Fractional heat equations with subcritical absorption having a measure as initial data

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Abstract

We study existence and uniqueness of weak solutions to (F) $\partial_t u+(-\Delta)^\alpha u+h(t,u)=0$ in $(0,\infty)\times\mathbb{R}^N$, with initial condition $u(0,\cdot)=\nu$ in $\mathbb{R}^N$, where $N\ge 2$, the operator $(-\Delta)^\alpha$ is the fractional Laplacian with $\alpha\in(0,1)$, ν is a bounded Radon measure and $h:(0,\infty)\times\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying a subcritical integrability condition.

In particular, if $h(t,u)=t^\beta u^p$ with $\beta>-1$ and $0<p<p^*_\beta:=1+\frac{2\alpha(1+\beta)}{N}$, we prove that there exists a unique weak solution u_k to (F) with $\nu=k\delta_0$, where δ_0 is the Dirac mass at the origin. We obtain that $u_k\to\infty$ in $(0,\infty)\times\mathbb{R}^N$ as $k\to\infty$ for $p\in(0,1]$ and the limit of u_k exists as $k\to\infty$ when $1<p<p^*_\beta$, we denote it by u_∞. When $1+\frac{2\alpha(1+\beta)}{N+2\alpha}:=p^{**}_\beta<p<p^*_\beta$, u_∞ is the minimal self-similar solution of $(F)_\infty$ $\partial_t u+(-\Delta)^\alpha u+t^\beta u^p=0$ in $(0,\infty)\times\mathbb{R}^N$ with the initial condition $u(0,\cdot)=0$ in $\mathbb{R}^N\setminus\{0\}$ and it satisfies $u_\infty(0,x)=0$ for $x\neq 0$. While if $1<p<p^{**}_\beta$, then $u_\infty\equiv U_p$, where U_p is the maximal solution of the differential equation $y'+t^\beta y^p=0$ on $\mathbb{R}_+$.

Contents

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Key words: Fractional heat equation, Radon measure, Dirac mass, Self-similar solution, Very singular solution

MSC2010: 35R06, 35K05, 35R11

1 Introduction

Let $h:(0,\infty)\times\mathbb{R}\to\mathbb{R}$ be a continuous function and $Q_\infty=(0,\infty)\times\mathbb{R}^N$ with $N\geq 2$. The first object of this paper is to consider existence and uniqueness of weak solutions to fractional heat equations

$$\begin{aligned} \partial_t u+(-\Delta)^\alpha u+h(t,u)=0 \quad &\text{in} \quad Q_\infty,\\ u(0,\cdot)=\nu \quad &\text{in} \quad \mathbb{R}^N, \end{aligned} \tag{1.1}$$

where ν belongs to the space $\mathfrak{M}^b(\mathbb{R}^N)$ of bounded Radon measures in $\mathbb{R}^N$ and $(-\Delta)^\alpha$ $(0<\alpha<1)$ is the fractional Laplacian defined by

$$(-\Delta)^\alpha u(t,x)=\lim_{\epsilon\to0^+}(-\Delta)^\alpha_\epsilon u(t,x),$$

where, for $\epsilon>0$,

$$(-\Delta)^\alpha_\epsilon u(t,x)=\int_{\mathbb{R}^N}\frac{u(t,x)-u(t,z)}{|z-x|^{N+2\alpha}}\chi_\epsilon(|x-z|)dz$$

and

$$\chi_\epsilon(r)=\begin{cases}0 & \text{if} \quad r\in[0,\epsilon],\\ 1 & \text{if} \quad r>\epsilon.\end{cases}$$

In a pioneering work, Brezis and Friedman [6] have studied the semilinear heat equation with measure as initial data

$$\begin{aligned} \partial_t u-\Delta u+u^p=0 \qquad &\text{in} \quad Q_\infty,\\ u(0,\cdot)=k\delta_0 \quad &\text{in} \quad \mathbb{R}^N, \end{aligned} \tag{1.2}$$

where $k > 0$ and δ_0 is the Dirac mass at the origin. They proved that if $1 < p < (N+2)/N$, then for every $k > 0$ there exists a unique solution u_k to (1.2). When $p \geq (N+2)/N$, problem (1.2) has no solution and even more, they proved that no nontrivial solution of the above equation vanishing on $\mathbb{R}^N \setminus \{0\}$ at $t = 0$ exists. When $1 < p < 1 + \frac{2}{N}$, Brezis, Peletier and Terman used a dynamical system technique in [7] to prove the existence of a *very singular solution* u_s to

$$\partial_t u - \Delta u + u^p = 0 \quad \text{in} \quad Q_\infty, \tag{1.3}$$

vanishing at $t = 0$ on $\mathbb{R}^N \setminus \{0\}$. This function u_s is self-similar, i.e. expressed under the form

$$u_s(t,x) = t^{-\frac{1}{p-1}} f\left(\frac{|x|}{\sqrt{t}}\right), \tag{1.4}$$

and f is uniquely determined by the following conditions

$$\begin{aligned}
&f'' + \left(\tfrac{N-1}{\eta} + \tfrac{1}{2}\eta\right) f' + \tfrac{1}{p-1} f - f^p = 0 \quad \text{on } \mathbb{R}_+ \\
&f > 0 \quad \text{and } f \text{ is smooth on } \mathbb{R}_+ \\
&f'(0) = 0 \quad \text{and} \quad \lim_{\eta\to\infty} \eta^{\frac{2}{p-1}} f(\eta) = 0.
\end{aligned} \tag{1.5}$$

Furthermore, it satisfies

$$f(\eta) = c_1 e^{-\eta^2} \eta^{\frac{2}{p-1}-N} \{1 - O(|x|^{-2})\} \quad \text{as } \eta \to \infty$$

for some $c_1 > 0$. Later on, Kamin and Peletier in [21] proved that the sequence of weak solutions u_k converges to the very singular solution u_s as $k \to \infty$. After that, Marcus and Véron in [23] studied the equation in the framework of the *initial trace* theory. They pointed out the role of the very singular solution of (1.3) in the study of the singular set of the initial trace, showing in particular that it is the unique positive solution of (1.3) satisfying

$$\lim_{t\to 0} \int_{B_\epsilon} u(t,x)dx = \infty, \qquad \forall \epsilon > 0,\ B_\epsilon = B_\epsilon(0), \tag{1.6}$$

and

$$\lim_{t\to 0} \int_K u(t,x)dx = 0 \qquad , \forall K \subset \mathbb{R}^N \setminus \{0\},\ K \text{ compact}. \tag{1.7}$$

If one replaces u^p by $t^\beta u^p$ with $p \in (1, 1 + \frac{2(1+\beta)}{N})$, these results were extended by Marcus and Véron ($\beta \geq 0$) in [24] and then Al Sayed and Véron ($\beta > -1$) in [1]. The initial data problem with measure and general absorption term

$$\begin{aligned}
\partial_t u - \Delta u + h(t,x,u) &= 0 \quad \text{in} \quad (0,\mathrm{T}) \times \Omega, \\
u &= 0 \quad \text{in} \quad (0,T) \times \partial\Omega, \\
u(0,\cdot) &= \nu \quad \text{in} \quad \Omega,
\end{aligned} \tag{1.8}$$

in a bounded domain Ω of $\mathbb{R}^N$, has been studied by Marcus and Véron in [24] in the framework of the initial trace theory. They proved that the following general integrability condition on h

$$
\begin{aligned}
&0 \le | h(t,x,r) | \le \tilde{h}(t) f(|r|) \qquad , \forall (x,t,r) \in \Omega \times \mathbb{R}_+ \times \mathbb{R} \\
&\int_0^T \tilde{h}(t) f(\sigma t^{\frac{N}{2}}) t^{-\frac{N}{2}} dt < \infty \qquad , \forall \sigma > 0 \\
&\text{either } \tilde{h}(t) = t^\alpha \text{ with } \alpha \ge 0 \text{ or } f \text{ is convex,}
\end{aligned}
\tag{1.9}
$$

in order that the problem has a unique solution for any bounded measure. In the particular case with $h(t,x,r) = t^\beta |u|^{p-1} u$, it is fulfilled if $1 < p < 1 + \frac{2(1+\beta)}{N}$ and $\beta > -1$, and the very singular solution exists in this range of values.

Motivated by a growing number of applications in physics and by important links on the theory of Lévy process, semilinear fractional equations has been attracted much interest in last few years, (see e.g. [8, 9, 10, 12, 14, 17, 18, 19]). Recently, in [15] we obtained the existence and uniqueness of a weak solution to semilinear fractional elliptic equation

$$
\begin{aligned}
(-\Delta)^\alpha u + f(u) = \nu \quad &\text{in} \quad \Omega, \\
u = 0 \quad &\text{in} \quad \Omega^c,
\end{aligned}
\tag{1.10}
$$

when ν is a Radon measure and f satisfies a subcritical integrability condition. In [14] we studied the the different types of isolated singularities when $f(u) = u^p$ where $1 < p < \frac{N}{N-2\alpha}$. In particular, assuming that $0 \in \Omega$, we proved that the sequence of solutions $\{u_k\}$ $(k \in \mathbb{N})$ of (1.10), with $\nu = k\delta_0$ converges to infinity when $k \to \infty$, if $p \in (0, 1 + \frac{2\alpha}{N})$ and it converges to a solution with a strong singularity at 0 if $p \in (1 + \frac{2\alpha}{N}, \frac{N}{N-2\alpha})$.

One purpose of this paper is to study the existence and uniqueness of weak solutions to semilinear fractional heat equation (1.1) in a measure framework. We first make precise the notion of weak solution of (1.1) that we will use in this note.

Definition 1.1 *We say that u is a weak solution of (1.1), if for any $T > 0$, $u \in L^1(Q_T)$, $h(t,u) \in L^1(Q_T)$ and*

$$
\begin{aligned}
&\int_{Q_T} (u(t,x)[-\partial_t \xi(t,x) + (-\Delta)^\alpha \xi(t,x)] + h(t,u)\xi(t,x))\, dxdt \\
&= \int_{\mathbb{R}^N} \xi(0,x) d\nu - \int_{\mathbb{R}^N} \xi(T,x) u(T,x) dx \qquad \forall \xi \in \mathbb{Y}_{\alpha,T},
\end{aligned}
\tag{1.11}
$$

where $Q_T = (0,T) \times \mathbb{R}^N$ and $\mathbb{Y}_{\alpha,T}$ is a space of functions $\xi : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfying (*i*) $\|\xi\|_{L^1(Q_T)} + \|\xi\|_{L^\infty(Q_T)} + \|\partial_t\xi\|_{L^\infty(Q_T)} + \|(-\Delta)^\alpha\xi\|_{L^\infty(Q_T)} < +\infty$;

(*ii*) *for* $t \in (0,T)$*, there exist* $M > 0$ *and* $\epsilon_0 > 0$ *such that for all* $\epsilon \in (0, \epsilon_0]$, $\|(-\Delta)^\alpha_\epsilon \xi(t,\cdot)\|_{L^\infty(\mathbb{R}^N)} \le M$.

Before stating our main theorems, we introduce the subcritical integrability condition for the nonlinearity h, that is,

(H) (*i*) The function $h : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and for any $t \in (0,\infty)$, $h(t,0) = 0$ and $h(t,r_1) \ge h(t,r_2)$ if $r_1 \ge r_2$.

(*ii*) There exist $\beta > -1$ and a continuous, nondecreasing function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|h(t,r)| \le t^\beta g(|r|) \qquad \forall (t,r) \in (0,\infty) \times \mathbb{R}$$

and

$$\int_1^{+\infty} g(s) s^{-1-p^*_\beta} ds < +\infty, \tag{1.12}$$

where

$$p^*_\beta = 1 + \frac{2\alpha(1+\beta)}{N}. \tag{1.13}$$

We denote by $H_\alpha : (0,\infty) \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+$ the heat kernel for $(-\Delta)^\alpha$ in $(0,\infty) \times \mathbb{R}^N$, by $\mathbb{H}_\alpha[\nu]$ the associated heat potential of $\nu \in \mathfrak{M}^b(\mathbb{R}^N)$, defined by

$$\mathbb{H}_\alpha[\nu](t,x) = \int_{\mathbb{R}^N} H_\alpha(t,x,y) d\nu(y)$$

and by $\mathcal{H}_\alpha[\mu]$ the Duhamel operator defined for $(t,x) \in Q_T$ and any $\mu \in L^1(Q_T)$ by

$$\mathcal{H}_\alpha[\mu](t,x) = \int_0^t \mathbb{H}_\alpha[\mu(s,.)](t-s,x) ds = \int_0^t \int_{\mathbb{R}^N} H_\alpha(t-s,x,y)\mu(s,y) dy ds.$$

Now we state our first theorem as follows.

Theorem 1.1 *Assume that* $\nu \in \mathfrak{M}^b(\mathbb{R}^N)$ *and the function* h *satisfies* (H)*. Then problem (1.1) admits a unique weak solution* u_ν *such that*

$$\mathbb{H}_\alpha[\nu] - \mathcal{H}_\alpha[h(.,\mathbb{H}_\alpha[\nu_+])] \le u_\nu \le \mathbb{H}_\alpha[\nu] - \mathcal{H}_\alpha[h(.,-\mathbb{H}_\alpha[\nu_-])] \quad \text{in } Q_\infty, \tag{1.14}$$

where ν_+ *and* ν_- *are respectively the positive and negative part in the Jordan decomposition of* ν*. Furthermore,*

(*i*) *if* ν *is nonnegative, so is* u_ν*;*

(*ii*) *the mapping: $\nu \mapsto u_\nu$ is increasing and stable in the sense that if $\{\nu_n\}$ is a sequence of positive bounded Radon measures converging to ν in the weak sense of measures, then $\{u_{\nu_n}\}$ converges to u_ν locally uniformly in Q_∞.*

According to Theorem 1.1, there exists a unique positive weak solution u_k to

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + t^\beta u^p = 0 \qquad &\text{in} \quad Q_\infty, \\ u(0,\cdot) = k\delta_0 \qquad &\text{in} \quad \mathbb{R}^N, \end{aligned} \tag{1.15}$$

where $\beta > -1$, $k > 0$ and $p \in (0, p^*_\beta)$. We observe that $u_k \to \infty$ in $(0,\infty) \times \mathbb{R}^N$ as $k \to \infty$ for $p \in (0,1]$, see Proposition 4.2 for details. Our next interest in this paper is to study the limit of u_k as $k \to \infty$ for $p \in (1, p^*_\beta)$, which exists since $\{u_k\}_k$ is an increasing sequence of functions, bounded by $\left(\frac{1+\beta}{p-1}\right)^{\frac{1}{p-1}} t^{-\frac{1+\beta}{p-1}}$, and we set

$$u_\infty = \lim_{k\to\infty} u_k \quad \text{in} \quad Q_\infty. \tag{1.16}$$

Actually, u_∞ and $\{u_k\}_k$ are classical solutions to equation

$$\partial_t u + (-\Delta)^\alpha u + t^\beta u^p = 0 \quad \text{in} \quad Q_\infty, \tag{1.17}$$

see Proposition 4.3 for details.

Definition 1.2 (*i*) *A solution u of (1.17) is called a self-similar solution if*

$$u(t,x) = t^{-\frac{1+\beta}{p-1}} u(1, t^{-\frac{1}{2\alpha}} x) \qquad (t,x) \in Q_\infty.$$

(*ii*) *A solution u of (1.17) is called a very singular solution if it vanishes on $\mathbb{R}^N \setminus \{0\}$ at $t = 0$ and*

$$\lim_{t\to 0^+} \frac{u(t,0)}{\Gamma_\alpha(t,0)} = +\infty,$$

where $\Gamma_\alpha := \mathbb{H}_\alpha[\delta_0]$ is the fundamental solution of

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u = 0 \qquad &\text{in} \quad Q_\infty, \\ u(0,\cdot) = \delta_0 \qquad &\text{in} \quad \mathbb{R}^N. \end{aligned} \tag{1.18}$$

We remark that for $p \in (1, p^*_\beta)$, a self-similar solution u of (1.17) is also a very singular solution, since

$$\lim_{t\to 0^+} \Gamma_\alpha(t,0) t^{\frac{N}{2\alpha}} = c_2, \tag{1.19}$$

for some $c_2 > 0$. For any self-similar solution u of (1.17), $v(\eta) := u(1, t^{-\frac{1}{2\alpha}} x)$ with $\eta = t^{-\frac{1}{2\alpha}} x$ is a solution of *the self-similar equation*

$$(-\Delta)^\alpha v - \frac{1}{2\alpha} \nabla v \cdot \eta - \frac{1+\beta}{p-1} v + v^p = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{1.20}$$

Since $\left(\frac{1+\beta}{p-1}\right)^{\frac{1}{p-1}}$ is a constant nonzero solution of (1.20), the function

$$U_p(t) := \left(\frac{1+\beta}{p-1}\right)^{\frac{1}{p-1}} t^{-\frac{1+\beta}{p-1}} \qquad t > 0 \tag{1.21}$$

is a flat self-similar solution of (1.17). It is actually the maximal solution of the ODE $y' + t^\beta y^p = 0$ defined on $\mathbb{R}_+$. Our next goal in this paper is to study non-flat self-similar solutions of (1.17).

Theorem 1.2 *Assume that $\beta > -1$, u_∞ is defined by (1.16) and*

$$p^{**}_\beta < p < p^*_\beta,$$

where $p^{**}_\beta = 1 + \frac{2\alpha(1+\beta)}{N+2\alpha}$. Then u_∞ is a very singular self-similar solution of (1.17) in Q_∞. Moreover, there exists $c_3 > 1$ such that

$$\frac{c_3^{-1}}{1+|x|^{N+2\alpha}} \le u_\infty(1,x) \le \frac{c_3 \ln(2+|x|)}{1+|x|^{N+2\alpha}} \qquad x \in \mathbb{R}^N. \tag{1.22}$$

When $p^{**}_\beta < p < p^*_\beta$ with $\beta > -1$, we observe that u_∞ and U_p are self-similar solutions of (1.17) and u_∞ is non-flat. Now we are ready to consider the uniqueness of non-flat self-similar solution of (1.17) with decay at infinity, precisely, we study the uniqueness of self-similar solution to

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + t^\beta u^p = 0 \quad \text{in} \quad Q_\infty, \\ \lim_{|x|\to\infty} u(1,x) = 0. \end{aligned} \tag{1.23}$$

We remark that if u is self-similar, then the assumption $\lim_{|x|\to\infty} u(1,x) = 0$ is equivalent to $\lim_{|x|\to\infty} u(t,x) = 0$ for any $t > 0$. Finally, we state the properties of u_∞ when $1 < p \le p^{**}_\beta$ as follows.

Theorem 1.3 *(i) Assume $1 < p < p^{**}_\beta$ and u_∞ is defined by (1.16). Then $u_\infty = U_p$, where U_p is given by (1.21).*
(ii) Assume $p = p^{**}_\beta$ and u_∞ is defined by (1.16). Then u_∞ is a self-similar solution of (1.17) such that

$$u_\infty(t,x) \ge \frac{c_4 t^{-\frac{N+2\alpha}{2\alpha}}}{1+|t^{-\frac{1}{2\alpha}}x|^{N+2\alpha}} \qquad (t,x) \in (0,1) \times \mathbb{R}^N, \tag{1.24}$$

for some $c_4 > 0$.

We note that Theorem 1.3 indicates that there exists no self-similar solution of (1.17) with an initial data $u(0,\cdot)$ vanishing in $\mathbb{R}^N \setminus \{0\}$ if $p \in (1, p_\beta^{**})$, since u_∞ is the least self-similar solution. In Theorem 1.3 part (*ii*), we do not know if the self-similar solution is flat or not. From the above theorems, we have the following result.

Theorem 1.4 *(i) Assume* $p_\beta^{**} < p < p_\beta^*$. *Then problem (1.20) admits a minimal positive solution* v_∞ *satisfying*

$$\lim_{|\eta|\to\infty} |\eta|^{\frac{2\alpha(1+\beta)}{p-1}} v_\infty(\eta) = 0. \tag{1.25}$$

Furthermore,

$$\frac{c_3^{-1}}{1+|\eta|^{N+2\alpha}} \le v_\infty(\eta) \le \frac{c_3 \ln(2+|\eta|)}{1+|\eta|^{N+2\alpha}} \qquad \forall \eta \in \mathbb{R}^N \tag{1.26}$$

(ii) Assume $1 < p < p_\beta^{**}$. *Then problem (1.20) admits no positive solution satisfying (1.25).*

The question of uniqueness of the very singular solution in the case $p_\beta^{**} < p < p_\beta^*$ remains an open problem.

It is worth comparing the above theorems with the results obtained by Nguyen and Véron [25] concerning the limit, when $k \to \infty$ of the solutions $u = u_k$ of

$$\begin{aligned} \partial_t u - \Delta u + u(\ln(u+1)))^\alpha = 0 \qquad &\text{in} \quad Q_\infty, \\ u(0,.) = k\delta_0 \quad &\text{in} \quad \mathbb{R}^N, \end{aligned} \tag{1.27}$$

where $\alpha > 0$. Note that $u_k > 0$ and the sequence $\{u_k\}$ is increasing. In this problem, they proved that the diffusion is dominating if $0 < \alpha \le 1$ and the limit of the u_k is infinite. If $1 < \alpha \le 2$ the absorption dominates, but the limit of the u_k is the maximal solution of the associated ODE, $y' + y(\ln(y+1)))^\alpha = 0$ on $\mathbb{R}_+$. Finally, if $\alpha > 2$ the limit of the u_k is a solution with a strong isolated singularity at $(0,0)$, which could be called a very singular solution, although it is not self-similar.

This paper is organized as follows. In Section 2 we introduce some properties of Marcinkiewicz spaces and Kato's type inequality for non-homogeneous problems. In Section 3 we prove Theorem 1.1. Section 4 is devoted to investigate the properties of solutions to (1.15). In Section 5 we give the proof of Theorem 1.2 and Theorem 1.3. Finally, we prove Theorem 1.4.

2 Linear estimates

2.1 The Marcinkiewicz spaces

We recall the definition and basic properties of the Marcinkiewicz spaces.

Definition 2.1 *Let $\Theta \subset \mathbb{R}^{N+1}$ be an open domain and μ be a positive Borel measure in Θ. For $\kappa > 1$, $\kappa' = \kappa/(\kappa-1)$ and $u \in L^1_{loc}(\Theta, d\mu)$, we set*

$$\|u\|_{M^\kappa(\Theta,d\mu)} = \inf\left\{c \in [0,\infty] : \int_E |u| d\mu \le c \left(\int_E d\mu\right)^{\frac{1}{\kappa'}}, \ \forall E \subset \Theta, \ E \text{ Borel set}\right\} \tag{2.1}$$

and

$$M^\kappa(\Theta, d\mu) = \{u \in L^1_{loc}(\Theta, d\mu) : \|u\|_{M^\kappa(\Theta,d\mu)} < \infty\}. \tag{2.2}$$

$M^\kappa(\Theta, d\mu)$ is called the Marcinkiewicz space of exponent κ or weak L^κ space and $\|.\|_{M^\kappa(\Theta,d\mu)}$ is a quasi-norm. The following property holds.

Proposition 2.1 *[3, 15] Assume that $1 \le q < \kappa < \infty$ and $u \in L^1_{loc}(\Theta, d\mu)$. Then there exists $c_5 > 0$ dependent of q, κ such that*

$$\int_E |u|^q d\mu \le c_5 \|u\|_{M^\kappa(\Theta,d\mu)} \left(\int_E d\mu\right)^{1-q/\kappa},$$

for any Borel set E of Θ.

Remark 2.1 If Ω is a smooth domain of $\mathbb{R}^N$, we denote by $H^\Omega_\alpha : (0,\infty) \times \Omega \times \Omega \to \mathbb{R}_+$ the heat kernel for $(-\Delta)^\alpha$ and, if $\nu \in \mathfrak{M}^b(\Omega)$, by $\mathbb{H}^\Omega_\alpha[\nu]$ the corresponding heat potential of ν defined by

$$\mathbb{H}^\Omega_\alpha[\nu](t,x) = \int_\Omega H^\Omega_\alpha(t,x,y) d\nu(y).$$

When $\Omega = \mathbb{R}^N$, by Fourier transform, it is clear that

$$H_\alpha(t,x,y) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{i(x-y)\cdot\zeta - t|\zeta|^{2\alpha}} d\zeta = H_\alpha(t, x-y, 0).$$

Furthermore, $\|H_\alpha(t,.,0)\|_{L^1}$ is independent of t. This implies

$$\|\mathbb{H}^\Omega_\alpha[\nu](t,.)\|_{L^p} \le \|\nu\|_{L^p}, \qquad \forall 1 \le p \le \infty, \ \forall \nu \in L^p(\mathbb{R}^N). \tag{2.3}$$

Since $\mathbb{H}^\Omega_\alpha[\nu](t+s,.) = \mathbb{H}^\Omega_\alpha[\mathbb{H}^\Omega_\alpha[\nu](s,.)](t,.)$ for all $t, s > 0$ (semigroup property) and $\nu \ge 0 \Longrightarrow \mathbb{H}^\Omega_\alpha[\nu](t,.) \ge 0$ the semigroup $\{\mathbb{H}^\Omega_\alpha[.](t,.)\}_{t\ge 0}$ is sub-Markovian. Furthermore, since the operator $(-\Delta)^\alpha$ is symmetric in $L^2(\mathbb{R}^N)$, the above semigroup is analytic in $L^p(\mathbb{R}^N)$ for all $1 \leq p < \infty$: if $1 < p < \infty$ it follows from a general result of Stein [27]) and for $p = 1$ it is a consequence of regularity result from fractional powers of operators theory (see e.g. [22]). For $1 \leq p < \infty$ the generator A_p of the semigroup in $L^p(\mathbb{R}^N)$ is the operator $-(-\Delta)^\alpha$ with domain

$$D(A_p) := \{\nu \in L^p(\mathbb{R}^N) : (-\Delta)^\alpha \nu \in L^p(\mathbb{R}^N)\}. \tag{2.4}$$

and $D(A_p)$ is dense since it contains $C_0^\infty(\mathbb{R}^N)$. If $p = \infty$, the natural space is the space $C_0(\mathbb{R}^N)$ of continuous functions in $\mathbb{R}^N$ tending to 0 at infinity. The domain of the corresponding operator A_{c_0} is

$$D(A_{c_0}) := \{\nu \in C_0(\mathbb{R}^N) : (-\Delta)^\alpha \nu \in C_0(\mathbb{R}^N)\}. \tag{2.5}$$

This operator is densely defined in $C_0(\mathbb{R}^N)$. In order to avoid confusion, $C_c(\mathbb{R}^N)$ (resp. $C_c^\infty(\mathbb{R}^N)$) denotes the space of continuous (resp. C^∞) functions in $\mathbb{R}^N$ with compact support. It is a dense subset of $C_0(\mathbb{R}^N)$.

The following regularizing effect $L^p(\mathbb{R}^N) \mapsto L^q(\mathbb{R}^N)$ ($1 \leq p \leq q \leq \infty$) is valid for any submarkovian semigroup of contractions in all $L^p(\mathbb{R}^N)$-spaces which has a self-adjoint generator in $L^2(\mathbb{R}^N)$ (see e.g. [26]).

Proposition 2.2 *Assume $1 \leq p \leq q \leq \infty$, $p \neq \infty$. Then for any $\nu \in L^p(\mathbb{R}^N)$, $\mathbb{H}_\alpha[\nu](t,.) \in L^q(\mathbb{R}^N) \cap D(A_q)$ for all $t > 0$ and there holds, for some positive constant $c = c(\alpha, N, p, q)$,*

$$\|\mathbb{H}_\alpha[\nu](t,.)\|_{L^q(\mathbb{R}^N)} \leq \frac{c}{t^{\frac{N}{2\alpha}(\frac{1}{p}-\frac{1}{q})}} \|\nu\|_{L^q(\mathbb{R}^N)}. \tag{2.6}$$

Note also that the function $(t,x) \mapsto \mathbb{H}_\alpha[\nu](t,x)$ is C^∞ in Q_∞ as a result of the analyticity on the semigroup $\{\mathbb{H}_\alpha[.](t)\}_{t>0}$.

Proposition 2.3 *For any $\beta > -1$ and $T > 0$, there exists $c_6 > 0$ dependent of N, α, β such that for $\nu \in \mathfrak{M}^b(\Omega)$,*

$$\|\mathbb{H}_\alpha^\Omega[|\nu|]\|_{M^{p^*_\beta}(Q_T^\Omega, t^\beta dxdt)} \leq c_6 \|\nu\|_{\mathfrak{M}^b(\Omega)}, \tag{2.7}$$

where p^*_β is defined by (1.13) and $Q_T^\Omega = (0,T) \times \Omega$.

In order to prove this proposition, we introduce some notations. For $\lambda > 0$ and $y \in \Omega$, let us denote

$$A_\lambda^\Omega(y) = \{(t,x) \in Q_T^\Omega : H_\alpha^\Omega(t,x,y) > \lambda\} \text{ and } m_\lambda^\Omega(y) = \int_{A_\lambda^\Omega(y)} t^\beta dxdt.$$

We also set $A_\lambda^{\mathbb{R}^N} = A_\lambda$ and $m_\lambda^{\mathbb{R}^N} = m_\lambda$.

Lemma 2.1 *There exists $c_7 > 0$ such that for any $\lambda > 1$,*

$$A_\lambda(y) \subset (0,\ c_7\lambda^{-\frac{2\alpha}{N}}] \times B_{c_7\lambda^{-\frac{1}{N}}}(y), \tag{2.8}$$

where $B_r(y)$ is the ball with radius r and center y in $\mathbb{R}^N$.

Proof. We observe that $H_\alpha(t,x,y) = t^{-\frac{N}{2\alpha}}\Gamma_\alpha(1,(x-y)t^{-\frac{1}{2\alpha}})$, where Γ_α is the fundamental solution of (1.18). From [4] (see also[13] for an analytic proof), there exists $c_8 > 0$ such that

$$\Gamma_\alpha(1,z) \le \frac{c_8}{1+|z|^{N+2\alpha}}.$$

This implies in particular

$$H_\alpha(t,x,y) \le \frac{c_8 t^{-\frac{N}{2\alpha}}}{1+\left(t^{-\frac{1}{2\alpha}}|x-y|\right)^{N+2\alpha}}. \tag{2.9}$$

On the one hand, for $(t,x) \in A_\lambda(y)$, we have that

$$t^{-\frac{N}{2\alpha}}\Gamma_\alpha(1,0) \ge t^{-\frac{N}{2\alpha}}\Gamma_\alpha(1,(x-y)t^{-\frac{1}{2\alpha}}) > \lambda,$$

which implies

$$t < \Gamma_\alpha^{\frac{2\alpha}{N}}(1,0)\lambda^{-\frac{2\alpha}{N}}. \tag{2.10}$$

On the other hand, letting $r = |x-y|$,

$$\frac{c_8 t}{t^{1+\frac{N}{2\alpha}} + r^{N+2\alpha}} \ge t^{-\frac{N}{2\alpha}}\Gamma_\alpha(1,(x-y)t^{-\frac{1}{2\alpha}}) > \lambda,$$

then

$$r \le (c_8 t\lambda^{-1})^{\frac{1}{N+2\alpha}}, \tag{2.11}$$

which, together with (2.10), implies

$$r \le c_9\lambda^{-\frac{1}{N}},$$

for some $c_9 > 0$. □

Proof of Proposition 2.3. By Lemma 2.1, there exists $c_{10} > 0$ such that

$$m_\lambda(y) \le c_{10}\lambda^{-1-\frac{2\alpha(1+\beta)}{N}}.$$

Clearly

$$H_\alpha^\Omega(t,x,y) \le H_\alpha(t,x,y), \tag{2.12}$$

then for any Borel set $E \subset Q_T^\Omega$ and $y \in \Omega$, we have that

$$\int_E H_\alpha^\Omega(t,x,y)t^\beta dxdt \le \lambda \int_E t^\beta dxdt + \int_{A_\lambda(y)} H_\alpha(t,x,y)t^\beta dxdt$$

and

$$\begin{aligned}\int_{A_\lambda(y)} H_\alpha(t,x,y)t^\beta dxdt &= -\int_\lambda^{+\infty} s dm_s(y) = \lambda m_\lambda(y) + \int_\lambda^{+\infty} m_s(y)ds \\ &\le c_{10}\lambda^{-\frac{2\alpha(1+\beta)}{N}} + c_{10}\int_\lambda^{+\infty} s^{-1-\frac{2\alpha(1+\beta)}{N}} ds \\ &\le c_{11}\lambda^{-\frac{2\alpha(1+\beta)}{N}},\end{aligned}$$

where $c_{11} = c_{10}\left(1 + \frac{N}{2\alpha(1+\beta)}\right)$. As a consequence, it follows

$$\int_E H_\alpha^\Omega(t,x,y)t^\beta dxdt \le \lambda \int_E t^\beta dxdt + c_{11}\lambda^{-\frac{2\alpha(1+\beta)}{N}}.$$

Taking $\lambda = (\int_E t^\beta dxdt)^{-\frac{N}{N+2\alpha(1+\beta)}}$, we obtain that

$$\int_E H_\alpha^\Omega(t,x,y)t^\beta dxdt \le (c_{11}+1)(\int_E t^\beta dxdt)^{\frac{2\alpha(1+\beta)}{N+2\alpha(1+\beta)}}. \tag{2.13}$$

Since, by Fubini's theorem,

$$\begin{aligned}\int_E \mathbb{H}_\alpha^\Omega[|\nu|](t,x)t^\beta dxdt &= \int_E\int_\Omega H_\alpha^\Omega(t,x,y)d|\nu(y)|t^\beta dxdt \\ &= \int_\Omega\int_E H_\alpha^\Omega(t,x,y)t^\beta dxdtd|\nu(y)|,\end{aligned}$$

together with (2.13), it yields

$$\int_E \mathbb{H}_\alpha^\Omega[|\nu|](t,x)t^\beta dxdt \le (c_{11}+1)\|\nu\|_{\mathfrak{M}^b(\Omega)}\left(\int_E t^\beta dxdt\right)^{\frac{2\alpha(1+\beta)}{N+2\alpha(1+\beta)}}.$$

Thus,

$$\|\mathbb{H}_\alpha^\Omega[|\nu|]\|_{M^{1+\frac{2\alpha(1+\beta)}{N}}(Q_T^\Omega, t^\beta dxdt)} \le (c_{11}+1)\|\nu\|_{\mathfrak{M}^b(\Omega)},$$

which ends the proof. □

2.2 The non-homogeneous problem

In this section we consider the linear non-homogeneous problem

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u &= \mu \quad \text{in} \quad Q_T, \\ u(0,\cdot) &= \nu \quad \text{in} \quad \mathbb{R}^N. \end{aligned} \tag{2.14}$$

If $\mu \in L^1(Q_T)$ and $\nu \in L^1(\mathbb{R}^N)$ a function u defined in Q_T is an *integral solution* of (2.14) in Q_T if it is expressed by Duhamel's formula, that is

$$u(t,x) = \mathbb{H}_\alpha[\nu](t,x) + \mathcal{H}_\alpha[\mu](t,x) \qquad \text{a.e. in } Q_T. \tag{2.15}$$

where, we denote by $\mathcal{H}_\alpha$ the operator of $L^1(Q_T)$ defined for all $(x,t) \in Q_T$ by

$$\mathcal{H}_\alpha[\mu](x,t) = \int_0^t \mathbb{H}_\alpha[\mu(.,s)](x,t-s)ds = \int_0^t \int_{\mathbb{R}^N} H_\alpha(t-s,x,y)\mu(s,y)dyds. \tag{2.16}$$

Notice that, by Duhamel's formula, there holds

$$\|u(t,\cdot)\|_{L^1(\mathbb{R}^N)} \le \|\mu\|_{L^1(Q_T)} + \|\nu\|_{L^1(\mathbb{R}^N)}, \quad \forall t \in (0,T), \tag{2.17}$$

and

$$\|u\|_{L^1(Q_T)} \le T(\|\mu\|_{L^1(Q_T)} + \|\nu\|_{L^1(\mathbb{R}^N)}). \tag{2.18}$$

The advantage of this notion of solution is that Duhamel's formula has a meaning as soon as μ and ν are integrable in their respective domains of definition. As for any continuous semigroup of bounded linear operators, a strong solution is an integral solution.

The following proposition is the Kato's type estimate which is essential tool to prove the uniqueness of solutions to (1.1). For $T > 0$, we denote $Q_T = (0,T) \times \mathbb{R}^N$.

Proposition 2.4 *Assume $\mu \in L^1(Q_T)$ and $\nu \in L^1(\mathbb{R}^N)$. Then there exists a unique weak solution $u \in L^1(Q_T)$ to the problem (2.14) and there exists $c_{12} > 0$ such that*

$$\int_{Q_T} |u|dxdt \le c_{12} \int_{Q_T} |\mu|dxdt + c_{12} \int_{\mathbb{R}^N} |\nu|dx. \tag{2.19}$$

Moreover, for any $\xi \in \mathbb{Y}_{\alpha,T}$, $\xi \ge 0$, we have that

$$\begin{aligned} \int_{Q_T} |u|(-\partial_t \xi + (-\Delta)^\alpha \xi)dxdt + \int_{\mathbb{R}^N} |u(T,x)|\xi(T,x)dx \\ \le \int_{Q_T} \xi \text{sign}(u)\mu dxdt + \int_{\mathbb{R}^N} \xi(0,x)|\nu|dx \end{aligned} \tag{2.20}$$

and

$$\begin{aligned}\int_{Q_T} u_+(-\partial_t\xi+(-\Delta)^\alpha\xi)dxdt+\int_{\mathbb{R}^N}u_+(T,x)\xi(T,x)dx\\ \le \int_{Q_T}\xi\mathrm{sign}_+(u)\mu dxdt+\int_{\mathbb{R}^N}\xi(0,x)\nu_+dx.\end{aligned}\tag{2.21}$$

In order to prove Proposition 2.4, we introduce the following notations. We say that $u: Q_T\to\mathbb{R}$ is in $C^{\sigma,\sigma'}_{t,x}(Q_T)$ for $\sigma,\sigma'\in(0,1)$ if

$$\|u\|_{C^{\sigma,\sigma'}_{t,x}(Q_T)}:=\|u\|_{L^\infty(Q_T)}+\sup_{Q_T}\frac{|u(t,x)-u(s,y)|}{|t-s|^\sigma+|x-y|^{\sigma'}}<+\infty$$

and $u\in C^{1+\sigma,2\alpha+\sigma'}_{t,x}(Q_T)$ if

$$\|u\|_{C^{1+\sigma,2\alpha+\sigma'}_{t,x}(Q_T)}:=\|u\|_{L^\infty(Q_T)}+\|\partial_t u\|_{C^{\sigma,\sigma'}_{t,x}(Q_T)}+\|(-\Delta)^\alpha u\|_{C^{\sigma,\sigma'}_{t,x}(Q_T)}<+\infty.$$

Lemma 2.2 *Let $\mu\in C^1(Q_T)\cap L^\infty(Q_T)$, $\nu\in L^\infty(\mathbb{R}^N)$ and u be an integral solution of problem (2.14), then there exists $\sigma\in(0,1)$ such that $u\in C^{1+\sigma,2\alpha+\sigma}_{t,x}$ in $(\epsilon,T)\times\mathbb{R}^N$ for any $\epsilon\in(0,T)$. In particular, if $\|D^2\nu\|_{L^\infty(\mathbb{R}^N)}+\|(-\Delta)^\alpha\nu\|_{C^{1-\alpha}_x(\mathbb{R}^N)}<\infty$, then $u\in C^{1+\sigma,2\alpha+\sigma}_{t,x}(Q_T)$.*

Proof. *Step 1.* When $\|D^2\nu\|_{L^\infty(\mathbb{R}^N)}+\|(-\Delta)^\alpha\nu\|_{C^{1-\alpha}_x(\mathbb{R}^N)}<\infty$, it follows directly by [9, $(A.1)$] that $u\in C^{1+\sigma,2\alpha+\sigma}_{t,x}(Q_T)$.

Step 2. When $\nu\in L^\infty(\mathbb{R}^N)$, we use [10, Theorem 6.1] to obtain that $u\in C^{\frac{\sigma}{2\alpha},\sigma}_{t,x}(Q_T)$ for some $\sigma>0$. For any $\epsilon\in(0,T)$, let $\eta:[0,T]\to[0,1]$ be a C^2 function such that $\eta=0$ in $[0,\frac{\epsilon}{4}]$ and $\eta=1$ in $[\epsilon,T]$ and $v=\eta u$ in Q_T. Since η does not depend on x, we obtain that v satifies,

$$\partial_t v+(-\Delta)^\alpha v=\eta\mu+\eta'(t)u,\qquad\forall(t,x)\in Q_T,$$

where $\eta\mu+\eta'(t)u\in C^{\frac{\sigma}{2\alpha},\sigma}_{t,x}(Q_T)$ and $v(0,\cdot)=0$ in $\mathbb{R}^N$, Then we apply the argument in Step 1 to obtain that $v\in C^{1+\sigma,2\alpha+\sigma}_{t,x}(Q_T)$. Therefore, u is $C^{1+\sigma,2\alpha+\sigma}_{t,x}$ in $(\epsilon,T)\times\mathbb{R}^N$. The proof is complete. □

Lemma 2.3 *(i) Let $\mu\in C^1(Q_T)\cap L^\infty(Q_T)$ and $\nu\in C^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, then problem (2.14) admits a unique classical solution u.*
(ii) Let $\mu\in C^1(Q_T)\cap L^\infty(Q_T)\cap L^1(Q_T)$, $\nu\in C^2(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)$ and u be the classical solution of (2.14), then u is $C^{1+\sigma,2\alpha+\sigma}_{t,x}$ in $(\epsilon,T)\times\mathbb{R}^N$ for any $\epsilon\in(0,T)$

and for any $\xi\in\mathbb{Y}_{\alpha,T}$*,*

$$\begin{aligned}&\int_{Q_T}u(t,x)[-\partial_t\xi(t,x)+(-\Delta)^\alpha\xi(t,x)]dxdt\\&\qquad=\int_{Q_T}\mu(t,x)\xi(t,x)dxdt+\int_{\mathbb{R}^N}\xi(0,x)\nu dx-\int_{\mathbb{R}^N}\xi(T,x)u(T,x)dx.\end{aligned}\tag{2.22}$$

Thus u *is a weak solution and it belongs to* $\mathbb{Y}_{\alpha,T}$*.*
(iii) Let $\tilde{\mu}\in C^1(Q_T)\cap L^\infty(Q_T)$ *and* $\nu\in C^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$*, then problem*

$$\begin{aligned}-\partial_t w+(-\Delta)^\alpha w=\tilde{\mu}\quad&\text{in}\quad Q_T,\\ w(T,\cdot)=\nu\quad&\text{in}\quad\mathbb{R}^N\end{aligned}\tag{2.23}$$

admits a unique classical solution $w\in C^{1+\sigma,2\alpha+\sigma}_{t,x}(Q_T)$ *for some* $\sigma\in(0,1)$*. Moreover, if* $\mu\in C^1(Q_T)\cap L^\infty(Q_T)\cap L^1(Q_T)$ *and* $\nu\in C^2(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)$*, then* ξ *is a weak solution and it belongs to* $\mathbb{Y}_{\alpha,T}$*.*

Proof. *(i)* By [10, Theorem 3.3, Theorem 6.1], if μ and ν are continuous and bounded, there exists a unique viscosity solution $u\in C(\overline{Q}_T)$. The higher regularity is provided by [10, Theorem 6.1] which asserts that there exist $\sigma>0$ and a positive constant c depending on N, $\tau\in(0,T)$ and α such that for all (t,x) and (s,y) belonging to $Q^{B_1}_{T-\tau}$, there holds

$$\frac{|\ u(t,x)-u(s,y)\ |}{(|x-y|+|t-s|^{\frac{1}{2\alpha}})^\sigma}\le c\left(\|u\|_{L^\infty(Q^{B_2}_T)}+\sup_{0\le t\le T}\|u(t,.)\|_{L^1(\mathbb{R}^N}+\|\mu\|_{L^\infty(Q_T)}\right)\tag{2.24}$$

where $Q^\Omega_T=(0,T)\times\Omega$. Thus $u\in C^{\frac{\sigma}{2\alpha},\sigma}_{t,x}(Q_T)$. By Lemma 2.2 the integral solution u belongs to $C^{1+\sigma',2\alpha+\sigma'}_{t,x}$ in $(\epsilon,T)\times\mathbb{R}^N$ for any $\epsilon\in(0,T)$ and some $\sigma'\in(0,\min\{\frac{\sigma}{2\alpha},\sigma\})$. Then u is a classical solution of (2.14) and thus a viscosity solution.
(ii) By the definition of $(-\Delta)^\alpha u$, $u(t,\cdot)\in L^1(\mathbb{R}^N)$ for all $t\in(0,T)$. As in [9, Appendix A.2] we have Duhamel formula, thus $u\in L^1(Q_T)$ and it is an integral solution.
We claim that $\|(-\Delta)^\alpha_\epsilon u(t,\cdot)\|_{L^\infty(\mathbb{R}^N)}$ *is uniformly bounded with respect to* $\epsilon\in(0,\epsilon_0)$*.* Since $u(t,\cdot)\in C^{2\alpha+\sigma}_x(\mathbb{R}^N)$ for some $\sigma\in(0,\min\{2-2\alpha,1\})$, then for $x\in\mathbb{R}^N$ and

$y \in B_1(0)$, $|u(x+y)+u(x-y)-2u(x)| \le \|u(t,\cdot)\|_{C_x^{2\alpha+\sigma}(\mathbb{R}^N)}|y|^{2\alpha+\sigma}$. Thus,

$$\begin{aligned}\||(-\Delta)_\epsilon^\alpha u(t,\cdot)|\|_{L^\infty(\mathbb{R}^N)} &\le \sup_{x\in\mathbb{R}^N}\left[\int_{\mathbb{R}^N\setminus B_1(0)}\frac{|u(x+y)-u(x)|}{|y|^{N+2\alpha}}dy\right.\\ &\quad\left.+\frac{1}{2}\int_{B_1(0)\setminus B_\epsilon(0)}\frac{|u(x+y)+u(x-y)-2u(x)|}{|y|^{N+2\alpha}}dy\right]\\ &\le 2\|u\|_{L^1(\mathbb{R}^N)}+\int_{B_1(0)}|y|^{\sigma-N}dy\|u(t,\cdot)\|_{C_x^{2\alpha+\sigma}(\mathbb{R}^N)}.\end{aligned}$$

Next we claim that

$$\int_{Q_T}\xi(-\Delta)_\epsilon^\alpha u dxdt = \int_{Q_T}u(-\Delta)_\epsilon^\alpha \xi dxdt \qquad \forall \xi\in\mathbb{Y}_{\alpha,T}. \tag{2.25}$$

Indeed, using the fact that for any $t>0$ there holds

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[u(t,z)-u(t,x)]\xi(t,x)}{|z-x|^{N+2\alpha}}\chi_\epsilon(|x-z|)dzdx\\ &\qquad = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[u(t,x)-u(t,z)]\xi(t,z)}{|z-x|^{N+2\alpha}}\chi_\epsilon(|x-z|)dzdx,\end{aligned}$$

then we have

$$\begin{aligned}&\int_{\mathbb{R}^N}\xi(t,x)(-\Delta)_\epsilon^\alpha u(t,x)dx\\ &= -\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left[\frac{(u(t,z)-u(t,x))\xi(t,x)}{|z-x|^{N+2\alpha}}+\frac{(u(t,x)-u(t,z))\xi(t,z)}{|z-x|^{N+2\alpha}}\right]\chi_\epsilon(|x-z|)dzdx\\ &= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[u(t,z)-u(t,x)][\xi(t,z)-\xi(t,x)]}{|z-x|^{N+2\alpha}}\chi_\epsilon(|x-z|)dzdx.\end{aligned}$$

Similarly,

$$\int_{\mathbb{R}^N}u(t,x)(-\Delta)_\epsilon^\alpha\xi(t,x)dx = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[u(t,z)-u(t,x)][\xi(t,z)-\xi(t,x)]}{|z-x|^{N+2\alpha}}\chi_\epsilon(|x-z|)dzdx.$$

Then (2.25) holds. Since u is $C_{t,x}^{1+\sigma,2\alpha+\sigma}$ in $(\epsilon,T)\times\mathbb{R}^N$ for any $\epsilon\in(0,T)$ and ξ belongs to $\mathbb{Y}_{\alpha,T}$, $(-\Delta)_\epsilon^\alpha\xi(t,\cdot)\to(-\Delta)^\alpha\xi(t,\cdot)$ and $(-\Delta)_\epsilon^\alpha u(t,\cdot)\to(-\Delta)^\alpha u(t,\cdot)$ as $\epsilon\to 0$ in $\mathbb{R}^N$ and $(-\Delta)_\epsilon^\alpha\xi(t,\cdot)$, $(-\Delta)_\epsilon^\alpha u(t,\cdot)\in L^\infty(\mathbb{R}^N)$ and $\xi(t,\cdot), u(t,\cdot)\in L^1(\mathbb{R}^N)$, then it follows by the Dominated Convergence Theorem that

$$\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^N}\xi(t,x)(-\Delta)_\epsilon^\alpha u(t,x)dx = \int_{\mathbb{R}^N}\xi(t,x)(-\Delta)^\alpha u(t,x)dx$$

and

$$\lim_{\epsilon\to0^+}\int_{\mathbb{R}^N}(-\Delta)^\alpha_\epsilon\xi(t,x)u(t,x)dx=\int_{\mathbb{R}^N}(-\Delta)^\alpha\xi(t,x)u(t,x)dx.$$

Combining this with (2.25), and letting $\epsilon\to0^+$, we have that

$$\int_{\mathbb{R}^N}\xi(t,x)(-\Delta)^\alpha u(t,x)dx=\int_{\mathbb{R}^N}(-\Delta)^\alpha\xi(t,x)u(t,x)dx,$$

integrating over $[0,T]$ and by (2.14), we conclude that (2.22) holds.

(iii) End of the proof. Let u be the weak solution of problem (2.14) obtained from (ii) with $\tilde\mu(T-t,.)=\mu(t,.)$ and

$$w(t,x)=u(T-t,x)\qquad (t,x)\in[0,T]\times\mathbb{R}^N.$$

Then w is a solution of (2.23) and for some $\sigma\in(0,1)$, w is $C^{1+\sigma,2\alpha+\sigma}_{t,x}(Q_T)$. On the contrary, if w is a solution of (2.23), then $u(t,x)=w(T-t,x)$ for $(t,x)\in[0,T]\times\mathbb{R}^N$ is a solution of (2.14), then the uniqueness holds since the solution of (2.14) is unique. Since $u\in C^{1+\sigma,2\alpha+\sigma}_{t,x}(Q_T)$, then $(-\Delta)^\alpha u(t,\cdot)\in C^\sigma_x$ and then $(-\Delta)^\alpha_\epsilon u(t,\cdot)$ is bounded, which implies $u\in\mathbb{Y}_{\alpha,T}$. □

Proof of Proposition 2.4. *Uniqueness.* Let $v\in L^1(Q_T)$ be a weak solution of

$$\begin{aligned}\partial_t v+(-\Delta)^\alpha v&=0\quad\text{in}\quad Q_T,\\ v(0,\cdot)&=0\quad\text{in}\quad \mathbb{R}^N.\end{aligned}\tag{2.26}$$

We claim that $v=0$ a.e. in Q_T.

In fact, let ω be a Borel subset of Q_T and $\eta_{\omega,n}$ be the solution of

$$\begin{aligned}-\partial_t u+(-\Delta)^\alpha u&=\zeta_n\quad\text{in}\quad Q_T,\\ u(T,\cdot)&=0\quad\text{in}\quad \mathbb{R}^N,\end{aligned}\tag{2.27}$$

where $\zeta_n:\bar Q_T\to[0,1]$ is a function $C^1_c(Q_T)$ such that

$$\zeta_n\to\chi_\omega\quad\text{in } L^\infty(\bar Q_T)\quad\text{as } n\to\infty.$$

Then $\eta_{\omega,n}\in\mathbb{Y}_{\alpha,T}$ by Lemma 2.3, and

$$\int_{Q_T}v\zeta_n dxdt=0.$$

Passing to the limit when $n\to\infty$, we derive

$$\int_\omega vdxdt=0.$$

This implies $v = 0$ a.e. in Q_T.

Existence and estimate (2.20). For $\delta > 0$, we define an even convex function ϕ_δ by

$$\phi_\delta(t) = \begin{cases} |t| - \frac{\delta}{2} & \text{if} \quad |t| \geq \delta, \\ \frac{t^2}{2\delta} & \text{if} \quad |t| < \delta/2. \end{cases} \tag{2.28}$$

Then for any $t, s \in \mathbb{R}$, $|\phi'_\delta(t)| \leq 1$, $\phi_\delta(t) \to |t|$ and $\phi'_\delta(t) \to \text{sign(t)}$ when $\delta \to 0^+$. Moreover,

$$\phi_\delta(s) - \phi_\delta(t) \geq \phi'_\delta(t)(s - t). \tag{2.29}$$

Let $\{\mu_n\}$, $\{\nu_n\}$ be two sequences of functions in $C_0^2(Q_T)$, $C_0^2(\mathbb{R}^N)$, respectively, such that

$$\lim_{n\to\infty} \int_{Q_T} |\mu_n - \mu| dxdt = 0, \quad \lim_{n\to\infty} \int_{\mathbb{R}^N} |\nu_n - \nu| dx = 0.$$

We denote by u_n the corresponding solution to (2.14) where μ, ν are replaced by μ_n, ν_n, respectively. By Lemma 2.2 and Lemma 2.3(ii), $u_n \in C_{t,x}^{1+\sigma, 2\alpha+\sigma}(Q_T) \cap L^1(Q_T)$ and then we use Lemma 2.3 in [15] and Lemma 2.3 (ii) to obtain that for any $\delta > 0$ and $\xi \in \mathbb{Y}_{\alpha,T}$, $\xi \geq 0$,

$$\begin{aligned} \int_{Q_T} \phi_\delta(u_n)[-\partial_t\xi + (-\Delta)^\alpha\xi] dxdt &+ \int_{\mathbb{R}^N} \xi(T, x)\phi_\delta(u_n(T, x))dx \\ &= \int_{Q_T} \xi[\partial_t\phi_\delta(u_n) + (-\Delta)^\alpha\phi_\delta(u_n)]dxdt + \textstyle\int_{\mathbb{R}^N} \xi(0, x)\phi_\delta(\nu_n)dx \\ &\leq \int_{Q_T} \xi\phi'_\delta(u_n)[\partial_t u_n + (-\Delta)^\alpha u_n]dxdt + \int_{\mathbb{R}^N} \xi(0, x)\phi_\delta(\nu_n)dx \\ &= \int_{Q_T} \xi\phi'_\delta(u_n)\mu_n dxdt + \int_{\mathbb{R}^N} \xi(0, x)\phi_\delta(\nu_n)dx. \end{aligned}$$

Letting $\delta \to 0^+$, we obtain

$$\begin{aligned} \int_{Q_T} |u_n|[-\partial_t\xi + (-\Delta)^\alpha\xi]dxdt &+ \int_{\mathbb{R}^N} \xi(T, x)|u_n(T, x)|dx \\ &\leq \int_{Q_T} \xi \text{sign}(u_n)\mu_n dxdt + \int_{\mathbb{R}^N} \xi(0, x)|\nu_n|dx. \end{aligned} \tag{2.30}$$

Let η_k be the solution of

$$\begin{aligned} -\partial_t u + (-\Delta)^\alpha u = \varsigma_k \quad &\text{in} \quad Q_T, \\ u(T, \cdot) = 0 \quad &\text{in} \quad \mathbb{R}^N, \end{aligned} \tag{2.31}$$

where $\varsigma_k : Q_T \to [0,1]$ is a C_0^2 function such that $\varsigma_k = 1$ in $(0,T)\times B_k(0)$. From the proof of Lemma 2.3, $\tilde{\eta}_k(t,x) := \eta_k(T-t,x)$ satisfies with $\tilde{\varsigma}_k(t,x) = \varsigma_k(T-t,x)$

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u = \tilde{\varsigma}_k \quad &\text{in} \quad Q_T, \\ u(0,\cdot) = 0 \quad &\text{in} \quad \mathbb{R}^N. \end{aligned}$$

By Lemma 2.2, $\tilde{\eta}_k \in C_{t,x}^{1+\sigma,2\alpha+\sigma}(Q_T)$ with some $\sigma \in (0,1)$ and

$$\begin{aligned} 0 \le \tilde{\eta}_k(t,x) &\le c_8 \int_t^T \int_{\mathbb{R}^N} \frac{(s-t)^{-\frac{N}{2\alpha}}}{1+|(s-t)^{-\frac{1}{2\alpha}}(y-x)|^{N+2\alpha}} dyds \\ &\le c_8 \int_t^T \int_{\mathbb{R}^N} \frac{dz}{1+|z|^{N+2\alpha}} ds \\ &= c_{13}(T-t). \end{aligned}$$

Taking $\xi = \eta_k$ in (2.30), we derive that

$$\int_{Q_T} |u_n| \chi_{(0,T)\times B_k(0)} dxdt \le c_{13}T \int_{Q_T} |\mu_n| dxdt + c_{13}T \int_{\mathbb{R}^N} |\nu_n| dx.$$

Then, letting $k \to \infty$, we have

$$\int_{Q_T} |u_n| dxdt \le c_{13}T \int_{Q_T} |\mu_n| dxdt + c_{13}T \int_{\mathbb{R}^N} |\nu_n| dx. \tag{2.32}$$

Similarly,

$$\int_{Q_T} |u_n - u_m| dx \le c_{13}T \int_{Q_T} |\mu_n - \mu_m| dxdt + c_{13}T \int_{\mathbb{R}^N} |\nu_n - \nu_m| dx. \tag{2.33}$$

Therefore, $\{u_n\}_n$ is a Cauchy sequence in $L^1(Q_T)$ and its limit u is a weak solution of (2.14). Letting $n \to \infty$, (2.20) and (2.19) follow by (2.30) and (2.32), respectively. The proof of (2.21) is similar. □

Remark 2.2 Other classes of uniqueness of solutions of the fractional heat equations exist. In [2] it is proved that any positive strong solution $u \in C([0,T)\times\mathbb{R}^N)$ can be represented by the convolution integral defined by

$$u(t,x) = \int_{\mathbb{R}^N} P_t(y) u(0,y) dy$$

where

$$P_t(x) = \frac{1}{t^{\frac{N}{2\alpha}}} e^{it^{-\frac{1}{2\alpha}} x.\xi - |\xi|^{2\alpha}}.$$

However the fact that $u \in L^1(Q_T)$ is a part of the definition of strong solution therein. Furthermore, the notion of weak solution used in this paper differs from ours.

3 Proof of Theorem 1.1

If $h(t,.)$ is monotone nondecreasing, for any $\lambda > 0$, $I + \lambda h(t,.)$ is an homeorphism of $\mathbb{R}$ and the inverse function $J_\lambda(t,.) = (I + \lambda h(t,.))^{-1}$ is a contraction. We define the Yosida approximation by

$$h_\lambda(t,.) = \frac{I - J_\lambda(t,.)}{\lambda}. \tag{3.1}$$

The function $h_\lambda(t,.)$ is monotone nondecreasing, vanishes at 0 as h does it and it is $\frac{1}{\lambda}$-Lipschitz continuous. Furthermore

$$rh_\lambda(t,r) \uparrow rh(t,r) \quad \text{as } \lambda \to 0, \qquad \forall r \in \mathbb{R}, \tag{3.2}$$

see [5, Chap 2, Prop. 2.6]. If u is a real valued function we will denote by $h \circ u$ and $h_\lambda \circ u$ respectively the functions $(t,x) \mapsto h(t,u(t,x))$ and $(t,x) \mapsto h_\lambda(t,u(t,x))$.

Lemma 3.1 *Assume that h satisfies (H)-(i), $\lambda > 0$ and $\phi \in L^1(\mathbb{R}^N)$. Then there exists a unique solution u_ϕ of*

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + h_\lambda \circ u = 0 \quad &\text{in} \quad Q_\infty, \\ u(0,\cdot) = \phi \quad &\text{in} \quad \mathbb{R}^N. \end{aligned} \tag{3.3}$$

Moreover,

$$\mathbb{H}_\alpha[\phi] - \mathcal{H}_\alpha[h_\lambda \circ \mathbb{H}_\alpha[\phi_+])] \le u_\phi \le \mathbb{H}_\alpha[\phi] - \mathcal{H}_\alpha[h_\lambda \circ (-\mathbb{H}_\alpha[\phi_-])] \quad \text{in } Q_T, \tag{3.4}$$

where $\phi_\pm = \max\{0, \pm\phi\}$ *and*

$$\|u_\phi(t,.) - u_\psi(t,.)\|_{L^1} \le \|\phi - \psi\|_{L^1}, \qquad \forall 1 \le p \le \infty. \tag{3.5}$$

(i) $u_\phi \ge 0$ *if* $\phi \ge 0$ *in* Ω;
(ii) the mapping $\phi \mapsto u_\phi$ *is increasing.*

Proof. Existence is a consequence of the Cauchy-Lipschitz-Picard theorem (see [11, Chap 4]): we write (3.3) under the integral form $u = \mathcal{T}[u] = \mathbb{H}_\alpha[\phi] - \mathcal{H}_\alpha[h_\lambda \circ u]$, i.e.

$$\mathcal{T}[u](t,.) = \mathbb{H}_\alpha[\phi](t,.) - \int_0^t \mathbb{H}_\alpha[h_\lambda \circ u](t-s,.)ds. \tag{3.6}$$

The space $C([0,\infty); L^1(\mathbb{R}^N))$ endowed with the norm

$$\|w\|_{C-L^1} = \sup\left\{ e^{-kt} \|w(t,.)\|_{L^1} : t \ge 0 \right\},$$

$(k > \lambda^{-1})$, is a Banach space. Since $u \mapsto h_\lambda(t,u)$ is $\frac{1}{\lambda}$-Lipschitz continuous, the mapping $\mathcal{T}$ is $\frac{1}{\lambda k}$-Lipschitz continuous in X_p. Thus it admits a unique fixed point u_ϕ which is an integral solution of (3.3).

$$u_\phi(t,.) = \mathbb{H}_\alpha[\phi](t,.) - \int_0^t \mathbb{H}_\alpha[h_\lambda \circ u_\phi](t-s,.)ds. \tag{3.7}$$

The semigroup $\{\mathbb{H}_\alpha[.](t,.)\}_{t\geq 0}$ is analytic in $L^1(\mathbb{R}^N)$ since it is generated by the fractional power of a closed operator. It follows from the classical regularity theory for analytic semigroups as exposed in [20, Sec 6] that u_ϕ is a strong solution of (3.3). Since it is continuous, it is also a weak solution in the sense that

$$\begin{aligned}\int_{Q_T} & (u_\phi[-\partial_t\xi + (-\Delta)^\alpha\xi] + \xi h_\lambda \circ u_\phi)\, dxdt \\ &= \int_{\mathbb{R}^N} \xi(0,x)\phi(x)dx - \int_{\mathbb{R}^N} \xi(T,x)u_\phi(T,x)dx \qquad \forall \xi \in \mathbb{Y}_{\alpha,T}.\end{aligned} \tag{3.8}$$

If $\phi_1, \phi_2 \in L^1(\mathbb{R}^N)$ and u_{ϕ_j} are the corresponding solutions of (3.3), it follows from the positivity of H_α that

$$(u_{\phi_2} - u_{\phi_1})_+ \leq (\mathcal{H}_\alpha[h_\lambda \circ u_{\phi_2} - h_\lambda \circ u_{\phi_1}])_+ \leq \frac{1}{\lambda}\mathcal{H}_\alpha[(u_{\phi_2} - u_{\phi_1})_+].$$

Therefore,

$$\|(u_{\phi_2}(t,.) - u_{\phi_1}(t,.))_+\|_{L^p} \leq \frac{1}{\lambda}\int_0^t \|(u_{\phi_2}(t-s) - u_{\phi_1}(t-s))_+\|_{L^p}ds,$$

and by Gronwall inequality

$$\|(u_{\phi_2}(t) - u_{\phi_1}(t))_+\|_{L^p} \leq e^{\frac{t}{\lambda}}\|(\phi_2 - \phi_1)_+\|_{L^p}.$$

This implies (i) and (ii). As a consequence,

$$-\mathbb{H}_\alpha[\phi_-] \leq -u_{\phi_-} \leq u_\phi \leq u_{\phi_+} \leq \mathbb{H}_\alpha[\phi_+]$$

and thus

$$h_\lambda \circ (-\mathbb{H}_\alpha[\phi_-]) \leq h_\lambda \circ (-u_{\phi_-}) \leq h_\lambda \circ u_\phi \leq h_\lambda \circ u_{\phi_+} \leq h_\lambda \circ \mathbb{H}_\alpha[\phi_+].$$

Jointly with (3.7) it yields (3.4). □

Notation. In the sequel, if $\eta \in L^1(Q_\tau)$ and $\tau \geq T$, we denote by $\xi_{\eta,\tau}$ the solution of

$$\begin{aligned} -\partial_t\xi_\eta + (-\Delta)^\alpha\xi_\eta &= \eta \qquad \text{in } Q_\tau, \\ \xi_\eta(\tau,.) &= 0. \end{aligned} \tag{3.9}$$

If $\eta \geq 0$, then $\xi_{\eta,\tau} \geq 0$; if $\eta \in C_0^\infty(\mathbb{R}^{N+1})$, then $\eta \in \mathbb{Y}_{\alpha,\tau}$; if $\eta_n = \eta(\frac{\cdot}{n})$, where $n \in \mathbb{N}_*$ and $\eta \in C_0^\infty(\mathbb{R}^{N+1})$ is nonnegative, $0 \leq \eta \leq 1$, with value 1 on B_1 and 0 on B_2^c, then $\xi_{\eta_n,\tau} \uparrow \tau - t$ as $n \to \infty$.

In the next lemma we prove that we can replace h_λ by h.

Lemma 3.2 *Assume that h satisfies (H)-(i) and $\phi \in L^1(\mathbb{R}^N)$. Then there exists a unique solution $u_\phi \in C([0,\infty); L^1(\mathbb{R}^N)$ of*

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + h \circ u = 0 \quad &\text{in} \quad Q_\infty, \\ u(0,\cdot) = \phi \quad &\text{in} \quad \mathbb{R}^N. \end{aligned} \tag{3.10}$$

Moreover inequality (3.5) and statements (i) and (ii) in Lemma 3.1 hold.

Proof. We denote by $u_{\lambda,\phi}$ the solution of (3.3).
Step 1- A priori estimate. Let $\phi \geq 0$. If we take $\xi = \xi_{\eta_n,\tau}$ in (3.8) and let $n \to \infty$, we derive

$$\int_{Q_T} (u_{\lambda,\phi} + (\tau - t) h_\lambda \circ u_{\lambda,\phi})\, dxdt + (\tau - T) \int_{\mathbb{R}^N} u_{\lambda,\phi}(T,x) dx = \tau \int_{\mathbb{R}^N} \phi(x) dx. \tag{3.11}$$

For $0 < \lambda < \lambda'$ we set $w = u_{\lambda,\phi} - u_{\lambda',\phi}$. It follows from (2.21) and inequality $h_{\lambda'} \circ u_{\lambda,\phi} \leq h_\lambda \circ u_{\lambda,\phi}$, that for any nonnegative ξ in $\mathbb{Y}_{\alpha,T}$,

$$\begin{aligned} &\int_{Q_T} \left(w_+[-\partial_t \xi + (-\Delta)^\alpha \xi] + \xi \left(h_\lambda \circ u_{\lambda,\phi} - h_\lambda \circ u_{\lambda',\phi} \right) \operatorname{sign}_+(w) \right) dxdt \\ &\qquad \leq \int_{Q_T} w_+ \left(h_{\lambda'} \circ u_{\lambda',\phi} - h_\lambda \circ u_{\lambda',\phi} \right) dxdt - \int_{\mathbb{R}^N} \xi(T,x) w_+(T,x) dx, \end{aligned}$$

Since $h_\lambda(t,.)$ is nondecreasing, we derive

$$\int_{Q_T} w_+[-\partial_t \xi + (-\Delta)^\alpha \xi] dxdt \leq 0 \qquad \forall \xi \in \mathbb{Y}_{\alpha,T},\ \xi \geq 0.$$

If $\eta \in C_0^\infty(\mathbb{R}^{N+1})$ is nonnegative, then $\xi_\eta \in \mathbb{Y}_{\alpha,T}$, $\xi_\eta \geq 0$ and

$$\int_{Q_T} w_+ \eta dxdt = 0.$$

This implies $u_{\lambda,\phi} \leq u_{\lambda',\phi}$.
Step 2- Truncation. We replace ϕ by $\phi_n = \inf\{\phi, n\}$ for $n \in \mathbb{N}_*$ and denote by u_{λ,ϕ_n} the corresponding solution of (3.3). By Step 1, the sequence $\{u_{\lambda,\phi_n}\}_{\lambda>0}$ is decreasing

and it converges to some nonnegative u_{ϕ_n} when $\lambda \downarrow 0$. Therefore $h_\lambda \circ u_{\lambda,\phi_n} \to h \circ u_{\phi_n}$ a.e. in Q_T. It follows from (3.11) and Fatou's lemma that

$$\int_{Q_T} (u_{\phi_n} + (\tau - t) h \circ u_{\phi_n})\, dxdt + (\tau - T) \int_{\mathbb{R}^N} u_{\phi_n}(.,T) dx = \tau \int_{\mathbb{R}^N} \phi_n(x) dx. \quad (3.12)$$

Since $0 \leq u_{\lambda,\phi_n} \leq n$, then $0 \leq h_\lambda \circ u_{\lambda,\phi_n} \leq h \circ u_{\lambda,\phi_n} \leq h(n)$ by (3.5). If $E \subset Q_T$ is a Borel set,

$$\int_E h_\lambda \circ u_{\lambda,\phi_n} dxdt \leq h(n)|E|.$$

By Vitali convergence theorem $h_\lambda \circ u_{\lambda,\phi_n} \to h \circ u_{\phi_n}$ in $L^1(Q_T)$. Therefore, we can let $\lambda \to 0$ in identity (3.8) and conclude that u_{ϕ_n} is a weak solution of (3.10) with initial data ϕ_n.

Step 3- Existence with ϕ bounded. If $\phi = \phi_+ - \phi_- \in L^1(\mathbb{R}^N)$, set $\phi_{+,n} = \inf\{\phi_+, n\}$ and $\phi_{-,n} = \inf\{\phi_-, n\}$. We denote by $u_{\lambda,\phi_{+,n}}$, $u_{\phi_{+,n}}$, $u_{\lambda,-\phi_{-,n}}$ and $u_{-\phi_{-,n}}$ the corresponding solutions of (3.3) and (3.10). Then

$$\begin{gathered} u_{\lambda,-\phi_{-,n}} \leq u_{\lambda,\phi_{+,n}-\phi_{-,n}} \leq u_{\lambda,\phi_{+,n}} \\ \text{which implies} \\ h_\lambda \circ u_{\lambda,-\phi_{-,n}} \leq h_\lambda \circ u_{\lambda,\phi_{+,n}-\phi_{-,n}} \leq h_\lambda \circ u_{\lambda,\phi_{+,n}}. \end{gathered} \quad (3.13)$$

Estimate (3.11) is valid under the form

$$\begin{aligned} \int_{Q_T} &\left(u_{\lambda,\phi_{+,n}} + (\tau - t) h_\lambda \circ u_{\lambda,\phi_{+,n}}\right) dxdt \\ &+ (\tau - T) \int_{\mathbb{R}^N} u_{\lambda,\phi_{+,n}}(.,T) dx = \tau \int_{\mathbb{R}^N} \phi_{+,n}(x) dx. \end{aligned} \quad (3.14)$$

and

$$\begin{aligned} \int_{Q_T} &\left(u_{\lambda,-\phi_{-,n}} + (\tau - t) h_\lambda \circ u_{\lambda,-\phi_{-,n}}\right) dxdt \\ &+ (\tau - T) \int_{\mathbb{R}^N} u_{\lambda,-\phi_{-,n}}(.,T) dx = -\tau \int_{\mathbb{R}^N} \phi_{-,n}(x) dx. \end{aligned} \quad (3.15)$$

Since $h_\lambda \circ u_{\lambda,\phi_{+,n}}$ and $h_\lambda \circ u_{\lambda,-\phi_{-,n}}$ are bounded in $L^1(Q_T)$ independently of λ and n, $h_\lambda \circ u_{\lambda,\phi_{+,n}-\phi_{-,n}}$ inherits the same property. Since

$$u_{\lambda,\phi_{+,n}-\phi_{-,n}} = \mathbb{H}_\alpha[\phi_{+,n} - \phi_{-,n}] - \mathcal{H}_\alpha[h_\lambda \circ u_{\lambda,\phi_{+,n}-\phi_{-,n}}],$$

it follows from [20, Sec 6] that $u_{\lambda,\phi_{+,n}-\phi_{-,n}}$ remains bounded in the interpolation space $Y_1 := L^1([0,T]; D(A_1)(\mathbb{R}^N)) \cap W^{s,1}([0,T]; L^1(\mathbb{R}^N))$, for any $s \in (0,1)$, where

$D(A_1)$ is defined in (2.4). Although a bounded subset K of Y_1 is not a relatively compact subset of $L^1(Q_T)$, for any ball $B \subset \mathbb{R}^N$, the set of restrictions to B of functions belonging to K is relatively compact in $L^1((0,T)\times B)$. Thus, there exists a subsequence $\{\lambda_k\}$ such that $\{u_{\lambda_k,\phi_{+,n}-\phi_{-,n}}\}$ converges a.e. to some function U_n. Furthermore $\{h_{\lambda_k}\circ u_{\lambda_k,\phi_{+,n}-\phi_{-,n}}\}$ converges a.e. to $h\circ U_n$. Since the sequences $\{u_{\lambda_k,-\phi_{-,n}}\}_{\lambda_k}$, $\{u_{\lambda_k,\phi_{+,n}}\}_{\lambda_k}$, $\{h_{\lambda_k}\circ u_{\lambda_k,-\phi_{-,n}}\}_{\lambda_k}$ and $\{h_{\lambda_k}\circ u_{\lambda_k,\phi_{+,n}}\}_{\lambda_k}$ are convergent in $L^1(Q_T)$ they are uniformly integrable. Because of (3.13) the same property is shared by the two sequences $\{u_{\lambda_k,\phi_{+,n}-\phi_{-,n}}\}_{\lambda_k}$ and $\{h_{\lambda_k}\circ u_{\lambda_k,\phi_{+,n}-\phi_{-,n}}\}_{\lambda_k}$. Letting λ_k to 0 in the identity

$$u_{\lambda_k,\phi_{+,n}-\phi_{-,n}}(t,.) = \mathbb{H}_\alpha[\phi_{+,n}-\phi_{-,n}](t,.) - \int_0^t \mathbb{H}_\alpha[h_{\lambda_k}\circ u_{\lambda_k,\phi_{+,n}-\phi_{-,n}}](t-s,.)ds \tag{3.16}$$

yields

$$U_n(t,.) = \mathbb{H}_\alpha[\phi_{+,n}-\phi_{-,n}](t,.) - \int_0^t \mathbb{H}_\alpha[h\circ U_n](t-s,.)ds. \tag{3.17}$$

This implies that U_n is an integral solution, thus a weak solution of (3.10) with initial data $\phi_{+,n}-\phi_{-,n} = \mathrm{sgn}(\phi)\inf\{n,|\phi|\}$ and then $U_n = u_{\phi_n}$.

Step 4- Existence with $\phi \in L^1(\mathbb{R}^N)$. By Kato's inequality (2.20), we obtain that

$$\int_{Q_T} \left(|u_{\phi_k}-u_{\phi_m}|(-\partial_t\xi + (-\Delta)^\alpha\xi) + \xi|h\circ u_{\phi_k} - h\circ u_{\phi_m}|\right) dxdt \\ + \int_{\mathbb{R}^N} |u_{\phi_k}(T,x)-u_{\phi_m}(T,x)|\xi(T,x)dx \le \int_{\mathbb{R}^N}\xi(0,x)|\phi_k-\phi_m|dx,$$

for $m,k\in\mathbb{N}_*$ and $\xi\in\mathbb{Y}_{\alpha,T}$, $\xi>0$. Taking $\xi=\xi_{\eta_n,\tau}$ as in (3.9) and letting $n\to\infty$ yields

$$\int_{Q_T} \left(|u_{\phi_k}-u_{\phi_m}| + (\tau-t)|h\circ u_{\phi_k} - h\circ u_{\phi_m}|\right) dxdt \\ + (\tau-T)\int_{\mathbb{R}^N} |u_{\phi_k}(T,.)-u_{\phi_m}(T,.)|dx \le \tau\int_{\mathbb{R}^N}|\phi_k-\phi_m|dx. \tag{3.18}$$

Since $\{\phi_m\}$ is a Cauchy sequence in $L^1(\mathbb{R}^N)$, $\{u_{\phi_m}\}$ and $\{h\circ u_{\phi_m}\}$ are also Cauchy sequences in $C(0,T;L^1(\mathbb{R}^N))$ and $L^1(Q_T)$ respectively. Set $U=\lim_{m\to\infty}u_{\phi_m}$, then it satisfies

$$\int_{Q_T} \left(U[-\partial_t\xi + (-\Delta)^\alpha\xi] + \xi h\circ U\right) dxdt \\ = \int_{\mathbb{R}^N}\xi(0,x)\phi(x)dx - \int_{\mathbb{R}^N}\xi(T,x)U(T,x)dx \qquad \forall\xi\in\mathbb{Y}_{\alpha,T}, \tag{3.19}$$

and it is also an integral solution of (3.10). Thus $u_\phi \in C([0,\infty); L^1(\mathbb{R}^N))$.

Finally, we end the proof of uniqueness which is a consequence of the inequality below

$$\begin{aligned}\int_{Q_T} &(|U-U'|+(\tau-t)|h\circ U - h\circ U'|)\,dxdt\\ &\qquad + (\tau-T)\int_{\mathbb{R}^N}|U(T,.)-U'(T,.)|dx \le \tau\int_{\mathbb{R}^N}|\phi-\phi'|dx,\end{aligned} \tag{3.20}$$

valid for two solutions U and U' of problem (3.10) with respective initial data ϕ and ϕ', the proof of which is the same as the one of (3.18). Notice also that statement (i) and (ii) as well as inequality (3.5) follows by the above approximations. □

Remark 3.1 *By the same method it can be proved that for any $p \in (1,\infty)$ and $\phi \in L^p(\mathbb{R}^N)$ (resp. $\phi \in C_0(\mathbb{R}^N)$) there exists a unique solution $u_\phi \in C([0,\infty); L^p(\mathbb{R}^N))$ (resp. $u_\phi \in C([0,\infty); C_0(\mathbb{R}^N))$) solution of (3.10). Furthermore (3.5) holds.*

Proof of Theorem 1.1. *Existence for $\nu \ge 0$.* We consider a sequence of nonnegative functions $\{\nu_n\}_n \subset C_0^2(\mathbb{R}^N)$ such that $\nu_n \to \nu$ as $n \to \infty$ in the weak sense of bounded measures, i.e.

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\zeta\nu_n dx = \int_{\mathbb{R}^N}\zeta d\nu \qquad \forall \zeta \in C(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N). \tag{3.21}$$

It follows from the Banach-Steinhaus theorem that $\|\nu_n\|_{\mathfrak{M}^b(\mathbb{R}^N)}$ is bounded independently of n and we assume that $\|\nu_n\|_{\mathfrak{M}^b(\mathbb{R}^N)} \le 2\|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)}$. By Lemma 3.1, we denote by u_{ν_n} the corresponding solution of (3.10) with initial data ν_n. Then u_n is nonnegative and satisfies that

$$0 \le u_{\nu_n} = \mathbb{H}_\alpha[\nu_n] - \mathcal{H}_\alpha[h\circ u_{\nu_n}] \le \mathbb{H}_\alpha[\nu_n] \quad \text{in } Q_T. \tag{3.22}$$

Jointly with (2.7) it implies

$$\|u_{\nu_n}\|_{M^{p^*_\beta}(Q_T,t^\beta dxdt)} \le c_5\|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)}. \tag{3.23}$$

We have also the following estimates from (2.9) and (3.12)

$$u_{\nu_n}(t,x) \le \mathbb{H}_\alpha[\nu_n](t,x) \le 2c_8 t^{-\frac{N}{2\alpha}}\|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)}, \qquad \forall (t,x)\in Q_T \tag{3.24}$$

and

$$\begin{aligned}\int_{Q_T}(u_{\nu_n}+(\tau-t)h\circ u_{\nu_n})\,dxdt + (\tau-T)\int_{\mathbb{R}^N}u_{\nu_n}(.,T)dx &= \tau\int_{\mathbb{R}^N}\nu_n(x)dx\\ &\le 2\tau\|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)}.\end{aligned} \tag{3.25}$$

As in the proof of Lemma 3.2-Step 3, using the regularizing properties of the semi-group $\mathbb{H}_\alpha[.](t)$ (see [20, Sec 6]) we infer that there exists a subsequence $\{u_{\nu_{n_k}}\}$ which converges a.e. in Q_T to some function U and $\{h\circ u_{\nu_{n_k}}\}$ converges a.e. to $h\circ U$.

For $\kappa>0$, we denote $S_\kappa=\{(t,x)\in Q_T: |u_{n_k}(t,x)|>\kappa\}$ and $\omega(\kappa)=\int_{S_\kappa}t^\beta dxdt$. Then for any Borel set $E\subset Q_T$

$$\begin{aligned}\iint_E h\circ u_{\nu_{n_k}}dxdt &\leq \iint_{E\cap\{u_{\nu_{n_k}}\leq\kappa\}} h\circ u_{\nu_{n_k}}dxdt+\iint_{E\cap S_\kappa} h\circ u_{\nu_{n_k}}dxdt\\ &\leq g(\kappa)\iint_E t^\beta dxdt+\iint_{\{S_\kappa} t^\beta g(u_{\nu_{n_k}})dxdt\\ &\leq g(\kappa)\iint_E t^\beta dxdt-\int_\kappa^\infty g(s)d\omega(s),\end{aligned}$$

where

$$\int_\kappa^\infty g(s)d\omega(s)=\lim_{M\to\infty}\int_\kappa^M g(s)d\omega(s).$$

By (2.1) and (3.23), $\omega(s)\leq c_{14}s^{-p^*_\beta}$, thus

$$\begin{aligned}-\int_\kappa^M g(s)d\omega(s) &= -\Big[g(s)\omega(s)\Big]_{s=\kappa}^{s=M}+\int_\kappa^M \omega(s)dg(s)\\ &\leq g(\kappa)\omega(\kappa)-g(M)\omega(M)+c_{14}\int_\kappa^M s^{-p^*_\beta}dg(s)\\ &\leq g(\kappa)\omega(\kappa)-g(M)\omega(M)+c_{14}\left(M^{-p^*_\beta}g(M)-\kappa^{-p^*_\beta}g(\kappa)\right)\\ &\qquad +\frac{c_{14}}{p^*_\beta+1}\int_\kappa^M s^{-1-p^*_\beta}g(s)ds.\end{aligned}$$

Since $\lim_{M\to\infty}M^{-p^*_\beta}g(M)=0$ by (1.12) and [15, Lemma 4.1] and $\omega(s)\leq c_{14}s^{-p^*_\beta}$, we derive $g(\kappa)\omega(\kappa)\leq c_{14}\kappa^{-p^*_\beta}g(\kappa)$ and then

$$-\int_\kappa^\infty g(s)d\omega(s)\leq\frac{c_{14}}{p^*_\beta+1}\int_\kappa^\infty s^{-1-p^*_\beta}g(s)ds.$$

The above quantity on the right-hand side tends to 0 when $\kappa\to\infty$. The conclusion follows: for any $\epsilon>0$ there exists $\kappa>0$ such that

$$\frac{c_{14}}{p^*_\beta+1}\int_\kappa^\infty s^{-1-p^*_\beta}g(s)ds\leq\frac{\epsilon}{2}$$

and there exists $\delta>0$ such that

$$\int_E t^\beta dxdt\leq\delta\Longrightarrow g(\kappa)\int_E t^\beta dxdt\leq\frac{\epsilon}{2}.$$

This means that $\{h_{n_k} \circ u_{\nu_{n_k}}\}$ is uniformly integrable in $L^1(Q_T)$ and by Vitali convergence theorem $h_{n_k} \circ u_{\nu_{n_k}} \to h \circ U$ in $L^1(Q_T)$. Letting $n_k \to \infty$ in the identity

$$u_{\nu_{n_k}}(t,.) = \mathbb{H}_\alpha[\nu_{n_k}](t,.) - \int_0^t \mathbb{H}_\alpha[h \circ u_{\nu_{n_k}}(s,.)](t-s,.)ds$$

for some $t > 0$ such that $u_{\nu_{n_k}}(t,.) \to U(t,.)$ a.e. in $\mathbb{R}^N$ yields

$$U(t,.) = \mathbb{H}_\alpha[\nu](t,.) - \int_0^t \mathbb{H}_\alpha[h \circ U(s,.)](t-s,.)ds.$$

This is valid for almost all $t > 0$ and implies that $U \in C([0,T]; L^1(\mathbb{R}^N))$, up to a modification on a set of $t > 0$ with zero measure. Moreover

$$\int_{Q_T} \left(u_{\nu_{n_k}}(-\partial_t \xi + (-\Delta)^\alpha \xi) + \xi h \circ u_{\nu_{n_k}} \right) dxdt$$
$$= \int_{\mathbb{R}^N} \xi(0,x)\nu_{n_k} dx - \int_{\mathbb{R}^N} u_{\nu_{n_k}}(T,x)\xi(T,x)dx.$$

where $\xi \in \mathbb{Y}_{\alpha,T}$ is arbitrary. Thus, using the continuity of $t \mapsto U(t,.)$ in $L^1(\mathbb{R}^N)$, we derive

$$\int_{Q_T} (U(-\partial_t \xi + (-\Delta)^\alpha \xi) + \xi h \circ U) \, dxdt$$
$$= \int_{\mathbb{R}^N} \xi(0,x)d\nu(x) - \int_{\mathbb{R}^N} U(T,x)\xi(T,x)dx.$$

From this we infer that U is a weak solution of (1.1).
Existence for general ν. For $\nu \in \mathfrak{M}^b(\mathbb{R}^N)$, a sequence $\{\nu_n\}$ in $C_0^2(\mathbb{R}^N)$ converge to ν in the weak sense of bounded measures. Because of the monotonicity of $h(t,\cdot)$,

$$-\mathbb{H}_\alpha[|\nu_n|] \le u_{-|\nu_n|} \le u_{\nu_n} \le u_{|\nu_n|} \le \mathbb{H}_\alpha[|\nu_n|].$$

Then by above analysis, the sequence $\{h \circ u_{-|\nu_n|})\}$ and $\{h \circ u_{|\nu_n|})\}$ are relatively compact in $L^1(Q_T^B)$ for any $T > 0$ and ball B and (3.23) holds for $\{u_{\nu_n}\}$. Therefore $\{u_{\nu_n}\}$ is relatively locally compact in $L^1(Q_T^B)$ and there exist some subsequence $\{u_{\nu_{n_k}}\}$ and $U \in L^1(Q_T)$ such that

$$u_{\nu_{n_k}} \to U \Longrightarrow h \circ u_{\nu_{n_k}} \to h \circ U \quad \text{as } k \to \infty \quad \text{a.e. in} \quad Q_T.$$

As in the previous case it implies that U is a weak solution of (1.1) and also an integral solution.

Uniqueness. Let u_1, u_2 be two weak solutions of (1.1) with the same initial ν and $w = u_1 - u_2$. Then

$$\partial_t w + (-\Delta)^\alpha w = h \circ u_2 - h \circ u_1 \quad \text{in} \ \ Q_T.$$

Since $h \circ u_2 - h \circ u_1 \in L^1(Q_T)$, then by (2.20), for $\xi \in \mathbb{Y}_{\alpha,T}$, $\xi \geq 0$, we have that

$$\int_{Q_T} |w|[-\partial_t \xi + (-\Delta)^\alpha \xi] dxdt + \int_{\mathbb{R}^N} |w(T,x)|\xi(T,x) dxdt \\ + \int_{Q_T} (h \circ u_2 - h \circ u_1) \mathrm{sign}(w) \xi dxdt \leq 0.$$

This implies $w = 0$ by monotonicity.

Statements (i) and (ii) and inequality (1.14) follows from the fact that the same relation holds for u_{ν_n} by Lemma 3.2.

Stability is proved by the same approach that existence. If $\{\nu_n\}$ converges to ν in the weak sense of measures, then $\|\nu_n\|_{\mathfrak{M}^b}$ is bounded independently of n. Since the distribution function of $h \circ u_{\nu_n}$ depends only on the supremum of $\|\nu_n\|_{\mathfrak{M}^b}$, this set of functions is uniformly integrable in Q_T. This, combined with local compactness of the set $\{u_{\nu_n}\}$ in $L^1(Q_T)$, implies the convergence of a subsequence $(u_{\nu_{n_k}}, h \circ u_{\nu_{n_k}})$ to $(u_\nu, h \circ u_\nu)$ where u_ν is the solution of (1.1). Because of uniqueness, all converging subsequences have the same limit, which imply the convergence of the whole sequence and stability. □

4 Dirac mass as initial data

In this section, we study the properties of solutions to (1.1) when $h(t,r) = t^\beta r^p$ with $\beta > -1$ and $0 < p < p^*_\beta$ and the initial data is $\nu = k\delta_0$ with $k > 0$.

Proposition 4.1 *Assume $0 < p < p^*_\beta$ and that u_k is the solution of (1.15), then there exists $c_{15} > 0$ such that*

$$\lim_{t \to 0^+} t^{\frac{N}{2\alpha}} u_k(t,0) = c_{15} k. \tag{4.1}$$

Proof. By (1.14) it follows that

$$u_k(t,0) \leq k\mathbb{H}_\alpha[\delta_0](t,0) = k\Gamma_\alpha(t,0), \qquad t > 0. \tag{4.2}$$

We claim that there exists $c_{16} > 0$ independent of k such that

$$u_k(t,0) \geq k\Gamma_\alpha(t,0) - c_{16} k^p t^{-\frac{N}{2\alpha}p + 1 + \beta}, \quad t \in (0, 1/2). \tag{4.3}$$

Indeed, from (1.14), it is infered that

$$u_k(t,0) \geq k\Gamma_\alpha(t,0) - k^p W(t,0), \qquad t \in (0,1/2),$$

where

$$W(t,x) = \int_0^t \mathbb{H}_\alpha[s^\beta(\mathbb{H}_\alpha^p[\delta_0]](t-s,x)ds, \qquad (t,x) \in Q_\infty.$$

For $t \in (0,1/4)$, there exists $c_{17}, c_{18} > 0$ such that

$$\begin{aligned}
W(t,0) &\leq c_{17} \int_0^t \int_{\mathbb{R}^N} \frac{(t-s)^{-\frac{N}{2\alpha}} s^\beta}{1 + ((t-s)^{-\frac{1}{2\alpha}}|y|)^{N+2\alpha}} \left(\frac{s^{-\frac{N}{2\alpha}}}{1 + (s^{-\frac{1}{2\alpha}}|y|)^{N+2\alpha}} \right)^p dyds \\
&\leq c_{17} \int_0^t \int_{\mathbb{R}^N} \frac{s^{\beta - \frac{N}{2\alpha}p} dzds}{\left(1 + \left((\frac{t-s}{s})^{\frac{1}{2\alpha}}|z|\right)^{(N+2\alpha)p}\right)(1+|z|^{N+2\alpha})} \\
&\leq c_{17} t^{\beta+1-\frac{Np}{2\alpha}} \int_0^1 \int_{\mathbb{R}^N} \frac{d\tau dZ}{\left(1 + (\frac{1-\tau}{\tau})^{\frac{(N+2\alpha)p}{2\alpha}} |Z|^{(N+2\alpha)p}\right)(1+|Z|^{N+2\alpha})} \\
&\leq c_{18} t^{\beta+1-\frac{Np}{2\alpha}}.
\end{aligned}$$

Combining (1.19) and $-\frac{N}{2\alpha}p + 1 + \beta > -\frac{N}{2\alpha}$, we obtain that

$$\lim_{t \to 0^+} t^{\frac{N}{2\alpha}} W(t,0) = 0.$$

Therefore, (4.1) holds. □

In what follows we consider the limit of the solution $\{u_k\}$ of (1.15) as $k \to \infty$ for $p \in (0,1]$.

Proposition 4.2 *Assume* $0 < p \leq 1$ *and that* u_k *is the solution of (1.15), then*

$$\lim_{k \to \infty} u_k = \infty \quad \text{in} \quad Q_\infty,$$

locally uniformly in Q_∞*.*

Proof. We observe that $\mathbb{H}_\alpha[\delta_0]$ and $\mathbb{H}_\alpha[t^\beta(\mathbb{H}_\alpha[\delta_0])^p]$ are positive in $(0,\infty) \times \mathbb{R}^N$. By (1.14), for $p \in (0,1)$ and $(t,x) \in (0,\infty) \times \mathbb{R}^N$, we have that

$$u_k \;\; \geq \;\; k\mathbb{H}_\alpha[\delta_0] - k^p W \Longrightarrow \lim_{k \to \infty} u_k = \infty.$$

For $p = 1$, it is obvious that $u_k = ku_1$ and $u_1 > 0$ in $(0,\infty) \times \mathbb{R}^N$, then

$$\lim_{k \to \infty} u_k = \infty \quad \text{in} \quad Q_\infty.$$

The proof is complete. □

Now we deal with the range $p \in (1, p^*_\beta)$.

Lemma 4.1 *Assume* $1 < p < p^*_\beta$ *and that* u_k *is the solution of (1.15). Then for any* $k > 0$,

$$0 \le u_k \le U_p \quad \text{in} \quad Q_\infty, \tag{4.4}$$

where U_p *is given by (1.21).*

Proof. Let $\{f_{n,k}\}$ be a sequence of nonnegative functions in $C^1_c(\mathbb{R}^N)$ which converges to $k\delta_0$ in the weak sense of measures as $n \to \infty$. We denote by $u_{n,k}$ the corresponding solution of (1.17) with initial data by $f_{n,k}$.

We claim that

$$u_{n,k} \le U_p \quad \text{in} \quad Q_\infty, \tag{4.5}$$

where, we recall it, U_p is the maximal solution of the ODE $y' + t^\beta y^p = 0$ on $\mathbb{R}_+$. Indeed this implies (4.4).

Step 1. We claim that

$$\lim_{|x|\to\infty} u_{n,k}(t,x) = 0, \qquad \forall t > 0. \tag{4.6}$$

From [13, 17], there exists $c_8 > 0$ such that for any $x, y \in \mathbb{R}^N$ and $t \in (0,\infty)$,

$$0 < \Gamma_\alpha(t, x-y) \le \frac{c_8 t^{-\frac{N}{2\alpha}}}{1 + (|x-y|t^{-\frac{1}{2\alpha}})^{N+2\alpha}}.$$

Then for $|x| > 1$,

$$\begin{aligned} 0 \le \mathbb{H}_\alpha[f_{n,k}](t,x) &\le c_8 t^{-\frac{N}{2\alpha}} \int_{\mathbb{R}^N} \frac{f_{n,k}(y)}{1 + (|x-y|t^{-\frac{1}{2\alpha}})^{N+2\alpha}} dy \\ &= c_8 \int_{\mathbb{R}^N} \frac{f_{n,k}(x - zt^{\frac{1}{2\alpha}})}{1 + |z|^{N+2\alpha}} dz \\ &= c_8 \left(\int_{\mathbb{R}^N \setminus B_R} \frac{f_{n,k}(x - zt^{\frac{1}{2\alpha}})}{1 + |z|^{N+2\alpha}} dz + \int_{B_R} \frac{f_{n,k}(x - zt^{\frac{1}{2\alpha}})}{1 + |z|^{N+2\alpha}} dz \right), \end{aligned}$$

where $R = \frac{1}{2}|x|t^{-\frac{1}{2\alpha}}$ and $B_R = \{z \in \mathbb{R}^N : |z| < R\}$. It is obvious that

$$|x - zt^{\frac{1}{2\alpha}}| \ge |x| - |z|t^{\frac{1}{2\alpha}} \ge |x|/2 \quad \text{for all } z \in B_R.$$

Then

$$\begin{aligned} \int_{B_R} \frac{f_{n,k}(x - zt^{\frac{1}{2\alpha}})}{1 + |z|^{N+2\alpha}} dz &\le \sup_{|y| \ge \frac{|x|}{2}} f_{n,k}(y) \int_{B_R} \frac{1}{1 + |z|^{N+2\alpha}} dz \\ &\le \sup_{|y| \ge \frac{|x|}{2}} f_{n,k}(y) \int_{\mathbb{R}^N} \frac{1}{1 + |z|^{N+2\alpha}} dz \\ &= c_{16} \sup_{|y| \ge \frac{|x|}{2}} f_{n,k}(y) \end{aligned}$$

and

$$\int_{\mathbb{R}^N\setminus B_R} \frac{f_{n,k}(x - zt^{\frac{1}{2\alpha}})}{1+|z|^{N+2\alpha}}dz \le \int_{\mathbb{R}^N\setminus B_R} \frac{\|f_{n,k}\|_{L^\infty(\mathbb{R}^N)}}{1+|z|^{N+2\alpha}}dz \le c_{18}R^{-2\alpha} = \frac{c_{18}t}{|x|^{2\alpha}},$$

for some $c_{18} > 0$ independent of x, t and R. Since $f_{n,k} \in C_0^1(\mathbb{R}^N)$, we have that

$$\lim_{|x|\to\infty} \sup_{|y|\ge \frac{|x|}{2}} f_{n,k}(y) = 0$$

and then for any $t > 0$, $0 \le u_{n,k}(t, x) \le \mathbb{H}_\alpha[f_{n,k}](t, x) \to 0$ as $|x| \to \infty$.

Step 2. We claim that (4.5) holds. By contradiction, if (4.5) is not verified, there exists $(t_0, x_0) \in (0,\infty) \times \mathbb{R}^N$ such that

$$(U_p - u_{n,k})(t_0, x_0) = \min_{(t,x)\in(0,\infty)\times\mathbb{R}^N} (U_p - u_{n,k})(t, x) < 0,$$

since $U_p(t) > 0 = \lim_{|x|\to\infty} u_{n,k}(t, x)$ for any $t \in (0,\infty)$, $U_p(0) = \infty > f_{n,k}(x) = u_{n,k}(0, x)$ for $x \in \mathbb{R}^N$ and $\lim_{t\to\infty} U_p(t) = \lim_{t\to\infty} u_{n,k}(t, x) = 0$ for $x \in \mathbb{R}^N$. Then $\partial_t(U_p - u_{n,k})(t_0, x_0) = 0$. Moreover,

$$\begin{aligned}(U_p - u_{n,k})(t_0, x_0) &= \min\{U_p(t_0) - u_{n,k}(t_0, x) : x \in \mathbb{R}^N\}\\ &= U_p(t_0) - \max\{u_{n,k}(t_0, x) : x \in \mathbb{R}^N\}\end{aligned}$$

and

$$u_{n,k}(t_0, x_0) = \max\{u_{n,k}(t_0, x) : x \in \mathbb{R}^N\} \Longrightarrow (-\Delta)^\alpha u_{n,k}(t_0, x_0) \ge 0.$$

Then

$$0 = \partial_t(U_p - u_{n,k})(t_0, x_0) - (-\Delta)^\alpha u_{n,k}(t_0, x_0) + t_0^\beta U_p^p(t_0) - t_0^\beta u_{n,k}^p(t_0, x_0) < 0,$$

which is impossible. Thus (4.5) holds. □

Proposition 4.3 *(i) Assume $0 < p < p_\beta^*$ and that u_k is the solution of (1.15). Then u_k is a classical solution of (1.17).*

(ii) Assume $1 < p < p_\beta^*$ and that u_∞ is defined by (1.16). Then u_∞ is a classical solution of (1.17).

Proof. (i) Since $u_k \le k\mathbb{H}_\alpha[\delta_0]$, it is infered that u_k is bounded in $(\epsilon,\infty) \times \mathbb{R}^N$ for $\epsilon > 0$. Let $\{g_{n,k}\}$ be a sequence of nonnegative functions in $C_0^1(\mathbb{R}^N)$ which converges to $k\delta_0$ as $n \to \infty$ and $u_{n,k}$ the corresponding solution of (1.17) with initial data $g_{n,k}$.

Then $\mathbb{H}_\alpha[g_{n,k}] \to k\mathbb{H}_\alpha[\delta_0]$ as $n \to \infty$ uniformly in $[\epsilon, \infty) \times \mathbb{R}^N$ for any $\epsilon > 0$ and by the Comparison Principle, there exists $c_{19} > 1$ such that

$$0 \le u_{n,k}(t,x) \le k\mathbb{H}_\alpha[g_{n,k}] \le c_{19} k\mathbb{H}_\alpha[\delta_0] \quad \text{in} \;\; [\epsilon, \infty) \times \mathbb{R}^N,$$

and there exists $\sigma \in (0,1)$ such that $\{u_{n,k}\}$ are uniformly bounded with respect to n in $C_{t,x}^{\frac{\sigma}{2\alpha},\sigma}((\epsilon,\infty)\times\mathbb{R}^N)$ with $\epsilon > 0$. Therefore, by the Arzela-Ascoli theorem, $u_{n,k}$ converges to u_k in $C_{t,x}^{\frac{\sigma'}{2\alpha},\sigma'}((\epsilon,\infty)\times\mathbb{R}^N)$ with $\sigma' \in (0,\sigma)$ and then u_k is a viscosity solution of (1.17) in $(\epsilon,\infty)\times\mathbb{R}^N$. By estimate $(A.1)$ in [9], u_k is in $C_{t,x}^{1+\sigma',2\alpha+\sigma'}((\epsilon,\infty)\times\mathbb{R}^N)$ and u_k is a classical solution of (1.17) in $(\epsilon,\infty)\times\mathbb{R}^N$.
(ii) The proof is the same as part (i), just replacing $u_k \le k\mathbb{H}_\alpha[\delta_0]$ by $u_\infty \le U_p$. □

5 Self-similar and very singular solutions

By Theorem 1.1 and (4.4), we see that $\{u_k\}$ is an increasing sequence of nonnegative functions bounded from above by U_p. Then for $p \in (1, p^*_\beta)$, there exists $u_\infty = \lim_{k\to\infty} u_k$, which is a classical solution of (1.17) by Proposition 4.3 (ii) and satisfies

$$u_\infty \le U_p \quad \text{in} \quad Q_\infty. \tag{5.1}$$

Proposition 5.1 *Assume* $1 < p < p^*_\beta$, *then* u_∞ *is a self-similar solution of (1.17).*

Proof. For $\lambda > 0$, we set

$$T_\lambda[u](t,x) = \lambda^{\frac{2\alpha(1+\beta)}{p-1}} u(\lambda^{2\alpha} t, \lambda x), \qquad (t,x) \in Q_\infty.$$

It is straightforward to verify that $T_\lambda[u_k]$ is the solution of

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + t^\beta u^p &= 0 && \text{in } Q_\infty, \\ u(0,.) &= \lambda^{\frac{2\alpha(1+\beta)}{p-1} - N} k\delta_0 && \text{in } \mathbb{R}^N. \end{aligned} \tag{5.2}$$

Because of uniqueness, $T_\lambda[u_k] = u_{k\lambda^{\frac{2\alpha(1+\beta)}{p-1}-N}}$. Letting $k \to \infty$ and using the continuity of $u \mapsto T_\lambda[u]$, we have that

$$\lim_{k\to\infty} T_\lambda[u_k] = T_\lambda[u_\infty] = u_\infty,$$

which implies that u_∞ is a self-similar solution (1.17). □

Let us denote

$$U_\infty(z) = u_\infty(1,z), \qquad z \in \mathbb{R}^N,$$

then U_∞ is a classical solution of (1.20). It is clear that the constant $(\frac{1+\beta}{p-1})^{\frac{1}{p-1}}$ is a constant positive solution of the self-similar equation (1.20). We observe that $N < \frac{2\alpha(1+\beta)}{p-1} < N+2\alpha$ when $1+\frac{2\alpha(1+\beta)}{N+2\alpha} < p < 1+\frac{2\alpha(1+\beta)}{N}$.

We prove below this fundamental result that u_∞ is the minimal self similar solution.

Proposition 5.2 *Assume that $1 < p < 1+\frac{2\alpha(1+\beta)}{N}$ and $\tilde{u}$ is a positive self-similar solution of (1.23). Then $u_\infty \le \tilde{u}$.*

Proof. For any $r > 0$, we have that

$$\begin{aligned}
\int_{B_r(0)} \tilde{u}(t,x)dx &= t^{-\frac{1+\beta}{p-1}} \int_{B_r(0)} \tilde{u}(1, t^{-\frac{1}{2\alpha}}x)dx \\
&= t^{-\frac{1+\beta}{p-1}+\frac{N}{2\alpha}} \int_{B_{t^{-\frac{1}{2\alpha}}r}(0)} \tilde{u}(1,z)dz \\
&\ge t^{-\frac{1+\beta}{p-1}+\frac{N}{2\alpha}} \int_{B_1(0)} \tilde{u}(1,z)dz \\
&\to +\infty \quad \text{as } t \to 0^+,
\end{aligned}$$

where last inequality holds for $t \in (0, r^{2\alpha}]$. Let $\{\epsilon_n\}$ be a sequence positive decreasing numbers converging to 0 as $n \to \infty$. For ϵ_n and $k > 0$, there exists $t_{n,k} > 0$ such that

$$\int_{B_{\epsilon_n}(0)} \tilde{u}(t_{n,k}, x)dx = k.$$

We observe that for any fixed k, $t_{n,k} \to 0$ as $n \to \infty$ since $\lim_{n\to\infty} \epsilon_n = 0$. Let $\eta_0 : \mathbb{R}^N \to [0,1]$ be a C^2 function such that $\operatorname{supp} \eta_0 \subset \bar{B}_2(0)$, $\eta_0 = 1$ in $B_1(0)$ and $\eta_n(x) = \eta_0(\epsilon_n^{-1}x)$ for $x \in \mathbb{R}^N$. Choosing $\{f_{n,k}\}$ be a sequence of C^2 functions such that

$$0 \le f_{n,k}(x) \le \eta_n(x)\tilde{u}(t_{n,k}, x), \qquad \forall x \in \mathbb{R}^N$$

and

$$f_{n,k} \to k\delta_0 \qquad \text{as} \quad n \to \infty.$$

Let $u_{n,k}$ be the solution of (1.1) with initial data $f_{n,k}$, then

$$u_{n,k}(t,x) \le u(t_{n,k}+t, x), \qquad \forall (t,x) \in Q_\infty$$

and by uniqueness of u_k, $\lim_{n\to\infty} u_{n,k} = u_k$, where u_k is the solution of (1.1) with initial data $k\delta_0$. Then for any k, we have $u_k \le \tilde{u}$ in Q_∞, which implies that

$$u_\infty \le \tilde{u} \quad \text{in} \quad Q_\infty.$$

$\square$

5.1 The case $1+\frac{2\alpha(1+\beta)}{N+2\alpha}<p<1+\frac{2\alpha(1+\beta)}{N}$

We define the function w_λ by

$$w_\lambda(t,x)=\lambda t^{-\frac{1+\beta}{p-1}}w(t^{-\frac{1}{2\alpha}}|x|),\qquad (t,x)\in Q_\infty, \tag{5.3}$$

where $w(s)=\frac{\ln(e+s^2)}{1+s^{N+2\alpha}}$.

Lemma 5.1 *Assume $1+\frac{2\alpha(1+\beta)}{N+2\alpha}<p<1+\frac{2\alpha(1+\beta)}{N}$, then there exists $\Lambda_0>0$ such that for $\lambda\geq\Lambda_0$,*

$$\partial_t w_\lambda(t,x)+(-\Delta)^\alpha w_\lambda(t,x)+t^\beta w_\lambda^p(t,x)\geq 0,\quad \forall (t,x)\in Q_\infty. \tag{5.4}$$

Proof. By direct computation, we have

$$\partial_t w_\lambda(t,x)=-\frac{\lambda(1+\beta)}{p-1}t^{-\frac{1+\beta}{p-1}-1}w(t^{-\frac{1}{2\alpha}}|x|)-\frac{\lambda}{2\alpha}t^{-\frac{1+\beta}{p-1}-\frac{1}{2\alpha}-1}|x|w'(t^{-\frac{1}{2\alpha}}|x|)$$

and

$$(-\Delta)^\alpha w_\lambda(t,x)=\lambda t^{-\frac{1+\beta}{p-1}-1}(-\Delta)^\alpha w(t^{-\frac{1}{2\alpha}}|x|),$$

which implies

$$\begin{aligned}&\partial_t w_\lambda(t,x)+(-\Delta)^\alpha w_\lambda(t,x)+t^\beta w_\lambda^p(t,x)\\ &\quad=\lambda t^{-\frac{1+\beta}{p-1}-1}\left[(-\Delta)^\alpha w(s)-\frac{1}{2\alpha}w'(s)s-\frac{1+\beta}{p-1}w(s)+\lambda^{p-1}w^p(s)\right],\end{aligned} \tag{5.5}$$

where $s=|z|$ with $z=t^{-\frac{1}{2\alpha}}x$. Next, for $s>0$, we have

$$-\frac{1}{2\alpha}w'(s)s-\frac{1+\beta}{p-1}w(s)=\left[\frac{N+2\alpha}{2\alpha}\frac{s^{N+2\alpha}}{1+s^{N+2\alpha}}-\frac{1+\beta}{p-1}-\frac{s^2(e+s^2)^{-1}}{\alpha\ln(e+s^2)}\right]w(s).$$

Since $\frac{N+2\alpha}{2\alpha}>\frac{1+\beta}{p-1}$, $\lim_{s\to\infty}\frac{s^{N+2\alpha}}{1+s^{N+2\alpha}}=1$ and $\lim_{s\to\infty}\frac{1}{\ln(e+s^2)}=0$, there exists $R_0>0$ and $\sigma_0>0$ such that

$$-\frac{1}{2\alpha}w'(s)s-\frac{1+\beta}{p-1}w(s)\geq\sigma_0 w(s),\qquad \forall s\geq R_0. \tag{5.6}$$

For $|z|>2$, and using the definition of the fractional Laplacian, we have

$$\begin{aligned}-(-\Delta)^\alpha w(|z|)&=\frac12\int_{\mathbb{R}^N}\left(\frac{\ln(e+|z+\tilde y|^2)}{1+|z+\tilde y|^{N+2\alpha}}+\frac{\ln(e+|z-\tilde y|^2)}{1+|z-\tilde y|^{N+2\alpha}}-\frac{2\ln(e+|z|^2)}{1+|z|^{N+2\alpha}}\right)\frac{d\tilde y}{|\tilde y|^{N+2\alpha}}\\ &=\frac{w(|z|)}{2|z|^{2\alpha}}\int_{\mathbb{R}^N}\frac{I_z(y)}{|y|^{N+2\alpha}}dy,\end{aligned} \tag{5.7}$$

where

$$I_z(y) = \frac{1+|z|^{N+2\alpha}}{1+|z|^{N+2\alpha}|e_z+y|^{N+2\alpha}} \frac{\ln(e+|z|^2|e_z+y|^2)}{\ln(e+|z|^2)} + \frac{1+|z|^{N+2\alpha}}{1+|z|^{N+2\alpha}|e_z-y|^{N+2\alpha}} \frac{\ln(e+|z|^2|e_z-y|^2)}{\ln(e+|z|^2)} - 2$$

and $e_z = \frac{z}{|z|}$.
We claim that there exists $c_{20} > 0$ such that

$$\int_{B_{\frac{1}{2}}(-e_z)\cup B_{\frac{1}{2}}(e_z)} \frac{I_z(y)}{|y|^{N+2\alpha}} dy \le \frac{c_{20}}{w(|z|)|z|^N}. \tag{5.8}$$

In fact, for $y \in B_{\frac{1}{2}}(-e_z)$, there exists $c_{21} > 0$ such that

$$\frac{1+|z|^{N+2\alpha}}{1+|z|^{N+2\alpha}|e_z-y|^{N+2\alpha}} \frac{\ln(e+|z|^2|e_z-y|^2)}{\ln(e+|z|^2)} \le c_{21}$$

and then

$$\begin{aligned} \int_{B_{\frac{1}{2}}(-e_z)} \frac{I_z(y)}{|y|^{N+2\alpha}} dy &\le \omega_N \int_0^{\frac{1}{2}} \frac{1+|z|^{N+2\alpha}}{1+(|z|r)^{N+2\alpha}} \frac{\ln(e+|z|^2r^2)}{\ln(e+|z|^2)} r^{N-1} dr + c_{22} \\ &\le \frac{\omega_N}{w(|z|)|z|^N} \int_0^{\infty} \frac{t^{N-1}\ln(e+t^2)}{1+t^{N+2\alpha}} dt + c_{22} \\ &\le \frac{c_{23}}{w(|z|)|z|^N}, \end{aligned}$$

where $c_{22}, c_{23} > 0$ and the last inequality holds since $w(|z|)|z|^N \to 0$ as $|z| \to \infty$. Thus,

$$\int_{B_{\frac{1}{2}}(e_z)} \frac{I_z(y)}{|y|^{N+2\alpha}} dy = \int_{B_{\frac{1}{2}}(-e_z)} \frac{I_z(y)}{|y|^{N+2\alpha}} dy \le \frac{c_{23}}{w(|z|)|z|^N}.$$

We claim that there exists $c_{24} > 0$ such that

$$\int_{B_{\frac{1}{2}}(0)} \frac{I_z(y)}{|y|^{N+2\alpha}} dy \le c_{24}. \tag{5.9}$$

Indeed, since the function I_z is C^2 in $\bar{B}_{\frac{1}{2}}(0)$, $I_z(0) = 0$ and $I_z(y) = I_z(-y)$, then $\nabla I_z(0) = 0$ and there exists $c_{34} > 0$ such that

$$|D^2 I_z(y)| \le c_{25} \qquad \forall y \in B_{\frac{1}{2}}(0).$$

Then we have

$$I_z(y) \le c_{25}|y|^2 \qquad \forall y \in B_{\frac{1}{2}}(0),$$

which implies

$$\int_{B_{\frac{1}{2}}(0)} \frac{I_z(y)}{|y|^{N+2\alpha}} dy \le c_{25} \int_{B_{\frac{1}{2}}(0)} \frac{|y|^2}{|y|^{N+2\alpha}} dy \le c_{24}.$$

We claim that there exists $c_{26} > 0$ such that

$$\int_A \frac{I_z(y)}{|y|^{N+2\alpha}} dy \le c_{26}, \tag{5.10}$$

where $A = \mathbb{R}^N \setminus (B_{\frac{1}{2}}(0) \cup B_{\frac{1}{2}}(e_z) \cup B_{\frac{1}{2}}(-e_z))$. In fact, for $y \in A$, we observe that there exists $c_{27} > 0$ such that $I_z(y) \le c_{27}$ and

$$\int_A \frac{I_z(y)}{|y|^{N+2\alpha}} dy \le \int_{\mathbb{R}^N \setminus B_{\frac{1}{2}}(0)} \frac{c_{27}}{|y|^{N+2\alpha}} \le c_{28},$$

for some $c_{28} > 0$. Therefore, by (5.5)-(5.10), there exists $c_{29} > 0$ such that

$$(-\Delta)^\alpha w(|z|) \ge -\frac{c_{29}}{1+|z|^{N+2\alpha}}, \qquad |z| \ge 2. \tag{5.11}$$

By (5.6) and (5.11), there exists $R_1 \ge R_0 + 2$ such that for $|z| > R_1$,

$$\begin{aligned}(-\Delta)^\alpha w(|z|) - \frac{1}{2\alpha} w'(|z|)|z| - \frac{1+\beta}{p-1} w(|z|) &\ge \sigma_0 w(|z|) - \frac{c_{29}}{1+|z|^{N+2\alpha}} \\ &= w(|z|)\left(\sigma_0 - \frac{c_{29}}{\ln(e+|z|^2)}\right) \\ &\ge 0.\end{aligned}$$

When $|z| \le R_1$, it is clear that there exists $c_{30} > 0$ such that

$$(-\Delta)^\alpha w(|z|) - \frac{1}{2\alpha} w'(|z|)|z| - \frac{1+\beta}{p-1} w(|z|) \ge -c_{30}.$$

Then there exists $\Lambda_0 > 0$ such that for $\lambda \ge \Lambda_0$,

$$(-\Delta)^\alpha w(|z|) - \frac{1}{2\alpha} w'(|z|)|z| - \frac{1+\beta}{p-1} w(|z|) + \lambda^{p-1} w^p(|z|) \ge 0, \qquad \forall z \in \mathbb{R}^N, \tag{5.12}$$

which, together with (5.5), implies that (5.4) holds. □

Next we prove that u_∞ is not a trivial flat solution when $1 + \frac{2\alpha(1+\beta)}{N+2\alpha} < p < p^*_\beta$.

Lemma 5.2 *Assume* $1+\frac{2\alpha(1+\beta)}{N+2\alpha}<p<1+\frac{2\alpha(1+\beta)}{N}$, *that* w_{Λ_0} *is given in (5.3) and* u_∞ *is given in (1.16). Then*

$$u_\infty(t,x)\le w_{\Lambda_0}(t,x)\qquad \forall (t,x)\in Q_\infty. \tag{5.13}$$

Moreover,

$$\lim_{t\to 0} u_\infty(t,\cdot)=0\quad \textit{uniformly on}\;\; B_\epsilon^c,\quad \forall\epsilon>0. \tag{5.14}$$

Proof. Let us denote

$$f_0(r)=\frac{k_0\ln(e+r^2)}{1+r^{N+2\alpha}},\quad \forall\, r\ge 0\quad\text{and}\quad f_{n,k}(x)=kn^Nf_0(n|x|),\quad \forall x\in\mathbb{R}^N,$$

where

$$k_0=\left[\omega_N\int_0^\infty \frac{\ln(e+r^2)}{1+r^{N+2\alpha}}r^{N-1}dr\right]^{-1}.$$

Then for any $\eta\in C_c(\mathbb{R}^N)$, we have that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} f_{n,k}\eta dx=k\lim_{n\to\infty}\int_{\mathbb{R}^N} f_0(|x|)\eta\left(\frac{x}{n}\right)dx=k\eta(0).$$

Let $t_n=n^{-2\alpha}$ and then

$$\begin{aligned}
w_{\Lambda_0}(t_n,x) &= \Lambda_0 t_n^{-\frac{1+\beta}{p-1}}\frac{\ln(e+(t_n^{-\frac{1}{2\alpha}}|x|)^2)}{1+(t_n^{-\frac{1}{2\alpha}}|x|)^{N+2\alpha}}=\Lambda_0 n^{\frac{2\alpha(1+\beta)}{p-1}}\frac{\ln(e+(n|x|)^2)}{1+(n|x|)^{N+2\alpha}}\\
&= \frac{\Lambda_0}{k_0}n^{\frac{2\alpha(1+\beta)}{p-1}-N}n^Nf_0(n|x|)\\
&\ge \frac{\Lambda_0}{k_0}\tilde n^{\frac{2\alpha(1+\beta)}{p-1}-N}n^Nf_0(n|x|)=f_{n,k_{\tilde n}}(x),
\end{aligned}$$

where $\tilde n\le n$ and $k_{\tilde n}=\Lambda_0\tilde n^{\frac{2\alpha(1+\beta)}{p-1}-N}$. We see that $k_{\tilde n}=\Lambda_0\tilde n^{\frac{2\alpha(1+\beta)}{p-1}-N}\to\infty$ as $\tilde n\to\infty$, since $\frac{2\alpha(1+\beta)}{p-1}-N>0$. Let $u_{n,k_{\tilde n}}$ be the solution of (1.17) with initial data $f_{n,k_{\tilde n}}$. By Lemma 5.1, $w_{\Lambda_0}(\cdot+t_n,\cdot)$ is a super-solution of (1.17) with initial data $w_{\Lambda_0}(t_n,\cdot)$, that is, for $(t,x)\in Q_\infty$,

$$\partial_t w_\lambda(t+t_n,x)+(-\Delta)^\alpha w_\lambda(t+t_n,x)+(t+t_n)^\beta w_\lambda^p(t+t_n,x)\ge 0.$$

By the Comparison Principle,

$$u_{n,k_{\tilde n}}(t,x)\le w_{\Lambda_0}(t+t_n,x),\qquad \forall (t,x)\in Q_\infty,$$

for any $\tilde n\le n$. Letting $n\to\infty$ we infer

$$u_{k_{\tilde n}}(t,x)\le w_{\Lambda_0}(t,x),\qquad \forall (t,x)\in Q_\infty, \tag{5.15}$$

where $u_{k_{\tilde{n}}}$ is the solution of (1.17) with $k_{\tilde{n}}\delta_0$ initial data. Thus (5.13) is obtained by letting $\tilde{n} \to \infty$. Finally (5.14) follows by the fact that

$$\lim_{t\to 0^+} w_{\Lambda_0}(t,x) = 0, \qquad \forall x \in \mathbb{R}^N \setminus \{0\},$$

which completes the proof. □

Lemma 5.3 *Assume* $1 < p < p^*_\beta$, *then there exists* $c_{31} > 0$ *such that*

$$u_\infty(t,x) \geq \frac{c_{31} t^{-\frac{1+\beta}{p-1}}}{1 + |t^{-\frac{1}{2\alpha}} x|^{N+2\alpha}}, \qquad \forall (t,x) \in (0,1) \times \mathbb{R}^N. \tag{5.16}$$

Proof. We divide the proof into two steps.
Step 1. Let $\sigma_0 = 1 + \beta - \frac{N}{2\alpha}(p-1) > 0$, $\eta(t) = 2 - t^{\sigma_0}$ for $t > 0$ and denote

$$v_\epsilon(t,x) = \epsilon\eta(t)\Gamma_\alpha(t,x),$$

where Γ_α is the fundamental solution of (1.17). In this step we prove that there exists $\epsilon_0 > 0$ such that

$$u_{k_0} \geq v_{\epsilon_0} \quad \text{in } (0,1) \times \mathbb{R}^N, \tag{5.17}$$

where $k_0 = 2\epsilon_0$ and u_{k_0} is the solution of (1.17) with initial data $k_0\delta_0$. Indeed,

$$\partial_t v_\epsilon(t,x) = \epsilon\eta'(t)\Gamma_\alpha(t,x) + \epsilon\eta(t)\partial_t\Gamma_\alpha(t,x)$$

and

$$(-\Delta)^\alpha v_\epsilon(t,x) = \epsilon\eta(t)(-\Delta)^\alpha\Gamma_\alpha(t,x).$$

Let $\Gamma_1(t^{-\frac{1}{2\alpha}}x) = \Gamma_\alpha(1, t^{-\frac{1}{2\alpha}}x)$, then there exists $\epsilon_0 > 0$ such that for any $\epsilon \leq \epsilon_0$ and $(t,x) \in (0,1) \times \mathbb{R}^N$, we have that

$$\begin{aligned}
&\partial_t v_\epsilon(t,x) + (-\Delta)^\alpha v_\epsilon(t,x) + t^\beta v_\epsilon^p(t,x) \\
&\quad = \epsilon\eta'(t) t^{-\frac{N}{2\alpha}}\Gamma_1(t^{-\frac{1}{2\alpha}}x) + \epsilon^p\eta^p(t) t^{-\frac{N}{2\alpha}p+\beta}\Gamma_1^p(t^{-\frac{1}{2\alpha}}x) \\
&\quad \leq -\epsilon\sigma_0 t^{-\frac{N}{2\alpha}-1+\sigma_0}\Gamma_1(t^{-\frac{1}{2\alpha}}x) + 2^p\epsilon^p t^{-\frac{N}{2\alpha}p+\beta}\Gamma_1^p(t^{-\frac{1}{2\alpha}}x) \leq 0,
\end{aligned}$$

the last inequality holds since $-\frac{N}{2\alpha} - 1 + \sigma_0 = -\frac{N}{2\alpha}p + \beta$ and Γ_1 is bounded. In particular, there holds

$$\partial_t v_{\epsilon_0}(t,x) + (-\Delta)^\alpha v_{\epsilon_0}(t,x) + t^\beta v_{\epsilon_0}^p(t,x) \leq 0, \qquad \forall (t,x) \in (0,1) \times \mathbb{R}^N. \tag{5.18}$$

Let $f_n(x) = v_{\epsilon_0}(t_n, x)$ with $t_n = n^{-2\alpha}$. Since $\lim_{t\to 0^+} \eta(t) = 2$, then we have that $f_n \to 2\epsilon_0\delta_0$ as $n \to \infty$ in the weak sense of measures. There exists $N_0 > 0$ such that $t_n \in (0, \frac{1}{8})$ for $n \geq N_0$. Let w_n be the solution of (1.17) with initial data f_n, then it infers that

$$w_n(t, x) \geq v_{\epsilon_0}(t + t_n, x), \qquad (t, x) \in (0, 1 - t_n) \times \mathbb{R}^N.$$

Because u_{k_0} is uniquely defined, there holds

$$w_n \to u_{k_0} \quad \text{as } \ n \to \infty \qquad \text{in } \ (0, 1) \times \mathbb{R}^N$$

and

$$\lim_{n\to\infty} v_{\epsilon_0}(t + t_n, x) = v_{\epsilon_0}(t, x), \qquad \forall (t, x) \in (0, 1) \times \mathbb{R}^N,$$

which imply (5.17).

Step 2. We claim that (5.16) holds. Since

$$v_{\epsilon_0}(t, x) \geq \epsilon_0 t^{-\frac{N}{2\alpha}} \Gamma_1(t^{-\frac{1}{2\alpha}} x), \qquad (t, x) \in (0, 1) \times \mathbb{R}^N,$$

then, along with the relation $T_\lambda[u_k] = u_{k\lambda^{\frac{2\alpha(1+\beta)}{p-1} - N}}$, we observe that for any $\lambda > 0$,

$$\begin{aligned} u_{k_0\lambda^{\frac{2\alpha(1+\beta)}{p-1} - N}}(t, x) &= \lambda^{\frac{2\alpha(1+\beta)}{p-1}} u_{k_0}(\lambda^{2\alpha} t, \lambda x) \\ &\geq \lambda^{\frac{2\alpha(1+\beta)}{p-1}} v_{\epsilon_0}(\lambda^{2\alpha} t, \lambda x) \\ &\geq \epsilon_0 \lambda^{\frac{2\alpha(1+\beta)}{p-1} - N} t^{-\frac{N}{2\alpha}} \Gamma_1(t^{-\frac{1}{2\alpha}} x). \end{aligned}$$

Let $\varrho = \lambda^{\frac{2\alpha(1+\beta)}{p-1} - N}$, $t_\varrho = (2\varrho)^{\frac{1}{\frac{N}{2\alpha} - \frac{1+\beta}{p-1}}}$ and $T_\varrho = \varrho^{\frac{1}{\frac{N}{2\alpha} - \frac{1+\beta}{p-1}}}$, then

$$0 < t_\varrho < T_\varrho \to 0 \quad \text{as } \ \varrho \to \infty.$$

For $(t, x) \in (t_\varrho, T_\varrho) \times \mathbb{R}^N$, we have that

$$u_{k_0\varrho}(t, x) \geq \epsilon_0 \varrho t^{-\frac{N}{2\alpha}} \Gamma_1(t^{-\frac{1}{2\alpha}} x) \geq \frac{\epsilon_0}{2} t^{-\frac{1+\beta}{p-1}} \Gamma_1(t^{-\frac{1}{2\alpha}} x),$$

then

$$u_\infty(t, x) \geq \frac{\epsilon_0}{2} t^{-\frac{1+\beta}{p-1}} \Gamma_1(t^{-\frac{1}{2\alpha}} x), \qquad \forall (t, x) \in (t_\varrho, T_\varrho) \times \mathbb{R}^N.$$

which implies (5.16) and completes the proof. □

Proof of Theorem 1.2. It follows from Proposition 5.1 and Lemma 5.2 that u_∞ is a nontrivial self-similar solution of (1.17) and (1.22) follows by (5.13), (5.16) and $\ln(e + |t^{-\frac{1}{2\alpha}} x|^2) \leq 2\ln(2 + |t^{-\frac{1}{2\alpha}} x|)$, which ends the proof. □

We have actually a stronger result which is a consequence of Theorem 1.4-(i) proved in next section:

Corollary 5.1 *Assume* $1 + \frac{2\alpha(1+\beta)}{N+2\alpha} < p < 1 + \frac{2\alpha(1+\beta)}{N}$. *Then either*

$$\tilde{u} > u_\infty \quad \text{in} \quad Q_\infty \tag{5.19}$$

or

$$\tilde{u} \equiv u_\infty \quad \text{in} \quad Q_\infty. \tag{5.20}$$

5.2 The case $1 < p < 1 + \frac{2\alpha(1+\beta)}{N+2\alpha}$

For $1 < p < 1 + \frac{2\alpha(1+\beta)}{N+2\alpha}$, it follows from Lemma 5.3 that

$$\lim_{t \to 0^+} u_\infty(t, x) = \infty, \qquad \forall x \in \mathbb{R}^N. \tag{5.21}$$

Proof of Theorem 1.3 (i)**.** Let $f_0 \in C_c(\mathbb{R}^N)$ be a nonnegative function such that

$$\text{supp} f_0 \subset B_1(0) \quad \text{and} \quad \max_{x \in B_1(0)} f_0 = 1.$$

Denote

$$f_{n,k}(x) = k n^{\theta N} f_0(n^\theta (x - x_0)),$$

where $k \leq n^\tau$ with $\tau = \frac{1}{2}(\frac{2\alpha(1+\beta)}{p-1} - N - 2\alpha) > 0$, $\theta = \frac{\tau}{N}$ and $x_0 \in \mathbb{R}^N$. Since $f_{n,k}(x) \leq n^\tau$ for $x \in B_1(x_0)$, $f_n(x) = 0$ for $x \in B_1^c(x_0)$ and

$$v_{\epsilon_0}(t_n, x) \geq \frac{c_{39} n^{\frac{2\alpha(1+\beta)}{p-1} - N - 2\alpha}}{(2 + |x_0|)^{N+2\alpha}}, \qquad \forall x \in B_1(x_0),$$

where $t_n = n^{-2\alpha}$. Then there exists $N_0 > 0$ such that for any $n \geq N_0$,

$$f_{n,k}(x) \leq v_{\epsilon_0}(t_n, x), \qquad \forall x \in B_1(x_0).$$

Since $n^{\theta N} f_0(n^\theta (x - x_0)) \to c_{41} \delta_{x_0}$, as $n \to \infty$ in weak sense of measures, for some $c_{41} > 0$.

Let $w_{n,k}$ be the solution of (1.17) with initial data $f_{n,k}$, then

$$w_{n,k}(0, x) = f_{n,k}(x) \leq v_{\epsilon_0}(t_n, x) \leq u_\infty(t_n, x), \qquad \forall x \in \mathbb{R}^N.$$

Therefore, by the Comparison Principle

$$w_{n,k}(t, x) \leq u_\infty(t + t_n, x), \qquad \forall (t, x) \in Q_\infty.$$

We observe that

$$\lim_{k \to \infty} [\lim_{n \to \infty} w_{n,k}(t, x)] = u_\infty(t, x - x_0), \qquad \forall (t, x) \in Q_\infty.$$

Thus, we derive that

$$u_\infty(t,x-x_0)\le u_\infty(t,x),\qquad \forall(t,x)\in Q_\infty. \tag{5.22}$$

Then $u_\infty(t,x-x_0)=u_\infty(t,x)$ for all $(t,x)\in Q_\infty$, which implies that u_∞ is independent of x. Combining (5.1) and (5.16), implies that

$$u_\infty=\left(\frac{1+\beta}{p-1}\right)^{\frac{1}{p-1}}t^{-\frac{1+\beta}{p-1}}.$$

The proof is complete. □

In the case of $p=1+\frac{2\alpha(1+\beta)}{N+2\alpha}$, it derive from Lemma 5.3 that

$$\liminf_{t\to0^+}u_\infty(t,x)\ge\lim_{t\to0^+}\frac{c_{40}t^{-\frac{1+\beta}{p-1}}}{1+|t^{-\frac{1}{2\alpha}}x|^{N+2\alpha}}=\frac{c_{40}}{|x|^{N+2\alpha}},\qquad\forall x\in\mathbb{R}^N.$$

Proof of Theorem 1.3 *(ii)***.** We note that u_∞ is a self-similar solution of (1.17). Moreover, we derive (1.24) by (5.16), which ends the proof. □

5.3 The self-similar equation

In this section we prove Theorem 1.4.

Proof of Theorem 1.4 *(i)***.** We set $v_\infty(\eta)=t^{\frac{1+\beta}{p-1}}u_\infty(1,\eta)$. Then relations (1.25) and (1.26) hold from Lemmas 5.2 and 5.3. Assume $\tilde v$ is another positive solution of (1.20). Then $(t,x)\mapsto t^{-\frac{1+\beta}{p-1}}\tilde v(t^{-\frac{1}{2\alpha}}x)$ is a positive self-similar solution of (1.23). By Proposition 5.2 it is larger than u_∞. Thus $v_\infty\le\tilde v$. Assume now that there exists $\eta_0\in\mathbb{R}^N$ such that $v_\infty(\eta_0)=\tilde v(\eta_0)$. and set $w=\tilde v-v_\infty$. Then

$$(-\Delta)^\alpha w(\eta_0)=\lim_{\epsilon\to0}(-\Delta)^\alpha_\epsilon w(\eta_0)=\lim_{\epsilon\to0}\int_{B^c_\epsilon(\eta_0)}\frac{w(\eta_0)-w(\eta)}{|\eta-\eta_0|^{N+2\alpha}}d\eta<0.$$

Since $\nabla w(\eta_0)$ we reach a contradiction. □

Proof of Theorem 1.4 *(ii)***.** It is a consequence of the equality

$$u_\infty=U_p\Longleftrightarrow v_\infty=\left(\frac{1+\beta}{p-1}\right)^{\frac{1}{p-1}}.$$

Open problem. We conjecture that in the case $1+\frac{2\alpha(1+\beta)}{N+2\alpha}<p<1+\frac{2\alpha(1+\beta)}{N}$, v_∞ is the unique positive solution of the self-similar equation satisfying (1.25). One step could be to prove that any positive solution $\tilde{v}$ satisfying (1.25) satisfies, for some $K > 1$,

$$\tilde{v} \leq K v_\infty \qquad \text{in } \mathbb{R}^N. \tag{5.23}$$

We also conjecture that v_∞ satisfies the following asymptotic behavior

$$v_\infty(\eta) = c_{N,p,\alpha,\beta}|\eta|^{-N-2\alpha} \qquad \text{as } |\eta| \to \infty. \tag{5.24}$$

Thus if any positive solution $\tilde{v}$ inherits the same property, the conclusion (and the uniqueness) follows.

Acknowledge: H. Chen is supported by National Natural Science Foundation of China, No:11401270 and the Project-sponsored by SRF for ROCS, SEM. L. Véron is supported by the MathAmsud collaboration program 13MATH-02 QUESP.

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