Well-posedness in energy space for the periodic modified Benjamin-Ono equation

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WELL-POSEDNESS IN ENERGY SPACE FOR THE PERIODIC MODIFIED BENJAMIN-ONO EQUATION

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Abstract. We prove that the periodic modified Benjamin-Ono equation is locally well-posed in the energy space $H^{1/2}$. This ensures the global well-posedness in the defocusing case. The proof is based on an $X^{s,b}$ analysis of the system after gauge transform.

Contents

1. Introduction, main results and notations

In this paper, we study the Cauchy problem for the modified Benjamin-Ono equation on the torus that reads

$$\begin{cases} \partial_t u + \mathcal{H}\partial_x^2 u = \mp u^2 u_x, \\ u(x,0) = u_0 \end{cases} \tag{1.1}$$

where $u(t,x) : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and $\mathcal{H}$ is the Hilbert transform

$$\widehat{\mathcal{H}f}(0) = 0, \quad \widehat{\mathcal{H}f}(k) = -i\text{sgn}(k)\hat{f}(k), \ k \in \mathbb{Z}^*.$$

This equation is called defocusing when there is a minus sign in front of the nonlinear term $u^2 u_x$ and focusing when it is a plus sign.
The Benjamin-Ono equation with the quadratic nonlinear term

$$\partial_t u + \mathcal{H}\partial_x^2 u = u u_x \tag{1.2}$$

was derived by Benjamin [2] and Ono [25] as a model for one-dimensional waves in deep water. On the other hand, the cubic nonlinearity is also of much interest for long wave models [1, 13].

There are at least the three following quantities preserved under the flow of the real-valued mBO equation (1.1)[1]

$$\int_{\mathbb{T}} u(t,x)dx = \int_{\mathbb{T}} u_0(x)dx, \tag{1.3}$$

$$\int_{\mathbb{T}} u(t,x)^2dx = \int_{\mathbb{T}} u_0(x)^2dx, \tag{1.4}$$

$$\int_{\mathbb{T}} \frac{1}{2}u\mathcal{H}u_x \mp \frac{1}{12}|u(t,x)|^4dx = \int_{\mathbb{T}} \frac{1}{2}u_0\mathcal{H}u_{0,x} \mp \frac{1}{12}|u_0(x)|^4dx\ . \tag{1.5}$$

These conservation laws provide a priori bounds on the solution. For instance, in the defocusing case we get from (1.4) and (1.5) that the $H^{1/2}$ norm of the solution remains bounded for all times if the initial data belongs to $H^{1/2}$. This is crucial in order to prove the well-posedness result. On the other hand the mBO equation is L^2-critical (in the sense that the $L^2(\mathbb{R})$-norm is preserved by the dilation symmetry of the equation). Therefore, in the focusing case, one expects that a phenomenon of blow-up in the energy space occurs[2]

The Cauchy problems for (1.1) and the Benjamin-Ono equation (1.2) have been extensively studied. For instance, in both real-line and periodic case, the energy method provides local well-posedness for BO and mBO in H^s for $s > 3/2$ [10]. In the real-line case, this result was improved by combination of energy method and the dispersive effects. For real-line BO equation, the result $s \geq 3/2$ by Ponce [26] was the first place of such combination as a consequence of the commutator estimates in [11], was later improved to $s > 5/4$ in [17], and $s > 9/8$ in [12]. Tao [27] obtained global well-posedness in H^s for $s \geq 1$ by using a gauge transformation as for the derivative Schrödinger equation and Strichartz estimates. This result was improved to $s \geq 0$ by Ionescu and Kenig [9], and to $s \geq 1/4$ (local well-posedness) by Burq and Planchon [4]. Their proof both used the Fourier restriction norm introduced in [3]. Recently, Molinet and Pilod [18] gave a simplified proof for $s \geq 0$ and obtained unconditional uniqueness for $s > 1/4$.

For the real-line mBO, this was improved to $s \geq 1$ by Kenig-Koenig [12] by the enhanced energy methods. Molinet and Ribaud [20] obtained analytic local well-posedness for the complex-valued mBO in H^s for $s > 1/2$ and $B^{1/2}_{2,1}$ with a small L^2 norm, improving the result of Kenig-Ponce-Vega [14] for $s > 1$. The smallness condition of $H^s(s > 1/2)$ results was later removed in [19] by using Tao's gauge transformation [27]. The result for $s = 1/2$ was obtained by Kenig and Takaoka [15] by using frequency dyadically localized gauge transformation. Their result is sharp in the sense that the solution map is not locally uniformly continuous in H^s for $s < 1/2$ (The failure of C^3 smoothness was obtained in [20]). Later, Guo [7] obtained the same result without using gauge transform under a smallness condition on the L^2 norm.

[1]In (1.5) the + corresponds to the defocusing case whereas the − corresponds the the focusing one.

[2]Progress in this direction can be found in [16] for the case on the real line.

In the periodic case, there is no smoothing effect for the equation. However, to overcome the loss of derivative, the gauge transform still applies. For the periodic BO equation, global well-posedness in H^1 was proved by Molinet and Ribaud [23], was later improved by Molinet to $H^{1/2}$ [22], and L^2 [21]. Molinet [24] also proved that the result in L^2 is sharp in the sense that the solution map fails to be continuous below L^2. For the periodic mBO (1.1), local well-posedness in H^1 was proved in [23]. Their proof used the Strichartz norm and gauge transform.

The purpose of this paper is to improve the well-posedness results for (1.1) to the energy space $H^{1/2}$ and, as a by-product, to prove that the solutions can be extended for all times in the defocusing case. The main result of this paper is

Theorem 1.1. *Let $s \geq 1/2$. For any intial data $u_0 \in H^s(\mathbb{T})$ there exists $T = T(\|u_0\|_{H^{1/2}}) > 0$ such that the mBO equation (1.1) admits a unique solution*

$$u \in C([-T,T]; H^s(\mathbb{T})) \quad \textit{with} \quad P_+(e^{iF(u)}) \in X_T^{\frac{1}{2},\frac{1}{2}} .$$

Moreover, the solution-map $u_0 \mapsto u$ is continuous from the ball of $H^{1/2}(\mathbb{T})$ of radius $\|u_0\|_{H^{1/2}}$, equipped with the $H^s(\mathbb{T})$-topology, with values in $C([-T,T]; H^s(\mathbb{T}))$. Finally, in the defocusing case, the solution can be extended for all times and belongs to $C(\mathbb{R}; H^s(\mathbb{T})) \cap C_b(\mathbb{R}; H^{1/2}(\mathbb{T}))$.

A very similar equation to mBO (1.1) is the derivative nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \partial_x^2 u = i(|u|^2 u)_x, & (t,x) \in \mathbb{R} \times \mathbb{T} \\ u(0,x) = u_0. \end{cases} \tag{1.6}$$

It has also attracted extensive attention. Local well-posedness for (1.6) in $H^{1/2}$ was proved by Herr [8]. There are several differences between (1.1) and (1.6). The first one is the integrability: (1.6) is integrable while (1.1) is not. The second one is the conservation laws: (1.1) has a conservation law at level $H^{1/2}$, and hence GWP in $H^{1/2}$ is much easier. The last one is the action of the gauge transform: let v be the function after gauge transform, (1.6) can be reduced to a clean equation which involves only v, while (1.1) can only reduce to a system that involves both u and v, and hence the gauge for (1.1) brings more technical difficulties.

We discuss now the ingredients in the proof of Theorem 1.1. Let u be a smooth solution to (1.1), define

$$w = T(u) := \frac{1}{\sqrt{2}} u(t, x - \int_0^t \frac{1}{2\pi} \int u^2(s,x)\,dx\,ds). \tag{1.7}$$

Then w solves the "Wicked order" mBO equation:

$$\begin{cases} \partial_t u + \mathcal{H}\partial_x^2 u = 2P_{\neq c}(u^2) u_x, \\ u(x,0) = u_0, \end{cases} \tag{1.8}$$

where $P_{\neq c} f = f - \frac{1}{2\pi}\int_{\mathbb{T}} f\,dx$. It is easy to see that T and its inverse T^{-1} are both continuous maps from $C((-T,T) : H^s)$ to $C((-T,T) : H^s)$ for $s \geq 0$. Therefore we will consider (1.8) instead of (1.1). Now, in order to overcome the loss of derivative, we will apply the method of gauge transform as in [23, 21, 22], which was first developed for BO equation by Tao [27]. As noticed above the equation satisfied by this gauge transform v involves terms with both u and v. One of the main difficulties is that the solution u does not share the same regularity in Bourgain'space as the gauge transform v. The main new ingredient is the use of the Marcinkiewicz multiplier theorem that enables us to treat the multiplication by u in Bourgain'space in a simple way.

1.1. **Notations.** For $A, B > 0$, $A \lesssim B$ means that there exists $c > 0$ such that $A \le cB$. When c is a small constant we use $A \ll B$. We write $A \sim B$ to denote the statement that $A \lesssim B \lesssim A$.

We denote the sum on $\mathbb{Z}$ by integral form $\int a(\xi)d\xi := \sum_{\xi\in\mathbb{Z}} a(\xi)$. For a 2π-periodic function ϕ, we define its Fourier transform on $\mathbb{Z}$ by

$$\hat{\phi}(\xi) := \int_{\mathbb{R}/2\pi\mathbb{Z}} e^{-i\xi x}\phi(x)dx, \ \forall\ \xi \in \mathbb{Z}.$$

We denote by $W(\cdot)$ the unitary group $W(t)u_0 := \mathcal{F}_x^{-1}e^{-it|\xi|\xi}\mathcal{F}_x u_0(\xi)$.

For a function $u(t,x)$ on $\mathbb{R}\times\mathbb{R}/(2\pi)\mathbb{Z}$, we define its space-time Fourier transform as follows, $\forall\ (\tau,\xi) \in \mathbb{R}\times\mathbb{Z}$

$$\hat{u}(\tau,\xi) := \mathcal{F}_{t,x}(u)(\tau,\xi) := \mathcal{F}(u)(\tau,\xi) = \int_{\mathbb{R}}\int_{\mathbb{R}/(2\pi)\mathbb{Z}} e^{-i(\tau t+\xi x)}u(t,x)dxdt.$$

Then define the Sobolev spaces H^s for (2π)-periodic function by

$$\|\phi\|_{H^s} := \|\langle\xi\rangle^s\hat{\phi}\|_{l^2_\xi} = \|J_x^s\phi(x)\|_{L^2_x},$$

where $\langle\xi\rangle := (1+|\xi|^2)^{\frac{1}{2}}$ and $\widehat{J_x^s\phi}(\xi) := \langle\xi\rangle^s\hat{\phi}(\xi)$. For $2 < q < \infty$ we define also the Sobolev type spaces H^s_q by

$$\|\phi\|_{H^s_q} := \|J^x_s\phi\|_{L^q}\ .$$

We will use the following Bourgain-type spaces denoted by $X^{s,b}$, $Z^{s,b}$ and Y^s of (2π)-periodic (in x) functions respectively endowed with the norm

$$\|u\|_{X^{s,b}} :=\|\langle\xi\rangle^s\langle\tau+|\xi|\xi\rangle^b\hat{u}(\tau,\xi)\|_{L^2_{\tau,\xi}},$$
$$\|u\|_{Z^{s,b}} :=\|\langle\xi\rangle^s\langle\tau+|\xi|\xi\rangle^b\hat{u}(\tau,\xi)\|_{L^2_\xi L^1_\tau},$$

and

$$\|u\|_{Y^s} := \|u\|_{X^{s,\frac{1}{2}}} + \|u\|_{Z^{s,0}}. \tag{1.9}$$

One can easily check that $u \mapsto \overline{u}$ an isometry in $X^{s,b}$ and $Z^{s,b}$ and that $Y^s \hookrightarrow Z^{s,0} \hookrightarrow C(\mathbb{R};H^s)$. We will also use the space-time Lebesgue spaces denoted by $L^p_tL^q_x$ of (2π)-periodic (in x) functions endowed with the norm

$$\|u\|_{L^p_tL^q_x} := \Big(\int_{\mathbb{R}}\|u(t,\cdot)\|^p_{L^q_x}dt\Big)^{1/p},$$

with the obvious modification for $p=\infty$. For any space-time function space B and any $T>0$, we denote by B_T the corresponding restriction in time space endowed with the norm

$$\|u\|_{B_T} := \inf_{v\in B}\{\|v\|_B, v(\cdot)\equiv u(\cdot)\ on\ (0,T)\}.$$

Let $\eta_0 : \mathbb{R}\to[0,1]$ denote an even smooth function supported in $[-8/5,8/5]$ and equal to 1 in $[-5/4,5/4]$. For $k\in\mathbb{N}^*$ let $\chi_k(\xi) = \eta_0(\xi/2^{k-1}) - \eta_0(\xi/2^{k-2})$,

$\eta_{\le k} = \eta_0(\xi/2^{k-1})$, and then let P_{2^k} and $P_{\le 2^k}$ denote the operators on $L^2(\mathbb{T})$ defined by

$$\widehat{P_1 u}(\xi) = \eta_0(2\xi),\ \widehat{P_{2^k} u}(\xi) = \chi_k(\xi)\widehat{u}(\xi)\,, k \in \mathbb{N}^*,\ \text{and}\ \widehat{P_{\le 2^k} u}(\xi) = \eta_{\le k}(\xi) u(\xi)\ .$$

By a slight abuse of notation we define the operators $P_{2^k}, P_{\le 2^k}$ on $L^2(\mathbb{R}\times\mathbb{T})$ by the formulas $\mathcal{F}(P_{2^k}u)(\tau,\xi) = \chi_k(\xi)\mathcal{F}(u)(\tau,\xi)$, $\mathcal{F}(P_{\le 2^k}u)(\tau,\xi) = \eta_{\le k}(\xi)\mathcal{F}(u)(\tau,\xi)$. We also define the projection operators $P_\pm f = \mathcal{F}^{-1}1_{\pm\xi>0}\mathcal{F}f$, $P_c f = \frac{1}{2\pi}\int_{\mathbb{T}} f dx$, $P_{\ne c} = I - P_c$, and $P^+_{2^k} = P_+ P_{2^k}$, $P^+_{\le 2^k} = P_+ P_{\le 2^k}$.

To simplify the notation, we use capitalized variables to describes the dyadic number, i.e. any capitalized variables such as N range over the dyadic number $2^{\mathbb{N}}$. Finally, for any $1 \le p \le \infty$ and any function space B we define the space-time function space $\widetilde{L^p_t B}$ by

$$\|u\|_{\widetilde{L^p_t B}} := \left(\sum_{k=0}^{\infty} \|P_{2^k} u\|^2_{L^p_t B}\right)^{\frac{1}{2}}\ .$$

It is worth noticing that Littlewood-Paley square function theorem ensures that $\widetilde{L^p_t L^p_x} \hookrightarrow L^p_t L^p_x$ for $2 \le p < \infty$.

2. Gauge transform

In this section, we introduce the gauge transform. Let $u \in C([-T,T] : H^\infty(\mathbb{T}))$ be a smooth solution to (1.8). Define the periodic primitive of $u^2 - \frac{1}{2\pi}\|u(t)\|_2^2$ with zero mean by

$$F = F(u) = \partial_x^{-1} P_{\ne c}(u^2) = \frac{1}{2\pi}\int_0^{2\pi}\int_\theta^x u^2(t,y) - \frac{1}{2\pi}\|u(t)\|^2_{L^2} dy d\theta.$$

Let

$$v = \mathcal{G}(u) := P_+(e^{-iF}u), \tag{2.1}$$

then we look for the equation that v solves. It holds

$$\begin{aligned} v_t =& P_+[e^{-iF}(-iF_t u + u_t)], \\ v_{xx} =& P_+[e^{-iF}(-F_x^2 u - iF_x u_x - i(F_x u)_x + u_{xx})]\ , \end{aligned}$$

and thus

$$\begin{aligned} v_t - iv_{xx} =& P_+[e^{-iF}(-iF_t u + i(F_x)^2 u - F_{xx} u)] \\ &+ P_+[e^{-iF}(u_t - iu_{xx} - 2F_x u_x)] := I + II. \end{aligned}$$

Using equation (1.8) we easily get

$$II = P_+[e^{-iF}(u_t + \mathcal{H}u_{xx} - 2iP_- u_{xx} - 2F_x u_x)] = -2iP_+[e^{-iF}P_- u_{xx}].$$

Next we compute I. Using again (1.8) and the conservation of the L^2-norm for smooth solutions, we have

$$\begin{aligned} F_t =&\partial_t\partial_x^{-1}(P_{\neq c}u^2) = \partial_x^{-1}\partial_t(u^2 - P_c u^2) = \partial_x^{-1}\partial_t u^2 \\ =&2\partial_x^{-1}\Big(-u\mathcal{H}u_{xx} + 2P_{\neq c}?u^2)uu_x\Big) \\ =&2\partial_x^{-1}\Big(-\partial_x(u\mathcal{H}u_x) + u_x\mathcal{H}u_x + P_{\neq c}(u^2)\partial_x P_{\neq c}(u^2)\Big) \\ =&P_{\neq c}\Big((P_{\neq c}u^2)^2\Big) - 2u\mathcal{H}u_x + 2P_c?u\mathcal{H}u_x) + 2\partial_x^{-1}(u_x\mathcal{H}u_x)\ . \end{aligned} \tag{2.2}$$

Noticing that $F_x = P_{\neq c}(u^2)$ we infer that

$$-iuP_{\neq c}\Big((P_{\neq c}u^2)^2\Big) + iu(F_x)^2 = iuP_c\Big((P_{\neq c}u^2)^2\Big)$$

and noticing that $F_{xx} = 2uu_x$,

$$2iuP_{\neq c}\Big(u\mathcal{H}\partial_x u\Big) - F_{xx}u = -4u^2P_-u_x?2iuP_c(u\mathcal{H}u_x)\ .$$

Moreover, following [19], we will use the symmetry of the term $\partial_x^{-1}(u_x\mathcal{H}u_x)$. Indeed, it is easy to check that $\partial_x^{-1}(u_x\mathcal{H}u_x) = -i\partial_x^{-1}(P_+u_x)^2 + i\partial_x^{-1}(P_-u_x)^2$ and thus setting

$$B(u,v) = -i\partial_x^{-1}(P_+u_xP_+v_x) + i\partial_x^{-1}(P_-u_xP_-v_x)\ , \tag{2.3}$$

we infer that $\partial_x^{-1}(u_x\mathcal{H}u_x) = B(u,u)$. We thus finally get

$$I = P_+\Big[e^{-iF}\Big(-4u^2P_-u_x - 2iuB(u,u) + 2iuP_c(u\mathcal{H}u_x) - iuP_c\Big((P_{\neq c}u^2)^2\Big)\Big]$$

which leads to

$$\begin{aligned} v_t - iv_{xx} =&P_+\Big[e^{-iF}\Big(-4u^2P_-u_x - 2iP_-u_{xx} - 2iuB(u,u) \\ &- 2iuP_c(u\mathcal{H}u_x) + iuP_c\Big((P_{\neq c}u^2)^2\Big)\Big)\Big]. \end{aligned} \tag{2.4}$$

Due to the projector P_+, P_-, we see formally that in the system (2.1)-(2.4) there is no high-low interaction of the form

$$P_{low}u^2 \cdot \partial_x P_{high}u.$$

Note that $u \to \mathcal{G}(u)$ can be "inverted" in Lebesgue space. This is the strategy used in [23] to prove well-posedness in H^1. To go below to $H^{1/2}$, we intend to use $X^{s,b}$ spaces. But $u \to \mathcal{G}(u)$ can not be well "inverted" in Bourgain'spaces and thus u will not have the same regularity as $\mathcal{G}(u)$ in these spaces. To handle this former difficulty, we will insert the "inverse" into some of the terms in (2.4). We first observe that

$$\begin{aligned} -2iP_+\Big(e^{-iF}P_-u_{xx}\Big) =& -2i\partial_xP_+(e^{-iF}P_-u_x) + 2P_+(e^{-iF}P_{\neq c}(u^2)P_-u_x) \\ =& -2\partial_xP_+(\partial_x^{-1}P_+(e^{-iF}P_{\neq c}(u^2))P_-u_x) \\ &+ 2P_+(e^{-iF}u^2P_-u_x) - 2P_c(u^2)P_+(e^{-iF}P_-u_x) \end{aligned}$$

and thus the sum of the first two terms of the right-hand side of (2.4) can be rewritten as

$$\begin{aligned} &-2P_+(e^{-iF}u^2P_-u_x) - 2\partial_xP_+(\partial_x^{-1}P_+(e^{-iF}u^2)P_-u_x) \\ &+ 2P_c(u^2)\Big(\partial_xP_+(\partial_x^{-1}P_+e^{-iF}P_-u_x) - P_+(e^{-iF}P_-u_x)\Big)\ . \end{aligned} \tag{2.5}$$

Now, let us denote

$$\begin{aligned} R(u) &:= [P_+, e^{-iF}]u = P_+(e^{-iF}u) - e^{-iF}P_+u \\ &= P_+(e^{-iF}P_-u) + P_+(e^{-iF}P_cu) - P_-(e^{-iF}P_+u) - P_c(e^{-iF}P_+u) \ . \end{aligned} \tag{2.6}$$

Formally, $R(u)$ is a commutator, and has one order higher regularity than $F_x = P_{\neq c}u^2$ (see Lemma 3.6). Then we get

$$v = e^{-iF}P_+u + R(u), \tag{2.7}$$

and thus $P_+u = e^{iF}v - e^{iF}[P_+, e^{-iF}]u$. Since u is real-valued, this leads to

$$P_-u = P_-(e^{-iF}\overline{v}) - P_-(e^{-iF}\overline{R(u)}) \ . \tag{2.8}$$

Substituting P_-u by the expression (2.8) in the two first terms of (2.5) we eventually get the following equation satisfied by v :

$$v_t - iv_{xx} = 2N^0(u,v) + 2N^1(u,v) - 2iP_+(e^{-iF}uB(u,u)) + G(u) \tag{2.9}$$

where

$$N^\nu(u,v) := -\partial_x^\nu P_+(\partial_x^{-\nu}P_+(e^{-iF}u^2)\partial_x P_-(e^{-iF}\overline{v})), \quad \nu = 0, 1.$$

and

$$\begin{aligned} G(u) := &P_+\Big(e^{-iF}\Big(-2iuP_c(u\mathcal{H}u_x) + uP_c((P_{\neq c}u^2)^2) + 2u^2\partial_x P_-(e^{-iF}\overline{R(u)})\Big)\Big) \\ &- 2\partial_x P_+\Big(\partial_x^{-1}P_+(e^{-iF})P_-u_x\Big) + 2i\partial_x P_+\Big((e^{-iF}\partial_x P_-(e^{-iF}\overline{R(u)})\Big) \\ &+ 2P_c(u^2)\Big(-P_+(e^{-iF}P_-u_x) + \partial_x P_+\Big(\partial_x^{-1}P_+(e^{-iF})P_-u_x\Big)\Big) \ . \end{aligned} \tag{2.10}$$

We will see that the worst terms of the right-hand side of (2.9) are the first two terms. Actually the use of Bourgain's spaces will be necessary to handle the first three terms of (2.9). On the other hand, $G(u)$ is a nice term that belongs to $L^2_t H^{1/2}_x$ as soon as $u \in L^\infty_t H^{1/2}$.

3. The main estimates and proof of Theorem 1.1

In this section, we present the main estimates. By combining all these estimates, we finish the proof of Theorem 1.1.

3.1. **Linear Estimates.** We list some linear estimates in this subsection. The first ones are the standard estimates for the linear solution, see [3] and [5].

Lemma 3.1. Let $s \in \mathbb{R}$. There exists $C > 0$ such that for all $f \in X^{s,-\frac{1}{2}+}$ and all $u_0 \in H^s$ we have

$$\|W(t)u_0\|_{Y^s_T} \leq C\|u_0\|_{H^s} \tag{3.1}$$

$$\Big\|\int_0^t W(t-\tau)f(\tau)d\tau\Big\|_{Y^s_T} \leq C\|f\|_{X_T^{s,-\frac{1}{2}+}} \ . \tag{3.2}$$

Next, we need some embedding properties of the space Y^s. The first one is the well-known estimate due to Bourgain [3]

$$\|v\|_{L^4_{t,x}} \lesssim \|v\|_{\widetilde{L^4_{t,x}}} \lesssim \|v\|_{X^{0,3/8}} \tag{3.3}$$

where the first inequality above follows from the Littlewood-Paley square function theorem. Note that (3.1) combined with (3.3) ensures that for $0 \le T \le 1$,

$$\|W(t)u_0\|_{L^4_{tx}} \lesssim \|u_0\|_{L^2} \ . \tag{3.4}$$

3.2. **Main Non-linear Estimates.**

Proposition 3.2 (Estimates of u). Let $T \in]0,1[$, $s \in [\frac{1}{2},1]$ and $(u_i,v_i) \in \Big(C^0_T H^s \cap \widetilde{L^4_T H^s_4}\Big) \times Y^s_T$, $i=1,2$, satisfying (1.8) and (2.1) on $]-T,T[$ with initial data $u_{i,0}$. Then for $u = u_i$

$$\|u\|_{\widetilde{L^4_T H^s_4}} \lesssim (1+\|u\|^4_{L^\infty_T H^{\frac{1}{2}}})\|v\|_{Y^s_T} + T^{\frac{1}{4}}(1+\|u\|^8_{L^\infty_T H^{\frac{1}{2}}})\|u\|_{L^\infty_T H^{\frac{1}{2}}}, \tag{3.5}$$

and for large $N \in \mathbb{N}$, we have

$$\begin{aligned}\|u\|_{L^\infty_T H^s} \lesssim & \|u_0\|_{H^s} + TN^2\|u\|^3_{L^\infty_T H^{1/2}} + \|v\|_{Y^s_T} \\ & + N^{-\frac{1}{4}}(\|u\|_{L^\infty_T H^{\frac{1}{2}}} + \|v\|_{Y^s_T})(1+\|u\|^8_{L^\infty_T H^{\frac{1}{2}}}).\end{aligned} \tag{3.6}$$

Moreover, we have

$$\begin{aligned}\|u_1-u_2\|_{\widetilde{L^4_T H^{1/2}_4}} \lesssim & (1+\|u\|^4_{L^\infty_T H^{1/2}})\|v_1-v_2\|_{Y^{\frac{1}{2}}_T} \\ & + \|u_1-u_2\|_{L^\infty_T H^{1/2}}\|v_1\|_{Y^{\frac{1}{2}}_T}\prod_{i=1}^2(1+\|u_i\|_{L^\infty_T H^{\frac{1}{2}}})^3 \\ & + T^{1/4}\|u_1-u_2\|_{L^\infty_T H^{\frac{1}{2}}}\prod_{i=1}^2(1+\|u_i\|_{L^\infty_T H^{\frac{1}{2}}})^8\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\|u_1-u_2\|_{L^\infty_T H^{1/2}} \lesssim & \|u_{1,0}-u_{2,0}\|_{H^{1/2}} + \prod_{i=1}^2(1+\|u_i\|_{L^\infty_T H^{\frac{1}{2}}} + \|v_i\|_{Y^{1/2}_T})^8 \\ & \times \Big(\|v_1-v_2\|_{Y^{1/2}_T} + (TN^2+N^{-1/4})\|u_1-u_2\|_{L^\infty_T H^{\frac{1}{2}}}\Big).\end{aligned} \tag{3.8}$$

Proposition 3.3 (Estimates of v). Let $0<T<1$, $s\in[\frac{1}{2},1]$ and $(u_i,v_i) \in \Big(C^0_t H^s \cap \widetilde{L^4_T H^s_4}\Big) \times Y^s_T$ satisfying (1.8), (2.1) and (2.9) on $]-T,T[$. Then for $(u,v)=(u_i,v_i)$ there exists $\nu>0$ and $q \in \mathbb{N}^*$ such that

$$\begin{aligned}\|v\|_{Y^s_T} \lesssim & (1+\|u_0\|^4_{H^{\frac{1}{2}}})\|u_0\|_{H^s} + T^\nu\Big((1+\|u\|^{q+1}_{L^\infty_T H^{\frac{1}{2}} \cap \widetilde{L^4 H^{1/2}_4}})\|v\|_{X^{s,1/2}} \\ & + (1+\|u\|^q_{L^\infty_T H^{\frac{1}{2}} \cap \widetilde{L^4 H^{1/2}_4}})\|v\|_{X^{1/2,1/2}}\|u\|_{L^\infty_T H^s \cap \widetilde{L^4 H^s_4}}\Big).\end{aligned} \tag{3.9}$$

and

$$\|v_1 - v_2\|_{Y_T^{\frac{1}{2}}} \lesssim (1 + \|u_0\|^4_{H^{\frac{1}{2}}})\|u_{1,0} - u_{2,0}\|_{H^s} \tag{3.10}$$

$$+ T^\nu \Big[\Big(1 + \sum_{i=1}^{2} \|u_i\|^{q+1}_{L^\infty_T H^{\frac{1}{2}} \cap \widetilde{L^4 H_4^{1/2}}}\Big) \|v_1 - v_2\|_{X^{s,1/2}}$$

$$+ \Big(1 + \sum_{i=1}^{2} \|u_i\|^{q}_{L^\infty_T H^{\frac{1}{2}} \cap \widetilde{L^4 H_4^{1/2}}}\Big) (\sum_{i=1}^{2} \|v_i\|_{X^{1/2,1/2}}) \|u_1 - u_2\|_{L^\infty_T H^s \cap \widetilde{L^4 H_4^s}} \Big]. \tag{3.11}$$

The rest of this subsection is devoted to proving Proposition 3.2, while the proof of Proposition 3.3 will be given in the next section.

3.3. **Proof of Proposition 3.2.** We start with recalling some technical lemmas that will be needed hereafter. We first recall the Sobolev multiplication laws.

Lemma 3.4. (a) Assume one of the following condition

$$s_1 + s_2 \geq 0, s \leq s_1, s_2, s < s_1 + s_2 - \frac{1}{2},$$

$$\text{or} \quad s_1 + s_2 > 0, s < s_1, s_2, s \leq s_1 + s_2 - \frac{1}{2}.$$

Then

$$\|fg\|_{H^s} \lesssim \|f\|_{H^{s_1}} \|g\|_{H^{s_2}}.$$

(b) For any $s \geq 0$, we have

$$\|fg\|_{H^s} \lesssim \|f\|_{H^s} \|g\|_{L^\infty} + \|g\|_{H^s} \|f\|_{L^\infty}.$$

Second, we state the classical fractional Leibniz rule estimate derived by Kenig, Ponce and Vega (See Theorems A.8 and A.12 in [14]).

Lemma 3.5. Let $0 < \alpha < 1$, $p,\ p_1,\ p_2 \in (1, +\infty)$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ and $\alpha_1,\ \alpha_2 \in [0, \alpha]$ with $\alpha = \alpha_1 + \alpha_2$. Then,

$$\big\|D_x^\alpha (fg) - f D_x^\alpha g - g D_x^\alpha f\big\|_{L^p} \lesssim \|D_x^{\alpha_1} g\|_{L^{p_1}} \|D_x^{\alpha_2} f\|_{L^{p_2}}. \tag{3.12}$$

Moreover, for $\alpha_1 = 0$, the value $p_1 = +\infty$ is allowed.

The next estimate is a frequency localized version of estimate (3.12) in the same spirit as Lemma 3.2 in [27]. It allows to share most of the fractional derivative in the first term on the right-hand side of (3.13).

Lemma 3.6. Let $\alpha, \beta \geq 0$ and $1 < q < \infty$. Then,

$$\big\|D_x^\alpha P_\mp \big(f P_\pm D_x^\beta g\big)\big\|_{L^q} \lesssim \|D_x^{\alpha_1} f\|_{L^{q_1}} \|D_x^{\alpha_2} g\|_{L^{q_2}}, \tag{3.13}$$

with $1 < q_i < \infty$, $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$ and $\alpha_1 \geq \alpha$, $\alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 = \alpha + \beta$.

Proof. See Lemma 3.2 in [21]. □

Finally we state the two following lemmas. The first one is a direct consequence of the continuous embeddings $H^{s+1/4} \hookrightarrow H_4^{1/2} \hookrightarrow L^\infty$ whereas the proof of the second one (in the real line case) can be found in [[19], Lemma 6.1].

Lemma 3.7. Let $s\in[1/2,1]$, $z\in L^\infty_T H^{s+\frac14}$ and $v\in \widetilde{L^4_T H^s_4}$ then

$$\|zv\|_{\widetilde{L^4_T H^s_4}}\lesssim \|z\|_{L^\infty_T H^{s+\frac14+}}\|v\|_{\widetilde{L^4_T H^s_4}} \tag{3.14}$$

Lemma 3.8. Let $v_1,v_2\in \widetilde{L^4 H^{1/2}_4}$ then

$$\|B(v_1,v_2)\|_{L^2}\lesssim \|D_x^{1/2}v_1\|_{L^4}\|D_x^{1/2}v_2\|_{L^4} \tag{3.15}$$

Let $k\in\mathbb{Z}^*$ with $|k|\le 10$. A direct computation gives

$$\partial_x(e^{ikF})=kie^{ikF}(u^2-P_c(u^2)), \tag{3.16}$$

Next by gathering the obvious estimates $\|e^{ikF}\|_{L^\infty_T L^2_x}\lesssim 1$ and $\|\partial_x(e^{ikF})\|_{L^\infty_T L^2_x}\lesssim P_c(u^2)+\|u\|^2_{L^\infty_T L^4_x}$, we get

$$\|e^{ikF}\|_{L^\infty_T H^1}\lesssim 1+\|u\|^2_{L^\infty_T H^{\frac12}}. \tag{3.17}$$

On the other hand, by Lemma 3.4, we have for any $s\in[1/2,1]$,

$$\|\partial_x(e^{ikF})\|_{L^\infty_T H^{s-}}\lesssim \|e^{ikF}\|_{L^\infty_T H^1}\|u^2-P_c(u^2)\|_{L^\infty_T H^{s-}}\lesssim (1+\|u\|^2_{L^\infty_T H^{\frac12}})\|u\|_{L^\infty_T H^{\frac12}}\|u\|_{L^\infty_T H^s}\ .$$

Gathering the above estimates leads for any $s\in[1/2,1]$ to

$$\|e^{ikF}\|_{L^\infty_T H^{s+1-}}\lesssim (1+\|u\|^2_{L^\infty_T H^{\frac12}})\Big(1+\|u\|_{L^\infty_T H^{\frac12}}\|u\|_{L^\infty_T H^s}\Big) \tag{3.18}$$

and, in view of (2.6) and Lemma 3.6, it holds

$$\begin{aligned}\|R(u)\|_{L^\infty_T H^{\frac54+}} &\lesssim \|e^{iF}\|_{L^\infty_T H^{\frac54+}_{2+}}\|u\|_{L^\infty_T L^{\infty-}_x} &(3.19)\\ &\lesssim \|e^{-iF}\|_{L^\infty_T H^{\frac32-}}\|u\|_{L^\infty_T H^{\frac12}}\\ &\lesssim (1+\|u\|^4_{L^\infty_T H^{\frac12}})\|u\|_{L^\infty_T H^{\frac12}}\ . &(3.20)\end{aligned}$$

Now, for $s\in[1/2,1]$, according to (2.7), (3.18)-(3.20) and Lemma 3.7?we easily get

$$\begin{aligned}\|P_+u\|_{\widetilde{L^4_T H^s_4}} &\lesssim \|e^{iF}v\|_{\widetilde{L^4_T H^s_4}}+\|e^{iF}R(u)\|_{\widetilde{L^4_T H^s_4}}\\ &\lesssim \|e^{iF}\|_{L^\infty_t H^{3/2-}}\|v\|_{\widetilde{L^4_T H^s_4}}+T^{\frac14}\|e^{iF}\|_{L^\infty_t H^{3/2-}}\|R(u)\|_{L^\infty_t H^{\frac54+}}\\ &\lesssim (1+\|u\|^4_{L^\infty_T H^{\frac12}})\Big(\|v\|_{Y^s_T}+T^{\frac14}(1+\|u\|^4_{L^\infty_T H^{\frac12}})\|u\|_{L^\infty_T H^{\frac12}}\Big)\end{aligned}$$

Estimate (3.5) follows by using that u is real valued and the conservation of the mean-value by (1.8).
Next, in order to get a better estimate of $\|u\|_{L^\infty_T H^s}$, $s\in[\frac12,1]$, we split u into a low frequency and a high frequency part. For low frequency, we use the equation for u, while for high frequency, we use $P_+u=e^{iF}v-e^{iF}R(u)$. For any $N=2^k\in\mathbb{N}$, and $s\in[\frac12,1]$, we have

$$\|u\|_{L^\infty_T H^s}\lesssim \|P_{\le k}u\|_{L^\infty_T H^s}+2\|P^+_{\ge k}u\|_{L^\infty_T H^s}.$$

By the equation of u, we have

$$P_{\le k}u=W(t)P_{\le k}u_0+\frac13\int_0^t W(t-\tau)P_{\le k}\partial_x(u^3)(\tau)d\tau,$$

that leads to

$$\|P_{\leq k}u\|_{L^\infty_T H^s} \lesssim \|u_0\|_{H^s} + T2^{2k}\|u\|^3_{L^\infty_T H^{1/2}}.$$

To estimate the term $\|P^+_{\geq k}u\|_{L^\infty_T H^s}$, we use $P_+u = e^{iF}v - e^{iF}R(u)$. By (3.17) -(3.20) we have

$$\begin{aligned}\|P^+_{\geq k}[e^{iF}R(u)]\|_{L^\infty_T H^s} &\lesssim N^{-1/4}\|e^{-iF}R(u)\|_{L^\infty_T H^{5/4}} \\ &\lesssim N^{-\frac{1}{4}}(1+\|u\|^8_{L^\infty_T H^{\frac{1}{2}}})\|u\|_{L^\infty_T H^{1/2}} .\end{aligned}$$

It remains to estimate $\|P^+_{\geq k}[e^{iF}v]\|_{H^s}$. By Lemma 3.4 we have

$$\begin{aligned}\|P^+_{\geq k}[e^{iF}v]\|_{L^\infty_t H^s} &\lesssim \|P^+_{\geq k}[P_{\leq k-5}(e^{iF})v]\|_{L^\infty_t H^s} + \|P^+_{\geq k}[P_{>k-5}(e^{iF})v]\|_{L^\infty_t H^s} \\ &\lesssim \|e^{iF}\|_{L^\infty_T L^\infty_x}\|v\|_{L^\infty_t H^s} + \|v\|_{L^\infty_t H^s}\|P_{\geq k-5}(e^{iF})\|_{L^\infty_t H^1} \\ &\lesssim \|v\|_{L^\infty_t H^s}\Big(1 + N^{-1/4}(1+\|u\|^4_{L^\infty_T H^{\frac{1}{2}}})\Big).\end{aligned}$$

Then (3.6) holds. For the difference estimates (3.7)-(3.8), the proofs are similar. We only need to observe that by the mean-value theorem, $|e^{ikF(u_1)} - e^{ikF(u_2)}| \leq |k(P_{\neq c}(u_1^2 - u_2^2)|$ and thus

$$\|e^{ikF(u_1)} - e^{ikF(u_2)}\|_{L^\infty_T L^2_x} \lesssim \|u_1 - u_2\|_{L^\infty_T H^{1/2}}(\|u_1\|_{L^\infty_T H^{1/2}} + \|u_2\|_{L^\infty_T H^{1/2}}) \tag{3.21}$$

and

$$\begin{aligned}\|\partial_x(e^{ikF(u_1)} - e^{ikF(u_2)})\|_{L^\infty_T L^2_x} &\lesssim \|u_1 - u_2\|_{L^\infty_T H^{1/2}}(\|u_1\|_{L^\infty_T H^{1/2}} + \|u_2\|_{L^\infty_T H^{1/2}}) \\ &\quad + \|P_{\neq c}(u_1^2)(e^{ikF(u_1)} - e^{ikF(u_2)})\|_{L^\infty_T L^2_x} \\ &\lesssim \|u_1 - u_2\|_{L^\infty_T H^{1/2}}(\|u_1\|_{L^\infty_T H^{1/2}} + \|u_2\|_{L^\infty_T H^{1/2}})^3 . \end{aligned} \tag{3.22}$$

3.4. **Proof of Theorem 1.1.** In this subsection, we prove Theorem 1.1. We will rely on the results obtained in [23]:

Lemma 3.9 ([23]). The mBO equation (1.1) is locally well-posed in H^s for $s \geq 1$. Moreover, the minimal length of the interval of existence is determined by $\|u_0\|_{H^1}$.

Now, fixing any $u_0 \in H^{1/2}(\mathbb{T})$, we choose $\{u_{0,n}\} \subset C^\infty(\mathbb{T})$, real-valued, such that $u_{0,n} \to u_0$ in $H^{1/2}$. We denote by u_n the solution of mBO emanating from $u_{0,n}$ given by Lemma 3.9 and $v_n = P_+(e^{-iF(u_n)}u_n)$.

Step 1. A priori estimate: we show that there exists $T = T(\|u_0\|_{H^{1/2}}) > 0$ such that u_n exists on $(-T, T)$.

It suffices to show that there exists a $T = T(\|u_0\|_{H^{1/2}}) > 0$, such that for any $n \in \mathbb{N}$, if $|t| \leq T$ and $u_n(t)$ exists, then

$$\|u_n(t)\|_{H^s} \leq C(\|u_{0,n}\|_{H^s}), \quad 1/2 \leq s \leq 1. \tag{3.23}$$

First we show (3.23) for $s = 1/2$. We may assume $\|u_{0,n}\|_{H^{1/2}} \leq 2\|u_0\|_{H^{1/2}}$, $\forall n \in \mathbb{N}$. Define the quantity $\|(u,v)\|_{F^s_T}$ by

$$\|(u,v)\|_{F^s_T} := \|u\|_{L^\infty_T H^s} + \|v\|_{X^{s,\frac{1}{2}}_T}.$$

Applying Proposition 3.2-3.3 to u_n, v_n (taking $s = 1/2$), we get

$$\|(u_n, v_n)\|_{F_T^{1/2}} \lesssim (1 + \|u_0\|_{H^{1/2}}^8)\|u_0\|_{H^{1/2}} + (T^{1/4}N^2 + N^{-1/4})\|(u_n, v_n)\|_{F_T^{1/2}}^9 + T^\nu\Big(1 + \|(u_n, v_n)\|_{F_T^{1/2}}^k\Big)\|(u_n, v_n)\|_{F_T^{1/2}} ,$$

for some $\nu > 0$ and $k \in \mathbb{N}^*$ and for any $N \geq 1$ and $0 < T < 1$. Therefore taking N large enough, we infer that there exits $T = T(\|u_0\|_{H^{1/2}}) > 0$ such that (3.23) holds for $s = 1/2$. Now, for $1/2 < s \leq 1$, we have

$$\|(u_n, v_n)\|_{F_T^s} \lesssim (1 + \|u_0\|_{H^{1/2}}^8)\|u_{0,n}\|_{H^s} + (T^{1/4}N^2 + N^{-1/4})\|(u_n, v_n)\|_{F_T^{1/2}}^8 \|(u_n, v_n)\|_{F_T^s} + T^\nu\Big(1 + \|(u_n, v_n)\|_{F_T^{1/2}}^k\Big)\|(u_n, v_n)\|_{F_T^s} ,$$

which yields (3.23) for some $T = T(\|u_0\|_{H^{1/2}}) > 0$ smaller if necessarily. This completes the Step 1.

Step 2. Next, we will show that u_n is a Cauchy sequence in $C([-T, T]; H^{1/2})$.

Applying the difference estimates in Proposition 3.2-3.3 to (u_n, v_n), arguing as in Step 1, we get

$$\|(u_n - u_m, v_n - v_m)\|_{F_T^{1/2}} \lesssim \|u_{0,n} - u_{0,m}\|_{H^{\frac{1}{2}}}. \tag{3.24}$$

Thus, (u_n, v_n) is a Cauchy sequence, and there exists $u \in C([-T, T]; H^{1/2})$ such that $\|u_n - u\|_{L_T^\infty H^{\frac{1}{2}}} \to 0$, $n \to \infty$. By classical compactness arguments, it is easy to check that u solves the mBO equation. Moreover, in view of (3.24) it is the only solution in the class $u \in L_T^\infty H^{1/2}$ with $P_+(e^{iF(u)}) \in X_T^{\frac{1}{2},\frac{1}{2}}$ and the solution-map $u_0 \mapsto u$ is continuous from $H^{\frac{1}{2}}(\mathbb{T})$ into $C([-T, T]; H^{1/2})$. At last, in the defocusing case using the conservation of $H^{\frac{1}{2}}$ norm of u, we get that u is global in time.

4. Proof of the estimates on v

In this section, we prove Proposition 3.3. We will work on the equation (2.9). By Lemma 3.1 and the trivial embedding $L_T^2 H^s \hookrightarrow X_T^{s,-\frac{1}{2}+}$, we infer that

$$\|v\|_{Y_T^s} \lesssim \|v(0)\|_{H^s} + T^\nu\Big(\|G(u)\|_{L_T^2 H^s} + \|N^0(u, v)\|_{X_T^{s,-\frac{1}{2}+}} + \|N^1(u, v)\|_{X_T^{s,-\frac{1}{2}+}} + \|P_+[-2ie^{-iF}uB(u, u)]\|_{X_T^{s,-\frac{1}{2}+}}\Big) \tag{4.1}$$

for some $\nu > 0$. Then to prove Proposition 3.3, we will estimate the terms of the right-hand side one by one.

4.1. Estimate on $G(u)$.

Lemma 4.1. Let $1/2 \leq s \leq 1$, $0 < T \leq 1$ and $u_i \in C([-T, T] : H^s) \cap \widetilde{L_T^4 H_4^s}$, $i = 1, 2$, be two solutions to (1.8) with initial data $u_{i,0}$. Then for $u = u_i$ we have

$$\|\mathcal{G}(u_{i,0})\|_{H^s} \lesssim (1 + \|u_{i,0}\|_{H^{1/2}}^4)\|u_{i,0}\|_{H^s}$$
$$\|G(u_i)\|_{L_T^2 H^s} \lesssim (1 + \|u_i\|_{L_T^\infty H^{1/2} \cap L_T^4 H_4^{1/2}}^{12})\|u_i\|_{L_T^\infty H^s \cap L_T^4 H_4^s} .$$

Moreover, it holds

$$\|\mathcal{G}(v_{1,0})-\mathcal{G}(v_{2,0})\|_{H^{1/2}}\lesssim\|u_{1,0}-u_{2,0}\|_{H^{1/2}}\prod_{i=1}^{2}(1+\|u_{i,0}\|_{H^{1/2}}^4)$$

$$\|G(u_1)-G(u_2)\|_{L^2_TH^{1/2}}\lesssim\|u_1-u_2\|_{L^\infty_TH^{1/2}\cap L^4_TH^{1/2}_4}\prod_{i=1}^{2}(1+\|u_i\|^{12}_{L^\infty_TH^{1/2}\cap L^4_TH^{1/2}_4}).$$

where the gauge transformation $\mathcal{G}$ and the function G are defined respectively in (2.1) and (2.10).

Proof. The estimates on $v_{i,0}=\mathcal{G}(u_{i,0})$ and its difference are similar to the estimates of u in the proof of Proposition 3.2. The estimates on G follow from the definition (2.10) of $G(u)$, Lemma 3.4 and Lemma 3.6. For instance we have

$$\begin{aligned}\|e^{-iF}P_+(u^2\partial_xP_-(e^{-iF}\overline{R(u)})\|_{H^s} &\lesssim\|e^{-iF}\|_{H^{\frac32-}}\|u^2\|_{H^s_4}\|\partial_xP_-(e^{-iF}\overline{R(u)})\|_{H^{0+}_4}\\ &\lesssim\|e^{-iF}\|^2_{H^{\frac32-}}\|u\|_{H^s_4}\|u\|_{H_4^{\frac12}}\|R(u)\|_{H^{\frac54+}}\\ &\lesssim\|u\|_{H^s_4}\|u\|_{H_4^{\frac12}}\|u\|_{H^{\frac12}}(1+\|u\|^4_{H^{1/2}})^3\end{aligned}$$

and

$$\left\|\partial_xP_+\Big(\partial_x^{-1}P_+(e^{-iF})P_-u_x\Big)\right\|_{H^s}\lesssim\|e^{-iF}\|_{H^1_4}\|u\|_{H^s_4}\lesssim(1+\|u\|^4_{H^{1/2}})\|u\|_{H^s_4}$$

□

4.2. **Estimates on suitable extensions of** u **and** $e^{-iF(u)}$ **.** Before proving the main multilinear estimates, we need to prove estimates on suitable extensions of u and $e^{-iF(u)}$.

Lemma 4.2. Let $1/2\le s\le 1$, $0<T\le 1$ and $u_1,u_2\in C([-T,T]:H^s)\cap\widetilde{L^4_TH^s_4}$ be two solutions to (1.8). Then for $i=1,2$

$$\|u_i\|_{(X^{s-1,1}\cap L^\infty_tH^s\cap\widetilde{L^4_tH^s_4})_T}\le(1+\|u_i\|^2_{L^\infty_TH^{\frac12}_x})\|u_i\|_{L^\infty_TH^s\cap\widetilde{L^4_TH^s_4}}. \tag{4.2}$$

Moreover, we have

$$\|u_1-u_2\|_{(X^{-\frac12,1}\cap L^\infty_tH^{\frac12}\cap\widetilde{L^4_tH^{\frac12}_4})_T}\lesssim\|u_1-u_2\|_{L^\infty_TH^{1/2}\cap\widetilde{L^4_TH^{1/2}_4}}\prod_{i=1}^{2}(1+\|u_i\|^2_{L^\infty_TH^{1/2}\cap L^4_TH^{1/2}_4}). \tag{4.3}$$

Proof. We consider $w(t)=W(-t)u(t)$ on the time interval $[-T,T]$ and extend w on $(-2,2)$ by setting $\partial_tw=0$ on $[-2,-2]\setminus[-T,T]$. Then it is clear that for any $\theta\in\mathbb{R}$,

$$\|\partial_tw\|_{L^2((-2,2):H^\theta)}=\|\partial_tw\|_{L^2_TH^\theta},\ \|w\|_{L^2((-2,2):H^\theta)}\lesssim\|w\|_{L^\infty_TH^\theta}$$

Now we define $\tilde u(t)=\eta(t)W(t)w(t)$. $\tilde u$ is clearly an extension of u outside $(-T,T)$ and it suffices to prove (4.2) with the $X^{s-1,1}$, $L^\infty_tH^s$ and $L^4_tH^s_4$-norms of $\tilde u$ in the left-hand side. First, using that $\partial_t w = 2W(-t)(P_{\neq c}(u^2)u_x)$, we get

$$\begin{aligned}\|\tilde u\|_{X^{s-1,1}} \lesssim & \|w\|_{L^2((-2,2):H^{s-1})} + \|\partial_t w\|_{L^2((-2,2):H^{s-1})} \\ \lesssim & \|u\|_{L^2((-2,2):H^{s-1})} + \|D_x^s(u^3)\|_{L^2_{Tx}} + \|u\|^2_{L^\infty_T L^2_x}\|D^s_x u\|_{L^2_{Tx}} \\ \lesssim & \|u\|_{L^2((-2,2):H^{s-1})} + \|D^s_x u\|_{L^4_{Tx}\cap L^\infty_T L^2_x}\|u\|^2_{L^\infty_T H^{\frac12}_x}\end{aligned}$$

where in the last step we used Lemma 3.5 together with $L^\infty_t H^{1/2}_x \hookrightarrow L^8_{tx}$. Second,

$$\|\tilde u\|_{L^\infty_t H^s} \lesssim \|\eta(t)W(t)w(t)\|_{L^\infty_t H^s}? \lesssim \|w\|_{L^\infty_T H^s} \lesssim \|u\|_{L^\infty_T H^s} \; .$$

Third, we notice that

$$\|\tilde u\|_{\widetilde{L^4_t H^s_4}} \lesssim \|u\|_{\widetilde{L^4(]-T,T[;H^s_4)}} + \|W(t)w(t)\|_{\widetilde{L^4(]-2,2[/]-T,T[;H^s_4)}}$$

with $w(t) = w(T)$ for all $t \in]T,2[$ and $w(t) = W(-T)$ for all $t \in]-2,-T[$. Therefore, in view of (3.4),

$$\|W(t)w(t)\|_{\widetilde{L^4(]T,2[;H^s_4)}} = \|W(t)w(T)\|_{\widetilde{L^4(]T,2[;H^s_4)}} \lesssim \|w(T)\|_{H^s} = \|u(T)\|_{H^s} \lesssim \|u\|_{L^\infty_T H^s} \; .$$

This completes the proof of (4.2). Finally the estimates for the difference is similar and thus will be omitted. □

Next, we prove the properties of the factor e^{ikF}.

Lemma 4.3. Let $1/2 \le s \le 1$, $0 < T \le 1$ and $u_1, u_2 \in C([-T,T] : H^s) \cap \widetilde{L^4_T H^s_4}$ be two solutions to (1.8). Then for $i = 1, 2$

$$\|e^{-iF(u_i)}\|_{(L^\infty_t H^{s+1-}\cap X^{-\frac12-,1})_T} \lesssim 1 + (1+\|u_i\|^6_{L^\infty_T H^{\frac12}\cap L^4_T H^{\frac12}_4})\|u_i\|_{L^\infty_T H^s \cap L^4_T H^s_4} \; . \qquad (4.4)$$

Moreover,

$$\begin{aligned}&\|e^{-iF(u_1)} - e^{-iF(u_2)}\|_{(L^\infty_t H^{\frac32-}\cap L^4_t H^{\frac32}\cap X^{-\frac12-,1})_T} \\ &\qquad \lesssim \|u_1 - u_2\|_{L^\infty_T H^{\frac12}\cap L^4_T H^{\frac12}_4} \prod_{i=1}^2 (1+\|u\|^6_{L^\infty_T H^{\frac12}\cap L^4_T H^{\frac12}_4}) \; . \qquad (4.5)\end{aligned}$$

Proof. We set $z(t) = W(-t)e^{-iF}$ on $]-T,T[$ and than extend z on $]-2,2[$ by setting $\partial_t z = 0$ on $[-2,-2]\backslash[-T,T]$. Then $\tilde w = \eta(t)W(t)z(t)$ is an extension of e^{-iF} outside $(-T,T)$. As in the previous lemma, for any $\theta \in \mathbb{R}$, it holds

$$\|\tilde w\|_{L^\infty_t H^\theta} \lesssim \|e^{-iF}\|_{L^\infty_T H^\theta}?$$

which together with (3.17)-(3.18) gives the estimate for the first term on the left-hand side of (4.4). Moreover,

$$\|\tilde w\|_{X^{-\frac12-,1}} \lesssim \|e^{-iF}\|_{L^2_T H^{-\frac12-}} + \|(\partial_t + \mathcal{H}\partial_x^2)e^{-iF}\|_{L^2_T H^{-\frac12-}}$$

.

with

$$(\partial_t + \mathcal{H}\partial_x^2)e^{-iF} = -ie^{-iF}F_t - i\mathcal{H}\Big(e^{-iF}\Big(2uu_x - i(P_{\neq c}(u^2))^2\Big)\Big)$$

According to the expression (2.2) of F_t, Lemma 3.4 and Lemma 3.8 , it holds

$$\|F_t\|_{L^2_T H^{-\frac{1}{2}-}} + \left\|2uu_x + ik(P_{\neq c}(u^2))^2\right\|_{L^2_T H^{-\frac{1}{2}-}} \lesssim \|u\|^4_{L^\infty_T H^{1/2}} + \|u\|^2_{L^4_T H^{1/2}_4}$$

which yields the desired result by using (3.17) and again Lemma 3.4.
For the difference estimate (4.5), the proof is similar by using (3.21)-(3.22). The details are omitted. □

4.3. **Multilinear estimates.** With Lemmas 4.2-4.3 in hand, the following proposition enables us to treat the worst term of (4.1), that is $N^\nu(u,v)$ with $\nu \in \{0,1\}$.

Proposition 4.4. Let $1/2 \le s \le 1$, $w_1, w_4 \in X^{-1/2-,1} \cap L^\infty_t H^{s+1-}_x$, $u_2, u_3 \in X^{s-1,1} \cap L^\infty_t H^s_x \cap \widetilde{L^4_t H^s_4}$ and $v_5 \in X^{1/2,1/2}$ with compact support in time. Then it holds

$$\begin{aligned}
&\left\|\partial_x^\nu P_+\Big(\partial_x^{-\nu} P_+\Big(w_1u_2u_3\Big)\partial_x P_-(w_4v_5)\Big)\right\|_{X^{s,-\frac{1}{2}+}} \\
&\lesssim \|w_1\|_{L^\infty_t H^{s+1-}} \|w_4\|_{L^\infty_T H^{1/2}_x} \|v_5\|_{X^{1/2,1/2}} \prod_{i=2}^{3} \|u_i\|_{L^\infty_t H^{1/2}_x} \\
&\quad + \|v_5\|_{X^{1/2,1/2}} \prod_{i=1,4} \Big(\|w_i\|_{X^{-1/2-,1}\cap L^\infty_t H^{3/2-}_x}\Big) \times \\
&\quad \sum_{2\le i\neq j\le 3} \Big(\|u_i\|_{X^{-1/2,1}\cap L^\infty_t H^{1/2}_x \cap \widetilde{L^4_t H^{1/2}_x}}\Big)\Big(\|u_j\|_{X^{s-1,1}\cap L^\infty_t H^s_x \cap \widetilde{L^4_t H^s_4}}\Big).
\end{aligned} \tag{4.6}$$

Proof. We want to prove that

$$\begin{aligned}
I :=& \left\|\partial_x^\nu P_+\Big(\partial_x^{-\nu} P_+\Big(w_1u_2u_3\Big)\partial_x P_-(w_4v_5)\Big)\right\|_{X^{s,-\frac{1}{2}+}} \\
=& \Bigg\| \sum_{N\ge 2, N_{123}\ge N, N_{45}\le N_{123}} \sum_{N_i,\ 1\le i\le 5} \\
&\partial_x^\nu P_N P_+\Big(\partial_x^{-\nu} P_{N_{123}}\Big(P_{N_1}w_1 P_{N_2}u_2 P_{N_3}u_3\Big)\partial_x P_{N_{45}} P_-(P_{N_4}w_4 P_{N_5}v_5)\Big)\Bigg\|_{X^{s,-1/2+}}.
\end{aligned}$$

By the triangle inequality we can separate this sum in different sums on disjoint subset of $(2^{\mathbb{N}})^8$. By symmetry we can assume that $N_2 \le N_3$.

1. $N_4 \geq 2^{-8}N_5$. Then $N_{45} \lesssim N_4$ and we can write by almost orthogonality

$$
\begin{aligned}
I \lesssim & \Big[\sum_{N_{123}}\Big(\sum_{N_4}\sum_{N_5\lesssim N_4}\sum_{2\leq N\leq N_{123}}\sum_{N_{45}\lesssim N_4} \\
& \qquad \Big\|\partial_x^{\nu}P_N P_+\Big(P_{N_{123}}\partial_x^{-\nu}\Big(w_1u_2u_3\Big)\partial_x P_{N_{45}}P_-(P_{N_4}w_4P_{N_5}v_5)\Big)\Big\|_{X^{s,-1/2+}}\Big)^2\Big]^{1/2} \\
\lesssim & \Big[\sum_{N_{123}}\Big(\sum_{N_4,N_5}\sum_{2\leq N\leq N_{123}}\sum_{N_{45}\lesssim N_4} N^s\Big\|\partial_x^{\nu}P_N P_+\Big(P_{N_{123}}\partial_x^{-\nu}\Big(w_1u_2u_3\Big) \\
& \qquad \cdot\partial_x P_{N_{45}}P_-(P_{N_4}w_4P_{N_5}v_5)\Big)\Big\|_{L^{4/3}_{tx}}\Big)^2\Big]^{1/2} \\
\lesssim & \Big[\sum_{N_{123}}\Big(N_{123}^s\|P_{N_{123}}(w_1u_2u_3)\|_{L^2_{tx}}\sum_{N\leq N_{123}}(\frac{N}{N_{123}})^s\Big)^2\Big]^{1/2} \\
& \sum_{N_4,N_5,N_{45}} N_{45}^{0-}N_4^{1+}N_5^{0-}\|P_{N_4}w_4\|_{L^8_{tx}}\|P_{N_5}v_5\|_{L^8_{tx}} \\
\lesssim & \|J_x^s(w_1u_2u_3)\|_{L^2_{tx}}\|w_4\|_{L^\infty_t H^{\frac{3}{2}-}_x}\|v_5\|_{L^8_{tx}} \\
\lesssim & (\|J_x^s w_1\|_{L^4_{tx}}+\|J_x^s u_2\|_{L^4_{tx}}+\|J_x^s u_3\|_{L^4_{tx}})(\|w_1\|_{L^8_{tx}}+\|u_2\|_{L^8_{tx}}+\|u_3\|_{L^8_{tx}})^2 \\
& \|w_4\|_{L^\infty_t H^{3/2-}_x}\|v_5\|_{X^{1/2,1/2}} \\
\lesssim & (\|w_1\|_{L^\infty_t H^{\frac{3}{2}-}}+\|u_2\|_{L^4_t H^s_4}+\|u_3\|_{L^4_t H^s_4})(\|w_1\|_{L^8_{tx}}+\|u_2\|_{L^8_{tx}}+\|u_3\|_{L^8_{tx}})^2 \\
& \|w_4\|_{L^\infty_t H^{3/2-}_x}\|v_5\|_{X^{1/2,1/2}}
\end{aligned}
$$

where in the second to the last step we used Lemma 3.12 .

2. $N_4 < 2^{-8}N_5$. Then $N_{45} \sim N_5$ so that we can drop the summation over N_{45} by replacing $P_{N_{45}}$ by $\tilde{P}_{N_5}$. Note that in this region the frequency projections force $N_5 \lesssim N_{123}$.

2.1. $N_4 \geq 2^{-8}N$. By almost orthogonality it yields

$$
\begin{aligned}
I \lesssim & \Big[\sum_{N_{123}}\Big(\sum_{N_5\lesssim N_{123}}\sum_{N_4\lesssim N_5}\sum_{2\leq N\leq N_4} N^s\Big\|\partial_x^{\nu}P_N P_+\Big(P_{N_{123}}\partial_x^{-\nu}\Big(w_1u_2u_3\Big) \\
& \qquad \cdot\partial_x\tilde{P}_{N_5}P_-(P_{N_4}w_4P_{N_5}v_5)\Big)\Big\|_{L^{4/3}_{tx}}\Big)^2\Big]^{1/2} \\
\lesssim & \Big[\sum_{N_{123}}\Big(N_{123}^s\|P_{N_{123}}(w_1u_2u_3)\|_{L^2_{tx}}\sum_{N_5\lesssim N_{123}}(\frac{N_5}{N_{123}})^{1/2}\|D_x^{1/2}P_{N_5}v_5\|_{L^4_{tx}}\Big)^2\Big]^{1/2} \\
& \sum_{N_4,N} N^{0-}N_4^{\frac{1}{2}+}\|P_{N_4}w_4\|_{L^\infty_{tx}} \\
\lesssim & \|J_x^s(w_1u_2u_3)\|_{L^2_{tx}}\|w_4\|_{L^\infty_t H^{3/2-}_x}\|D_x^{1/2}v_5\|_{L^4_{tx}} \\
\lesssim & (\|w_1\|_{L^\infty_t H^{\frac{3}{2}-}}+\|u_2\|_{L^4_t H^s_4}+\|u_3\|_{L^4_t H^s_4})(\|w_1\|_{L^8_{tx}}+\|u_2\|_{L^8_{tx}}+\|u_3\|_{L^8_{tx}})^2 \\
& \|w_4\|_{L^\infty_t H^{3/2-}_x}\|v_5\|_{X^{1/2,1/2}}
\end{aligned}
$$

2.2. $N_4 < 2^{-8}N$.
2.2.1. $N_1 \geq 2^{-8}N_{123}$. Then we get

$$I \lesssim \sum_{N\geq 2} N^{s+\nu} \sum_{N_{123}\geq N} N_{123}^{-\nu} \sum_{N_1 \gtrsim N_{123}} \sum_{N_2,N_3,N_4,N_5 \lesssim N_1} \|P_{N_1}w_1 P_{N_2}u_2 P_{N_3}u_3\|_{L^2_{tx}}$$
$$\cdot N_5^{1/2}\|D_x^{1/2}v_5\|_{L^4_{tx}}\|P_{N_4}w_4\|_{L^\infty_{tx}}$$
$$\lesssim \sum_{N\geq 2} N^{0-} \sum_{N_{123}\geq N} N_{123}^{0-} \sum_{N_1} N_1^{s+\frac{1}{2}+}\|P_{N_1}w_1\|_{L^4_{tx}}(\sum_{N_4} N_4^{0-}\|w_4\|_{L^\infty_{tx}})$$
$$(\sum_{N_5} N_5^{0-}\|D_x^{1/2}v_5\|_{L^4_{tx}})\Big)\prod_{i=2}^{3}\Big(\sum_{N_i} N_i^{0-}\|P_{N_i}u_i\|_{L^8_{tx}}\Big)$$
$$\lesssim \|w_1\|_{L^\infty_t H^{s+1-}}\|w_4\|_{L^\infty_T H^{1/2}_x}\|v_5\|_{X^{1/2,1/2}}\prod_{i=2}^{3}\|u\|_{L^\infty_t H^{1/2}_x}$$

2.2.2 $N_1 < 2^{-8}N_{123}$. Then we have $N_3 \sim N_{123} \sim N_{max}$. Since in this case it always hods $2^{-3} \leq N_3/N_{123} \leq 2^3$, by a slight abuse of notation we can drop the summation over N_{123} by replacing $P_{N_{123}}$ by $\tilde{P}_{N_3}$.
2.2.2.1 $N_1 \geq 2^{-5}N_5$. Then by almost orthogonality we get

$$I \lesssim \Big[\sum_{N_3\geq 2}\Big(\sum_{2\leq N\lesssim N_3} N^{s+\nu}N_3^{-\nu}\sum_{N_1,N_2}\|P_{N_1}w_1P_{N_2}u_2P_{N_3}u_3\|_{L^{\frac{8}{3}}_{tx}}$$
$$\cdot\sum_{N_5\lesssim N_1} N_5\|v_5\|_{L^8_{tx}}\sum_{N_4}\|P_{N_4}w_4\|_{L^\infty_{tx}}\Big)^2\Big]^{1/2}$$
$$\lesssim \Big[\sum_{N_3\geq 2}\Big(N_3^s\|P_{N_3}u_3\|_{L^4_{tx}}\sum_{N\lesssim N_3}(\frac{N}{N_3})^{s+\nu}\Big)^2\Big]^{1/2}\sum_{N_1} N_1^{1+}\|P_{N_1}w_1\|_{L^4_{tx}}$$
$$\sum_{N_3} N_2^{0-}\|D_x^{0+}P_{N_2}u_2\|_{L^8_{tx}}\sum_{N_4} N_4^{0-}\|D_x^{0+}P_{N_4}w_4\|_{L^\infty_{tx}}\sum_{N_5} N_5^{0-}\|D_x^{0+}P_{N_5}v_5\|_{L^8_{tx}}$$
$$\lesssim \|w_1\|_{L^\infty_t H^{3/2-}}\|u_3\|_{\widetilde{L^4_t H^s_4}}\|u_2\|_{L^\infty_t H^{1/2}}\|w_4\|_{L^\infty_t H^{1/2}}\|v_5\|_{X^{1/2,1/2}}$$

2.2.2.2. $N_1 < 2^{-5}N_5$ and $N_1 \geq 2^{-5}N$. Then it holds

$$I \lesssim \sum_{N_2}\|P_{N_2}u_2\|_{L^4_{tx}}\sum_{N_1} N_1^{\frac{1}{2}+}\|P_{N_1}w_1\|_{L^\infty_{tx}}\sum_{N_3} N_3^{s-}\|P_{N_3}u_3\|_{L^4_{tx}}$$
$$\sum_{N_4}\|P_{N_4}w_4\|_{L^\infty_{tx}}\sum_{N_5} N_5^{1/2-}\|P_{N_5}v_5\|_{L^4_{tx}}$$
$$\lesssim \|w_1\|_{L^\infty_t H^{3/2-}_x}\|u_2\|_{L^\infty_t H^{1/2}}\|u_3\|_{\widetilde{L^4_t H^s_4}}\|v_5\|_{X^{1/2,1/2}}\|w_4\|_{L^\infty_t H^{3/2-}_x}.$$

2..2.2.3 $N_1 < 2^{-5}(N_5 \wedge N)$ and $N_2 \geq 2^{-5}(N_5 \wedge N)$.
2.2.2.3.1 $N_2 \geq 2^{-7}N$. Then either $N_3 \sim N$ and then $N_3 \sim N \sim N_2$ which leads to

$$\begin{aligned}
I \lesssim & \Big[\sum_{N_3 \geq 2} \Big(\sum_{N_2 \sim N_3} \sum_{\substack{N_i \lesssim N_2 \\ i \in \{1,4,5\}}} \Big\| \partial_x^{\nu} \tilde{P}_{N_3} P_+ \Big(\tilde{P}_{N_3} \partial_x^{-\nu} \Big(P_{N_1} w_1 P_{N_2} u_2 P_{N_3} u_3 \Big) \\
& \qquad \cdot \partial_x \tilde{P}_{N_5} P_- (P_{N_4} w_4 P_{N_5} v_5) \Big) \Big\|_{X^{s,-1/2}} \Big)^2 \Big]^{1/2} \\
\lesssim & \Big[\sum_{N_3} \Big(\sum_{N_2 \sim N_3} \sum_{\substack{N_i \lesssim N_2 \\ i \in \{1,4,5\}}} N_3^s \Big\| P_{N_1} w_1 P_{N_2} u_2 P_{N_3} u_3 \Big\|_{L^2_{tx}} N_5 \| P_{N_4} w_4 P_{N_5} v_5 \|_{L^4_{tx}} \Big)^2 \Big]^{1/2} \\
\lesssim & \Big[\sum_{N_3} N_3^{2s} \| P_{N_3} u_3 \|^2_{L^4_{tx}} \Big(\sum_{N_1, N_4} \| P_{N_1} w_1 \|_{L^\infty_{tx}} \| P_{N_4} w_4 \|_{L^\infty_{tx}} \sum_{N_2 \sim N_3} \sum_{N_5 \lesssim N_3} \Big(\frac{N_3}{N_2} \Big)^{1/2} \Big(\frac{N_5}{N_3} \Big)^{1/2} \\
& \qquad \| D_x^{1/2} P_{N_2} u_2 \|_{L^4_{tx}} \| P_{N_5} D_x^{1/2} v_5 \|_{L^4} \Big)^2 \Big]^{1/2} \\
\lesssim & \| u_3 \|_{\widetilde{L^4_t H^s_4}} \| D_x^{1/2} u_2 \|_{L^4_{tx}} \| D_x^{1/2} v_5 \|_{L^4_{tx}} \| w_1 \|_{L^\infty_t H^{3/2-}} \| w_4 \|_{L^\infty_t H^{3/2-}}
\end{aligned}$$

or $N_3 \sim N_5$ and then we get

$$\begin{aligned}
I \lesssim & \Big[\sum_{N} \Big(\sum_{N_2 \gtrsim N} \sum_{N_3 \gtrsim N} \sum_{N_i, i \in \{1,4\}} N^{s+\nu} N_3^{1/2-\nu} \| P_{N_1} w_1 \|_{L^\infty_{tx}} \| P_{N_2} u_2 \|_{L^4_{tx}} \\
& \qquad \| P_{N_3} u_3 \|_{L^4_{tx}} \| P_{N_4} w_4 \|_{L^\infty_{tx}} \| P_{N_3} D_x^{1/2} v_5 \|_{L^4_{tx}} \Big)^2 \Big]^{1/2} \\
\lesssim & \| w_1 \|_{L^\infty_t H^{1/2}_x} \| w_4 \|_{L^\infty_t H^{1/2}_x} \sum_{N_3} N_3^s \| P_{N_3} u_3 \|_{L^4_{tx}} \| P_{N_3} D_x^{1/2} v_5 \|_{L^4_{tx}} \\
& \qquad \Big[\sum_{N} \Big(\sum_{N_2 \gtrsim N} \Big(\frac{N}{N_2} \Big)^{1/2} N_2^{1/2} \| P_{N_2} u_2 \|_{L^4_{tx}} \Big)^2 \Big]^{1/2} \\
\lesssim & \| u_3 \|_{\widetilde{L^4_t H^s_4}} \| u_2 \|_{\widetilde{L^4_t H^{1/2}_4}} \| D_x^{1/2} v_5 \|_{\tilde{L}^4_{tx}} \| w_1 \|_{L^\infty_t H^{3/2-}} \| w_4 \|_{L^\infty_t H^{3/2-}}
\end{aligned}$$

where, in the last step, we used Cauchy-Schwarz in N_3 and that by discrete Young inequality

$$\Big\| \sum_{k \in \mathbb{Z}} (2^{k-k_2})^{1/2} \chi_{\{k-k_2 \leq 5\}} \| J_x^{1/2} P_{2^{k_2}} u_2 \|_{L^4} \Big\|_{l^2(\mathbb{N})} \lesssim \| J_x^{1/2} u_2 \|_{\tilde{L}^4_{tx}} \,.$$

2.2.2.3.2. $N_2 < 2^{-7}N$. Then $N_2 \geq 2^{-5}N_5$ since we must have $N_5 \leq 2^{-3}N_3$. This forces $N_3 \sim N$ so that we get

$$
\begin{aligned}
I \lesssim & \Big[\sum_{N_3}\Big(\sum_{N_2}\sum_{N_5\lesssim N_2}\sum_{N_1,N_4} N_3^s\Big\|P_{N_1}w_1P_{N_2}u_2P_{N_3}u_3\Big\|_{L^2_{tx}} N_5\|P_{N_4}w_4P_{N_5}v_5\|_{L^4_{tx}}\Big)^2\Big]^{1/2} \\
\lesssim & \|w_1\|_{L^\infty_t H^{1/2}_x}\|w_4\|_{L^\infty_t H^{1/2}_x}\Big[\sum_{N_3} N_3^{2s}\|P_{N_3}u_3\|^2_{L^4_{tx}}\Big]^{1/2} \\
& \sum_{N_2}\sum_{N_5\lesssim N_2}\Big(\frac{N_5}{N_2}\Big)^{1/2}\|J_x^{1/2}P_{N_2}u_2\|_{L^4_{tx}}\|P_{N_5}D_x^{1/2}v_5\|_{L^4} \\
\lesssim & \|u_3\|_{\widetilde{L^4_tH^s_4}}\|u_2\|_{\widetilde{L^4_tH^{1/2}_4}}\|D_x^{1/2}v_5\|_{\tilde{L}^4_{tx}}\|w_1\|_{L^\infty_t H^{3/2-}}\|w_4\|_{L^\infty_t H^{3/2-}}
\end{aligned}
$$

where in the last step we used the discrete Young inequality.

2.2.2.4. $(N_1 \vee N_2) < 2^{-5}(N \wedge N_5)$. Here it is worth noticing that we can assume that $(N\wedge N_5) \geq 2^4$ and the result follows directly from the lemma below and the proof of the proposition is completed. □

Lemma 4.5. Under the same hypotheses on u_i as in Proposition, it holds

$$
\begin{aligned}
J := & \Big[\sum_{N\geq 2^4}\Big(\sum_{(N_i)_{1\leq i\leq 5}\in\Lambda_N}\Big\|\partial_x^{\nu}P_NP_+\Big(\partial_x^{-\nu}\tilde{P}_{N_3}\Big(P_{N_1}w_1P_{N_2}u_2P_{N_3}u_3\Big) \\
& \cdot\partial_x\tilde{P}_{N_5}P_-(P_{N_4}w_4P_{N_5}v_5)\Big)\Big\|_{X^{s,-1/2+}}\Big)^2\Big]^{1/2} \lesssim \prod_{i=1}^3\|u_i\|_Z
\end{aligned}
$$

where

$$
\begin{aligned}
\Lambda_N := \Big\{ & (N_1,N_2,N_3,N_4,N_5)\in(2^{\mathbb{N}}\cup\{0\})^5,\ N_3\geq 2^{-3}N, \\
& 2^4 < N_5 \leq 4N_3,\ (N_1\vee N_2\vee N_4) < 2^{-5}(N\wedge N_5)\Big\}.
\end{aligned}
$$

Proof. It is worth noticing that, thanks to the frequency projections, $N_3 \sim N_{max}$ and the resonance relation yields

$$
|\sigma_{max}| \gtrsim |\xi\xi_5| \geq 2^{-2}NN_5 \gtrsim (N\wedge N_5)N_3 \tag{4.7}
$$

for all the contributions in J. First we can easily treat the contribution of the region $\{(\tau,\xi),\ \langle\tau-\xi|\xi|\rangle \geq 2^{-2}NN_5\}$. Indeed, we then get

$$
\begin{aligned}
J \ \lesssim \ & \sum_{N\geq 2^4}\ \sum_{(N_1,N_2,N_3,N_4,N_5)\in\Lambda_N}(NN_5)^{-\frac{1}{2}+}N_5^{1/2}N^{s+\nu}N_3^{-s-\nu} \\
& \|P_{N_3}D_x^su_3\|_{L^4_{tx}}\|D_x^{1/2}P_{N_5}v_5\|_{L^4_{tx}}\|P_{N_2}u_2\|_{L^\infty_{tx}}\prod_{i=1,4}\|P_{N_i}w_i\|_{L^\infty_{tx}} \\
\lesssim \ & \sum_{N\geq 2^4}N^{-\frac{1}{2}+}\|u_3\|_{L^4_tH^s_4}\|D_x^{1/2}v_5\|_{L^4_{tx}}\|u_2\|_{L^\infty_tH^{1/2}_x}\|w_1\|_{L^\infty_tH^{1/2}_x}\|w_4\|_{L^\infty_tH^{1/2}_x}
\end{aligned}
$$

which is acceptable. Therefore in the sequel we can assume that $\langle\tau-\xi|\xi|\rangle < 2^{-2}NN_5\}$. Now, for any fixed couple $(N,N_5)\in(2^{\mathbb{N}})^2$, we split any function $z\in L^2_{tx}$ into two parts related to the value of σ by setting

$$
z = \mathcal{F}^{-1}\Big(\eta_{2^{-4}NN_5}(\tau-\xi|\xi|)\widehat{z}\Big)+\mathcal{F}^{-1}\Big((1-\eta_{2^{-4}NN_5}(\tau-\xi|\xi|))\widehat{z}\Big) := \tilde{z}+\tilde{\tilde{z}}\ .
$$

1. Contribution of $\tilde{\tilde{v}}_5$. We now control the contribution of $\tilde{\tilde{v}}_5$ to J in the following way : either $N \sim N_3 \sim N_{max}$ and we write

$$J \lesssim \sum_{N \geq 2^4} \sum_{N_3 \sim N} \sum_{\substack{N_i \leq 2^5 N_3 \\ i=1,3,4,5}} N^s (NN_5)^{-1/2} N_5^{1/2} N_3^{-s}$$

$$\|P_{N_5} \tilde{\tilde{v}}_5\|_{X^{1/2,1/2}} \|P_{N_3} D_x^s u_3\|_{L^4_{tx}} \|P_{N_2} u_2\|_{L^\infty_{tx}} \prod_{i=1,4} \|P_{N_i} w_i\|_{L^\infty_{tx}}$$

$$\lesssim (\sum_N N^{-\frac{1}{2}+}) \|\tilde{\tilde{v}}_5\|_{X^{1/2,1/2}} \|u_3\|_{L^4_t H^s_4} \|u_2\|_{L^\infty_t H^{1/2}_x} \prod_{i=1,4} \|w_i\|_{L^\infty_t H^{1/2}_x}$$

or $N_5 \sim N_3 \sim N_{max}$ and we write

$$J \lesssim \sum_{N \geq 2^4} \sum_{N_3} \sum_{N_5 \sim N_3} \sum_{\substack{N_i \leq 2^{-5} N \\ i=1,2,4}} N^s (NN_5)^{-1/2} N_5^{1/2} N_3^{-s}$$

$$\|P_{N_5} \tilde{\tilde{v}}_5\|_{X^{1/2,1/2}} \|P_{N_3} D_x^s u_3\|_{L^4_{tx}} \|P_{N_2} u_2\|_{L^\infty_{tx}} \prod_{i=1,4} \|P_{N_i} w_i\|_{L^\infty_{tx}}$$

$$\lesssim (\sum_N N^{-\frac{1}{2}+}) \|v_5\|_{X^{1/2,1/2}} \|u_3\|_{\widetilde{L^4_t H^s_4}} \|u_2\|_{L^\infty_t H^{1/2}_x} \prod_{i=1,4} \|w_i\|_{L^\infty_t H^{1/2}_x}$$

where we apply Cauchy-Schwarz in $N_3 \sim N_5$ in the last step.
2. Contribution of $\tilde{v}_5$.
2.1 Contribution of $\tilde{\tilde{w}}_1$. We easily get

$$J \lesssim \sum_N \sum_{(N_1,N_2,N_3,N_4,N_5) \in \Lambda_N} ((N \wedge N_5) N_3)^{-1} N_1^{\frac{1}{2}+} N_5^{1-s} N^s$$

$$\|P_{N_1} \tilde{\tilde{w}}_1\|_{X^{-1/2-,1}} \|P_{N_3} u_3\|_{L^\infty_{tx}} \|P_{N_5} D_x^s \tilde{v}_5\|_{L^4_{tx}} \|P_{N_2} u_2\|_{L^\infty_{tx}} \|P_{N_4} w_4\|_{L^\infty_{tx}}$$

$$\lesssim \sum_N N^{-\frac{1}{2}+} \|w_1\|_{X^{-1/2-,1}} \|u_3\|_{L^\infty_t H^{\frac{1}{2}}_x} \|v_5\|_{X^{s,1/2}} \|u_2\|_{L^\infty_t H^{1/2}_x} \|w_4\|_{L^\infty_t H^{1/2}_x}$$

which is acceptable.
2.2 Contribution of $\tilde{w}_1$. To treat this contribution we will extensively use the following lemma which is a direct application of the Marcinkiewicz multiplier theorem.

Lemma 4.6. For any $p \in]1, +\infty[$ there exists $C_p > 0$ such that for all $N \geq 1$ and all $L \geq N^2$,

$$\Big\| \mathcal{F}^{-1}_{tx} \Big(\phi_N(\xi) \eta_L(\tau \mp \xi^2) f(\tau, \xi) \Big) \Big\|_{L^p_{tx}} \leq C_p \|f\|_{L^p_{tx}}, \quad \forall f \in L^p(\mathbb{R}^2) . \tag{4.8}$$

Proof. By Marcinkiewicz multiplier theorem (see for instance ([6],Corollary 5.2.5 page 361)), it suffices to check that

$$\Big| \partial^{(\alpha_1,\alpha_2)}_{\tau,\xi} \Big(\phi_N(\xi) \eta_L(\tau \mp \xi^2) \Big) \Big| \lesssim |\xi|^{\alpha_1} |\tau|^{\alpha_2} \text{ for } |\alpha| \leq 2.$$

But this follows directly from the fact that for $N^2 \leq L$,

$$\frac{d}{d\xi} \Big(\phi_N(\xi) \eta_L(\tau \mp \xi^2) \Big) = O(N^{-1}) \text{ and } \frac{d}{d\tau} \Big(\phi_N(\xi) \eta_L(\tau \mp \xi^2) \Big) = O(L^{-1}) .$$

□

It is worth noticing that on Λ_N, with $N \geq 2^4$, it holds $N_i^2 \leq 2^{-2}NN_5$ for $i \in \{1,2,4\}$. Hence, in view of (4.8), for any $1 < p < \infty$, setting $(z_1, z_2, z_4) = (w_1, u_2, w_4)$ it holds

$$\|P_{N_i} P_{\mp} \tilde{z}\|_{L^p_{tx}} \leq C_p \, \|P_{N_i} P_{\mp} z\|_{L^p_{tx}} \, .$$

and thus by the continuity of the Hilbert transform in L^p, $1 < p < \infty$,

$$\|P_{N_i} \tilde{z}\|_{L^p_{tx}} \leq \tilde{C}_p \, \|P_{N_i} z\|_{L^p_{tx}} \, . \tag{4.9}$$

We separate the contribution of $\tilde{w}_1$ in different sub-contributions.
2.2.1 Contribution of $\tilde{\tilde{u}}_2$. Then we write

$$\begin{aligned} J \lesssim & \sum_N \sum_{(N_1,N_2,N_3,N_4,N_5)\in\Lambda_N} ((N\wedge N_5)N_3)^{-1} N_2^{\frac{1}{2}+} N_5 N^s N_3^{-s+1/6} \\ & \|P_{N_2}\tilde{\tilde{u}}_2\|_{X^{-1/2-,1}} \|P_{N_3} D_x^{s-1/6} u_3\|_{L^6_{tx}} \|P_{N_5}\tilde{v}_5\|_{L^{24}_{tx}} \|P_{N_1}\tilde{w}_1\|_{L^{24}_{tx}} \|P_{N_2} u_2\|_{L^\infty_{tx}} \|P_{N_4} w_4\|_{L^\infty_{tx}} \\ \lesssim & \sum_N N^{-\frac{1}{6}} \|w_1\|_{L^{24}_{tx}} \|D_x^s u_3\|_{L^6_t L^3_x} \|v_5\|_{X^{1/2,1/2}} \|u_2\|_{X^{-1/2-,1}} \|w_4\|_{L^\infty_t H^{1/2}_x} \\ \lesssim & \|w_1\|_{L^\infty_t H^{1/2}_x} \|u_3\|_{L^\infty_t H^s \cap L^4_t H^s_4} \|v_5\|_{X^{1/2,1/2}} \|u_2\|_{X^{-1/2-,1}} \|w_4\|_{L^\infty_t H^{1/2}_x} \end{aligned}$$

where we used Sobolev inequalities and (4.9) in the last to the last step.
2.2.2 Contribution of $\tilde{u}_2$.
2.2.2.1 Contribution of $\tilde{\tilde{w}}_4$. This subcontribution can be estimated in the same way by

$$\begin{aligned} J \lesssim & \sum_N \sum_{(N_1,N_2,N_3,N_4,N_5)\in\Lambda_N} ((N\wedge N_5)N_3)^{-1} N_4^{\frac{1}{2}+} N_5 N^s N_3^{-s+1/6} \\ & \|P_{N_2}\tilde{\tilde{w}}_4\|_{X^{-1/2-,1}} \|P_{N_3} D_x^{s-1/6} u_3\|_{L^6_{tx}} \|P_{N_5}\tilde{v}_5\|_{L^{36}_{tx}} \|P_{N_1}\tilde{w}_1\|_{L^{36}_{tx}} \|P_{N_2}\tilde{u}_2\|_{L^{36}_{tx}} \\ \lesssim & \|w_1\|_{L^\infty_t H^{1/2}_x} \|u_3\|_{L^\infty_t H^s \cap L^4_t H^s_4} \|v_5\|_{X^{1/2,1/2}} \|u_2\|_{L^\infty_t H^{1/2}_x} \|w_4\|_{X^{-1/2-,1}} \end{aligned}$$

2.2.2.2 Contribution of $\tilde{w}_4$. Since $\max(|\sigma_i|) < 2^{-2}NN_5$, we only have to consider $\tilde{\tilde{u}}_3$ in this contribution. Either $N \sim N_3$ and then

$$\begin{aligned} J^2 \lesssim & \sum_{N_3} \Big(\sum_{(N_1,N_2,N_3,N_4,N_5)\in\Lambda_{N_3}} N_3^s (N_3N_5)^{-1} N_5^{2/3} N_3^{1-s} \\ & \|P_{N_3}\tilde{\tilde{u}}_3\|_{X^{s-1,1}} \|P_{N_5} D_x^{1/3}\tilde{v}_5\|_{L^6_{tx}} \|P_{N_2}\tilde{u}_2\|_{L^{36}_{tx}} \prod_{i=1,4} \|P_{N_i}\tilde{w}_i\|_{L^{36}_{tx}} \Big)^2 \\ \lesssim & \Big(\sum_{N_3} \|P_{N_3} u_3\|_{X^{s-1,1}} \Big)^2 \|v_5\|^2_{X^{1/2,1/2}} \|u_2\|^2_{L^\infty_t H^{1/2}_x} \prod_{i=1,4} \|w_i\|^2_{L^\infty_t H^{1/2}_x} \end{aligned}$$

or $N_5 \sim N_3$. In this last case we first notice that $X^{0,3/8} \hookrightarrow L^4_{tx}$ and that for any fixed $2 < p < \infty$, $X_T^{0,1/2-} \hookrightarrow L^p_T L^2_x$. Therefore by interpolation, Sobolev inequalities and duality we infer that $L^{\frac{6}{5}}_{Tx} \hookrightarrow X_T^{-\frac{1}{6}-,-1/2+}$. We thus get

$$
\begin{aligned}
J^2 \lesssim & \sum_N \Big(\sum_{N_3} \sum_{(N\vee N_1 \vee N_2 \vee N_4)\lesssim N_3} N^s N^{\frac{1}{6}+} (NN_3)^{-1} N_3^{1/2} N_3^{1-s} \\
& \|P_{N_3} \tilde{\tilde{u}}_3\|_{X^{s-1,1}} \|P_{N_3} D_x^{1/2} \tilde{v}_5\|_{L^4_{tx}} \|P_{N_2}\tilde{u}_2\|_{L^{36}_{tx}} \prod_{i=1,4} \|P_{N_i}\tilde{w}_i\|_{L^{36}_{tx}} \Big)^2 \\
\lesssim & \sum_N N^{-\frac{1}{6}+} \Big(\sum_{N_3} \|P_{N_3} u_3\|_{X^{s-1,1}} \|P_{N_3} D_x^{1/2} \tilde{v}_5\|_{L^4_{tx}} \Big)^2 \|u_2\|^2_{L^\infty_t H^{1/2}_x} \prod_{i=1,4} \|w_i\|^2_{L^\infty_t H^{1/2}_x} \\
\lesssim & \|u_3\|^2_{X^{s-1,1}} \|v_5\|^2_{X^{1/2,1/2}} \|u_2\|^2_{L^\infty_t H^{1/2}_x} \prod_{i=1,4} \|w_i\|^2_{L^\infty_t H^{1/2}_x}
\end{aligned}
$$

where we apply Cauchy-Scwharz in N_3 in the last step. □

Finally, this last proposition together with Lemmas 4.3 enables to treat the term containing $B(u,u)$ in (4.1).

Proposition 4.7. Let $1/2 \le s \le 1$, $w_1 \in X^{-1/2-,1} \cap L^\infty_t H^{s+1-}_x$, and $u_i \in X^{s-1,1} \cap \widetilde{L^4_t H^s_4} \cap L^\infty_t H^s$, $i = 2,3,4$. with compact support in time such that u_2 and u_3 are real-valued. Then it holds

$$
\begin{aligned}
\Big\| w_1 u_2 B(u_3,u_4) \Big\|_{X^{s,-\frac{1}{2}+}} \lesssim & \|w_1\|_{X^{-1/2-,1}} \|u_2\|_{L^\infty_t H^{\frac{1}{2}}} \|u_3\|_{L^\infty_t H^{\frac{1}{2}}} \|u_4\|_{L^4_t H^s_4} \\
& + \|w_1\|_{L^\infty_t H_x^{\frac{3}{2}-}} \sum_{(2\le i\ne j\ne q\le 4)} \|u_i\|_{X^{-\frac{1}{2}-,1} \cap \widetilde{L^4_t H_4^{\frac{1}{2}}} \cap L^\infty_t H^{\frac{1}{2}}} \\
& \cdot \|u_j\|_{X^{-\frac{1}{2},1} \cap \widetilde{L^4_t H_4^{\frac{1}{2}}} \cap L^\infty_t H^{\frac{1}{2}}} \|u_q\|_{X^{s-1,1} \cap \widetilde{L^4_t H^s_4} \cap L^\infty_t H^s} . \quad (4.10)
\end{aligned}
$$

Proof. Recall that $B(u,v) = -i\partial_x^{-1}(P_+ u_x P_+ v_x) + i\partial_x^{-1}(P_- u_x P_- v_x)$. By symmetry it thus suffices to estimate

$$
\begin{aligned}
I := & \Big\| w_1 u_2 \partial_x^{-1} (P_+ \partial_x u_3 P_+ \partial_x u_4) \Big\|_{X^{s,-\frac{1}{2}+}} \\
= & \Big\| \sum_{N\ge 1, N_1, N_2, N_3, N_4, N_{34} \ge (N_3 \vee N_4)/2} P_N \Big(P_{N_1} w_1 P_{N_2} u_2 \\
& \cdot P_{N_{34}} \partial_x^{-1} (P_+ \partial_x P_{N_3} u_3 P_+ \partial_x P_{N_4} u_4) \Big) \Big\|_{X^{s,-\frac{1}{2}+}} .
\end{aligned}
$$

By symmetry we can assume that $N_3 \le N_4$ and thus we must have $N_{34} \sim N_4$. We can thus drop the summation over N_{34} and replace $P_{N_{34}}$ by $\tilde{P}_{N_4}$.

By the triangle inequality we can separate this sum in different sums on disjoint subset of $(2^{\mathbb{N}})^5$.

1. $N_1 \geq 2^{-8}N$. Then we have

$$
\begin{aligned}
I \lesssim & \sum_N \sum_{N_1 \geq 2^{-8}N} \sum_{N_2,N_3,N_4} N^s \Big\| P_N\Big(P_{N_1}w_1 P_{N_2}u_2 \tilde{P}_{N_4}\partial_x^{-1}(P_+\partial_x P_{N_3}u_3 P_+\partial_x P_{N_4}u_4)\Big)\Big\|_{L^{4/3}_{tx}} \\
\lesssim & \sum_{N\leq 2^8 N_1} \sum_{N_2,N_3,N_4} \|P_{N_1}w_1\|_{L^\infty_t H^{s+}_8} \|P_{N_2}u_2\|_{L^\infty_t L^8_x} ? \prod_{i=3}^4 \|P_{N_i}u_i\|_{L^4_t H^{\frac12 -}_4} ? \\
\lesssim & \|w_1\|_{L^\infty_t H^{\frac32 -}_x} \|u_2\|_{L^\infty_t H^{\frac12}_x} \prod_{i=3}^4 \|P_{N_i}u_i\|_{L^4_t H^{\frac12}_4} ? \,.
\end{aligned}
$$

2. $N_1 < 2^{-8}N$ and $N_1 \geq 2^{-5}N_3$. Then either $N_4 \gtrsim N \vee N_2$ and it holds

$$
\begin{aligned}
I \lesssim & \Big[\sum_N \Big(\sum_{N_4 \gtrsim N} \sum_{N_1 < 2^{-8}N} \sum_{N_3 \leq 2^5 N_1} \sum_{N_2} N^s \Big\| P_N\Big(P_{N_1}w_1 P_{N_2}u_2 \\
& \cdot \tilde{P}_{N_4}\partial_x^{-1}(P_+\partial_x P_{N_3}u_3 P_+\partial_x P_{N_4}u_4)\Big)\Big\|_{L^{4/3}_{tx}}\Big)^2\Big]^{\frac12} \\
\lesssim & \Big(\sum_N \Big(\sum_{N_4 \gtrsim N} (\frac{N}{N_4})^{2s} \|J^s_x u_4\|^2_{L^4_{tx}}\Big)^{\frac12} \sum_{N_1,N_2,N_3} \|P_{N_1}w_1\|_{L^\infty_t H^1_4} \|P_{N_2}u_2\|_{L^\infty_t L^8_x} \|P_{N_2}u_3\|_{L^\infty_t L^8_x} ? \\
\lesssim & \|w_1\|_{L^\infty_t H^{\frac32 -}_x} \|u_2\|_{L^\infty_t H^{\frac12}_x} \|u_3\|_{L^\infty_t H^{\frac12}_x} \|u_4\|_{\widetilde{L^4_t H^s_4}} \,.
\end{aligned}
$$

or $N_2 \gtrsim N \vee N_4$ and it holds

$$
\begin{aligned}
I \lesssim & \Big[\sum_N \Big(\sum_{N_2 \gtrsim N} \sum_{N_1 < 2^{-8}N} \sum_{N_3 \leq 2^5 N_1} \sum_{N_4} N^s \Big\| P_N\Big(P_{N_1}w_1 P_{N_2}u_2 \\
& \cdot \tilde{P}_{N_4}\partial_x^{-1}(P_+\partial_x P_{N_3}u_3 P_+\partial_x P_{N_4}u_4)\Big)\Big\|_{L^{4/3}_{tx}}\Big)^2\Big]^{\frac12} \\
\lesssim & \Big(\sum_N \Big(\sum_{N_2 \gtrsim N} (\frac{N}{N_2})^{2s} \|J^s_x u_2\|^2_{L^4_{tx}}\Big)^{\frac12} \sum_{N,N_3,N_4} \|P_{N_1}w_1\|_{L^\infty_t H^1_4} ? \prod_{i=3}^4 \|P_{N_i}v_i\|_{L^\infty_t L^8_x} ? \\
\lesssim & \|w_1\|_{L^\infty_t H^{\frac32 -}_x} \|u_2\|_{\widetilde{L^4_t H^s_4}} \prod_{i=3}^4 \|u_i\|_{L^\infty_t H^{\frac12}_x} \,.
\end{aligned}
$$

3. $N_1 < (2^{-8}N \wedge 2^{-5}N_3)$.
3.1 $N_2 \geq 2^{-8}N$. Then we have

$$\begin{aligned} I \lesssim & \Big(\sum_N \Big[\sum_{N_2\geq 2^{-8}N}\sum_{N_3,N_4}\sum_{N_1} N^s \Big\| P_N\Big(P_{N_1}w_1 P_{N_2}u_2 \\ & \quad \cdot \tilde{P}_{N_3}\partial_x^{-1}(P_+\partial_x P_{N_3}u_3 P_+\partial_x P_{N_4}u_4)\Big)\Big\|_{L^{4/3}_{tx}}\Big]^2\Big)^{1/2} \\ \lesssim & \|w_1\|_{L^\infty_t H^1_x}\Big(\sum_N \Big[\sum_{N_2\geq 2^{-8}N}(\frac{N}{N_2})^s N_2^s \|P_{N_2}u_2\|_{L^4_{tx}}\Big]^2\Big)^{1/2} \\ & \cdot \sum_{N_4} N_4^{-1}\|\tilde{P}_{N_4}(P_+\partial_x P_{N_3}u_3 P_+\partial_x P_{N_4}u_4)\|_{L^2_{tx}} \\ \lesssim & \|w_1\|_{L^\infty_t H^1_x}\|u_2\|_{\widetilde{L^4_t H^s_4}}\sum_{N_4}\|P_{N_4}D_x^{1/2}u_4\|_{L^4_{tx}}\sum_{N_3\leq N_4}(\frac{N_3}{N_4})^{1/2}\|P_{N_3}D_x^{1/2}u_3\|_{L^4_{tx}} \\ \lesssim & \|w_1\|_{L^\infty_t H^1_x}\|u_2\|_{\widetilde{L^4_t H^s_4}}\|D_x^{1/2}u_3\|_{\tilde{L}^4_{tx}}\|D_x^{1/2}u_4\|_{\tilde{L}^4_{tx}} \; . \end{aligned}$$

where we use two times the discret Young inequality.
3.2 $N_2 < 2^{-8}N$ and $N_2 \geq 2^{-5}N_3$. Then we must have $N \sim N_4$ and thus

$$\begin{aligned} I \lesssim & \Big(\sum_{N_4}\Big[\sum_{N_1,N_3}\sum_{N_2\geq 2^{-5}N_3} N_2^{-1/2}N_3^{1/2}\|P_{N_1}w_1\|_{L^\infty_{tx}}\|P_{N_2}J_x^{1/2}u_2\|_{L^4_{tx}} \\ & \quad \cdot \|P_{N_3}D_x^{1/2}u_3\|_{L^4_{tx}}\|P_{N_4}D_x^s u_4\|_{L^4_{tx}}\Big]^2\Big)^{1/2} \\ \lesssim & \|w_1\|_{L^\infty_t H^1_x}\Big(\sum_{N_4}\|D_x^s P_{N_4}u_4\|^2_{L^4_{tx}}\Big)^{1/2} \\ & \cdot \sum_{N_2}\|P_{N_2}J_x^{1/2}u_2\|_{L^4_{tx}}\sum_{N_3\leq 2^5 N_2}(\frac{N_3}{N_2})^{1/2}\|P_{N_3}D_x^{1/2}u_3\|_{L^4_{tx}} \\ \lesssim & \|w_1\|_{L^\infty_t H^1_x}\|u_2\|_{\widetilde{L^4_t H^{1/2}_4}}\|u_3\|_{\widetilde{L^4_t H^{1/2}_4}}\|u_4\|_{\widetilde{L^4_t H^s_4}} \; . \end{aligned}$$

3.3 $N_2 < (2^{-8}N \wedge 2^{-5}N_3)$. Then $N \sim N_4$ and the resonance relation yields

$$|\sigma_{max}| \gtrsim |\xi_3\xi_4| \geq 2^{-2}N_3N_4 \; . \tag{4.11}$$

First we can easily treat the contribution of the region $\{(\tau,\xi),\ \langle \tau-\xi|\xi|\rangle \geq 2^{-2}N_3N_4\}$. Indeed, we then get

$$\begin{aligned} I \lesssim & \sum_{N_4}\sum_{N_3\lesssim N_4}\sum_{N_1\vee N_2\lesssim N_3} N_3^{1/2}(N_3N_4)^{-1/2+}\|w_1\|_{L^\infty_t H^1_x}\|P_{N_2}u_2\|_{L^\infty_{tx}} \\ & \cdot \|P_{N_3}D_x^{1/2}u_3\|_{L^4_{tx}}\|P_{N_4}J_x^s u_4\|_{L^4_{tx}} \\ \lesssim & \|w_1\|_{L^\infty_t H^1_x}\|D_x^{1/2}u_3\|_{L^4_{tx}}\|u_4\|_{L^4_t H^s_4}\sum_{N_2}N_2^{-1/2+}\|P_{N_2}u_2\|_{L^\infty_{tx}} \\ \lesssim & \|w_1\|_{L^\infty_t H^1_x}\|u_2\|_{L^\infty_t H^{1/2}_x}\|u_3\|_{L^4_t H^{\frac12}_4}\|u_4\|_{L^4_t H^s_4} \; . \end{aligned}$$

which is acceptable. Therefore in the sequel we can assume that $\langle \tau - \xi|\xi|\rangle < 2^{-2}N_3N_4\}$. We now split v_1, u_2 and u_3 into two parts related to the value of σ_i by setting

$$z=\mathcal{F}^{-1}\Big(\eta_{2^{-4}N_3N_4}(\tau-\xi|\xi|)\widehat{z}\Big)+\mathcal{F}^{-1}\Big((1-\eta_{2^{-4}N_3N_4}(\tau-\xi|\xi|))\widehat{z}\Big):=\tilde{z}+\tilde{\tilde{z}}\ .$$

It is worth noticing that in this region $N_i^2 << N_3N_4$ for $i=1,2,3$. Therefore Lemma 4.8 holds for $\tilde{w}_1$, $\tilde{u}_2$ and $\tilde{u}_3$.

3.3.1. Contribution of $\tilde{\tilde{w}}_1$. We first control the contribution of $\tilde{\tilde{w}}_1$ to I in the following way :

$$\begin{aligned}I\lesssim&\sum_{N_4}\sum_{N_3\lesssim N_4}\sum_{N_1\vee N_2\lesssim N_3}N_1^{1/2+}N_3(N_3N_4)^{-1}\|P_{N_1}\tilde{\tilde{w}}_1\|_{X^{-1/2-,1}}\\&\cdot\|P_{N_2}u_2\|_{L^\infty_{tx}}\|P_{N_3}u_3\|_{L^\infty_{tx}}\|P_{N_4}J_x^su_4\|_{L^4_{tx}}\\\lesssim&\|w_1\|_{X^{-1/2-,1}}\|u_2\|_{L^\infty_tH^{\frac12}}\|u_3\|_{L^\infty_tH^{\frac12}}\|u_4\|_{L^4_tH^s_4}\ .\end{aligned}$$

3.3.2. Contribution of $\tilde{w}_1$.

3.3.2.1 Contribution of $\tilde{\tilde{u}}_2$. In the same way, using Sobolev inequality, we have

$$\begin{aligned}I\lesssim&\sum_{N_4}\sum_{N_3\lesssim N_4}\sum_{N_1\vee N_2\lesssim N_3}N_1^{1/2+}N_3(N_3N_4)^{-1}N_4^{1/6}\|P_{N_1}\tilde{w}_1\|_{L^{12}_{tx}}\|P_{N_2}\tilde{\tilde{u}}_2\|_{X^{-1/2-,1}}\\&\|P_{N_3}u_3\|_{L^\infty_{tx}}\|P_{N_4}J_x^{s-1/6}u_4\|_{L^6_{tx}}\\\lesssim&\|w_1\|_{L^\infty_tH^{\frac12}}\|u_2\|_{X^{-1/2-,1}}\|u_3\|_{L^\infty_tH^{\frac12}}\|u_4\|_{L^\infty_tH^s\cap L^4_tH^s_4}\ .\end{aligned}$$

3.3.2.2 Contribution of $\tilde{u}_2$.

3.2.2.2.1 Contribution of $\tilde{\tilde{u}}_3$.

$$\begin{aligned}I\lesssim&\sum_{N_4}\sum_{N_3\lesssim N_4}\sum_{N_1\vee N_2\lesssim N_3}N_3^{3/2}(N_3N_4)^{-1}N_4^{1/6}\|P_{N_1}\tilde{w}_1\|_{L^{24}_{tx}}\\&\cdot\|P_{N_1}\tilde{u}_2\|_{L^{24}_{tx}}\|P_{N_3}\tilde{u}_3\|_{X^{-1/2,1}}\|P_{N_4}J_x^{s-1/6}u_4\|_{L^6_{tx}}\\\lesssim&\|w_1\|_{L^\infty_tH^{\frac12}}\|u_2\|_{L^\infty_tH^{\frac12}}\|u_3\|_{X^{-1/2,1}}\|u_4\|_{L^\infty_tH^s\cap L^4_tH^s_4}\ .\end{aligned}$$

3.2.2.2.2 Contribution of $\tilde{u}_3$. Since $\max(|\sigma_i|)\geq 2^{-2}N_3N_4$, it remains to treat the subcontribution of $\tilde{\tilde{u}}_4$. We easily obtain

$$\begin{aligned}I^2\lesssim&\sum_{N_4}\Big(\sum_{N_3\lesssim N_4}\sum_{N_1\vee N_2\lesssim N_3}N_3^{2/3}N_4(N_3N_4)^{-1}\|P_{N_1}\tilde{w}_1\|_{L^{24}_{tx}}\\&\cdot\|P_{N_1}\tilde{u}_2\|_{L^{24}_{tx}}\|P_{N_3}D_x^{1/3}P_+\tilde{u}_3\|_{L^6_{tx}}\|P_{N_4}\tilde{\tilde{u}}_4\|_{X^{s-1,1}}\Big)^2\\\lesssim&\Big(\|w_1\|_{L^\infty_tH^{\frac12}}\|u_2\|_{L^\infty_tH^{\frac12}}\|u_3\|_{L^\infty_tH^{\frac12}\cap L^4_tH^{\frac12}_4}\|u_4\|_{X^{s-1,1}}\Big)^2\ .\end{aligned}$$

Therefore, we complete the proof. □

Acknowledgment. Z. Guo is supported in part by NNSF of China (No. 11001003, No. 11271023).

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