any $0 < |x| < R$,

$$\begin{aligned} -\Delta_N U + e^U - 1 &= -(N-1)|U^{'}(|x|)|^{N-2} \left( U^{''}(|x|) + \frac{1}{|x|} U^{'}(|x|) \right) + e^U - 1 \\ &= -\frac{(N-1)R^{N-1}}{(R+|x|)^N |x|^{N-1}} + \frac{N-1}{2^{N+1}} \frac{1}{R^N} \left( \frac{R}{|x|} + 1 \right) - 1 \\ &\leq -\frac{(N-1)R^{N-1}}{(2R)^N |x|^{N-1}} + \frac{N-1}{2^{N+1}} \frac{1}{R^N} \frac{2R}{|x|} \\ &\leq -1. \end{aligned}$$

Hence, if $u \in C^1(\Omega)$ is the maximal solution of

$$-\Delta_N u + e^u - 1 = 0 \ \text{ in } \Omega$$

and $R = 2\text{diam}(\Omega)$, then $u(x) \geq U(|x-y|)$ for any $x \in \Omega$ and $y \in \partial\Omega$. Therefore, $u$ is a large solution and satisfies

$$u(x) \geq \log \left( \frac{N-1}{2^{N+1}} \frac{1}{R^N} \left( \frac{R}{\rho(x)} + 1 \right) \right) \quad \forall \, x \in \Omega.$$

Now, we prove Theorem **1** in the case $p < N$ and Theorem **2**. Let $u, v \in C^1(\Omega)$ be the maximal solutions of

$$\begin{aligned} &(i) \qquad -\Delta_p u + e^u - 1 = 0 \qquad \text{in } \Omega, \\ &(ii) \qquad -\Delta_p v + v^q = 0 \qquad \text{in } \Omega. \end{aligned}$$

Fix $x_0 \in \partial\Omega$. We can assume that $x_0 = 0$. Let $\delta \in (0, 1/12)$. For $z_0 \in \overline{B_\delta} \cap \Omega$. Set $K = \Omega^c \cap \overline{B_{1/4}(z_0)}$. Let $U_1, U_2 \in C^1(K^c)$ be the maximal solutions of (3.2) and (3.3) respectively. We have $u \geq U_1$ and $v \geq U_2$ in $\Omega$. By Theorem 3.1,

$$
\begin{aligned}
U_1(z_0) &\geq -c_1 + c_2 \int_\delta^1 \left( \frac{\mathcal{H}_1^{N-p}(K \cap B_r(z_0))}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\
&\geq -c_1 + c_2 \int_\delta^1 \left( \frac{\mathcal{H}_1^{N-p}(K \cap B_{r-|z_0|})}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \quad (\text{since } B_{r-|z_0|} \subset B_r(z_0))) \\
&\geq -c_1 + c_2 \int_{2\delta}^1 \left( \frac{\mathcal{H}_1^{N-p}(K \cap B_{\frac{r}{2}})}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\
&\geq -c_1 + c_3 \int_\delta^{1/2} \left( \frac{\mathcal{H}_1^{N-p}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}.
\end{aligned}
$$

We deduce

$$
\inf_{B_\delta \cap \Omega} u \geq \inf_{B_\delta \cap \Omega} U_1 \geq -c_1 + c_3 \int_\delta^{1/2} \left( \frac{\mathcal{H}_1^{N-p}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \to \infty \quad \text{as } \delta \to 0.
$$

Similarly, we also obtain

$$
\inf_{B_\delta \cap \Omega} v \geq -c_4 + c_5 \int_\delta^{1/2} \left( \frac{C_{p, \frac{q_1}{q_1 - p + 1}}(K \cap B_r)}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \to \infty \quad \text{as } \delta \to 0.
$$

Therefore, $u$ and $v$ satisfy (1.10) and (1.12) respectively. This completes the proof.

# 5 Large solutions of quasilinear Hamilton-Jacobi equations

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ with $N \geq 2$. In this section we use our previous results to give sufficient conditions for existence of solutions to the problem

$$
\begin{aligned}
-\Delta_p u + a\,|\nabla u|^q + bu^s = 0 \qquad &\text{in } \Omega, \\
\lim_{\rho(x) \to 0} u(x) = \infty, &
\end{aligned} \tag{5.1}
$$

where $a > 0, b > 0$ and $1 \leq q < p \leq 2$, $q > p - 1, s \geq p - 1$.

First we have the result of existence solutions to equation (5.1).

**Proposition 5.1** *Let $a > 0, b > 0$ and $q > p - 1, s \geq p - 1$, $1 \leq q \leq p$ and $1 < p \leq 2$. There exists a maximal nonnegative solution $u \in C^1(\Omega)$ to equation*

$$
-\Delta_p u + a\,|\nabla u|^q + bu^s = 0 \quad \text{in } \Omega, \tag{5.2}
$$

*which satisfies*

$$
u(x) \leq c(N, p, s) b^{-\frac{1}{s-p+1}} d(x, \partial\Omega)^{-\frac{p}{s-p+1}} \quad \forall x \in \Omega, \tag{5.3}
$$

*if $s > p - 1$,*

$$u(x) \leq c(N,p,q)\left(a^{-\frac{1}{q-p+1}}d(x,\partial\Omega)^{-\frac{p-q}{q-p+1}} + a^{-\frac{1}{q-p+1}}b^{-\frac{1}{p-1}}d(x,\partial\Omega)^{-\frac{q}{(p-1)(q-p+1)}}\right) \quad \forall x \in \Omega, \tag{5.4}$$

*if* $p-1 < q < p$ *and* $s = p-1$, *and*

$$u(x) \leq c(N,p)a^{-1}b^{-\frac{1}{p-1}}d(x,\partial\Omega)^{-\frac{p}{p-1}} \quad \forall x \in \Omega, \tag{5.5}$$

*if* $q = p$ *and* $s = p-1$.

**Proof.** Case $s = p-1$ and $p-1 < q < p$. We consider

$$U_1(x) = U_1(|x|) = c_1\left(\frac{R^{p'} - |x|^{p'}}{p'R^{p'-1}}\right)^{-\frac{p-q}{q-p+1}} + c_2 \in C^1(B_R(0)).$$

with $p' = \frac{p}{p-1}$ and $c_1, c_2 > 0$. We have

$$U_1^{'}(|x|) = \frac{c_1(p-q)}{q-p+1}\frac{|x|^{p'-1}}{R^{p'-1}}\left(\frac{R^{p'} - |x|^{p'}}{p'R^{p'-1}}\right)^{-\frac{1}{q-p+1}},$$

$$U_1^{''}(|x|) = \frac{c_1(p-q)(p'-1)}{q-p+1}\frac{|x|^{p'-2}}{R^{p'-1}}\left(\frac{R^{p'} - |x|^{p'}}{p'R^{p'-1}}\right)^{-\frac{1}{q-p+1}} + \frac{c_1(p-q)}{(q-p+1)^2}\left(\frac{|x|^{p'-1}}{R^{p'-1}}\right)^2\left(\frac{R^{p'} - |x|^{p'}}{p'R^{p'-1}}\right)^{-\frac{1}{q-p+1}-1}$$

and

$$A = -\Delta_p U_1 + a|\nabla U_1|^q + bU_1^{p-1} \geq -\Delta_p U_1 + a|\nabla U_1|^q + bc_2^{p-1}.$$

Thus, for all $x \in B_R(0)$

$$\begin{aligned}
A &\geq -(p-1)|U_1^{'}(|x|)|^{p-2}U_1^{''}(|x|) - \frac{N-1}{|x|}|U_1^{'}(|x|)|^{p-2}U_1^{'}(|x|) + a|U_1^{'}(|x|)|^q + bc_1^{p-1} \\
&= \left(\frac{c_1(p-q)(p'-1)}{q-p+1}\right)^{p-1}\left(\frac{R^{p'} - |x|^{p'}}{p'R^{p'-1}}\right)^{-\frac{q}{q-p+1}}\left\{-(p-1)\frac{p'-1}{p'}\left(1 - \left(\frac{|x|}{R}\right)^{p'}\right)\right. \\
&\quad - \frac{1}{q-p+1}\left(\frac{|x|}{R}\right)^{p'} - \frac{N-1}{p'}\left(\frac{|x|}{R}\right)^{p'}\left(1 - \left(\frac{|x|}{R}\right)^{p'}\right) \\
&\quad \left. + a\left(\frac{c_1(p-q)}{q-p+1}\right)^{q-p+1}\left(\frac{|x|}{R}\right)^{\frac{q}{q-p+1}}\right\} + bc_2^{p-1} \\
&\geq \left(\frac{c_1(p-q)(p'-1)}{q-p+1}\right)^{p-1}\left(\frac{R^{p'} - |x|^{p'}}{p'R^{p'-1}}\right)^{-\frac{q}{q-p+1}} \\
&\quad \times\left\{-\frac{N(p-1)}{p} - \frac{1}{q-p+1} + a\left(\frac{c_1(p-q)}{q-p+1}\right)^{q-p+1}\left(\frac{|x|}{R}\right)^{\frac{q}{q-p+1}}\right\} + bc_2^{p-1}.
\end{aligned}$$

Clearly, one can find $c_1 = c_2(N,p,q)a^{-\frac{1}{q-p+1}} > 0$ and $c_3 = c_3(N,p,q) > 0$ such that

$$A \geq -c_3 a^{-\frac{p-1}{q-p+1}} R^{-\frac{q}{q-p+1}} + bc_2^{p-1}.$$

Choosing $c_2 = c_3^{\frac{1}{p-1}} a^{-\frac{1}{q-p+1}} b^{-\frac{1}{p-1}} R^{-\frac{q}{(p-1)(q-p+1)}}$, we get

$$-\Delta_p U_1 + a|\nabla U_1|^q + bU_1^{p-1} \geq 0 \quad \text{in } B_R(0). \tag{5.6}$$

Likewise, we can verify that the function $U_2$ below

$$U_2(x) = c_4 a^{-1} \log\left(\frac{R^{p'}}{R^{p'} - |x|^{p'}}\right) + c_4 a^{-1} b^{-\frac{1}{p-1}} R^{-\frac{p}{p-1}}$$

belongs to $C^1_+(B_R(0))$ and satisfies

$$-\Delta_p U_2 + a|\nabla U_2|^p + bU_2^{p-1} \geq 0 \quad \text{in } B_R(0). \tag{5.7}$$

While, if $s > p-1$,

$$U_3(x) = c_5 b^{-\frac{1}{s-p+1}} \left(\frac{R^\beta - |x|^\beta}{\beta R^{\beta-1}}\right)^{-\frac{p}{s-p+1}}$$

belongs to $C^1(B_R(0))$ and verifies

$$-\Delta_p U_3 + bU_3^s \geq 0 \quad \text{in } B_R(0), \tag{5.8}$$

for some positive constants $c_4 = c_4(N,p,q)$, $c_5 = c_5(N,p,s)$ and $\beta = \beta(N,p,q) > 1$.
We emphasize the fact that with the condition $1 < p \leq 2$ and $q \geq 1$, equation (5.2) satisfies a comparison principle, see [17, Theorem 3.5.1, corollary 3.5.2]. Take a sequence of smooth domains $\Omega_n$ satisfying $\Omega_n \subset \overline{\Omega}_n \subset \Omega_{n+1}$ for all $n$ and $\bigcup\limits_{n=1}^{\infty} \Omega_n = \Omega$. For each $n, k \in \mathbb{N}^*$, there exist nonnegative solution $u_{n,k} = u \in W_k^{1,p}(\Omega_n) := W_0^{1,p}(\Omega_n) + k$ of equation (5.2) in $\Omega_n$.
Since $-\Delta_p u_{k,n} \leq 0$ in $\Omega_n$, so using the maximum principle we get $u_{n,k} \leq k$ in $\Omega_n$ for all $n$. Thus, by standard regularity (see [4] and [11]), $u_{n,k} \in C^{1,\alpha}(\overline{\Omega_n})$ for some $\alpha \in (0,1)$. It follows from the comparison principle and (5.6)-(5.8), that

$$u_{n,k} \leq u_{n,k+1} \qquad \text{in } \Omega_n$$

and (5.3)-(5.5) are satisfied with $u_{n,k}$ and $\Omega_n$ in place of $u$ and $\Omega$ respectively. From this, we derive uniform local bounds for $\{u_{n,k}\}_k$, and by standard interior regularity (see [4]) we obtain uniform local bounds for $\{u_{n,k}\}_k$ in $C^{1,\eta}_{loc}(\Omega_n)$. It implies that the sequence $\{u_{n,k}\}_k$ is pre-compact in $C^1$. Therefore, up to a subsequence, $u_{n,k} \to u_n$ in $C^1(\Omega_n)$. Hence, we can verify that $u_n$ is a solution of (5.2) and satisfies (5.3)-(5.5) with $u_n$ and $\Omega_n$ replacing $u$ and $\Omega$ and $u_n(x) \to \infty$ as $d(x, \Omega_n) \to 0$.
Next, since $u_{n,k} \geq u_{n+1,k}$ in $\Omega_n$ there holds $u_n \geq u_{n+1}$ in $\Omega_n$. In particular, $\{u_n\}$ is uniformly locally bounded in $\Omega$. Arguing as above, we obtain $u_n \to u$ in $C^1(\Omega)$, thus $u$ is a solution of (5.2) in $\Omega$ and satisfies (5.3)-(5.5). Clearly, $u$ is the maximal solution of (5.2). ■

**Theorem 5.2** *Let $q_1 > p-1$ and $1 < p \leq 2$. Assume that equation* (1.12) *admits a solution with $q = q_1$. Then for any $a > 0, b > 0$ and $q \in (p-1, \frac{pq_1}{q_1+1})$, $s \in [p-1, q_1)$ equation* (5.2) *has a large solution satisfying* (5.3) *and* (5.4).

**Proof.** Assume that equation (1.12) admits a solution $v$ with $q = q_1$ and set $v = \beta w^\sigma$ with $\beta > 0, \sigma \in (0,1)$, then $w > 0$ and

$$-\Delta_p w + (-\sigma+1)(p-1)\frac{|\nabla w|^p}{w} + \beta^{q_1-p+1}\sigma^{-p+1}w^{\sigma(q_1-p+1)+p-1} = 0 \text{ in } \Omega.$$

If we impose $\max\{\frac{s-p+1}{q_1-p+1}, \left(\frac{q}{p-q} - p + 1\right)\frac{1}{q_1-p+1}\} < \sigma < 1$, we can see that

$$(-\sigma+1)(p-1)\frac{|\nabla w|^p}{w} + \beta^{q_1-p+1}\sigma^{-p+1}w^{\sigma(q_1-p+1)+p-1} \geq a|\nabla w|^q + bw^s \text{ in } \{x : w(x) \geq M\},$$

where a positive constant $M$ depends on $p, q_1, q, s, a, b$. Therefore

$$-\Delta_p w + a\,|\nabla w|^q + bw^s \leq 0 \quad \text{in} \quad \{x : w(x) \geq M\}.$$

Now we take an open subset $\Omega'$ of $\Omega$ with $\overline{\Omega'} \subset \Omega$ such that the set $\{x : w(x) \geq M\}$ contains $\Omega\backslash\overline{\Omega'}$. So $w$ is a subsolution of $-\Delta_p u + a\,|\nabla u|^q + bu^s = 0$ in $\Omega\backslash\overline{\Omega'}$ and the same property holds with $w_\varepsilon := \varepsilon w$ for any $\varepsilon \in (0,1)$. Let $u$ be as in Proposition 5.1. Set $\min\{u(x) : x \in \partial\Omega'\} = \theta_1 > 0$ and $\max\{w(x) : x \in \partial\Omega'\} = \theta_2 \geq M$. Thus $w_\varepsilon < u$ on $\partial\Omega'$ with $\varepsilon < \min\{\frac{\theta_1}{\theta_2}, 1\}$. Hence, from the construction of $u$ in the proof of Proposition 5.1 and the comparison principle, we obtain $w_\varepsilon \leq u$ in $\Omega\backslash\overline{\Omega'}$. This implies the result. ■

**Remark 5.3** *From the proof of above Theorem, we can show that under the assumption as in Proposition 5.1, equation* (5.2) *has a large solution in* $\Omega$ *if and only if equation* (5.2) *has a large solution in* $\Omega\backslash K$ *for some a compact set* $K \subset \Omega$ *with smooth boundary.*

Now we deal with (5.1) in the case $q = p$.

**Theorem 5.4** *Assume that equation* (5.2) *has a large solution in* $\Omega$ *for some* $a, b > 0$, $s > p-1$ *and* $q = p > 1$. *Then for any* $a_1, b_1 > 0$ *and* $q_1 > p-1, s_1 \geq p-1$, $1 \leq q_1 \leq p \leq 2$, *equation* (5.2) *also has a large solution* $u$ *in* $\Omega$ *with parameters* $a_1, b_1, q_1, s_1$ *in place of* $a, b, q, s$ *respectively, and it satisfies* (5.3)-(5.5).

**Proof.** For $\sigma > 0$ we set $u = v^\sigma$ thus

$$-\Delta_p v - (\sigma-1)(p-1)\frac{|\nabla v|^p}{v} + a\sigma v^{\sigma-1}\,|\nabla v|^p + b\sigma^{-p+1}v^{(s-p+1)\sigma+p-1} = 0.$$

Choose $\sigma = \frac{s_1-p+1}{s-p+1} + 2$, it is easy to see that

$$-\Delta_p v + a_1|\nabla v|^{q_1} + b_2 v^{s_1} \leq 0 \text{ in } \{x : v(x) \geq M\},$$

for some a positive constant $M$ only depending on $p, s, a, b, a_1, b_1, q_1, s_1$. Similarly as in the proof of Theorem 5.2, we get the result as desired. ■

**Remark 5.5** *If we set* $u = e^v$ *then* $v$ *satisfies*

$$-\Delta_p v + be^{(s-p+1)v} = |\nabla v|^p\,(p-1-ae^v) \qquad \textit{in } \Omega.$$

*From this, we can construct a large solution of*

$$-\Delta_p u + be^{(s-p+1)u} = 0 \qquad \textit{in } \Omega\backslash K,$$

*for any a compact set $K \subset \Omega$ with smooth boundary such that $v \geq ln\left(\frac{p-1}{a}\right)$ in $\Omega \backslash K$. In case $p = 2$, It would be interesting to see what Wiener type criterion is implied by the existence as such a large solution. We conjecture that this condition must be*

$$\int_0^1 \frac{\mathcal{H}_1^{N-2}(B_r(x) \cap \Omega^c)}{r^{N-2}} \frac{dr}{r} = \infty \qquad \forall x \in \partial\Omega.$$

We now consider the function

$$U_4(x) = c\left(\frac{R^\beta - |x|^\beta}{\beta R^{\beta-1}}\right)^{\frac{p}{\gamma+p-1}} \quad \text{in } B_R(0), \gamma > 0.$$

As in the proof of proposition 5.1, it is easy to check that there exist positive constants $\beta$ large enough and $c$ small enough so that inequality $\Delta_p U_4 + U_4^{-\gamma} \geq 0$ holds.
From this, we get the existence of minimal solution to equation

$$\Delta_p u + u^{-\gamma} = 0 \qquad \text{in } \Omega. \tag{5.9}$$

**Proposition 5.6** *Assume $\gamma > 0$. Then there exists a minimal solution $u \in C^1(\Omega)$ to equation* (5.9) *and it satisfies $u(x) \geq Cd(x, \partial\Omega)^{\frac{p}{\gamma+p-1}}$ in $\Omega$.*

We can verify that if the boundary of $\Omega$ is satisfied (1.3), then above minimal solution $u$ belongs to $C(\overline{\Omega})$, vanishes on $\partial\Omega$ and it is therefore a solution to the quenching problem

$$\begin{aligned} \Delta_p u + u^{-\gamma} &= 0 \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{in } \partial\Omega. \end{aligned} \tag{5.10}$$

**Theorem 5.7** *Let $\gamma > 0$. Assume that there exists a solution $u \in C(\overline{\Omega})$ to problem* (5.10). *Then, for any $a, b > 0$ and $q > p-1, s \geq p-1$, $1 \leq q \leq p \leq 2$, equation* (5.2) *admits a large solution in $\Omega$ and it satisfies* (5.3)-(5.5).

**Proof.** We set $u = e^{-\frac{a}{p-1}v}$, then $v$ is a large solution of

$$-\Delta_p v + a\,|\nabla v|^p + \left(\tfrac{p-1}{a}\right)^{p-1} e^{\frac{a}{p-1}(\gamma+p-1)v} = 0 \qquad \text{in } \Omega.$$

So

$$-\Delta_p v + a\,|\nabla v|^q + bv^s \leq 0 \qquad \text{in } \{x : v(x) \geq M\},$$

for some a positive constant $M$ only depending on $p, q, s, a, b, \gamma$. Similarly to the proof of Theorem 5.2, we get the result as desired. ∎

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