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EIGENVALUES OF THE SUB-LAPLACIAN AND DEFORMATIONS OF CONTACT STRUCTURES ON A COMPACT CR MANIFOLD

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Abstract. Given a compact strictly pseudoconvex CR manifold M, we study the differentiability of the eigenvalues of the sub-Laplacian $\Delta_{b,\theta}$ associated with a compatible contact form (i.e. a pseudo-Hermitian structure) θ on M, under conformal deformations of θ. As a first application, we show that the property of having only simple eigenvalues is generic with respect to θ, i.e. the set of structures θ such that all the eigenvalues of $\Delta_{b,\theta}$ are simple, is residual (and hence dense) in the set of all compatible positively oriented contact forms on M. In the last part of the paper, we introduce a natural notion of critical pseudo-Hermitian structure of the functional $\theta \mapsto \lambda_k(\theta)$, where $\lambda_k(\theta)$ is the k-th eigenvalue of the sub-Laplacian $\Delta_{b,\theta}$, and obtain necessary and sufficient conditions for a pseudo-Hermitian structure to be critical.

1. Introduction

Let M be a compact strictly pseudoconvex CR manifold of real dimension $2n+1$. A pseudo-Hermitian structure on M is a contact form $\theta \in \Gamma(T^*M)$ whose kernel coincides with the horizontal distribution of M. The strict pseudoconvexity of M means that the Levi form associated to such a contact form is either positive definite or negative definite. We denote by $\mathcal{P}_+(M)$ the set of all pseudo-Hermitian structures with positive definite Levi form on M.

To every pseudo-Hermitian structure $\theta \in \mathcal{P}_+(M)$ we associate its sub-Laplacian $\Delta_{b,\theta}$ (or simply Δ_b if there is no risk of confusion) which is a sub-elliptic operator of order $1/2$, and denote by

$$0 = \lambda_0(\theta) < \lambda_1(\theta) \le \lambda_2(\theta) \le \cdots \le \lambda_k(\theta) \le \cdots \to \infty$$

the nondecreasing sequence of eigenvalues of $\Delta_{b,\theta}$.

Several works published in recent years are devoted to the study of the sub-Laplacian and the investigation of its spectral properties, see for instance [3, 5, 4, 6, 8, 9, 10, 14, 18, 19, 21, 25, 26, 27, 28, 30]. The aim of most of them is to extend to the CR context some of the spectral geometric results established in the Riemannian setting for the Laplace-Beltrami operator.

In our previous paper [5], we discussed the continuity of the eigenvalues $\lambda_k(\theta)$, as functions on the set $\mathcal{P}_+(M)$ that we have endowed with a natural

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metric topology. In the present paper, we start by studying the differentiability of the spectrum of the sub-Laplacian $\Delta_{b,\theta}$ under one-parameter deformations of the contact structure θ. We apply classical perturbation theory of selfadjoint operators to get a differentiability result (Theorem 3.2). Moreover, we prove that if $\theta(t) \in \mathcal{P}_+(M)$ is an analytic deformation of a contact structure θ, then the function $t \mapsto \lambda_k(\theta(t))$, which is not differentiable if $\lambda_k(\theta)$ is not simple, admits left-sided and right-sided derivatives at $t = 0$, and relate these derivatives to the eigenvalues of an explicit symmetric operator acting on the $\lambda_k(\theta)$-eigenspace (Theorem 3.3).

In the second part of the paper we use these facts to show that the property of having only simple eigenvalues is generic for the sub-Laplacians on a given compact strictly pseudoconvex CR manifold M. Indeed, we prove that the set of contact structures $\theta \in \mathcal{P}_+(M)$ such that all the eigenvalues of $\Delta_{b,\theta}$ are simple, is a residual set in the complete metric space $\mathcal{P}_+(M)$ (see Theorem 4.1). Our proof relies on an eigenvalue splitting technique (Proposition 4.1) used by many authors in the Riemannian setting (see [1, 7, 13]; see also [31] for a different approach).

The last section is devoted to the notion of critical pseudo-Hermitian structure. Despite the lack of differentiability of the eigenvalues $\lambda_k(\theta)$ upon analytic deformations $\theta(t) \in \mathcal{P}_+(M)$ of the pseudo-Hermitian structure, a natural notion of criticality can be defined using the existence of left-sided and right-sided derivatives of $\lambda_k(\theta(t))$ at $t = 0$ (see Definition 5.1). Since $\lambda_k(\theta)$ is not invariant under scaling of the pseudo-Hermitian structure, we restrict ourselves to the deformations that preserve the global volume $\mathrm{vol}(\theta) = \int_M \theta \wedge (d\theta)^n$. We give necessary and sufficient conditions for a pseudo-Hermitian structure to be a critical point of the functional $\theta \in \mathcal{P}_+(M) \mapsto \lambda_k(\theta)$, under the volume-preserving constraint. In particular, we will see that the criticality condition is strongly related to the existence of a finite family of $\lambda_k(\theta)$-eigenfunctions $v_1, \cdots, v_d$, satisfying $v_1^2 + \cdots + v_d^2 = 1$ (Corollary 5.1). This last condition is satisfied for instance by the first positive eigenvalue of the standard CR sphere $\mathbb{S}^{2n+1}$ (see [30, Proposition 4.4]).

2. Preliminaries

Let M be a compact connected orientable CR manifold of CR dimension n (and real dimension $2n + 1$). Such a manifold M is equipped with a pair (H, J), where H is a sub-bundle of the tangent bundle TM of real rank $2n$ (often called Levi distribution) and J is an integrable complex structure on H which means that, $\forall X, Y \in \Gamma(H)$,

$$[X, Y] - [JX, JY] \in \Gamma(H)$$

and

$$[JX, Y] + [X, JY] = J\left([X, Y] - [JX, JY]\right).$$

Since M is orientable, there exists a nonzero 1-form $\theta \in \Gamma(T^*M)$ whose kernel coincides with H. Such a 1-form, called *pseudo-Hermitian* structure on M, is of course not unique. Actually, the set of pseudo-Hermitian structures on M consists in all the forms $\pm e^u\theta$, $u \in C^\infty(M)$.

To each pseudo-Hermitian structure θ we associate its ***Levi form*** G_θ defined on H by

$$G_\theta(X,Y) = -d\theta(JX,Y) = \theta([JX,Y]).$$

The integrability of J implies that G_θ is symmetric and J-invariant. The CR manifold M is said to be ***strictly pseudoconvex*** if the Levi form G_θ of a pseudo-Hermitian structure θ is either positive definite or negative definite. Of course, this condition does not depend on the choice of θ. In all the sequel, we assume that M is strictly pseudoconvex and denote by $\mathcal{P}_+(M)$ the set of all pseudo-Hermitian structures with positive definite Levi form on M. Every $\theta \in \mathcal{P}_+(M)$ is in fact a contact form which induces on M the following volume form

$$\psi_\theta = \frac{1}{2^n\, n!}\,\theta \wedge (d\theta)^n.$$

The associated divergence div_θ is defined, for every smooth vector field Z on M, by

$$\mathcal{L}_Z\psi_\theta = \mathrm{div}_\theta(Z)\,\psi_\theta.$$

We denote by $L^2(M)$ the set of squared integrable functions on M with respect to ψ_θ. A function $u \in L^2(M)$ is *weakly differentiable* (w.d.) along H if there is $Y_u \in \Gamma(H)$ such that $|Y_u|_{G_\theta} = G_\theta(Y_u,Y_u)^{\frac{1}{2}} \in L^1_{loc}(M)$ and

$$\int_M G_\theta(Y_u,X)\,\psi_\theta = -\int_M u\,\mathrm{div}_\theta(X)\,\psi_\theta$$

for every $X \in \Gamma^\infty(H)$. Such Y_u is unique up to a set of measure zero and is denoted by $Y_u = \nabla^H u$ and called *weak horizontal gradient* of u. It is easy to check that if u is differentiable, then $\forall X \in \Gamma^\infty(H)$, $du(X) = G_\theta(X,\nabla^H u)$. Let

$$\mathcal{D}(\nabla^H) = \left\{u \in L^2(M)\ :\ u\ is\ (w.d.)\ along\ H\ and\ \nabla^H u \in L^2(H)\right\},$$

where $L^2(H)$ stands for the set of squared integrable sections of H with respect to the inner product G_θ and the volume element ψ_θ. Then we may regard the weak horizontal gradient as a linear operator

$$\nabla^H : \mathcal{D}(\nabla^H) \subset L^2(M) \to L^2(H).$$

As $C^\infty(M) \subset \mathcal{D}(\nabla^H)$ it follows that $\mathcal{D}(\nabla^H)$ is a dense subspace of $L^2(M)$. Let

$$(\nabla^H)^* : \mathcal{D}[(\nabla^H)^*] \subset L^2(H) \to L^2(M)$$

be the adjoint of ∇^H. Then $\Gamma^\infty(H) \subset \mathcal{D}[(\nabla^H)^*]$ and, for all $X \in \Gamma^\infty(H)$, one has

$$(\nabla^H)^*X = -\mathrm{div}_\theta(X).$$

In particular, $(\nabla^H)^*$ is densely defined in $L^2(H)$. The sub-Laplacian Δ_b, or $\Delta_{b,\theta}$ if it is necessary to avoid confusion, is given by

$$\mathcal{D}(\Delta_b) = \{u \in \mathcal{D}(\nabla^H) : \nabla^H u \in \mathcal{D}[(\nabla^H)^*]\},$$
$$\Delta_b = (\nabla^H)^* \circ \nabla^H = -\mathrm{div}_\theta \circ \nabla^H.$$

Note that

$$(\Delta_b u, u)_{L^2(M)} = \|\nabla^H u\|^2_{L^2(H)} \geq 0$$

for any $u \in \mathcal{D}(\Delta_b)$. Moreover, the sub-Laplacian is symmetric, i.e.

(1) $\mathcal{D}(\Delta_b)$ is dense in $L^2(M)$.
(2) $\mathcal{D}(\Delta_b) \subset \mathcal{D}(\Delta_b^*)$ and $(\Delta_b u, v)_{L^2(M)} = (u, \Delta_b v)_{L^2(M)}$ $\forall\, u, v \in \mathcal{D}(\Delta_b)$.

The operator Δ_b is also known to be subelliptic of order $\varepsilon = 1/2$. Indeed, one has (cf. [15, Theorem 2.1]) for any $u \in C^\infty(M)$,

$$\|u\|^2_{H^{1/2}(M)} \leq C\left((\Delta_b u, u)_{L^2(M)} + \|u\|^2_{L^2(M)}\right), \tag{2.1}$$

for some constant C independent of u. It is worth noticing that Δ_b can be seen as the real part of the Kohn Laplacian acting on functions $\Box_b = \bar\partial_b^* \bar\partial_b$, where $\bar\partial_b u$ is the projection of du onto $T^*_{(0,1)}M$. Indeed, we have (cf. [24, Theorem 2.3]) $\Box_b = \Delta_b + \mathrm{i}\, nT$, where T is the unique vector field satisfying $T \rfloor \theta = 1$ and $T \rfloor d\theta = 0$.

Lemma 2.1. *The space $H^{1/2}(M) = W^{1/2,2}(M)$ admits a compact embedding into $L^2(M)$.*

The proof of this Lemma uses standard arguments (see [4]).

Lemma 2.2. *The operator $(\Delta_b + I)^{-1} : \mathcal{D}((\Delta_b + I)^{-1}) \subset L^2(M) \to L^2(M)$ is compact.*

Proof. Based on the estimate (2.1) one has $\mathrm{Ker}(\Delta_b + I) = \{0\}$. Consequently,

$$\Delta_b + I : C^\infty(M) \to \mathcal{R}(\Delta_b + I) \subset C^\infty(M)$$

is invertible, where $\mathcal{R}(A)$ denotes the range of the operator A. Therefore, we may consider the inverse

$$(\Delta_b + I)^{-1} : \mathcal{D}\left((\Delta_b + I)^{-1}\right) = \mathcal{R}(\Delta_b + I) \subset L^2(M) \to H^{1/2}(M).$$

Let $v \in \mathcal{D}\left((\Delta_b + I)^{-1}\right)$ and let us apply (2.1) to the function $u = (\Delta_b + I)^{-1}(v)$ followed by the Cauchy-Schwartz inequality

$$\|(\Delta_b + I)^{-1} v\|^2_{H^{1/2}(M)} \leq C\left(v\,,\, (\Delta_b + I)^{-1} v\right)_{L^2(M)} \leq C\|v\|_{L^2(M)}\, \|(\Delta_b + I)^{-1} v\|_{L^2(M)}\,.$$

Moreover, there is a continuous embedding $H^{1/2}(M) \to L^2(M)$ so that

$$\|u\|_{L^2(M)} \leq C' \|u\|_{H^{1/2}(M)}\,, \quad u \in H^{1/2}(M),$$

for some constant $C' > 0$ independent of u. Thus,

$$\|(\Delta_b + I)^{-1} v\|^2_{H^{1/2}(M)} \leq C'' \|v\|_{L^2(M)} \|(\Delta_b + I)^{-1} v\|_{H^{1/2}(M)}$$

(with $C'' = CC'$) or

$$\|(\Delta_b + I)^{-1} v\|_{H^{1/2}(M)} \leq C'' \|v\|_{L^2(M)}$$

which proves the continuity of the operator $(\Delta_b + I)^{-1}$. Finally, by Lemma 2.1, the embedding $H^{1/2}(M) \to L^2(M)$ is compact. Hence, $(\Delta_b + I)^{-1} : \mathcal{D}((\Delta_b + I)^{-1}) \subset L^2(M) \to L^2(M)$ is compact (as the composition of a compact operator with a continuous operator). □

Corollary 2.1. *The spectrum $\sigma(\Delta_b)$ of the sub-Laplacian is discrete and consists of eigenvalues of finite multiplicity.*

3. Differentiability of eigenvalues with respect to 1-parameter deformations of the pseudo-Hermitian structure

We start by recalling the needed notions of functional analysis, cf. e.g. A. Kriegl & P.W. Michor [22, 23] and T. Kato [20]. Let $\mathcal{H}$ be a Hilbert space and $\{A(t)\}_{t\in\mathbb{R}}$ a family of linear operators $A(t) : \mathcal{D}(A(t)) \subset \mathcal{H} \to \mathcal{H}$. We say that $A(t)$ is a *real analytic* (respectively C^∞, or $C^{k,\alpha}$) family of selfadjoint operators if there is a dense subspace $V \subset \mathcal{H}$ such that
i) $\mathcal{D}(A(t)) = V$ and $A(t)$ is selfadjoint for any $t \in \mathbb{R}$ and
ii) the function $t \in \mathbb{R} \longmapsto (A(t)u, v)_{\mathcal{H}} \in \mathbb{C}$ is real analytic (respectively C^∞, or $C^{k,\alpha}$) for every $u \in V$ and $v \in \mathcal{H}$.
If this is the case then (by a result in [22]) the (vector valued) function

$$t \in \mathbb{R} \longmapsto A(t)u \in \mathcal{H},$$

is of the same class for every $u \in V$.

A sequence $\{\lambda_\nu\}_{\nu\geq 1}$ of scalar functions $\lambda_\nu : \mathbb{R} \to \mathbb{C}$ is said to *parametrize the eigenvalues* of $\{A(t)\}_{t\in\mathbb{R}}$ if for any $t \in \mathbb{R}$ and any $\lambda \in \sigma(A(t))$, the cardinality of the set $\{\nu \geq 1 : \lambda_\nu(t) = \lambda\}$ equals the multiplicity of λ.

We shall make use of the following result, which is referred hereafter as the Rellich-Alekseevsky-Kriegl-Losik-Michor theorem (cf. F. Rellich [29] for statement (i), D. Alekseevski & A. Kriegl & M. Losik & P.W. Michor [2] for statement (ii), and A. Kriegl & P.W. Michor [23] for statements (iii)-(iv)).

Theorem 3.1. *Let $t \in \mathbb{R} \mapsto A(t)$ be a curve of unbounded selfadjoint operators in a Hilbert space $\mathcal{H}$, with common domain of definition and compact resolvent. Then*

(i) *If $A(t)$ is real analytic in $t \in \mathbb{R}$, then the eigenvalues and the eigenvectors of $A(t)$ may be parameterized real analytically in t.*

(ii) *If $A(t)$ is C^∞ in $t \in \mathbb{R}$ and if no two unequal continuously parameterized eigenvalues meet of infinite order at any $t \in \mathbb{R}$, then the eigenvalues and eigenvectors can be parameterized C^∞ in t on the whole parameter domain.*

(iii) *If $A(t)$ is C^∞ in $t \in \mathbb{R}$, then the eigenvalues of $A(t)$ may be parameterized C^2 in t.*

(iv) *If $A(t)$ is $C^{k,\alpha}$ in $t \in \mathbb{R}$ for some $\alpha > 0$, then the eigenvalues of $A(t)$ may be parameterized C^1 in t.*

Among the applications to statements (i) and (iii) in Theorem 3.1 as proposed in [23], one may consider a compact manifold M and a smooth curve

$t \mapsto g_t$ of smooth Riemannian metrics on M. If moreover $t \mapsto \Delta_{g_t}$ is the corresponding smooth curve of Laplace-Beltrami operators on $L^2(M)$ then (by (iii) in Theorem 3.1) the eigenvalues may be parameterized C^2 in t. This was exploited by A. El Soufi & S. Ilias, [16]-[17], who discussed an array of related questions such as critical points of the functional $g \in \mathcal{M}(M) \mapsto \lambda_k(g)$, or suitable deformations of $g \in \mathcal{M}(M)$ producing quantitative variations of λ_k. Here $\mathcal{M}(M)$ is the set of all Riemannian metrics on M.

Let M be a compact strictly pseudoconvex CR manifold and let θ be a pseudo-Hermitian structure on M with positive definite Levi form. Let

$$\theta(t) = e^{u_t}\,\theta, \quad t \in \mathbb{R},$$

be an *analytic deformation* of θ, i.e. $\{u_t\}_{t\in\mathbb{R}}$ is a family of real valued C^∞ functions which is analytic with respect to t and $u_0 = 0$. Here $C^\infty(M,\mathbb{R})$ is thought of as organized as a real Fréchet space and the vector valued function

$$u : \mathbb{R} \to C^\infty(M,\mathbb{R}), \quad u(t) = u_t\,, \quad t \in \mathbb{R},$$

is assumed to be of class C^ω. For a theory of power series in Fréchet spaces we shall use Appendix B in [11].

Let Δ_b be the sub-Laplacian on M associated with θ and denote for each t, by $\Delta_{b,t}$ the sub-Laplacian associated with $\theta(t)$.

Theorem 3.2. *Let $\theta(t) = e^{u_t}\,\theta$ be an analytic deformation of θ and let $\lambda \in \sigma(\Delta_b)$ be an eigenvalue of multiplicity m. There exist a positive real number ε, a family of m real analytic functions $\{\Lambda_i\}_{1\le i\le m} \subset C^\omega((-\varepsilon,\varepsilon),\mathbb{R})$, and m families of C^∞ functions $\{v_i(t)\}_{|t|<\varepsilon} \in C^\infty(M,\mathbb{R})$, $1 \le i \le m$, such that each $v_i : (-\varepsilon,\varepsilon) \to C^\infty(M,\mathbb{R})$ is real analytic in t and*
(1) *$\Lambda_i(0) = \lambda$, $1 \le i \le m$,*
(2) *$\Delta_{b,t} v_i(t) = \Lambda_i(t)\, v_i(t)$, $1 \le i \le m$, $t \in (-\varepsilon,\varepsilon)$*
(3) *$\{v_i(t) : 1 \le i \le m\}$ is orthonormal in $L^2(M,\psi_{\theta(t)})$, $t \in (-\varepsilon,\varepsilon)$.*

Proof. The proof relies on the Rellich-Alekseevsky-Kriegl-Losik-Michor theorem (cf. Theorem 3.1 above). To this end we introduce the family of operators $U_t : L^2(M,\psi_\theta) \to L^2(M,\psi_{\theta(t)})$,

$$U_t v = e^{-(n+1)u_t/2}\, v, \quad v \in L^2(M,\psi_\theta).$$

The family $\{U_t\}_{t\in\mathbb{R}}$ is a real analytic family of unitary operators , i.e.

$$\|U_t\, v\|_{L^2(M,\psi_{\theta(t)})} = \|v\|_{L^2(M,\psi_\theta)},$$

and $U_t^{-1}\, v = e^{(n+1)u_t/2}\, v$. Moreover, let $A(t)$ be the family of operators

$$A(t) = U_t^{-1} \circ \Delta_{b,t} \circ U_t : L^2(M,\psi_\theta) \to L^2(M,\psi_\theta).$$

Then

$$\Delta_{b,t} v(t) = \lambda\, v(t) \Longleftrightarrow A(t)\left(U_t^{-1}\, v(t)\right) = \lambda\, U_t^{-1}\, v(t).$$

In particular, the spectrum of $\Delta_{b,t}$ coincides with that of $A(t)$. Let us show that the family $\{A(t)\}_{t\in\mathbb{R}}$ is analytic in t. Indeed, the dense subspace $\mathcal{D}(\Delta_b) \subset L^2(M,\psi_\theta)$ is the domain of $A(t)$ and, as we shall check in a moment, $A(t) \subset$

$A(t)^*$. Indeed, the sub-Laplacians Δ_b and $\Delta_{b,t} = \Delta_{b,\theta(t)}$ are related by (see [8, Proposition 5] or [30, Lemma 1.8])

$$\Delta_{b,t}v = e^{-u_t}\left(\Delta_b v - nG_\theta(\nabla^H u_t, \nabla^H v)\right), \quad v \in C^2(M). \tag{3.1}$$

Then, for each $v \in \mathcal{D}(\Delta_b)$,

$$A(t)v = (U_t^{-1} \circ \Delta_{b,t} \circ U_t)v = \cdots =$$

$$= e^{-u_t}\left[\Delta_b v + G_\theta(\nabla^H u_t, \nabla^H v) - \frac{n+1}{2}\left(\Delta_b u_t - \frac{(n-1)}{2}|\nabla^H u_t|^2\right)v\right].$$

Finally, the family $\{A(t)\}_{t\in\mathbb{R}}$ is an analytic curve of self-adjoint operators in $L^2(M, \psi_\theta)$ with common domain of definition and with compact resolvent. Therefore, we can apply Theorem 3.1 (i) to deduce that the eigenvalues and the eigenvectors of $A(t)$ depend analytically in t, i.e., there exists m analytic families of vectors $v_i(t)$ and m real analytic valued functions $\Lambda_i(t)$ in t satisfying (1), (2) and (3) of Theorem 3.2. □

For any $\theta \in \mathcal{P}_+(M)$, the set of all pseudo-Hermitian structures with positive definite Levi form on M, let

$$0 = \lambda_0(\theta) < \lambda_1(\theta) \leq \lambda_2(\theta) \leq \cdots \leq \lambda_k(\theta) \leq \cdots$$

be the spectrum of the sub-Laplacian $\Delta_b = \Delta_{b,\theta}$ of (M, θ). For every $k \in \mathbb{N}$, let

$$E_k(\theta) = \mathrm{Ker}\ (\Delta_b - \lambda_k(\theta)I)$$

be the eigenspace of Δ_b corresponding to the eigenvalue $\lambda_k(\theta)$. Also let $\Pi_k : L^2(M, \psi_\theta) \to E_k(\theta)$ be the orthogonal projection on $E_k(\theta)$. Let us fix $k \in \mathbb{N}$ and consider the functional $\theta \in \mathcal{P}_+(M) \longmapsto \lambda_k(\theta) \in \mathbb{R}$. This functional is continuous (with respect to an appropriate metric topology on $\mathcal{P}_+(M)$, as shown in [5]) but not differentiable in general. However, one has the following

Theorem 3.3. *Let M be a compact strictly pseudoconvex CR manifold and let $\theta \in \mathcal{P}_+(M)$. Let $\theta(t) = e^{u_t}\,\theta$, $t \in (-\varepsilon, \varepsilon)$, be an analytic deformation of θ. Then, for every positive $k \in \mathbb{N}$,*

(1) *The function $t \in (-\varepsilon, \varepsilon) \longmapsto \lambda_k(\theta(t))$ admits left and right derivatives at $t = 0$.*

(2) *The derivatives $\frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^-}$ and $\frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^+}$ are eigenvalues of the operator $\Pi_k \circ \Delta_b' : E_k(\theta) \to E_k(\theta)$ where, $\forall v \in \mathbb{C}^\infty(M)$,*

$$\Delta_b' v = \frac{d}{dt}\Delta_{b,t}v\Big|_{t=0} = -f\Delta_b v - n\, G_\theta\left(\nabla^H f, \nabla^H v\right)$$

with $f = \frac{d}{dt}u_t\Big|_{t=0}$.

(3) *If $\lambda_k(\theta) > \lambda_{k-1}(\theta)$, then $\frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^-}$ and $\frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^+}$ are the greatest and the least eigenvalues of $\Pi_k \circ \Delta_b'$ on $E_k(\theta)$, respectively.*

(4) *If $\lambda_k(\theta) < \lambda_{k+1}(\theta)$ then $\frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^-}$ and $\frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^+} \in \mathbb{R}$ are the smallest and the greatest eigenvalue of $\Pi_k \circ \Delta_b'$ on $E_k(\theta)$, respectively.*

Proof. Let us denote by m the dimension of $E_k(\theta)$. We apply Theorem 3.2 with $\lambda = \lambda_k(\theta)$ to derive the existence of m real analytic functions $\{\Lambda_i\}_{1\le i\le m} \subset C^\omega((-\varepsilon,\varepsilon),\mathbb{R})$ and m analytic families of functions $\{v_i(t)\}_{|t|<\varepsilon} \in C^\infty(M,\mathbb{R})$, $1 \le i \le m$, satisfying (1)-(3) of Theorem 3.2. Since $t \mapsto \lambda_k(\theta(t))$ and $t \mapsto \Lambda_i(t)$ are continuous and $\Lambda_1(0) = ... = \Lambda_m(0) = \lambda_k(\theta)$, one deduces that $\lambda_k(\theta(t)) \in \{\Lambda_1(t),\cdots,\Lambda_m(t)\}$ for sufficiently small t. Since, moreover, $\forall i \le m$, $t \mapsto \Lambda_i(t)$ is analytic, there exist $\delta > 0$ and two integers p, $q \le m$ such that

$$\lambda_k(\theta(t)) = \begin{cases} \Lambda_p(t) & \text{for } t \in (-\delta, 0) \\ \Lambda_q(t) & \text{for } t \in (0,\delta). \end{cases}$$

Therefore, the function $t \longmapsto \lambda_k(\theta(t))$ admits left and right derivatives at $t = 0$ with

$$\frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^-} = \Lambda'_p(0) \ \text{ and } \ \frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^+} = \Lambda'_q(0).$$

Now, one has for all $i \le m$ and $t \in (-\delta,\delta)$, $\Delta_{b,t} v_i(t) = \Lambda_i(t) v_i(t)$. Differentiating at $t = 0$, we get

$$\Delta'_b v_i + \Delta_b v'_i = \Lambda'_i(0) v_i + \lambda_k(\theta) v'_i \tag{3.2}$$

where $v_i = v_i(0)$ and $v'_i = \frac{d}{dt} v_i(t)\big|_{t=0}$. Multiplication by v_j and integration by parts yield

$$\int_M v_j \Delta'_b v_i \psi_\theta = \begin{cases} \Lambda'_i(0) & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases}$$

Since $\{v_1,\cdots,v_m\}$ is an orthonormal basis of $E_k(\theta)$ with respect to the inner product of $L^2(M,\psi_\theta)$, we deduce that

$$(\Pi_k \circ \Delta'_b) v_i = \Lambda'_i(0) v_i.$$

That is $\Lambda'_1(0),\cdots,\Lambda'_m(0)$ are the eigenvalues of $\Pi_k \circ \Delta'_b : E_k(\theta) \to E_k(\theta)$. Differentiating the identity (3.1) at $t = 0$ we get

$$\Delta'_b v = -f \Delta_b v - n G_\theta\left(\nabla^H v, \nabla^H f\right).$$

Assume now $\lambda_k(\theta) > \lambda_{k-1}(\theta)$. For any $i \le m$, one then has $\Lambda_i(0) = \lambda_k(\theta) > \lambda_{k-1}(\theta)$. By continuity, we necessarily have $\Lambda_i(t) > \lambda_{k-1}(\theta(t))$ for sufficiently small t. Hence, there exists $\eta > 0$ such that, $\forall\, |t| < \eta$ and $\forall i \le m$, $\Lambda_i(t) \ge \lambda_k(\theta(t))$, which means that $\lambda_k(\theta(t)) = \min\{\Lambda_1(t),\cdots,\Lambda_m(t)\}$. This implies that

$$\frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^-} = \max\{\Lambda'_1(0),\cdots,\Lambda'_m(0)\}$$

and

$$\frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^+} = \min\{\Lambda'_1(0),\cdots,\Lambda'_m(0)\}$$

which proves (3).

Smilarily, if $\lambda_k(\theta) < \lambda_{k+1}(\theta)$, one has, for sufficiently small t, $\Lambda_i(t) \le \lambda_k(\theta(t))$ which means that $\lambda_k(\theta(t)) = \max\{\Lambda_1(t),\cdots,\Lambda_m(t)\}$ and, then,

$$\frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^+} = \max\{\Lambda'_1(0),\cdots,\Lambda'_m(0)\}$$

and

$$\frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^-} = \min\left\{\Lambda_1'(0),\cdots,\Lambda_m'(0)\right\}.$$

□

Corollary 3.1. *Let M be a compact strictly pseudoconvex CR manifold and let $\theta \in \mathcal{P}_+(M)$. Let $\theta(t) = e^{u_t}\,\theta$, $t \in (-\varepsilon,\varepsilon)$, be an analytic deformation of θ and set $f = \frac{d}{dt}u_t\big|_{t=0}$. For every positive integer k, let $Q_{f,k} : E_k(\theta) \to \mathbb{R}$ be the quadratic form given by*

$$Q_{f,k}(v) = -\int_M \left(\lambda_k(\theta)v^2 + \frac{n}{2}\Delta_b v^2\right) f\,\psi_\theta.$$

(1) *If $Q_{f,k}$ is positive definite on $E_k(\theta)$, then there exists $\varepsilon > 0$ such that $\lambda_k(\theta(-t)) < \lambda_k(\theta) < \lambda_k(\theta(t))$ for all $t \in (0,\varepsilon)$.*

(2) *Assume that $\lambda_k(\theta) > \lambda_{k-1}(\theta)$. If $Q_{f,k}$ takes negative values somewhere in $E_k(\theta)$, then $\lambda_k(\theta(t)) < \lambda_k(\theta)$ for all $t \in (0,\varepsilon)$, for some $\varepsilon > 0$.*

(3) *Assume that $\lambda_k(\theta) < \lambda_{k+1}(\theta)$. If $Q_{f,k}$ takes positive values somewhere in $E_k(\theta)$, then $\lambda_k(\theta(t)) > \lambda_k(\theta)$ for all $t \in (0,\varepsilon)$, for some $\varepsilon > 0$.*

Proof. First, we have with the notations of Theorem 3.3, $\forall v \in E_k(\theta)$,

$$Q_{f,k}(v) = \int_M v\Delta_b' v\,\psi_\theta. \tag{3.3}$$

Indeed, $\forall v \in E_k(\theta)$,

$$\begin{aligned}\int_M v\Delta_b' v\,\psi_\theta &= -\int_M v\left(f\Delta_b v + nG_\theta(\nabla^H v,\nabla^H f)\right)\psi_\theta\\ &= -\int_M \left(f\lambda_k(\theta)v^2 + \frac{n}{2}G_\theta(\nabla^H v^2,\nabla^H f)\right)\psi_\theta\\ &= -\int_M \left(\lambda_k(\theta)v^2 + \frac{n}{2}\Delta_b v^2\right) f\,\psi_\theta = Q_{f,k}(v).\end{aligned}$$

Now, if $Q_{f,k}$ is positive definite on $E_k(\theta)$, then, thanks to (3.3), all the eigenvalues of the operator $\Pi_k \circ \Delta_b' : E_k(\theta) \to E_k(\theta)$ are positive. Applying Theorem 3.3 (2), it follows that both $\frac{d}{dt}\lambda_k(\theta(t))\big|_{t=0^+}$ and $\frac{d}{dt}\lambda_k(\theta(t))\big|_{t=0^-}$ are positive and that there exists $\varepsilon > 0$ such that $\lambda_k(\theta(-t)) < \lambda_k(\theta) < \lambda_k(\theta(t))$ for all $t \in (0,\varepsilon)$.

Assume that $\lambda_k(\theta) > \lambda_{k-1}(\theta)$ and that there exists $v \in E_k(\theta)$ such that $Q_{f,k}(v) < 0$. This implies that the operator $\Pi_k \circ \Delta_b' : E_k(\theta) \to E_k(\theta)$ has at least one negative eigenvalue. Applying Theorem 3.3 (3), we deduce that $\frac{d}{dt}\lambda_k(\theta(t))\big|_{t=0^+}$ is negative and that there exists $\varepsilon > 0$ such that $\lambda_k(\theta(t)) < \lambda_k(\theta)$ for all $t \in (0,\varepsilon)$.

The last part of the corollary can be proved using similar arguments.

□

4. Generic simplicity of sub-Laplacian eigenvalues

Let M be a compact stirctly pseudoconvex CR manifold and denote by $\mathcal{P}_+(M)$ the set of all pseudo-Hermitian structures with positive definite Levi form on M. In [5], we defined a complete distance on $\mathcal{P}_+(M)$ so that the eigenvalues of the sub-Laplacian $\theta \in \mathcal{P}_+(M) \mapsto \lambda_k(\theta)$ are continuous. This distance is defined as follows : We fix a form $\theta \in \mathcal{P}_+(M)$. Given $\theta_1 = e^{u_1}\theta$ and $\theta_2 = e^{u_2}\theta$ in $\mathcal{P}_+(M)$, we set

$$d(\theta_1, \theta_2) = d_{C^\infty}(u_1, u_2) + \rho(G_{\theta_1}, G_{\theta_2})$$

where d_{C^∞} is the distance function associated with the canonical Frechet structure of $C^\infty(M)$ and

$$\rho(G_{\theta_1}, G_{\theta_2}) = \inf\{\delta > 0 \ : \ e^{-\delta}G_{\theta_1}(X,X) \le G_{\theta_2}(X,X) \le e^{\delta}G_{\theta_1}(X,X)\,,\ \forall X \in H\}.$$

In [5], we proved that $(\mathcal{P}_+(M), d)$ is a complete metric space and that if $\rho(G_{\theta_1}, G_{\theta_2}) < \varepsilon$, then, $\forall k \ge 1$,

$$e^{-\varepsilon} \le \frac{\lambda_k(\theta_1)}{\lambda_k(\theta_2)} \le e^{\varepsilon}.$$

In the sequel, we denote by $\mathcal{J}$ the set of all elements $\theta \in \mathcal{P}_+(M)$ such that all the eigenvalues of the sub-Laplacian $\Delta_{b,\theta}$ have multiplicity one, that is,

$$\mathcal{J} = \{\theta \in \mathcal{P}_+(M) \ : \ 0 < \lambda_1(\theta) < \lambda_2(\theta) < \cdots < \lambda_k(\theta) < \cdots\}$$

Our main aim in this section is to prove the following

Theorem 4.1. *The set $\mathcal{J}$ is a residual set in $(\mathcal{P}_+(M), d)$, i.e., a countable intersection of open dense subsets. In particular, $\mathcal{J}$ is dense in $(\mathcal{P}_+(M), d)$.*

The proof of this theorem relies on the following proposition which is a consequence of Theorem 3.3.

Proposition 4.1. *Let M be a compact strictly pseudoconvex CR manifold and let $\theta \in \mathcal{P}_+(M)$. Let $\lambda \in \sigma(\Delta_{b,\theta})$ be an eigenvalue of multiplicity $m \ge 2$ and let $k \in \mathbb{N}$ be such that*

$$\lambda = \lambda_k(\theta) = \lambda_{k+1}(\theta) = \cdots = \lambda_{k+m-1}(\theta).$$

There exist $f \in C^\infty(M)$ and $\varepsilon > 0$ such that $\theta(t) = e^{tf}\theta$ satisfies for all $t \in (0, \varepsilon)$,

$$\lambda_k(\theta(t)) < \lambda_{k+m-1}(\theta(t)).$$

Proof. Let $E = E_k(\theta) = E_{k+m-1}(\theta)$ be the eigenspace of $\Delta_{b,\theta}$ corresponding to the eigenvalue λ and let $\Pi : L^2(M) \to E$ be the orthogonal projection on E. For every $f \in C^\infty(M)$, we denote by $L_f : E \to E$ the operator defined by

$$L_f v = \Pi \circ \Delta_b' v = -\Pi\left[\lambda f v + nG_\theta\left(\nabla^H v, \nabla^H f\right)\right].$$

From the definition of the integers k and m, one has $\lambda_k(\theta) > \lambda_{k-1}(\theta)$ and $\lambda_{k+m-1}(\theta) < \lambda_{k+m}(\theta)$. Therefore, Theorem 3.3 tells us that, for any $f \in C^\infty(M)$, $\frac{d}{dt}\lambda_k(e^{tf}\theta)\big|_{t=0^+}$ and $\frac{d}{dt}\lambda_{k+m-1}(e^{tf}\theta)\big|_{t=0^+}$ represent the smallest and the largest eigenvalues of the operator $L_f : E \to E$, respectively.

Therefore, it suffices to prove the existence of a function $f \in C^\infty(M)$ so that the operator L_f has at least two distinct eigenvalues (i.e. L_f is not proportional to the identity of E). Indeed, in this case, we would have $\frac{d}{dt}\lambda_k(e^{tf}\theta)\Big|_{t=0^+} < \frac{d}{dt}\lambda_{k+m-1}(e^{tf}\theta)\Big|_{t=0^+}$ which implies the conclusion of the proposition.

Thanks to (3.3), one has, $\forall v, w \in E$

$$\Big(L_f v, w\Big)_{L^2(M)} = \cdots = -\int_M \Big(\frac{n}{2}\Delta_b(vw) + \lambda vw\Big) f\, \psi_\theta.$$

Let $\{u_1, u_2\} \subset E$ be a pair of functions with $\|u_1\|_{L^2(M)} = \|u_2\|_{L^2(M)}$ and $u_1^2 \neq u_2^2$ (recall that E is of dimension at least 2) and set $v = u_1 - u_2$ and $w = u_1 + u_2$ so that $(v, w)_{L^2(M)} = 0$. The function

$$f_0 = \frac{n}{2}\Delta_b(vw) + \lambda vw = \frac{n}{2}\Delta_b(u_1^2 - u_2^2) + \lambda(u_1^2 - u_2^2) \tag{4.1}$$

is such that

$$\Big(L_{f_0} v, w\Big)_{L^2(M)} = -\int_M \Big(\frac{n}{2}\Delta_b(u_1^2 - u_2^2) + \lambda(u_1^2 - u_2^2)\Big)^2 \psi_\theta$$

which does not vanish since Δ_b has no negative eigenvalues. Thus, L_{f_0} cannot be proportional to the identity of E.

□

Proof of Theorem 4.1. For every positive integer k, let $\mathcal{J}_k$ be the subset of $\mathcal{P}_+(M)$ defined by

$$\mathcal{J}_k = \{\theta \in \mathcal{P}_+(M) \ : \ 0 < \lambda_1(\theta) < \lambda_2(\theta) < \cdots < \lambda_k(\theta)\}.$$

We have $\mathcal{P}_+(M) = \mathcal{J}_1 \supset \mathcal{J}_2 \supset \ldots \supset \mathcal{J}_k \supset \ldots$ and

$$\mathcal{J} = \bigcap_{k=1}^{\infty} \mathcal{J}_k.$$

According to Baire's category theorem, it suffices to prove that each $\mathcal{J}_k$ is an open dense subset of $\mathcal{P}_+(M)$.

The fact that $\mathcal{J}_k$ is open follows immediately from the continuity of the eigenvalues $\theta \in \mathcal{P}_+(M) \mapsto \lambda_i(\theta)$, $i \leq k$.

Let us prove that, for any $k \geq 1$, $\mathcal{J}_{k+1}$ is a dense subset of $\mathcal{J}_k$. An obvious recursion would then imply that each $\mathcal{J}_k$ is a dense subset of $\mathcal{P}_+(M)$. So, let $\theta \in \mathcal{J}_k \setminus \mathcal{J}_{k+1}$ and let η be any positive real number. Thus, one has

$$\lambda_1(\theta) < \lambda_2(\theta) < \cdots < \lambda_{k-1}(\theta) < \lambda_k(\theta) = \lambda_{k+1}(\theta) = \cdots = \lambda_{k+m-1}(\theta) < \lambda_{k+m}(\theta),$$

where m is the multiplicity of $\lambda_k(\theta)$. Using Proposition 4.1 and the continuity of the eigenvalues, one can find $f \in C^\infty(M)$ and $\varepsilon > 0$ such that the form $\theta(t) = e^{tf}\theta$ satisfies, for every $t \in (0, \varepsilon)$,

$$\lambda_1(\theta(t)) < \lambda_2(\theta(t)) < \cdots < \lambda_{k-1}(\theta(t)) < \lambda_k(\theta(t)) < \lambda_{k+m-1}(\theta(t))$$

which means that $\theta(t)$ belongs to $\mathcal{J}_k$ and the multiplicity of $\lambda_k(\theta)$ is at most $m-1$. Choosing $t_1 \in (0, \varepsilon)$ sufficiently small, one gets a form $\theta_1 = \theta(t_1) \in \mathcal{J}_k$ such that the multiplicity of $\lambda_k(\theta_1)$ is at most $m - 1$ and $d(\theta_1, \theta) < \eta/m$.

Repeating this argument at most $m-1$ times, we prove the existence of a 1-form $\hat\theta \in \mathcal{J}_{k+1}$ such that $d(\hat\theta,\theta) < \eta$.

□

5. Critical pseudo-Hermitian structures

The content of this section is patterned after the article [17] by Ilias and the third author dealing with Laplacian eigenvalues in the Riemannian setting. For the sake of completeness, we shall give self-contained proofs of the results we obtain in the CR context.

Let M be a compact strictly pseudoconvex CR manifold. For every positive integer k we consider the map $\theta \in \mathcal{P}_+(M) \mapsto \lambda_k(\theta) \in \mathbb{R}$, where, as before, $\mathcal{P}_+(M)$ denotes the set of all pseudo-Hermitian structures with positive definite Levi form on M, and $\lambda_k(\theta)$ is the k-th eigenvalue of the sub-Laplacian associated to θ. Since the eigenvalues are not invariant under scaling, we restrict λ_k to the subset

$$\mathcal{P}_{+,0}(M) = \{\theta \in \mathcal{P}_+(M) \ : \ \mathrm{vol}(\theta) = 1\}$$

where $\mathrm{vol}(\theta) = \int_M \psi_\theta$ is the volume of M with respect to ψ_θ.

Thanks to Theorem 3.3, one can introduce the following

Definition 5.1. *A pseudo-Hermitian structure θ is said to be critical for the functional λ_k restricted to $\mathcal{P}_{+,0}(M)$ if, for any analytic deformation $\{\theta(t) = e^{u_t}\theta\} \subset \mathcal{P}_{+,0}(M)$ of θ, we have*

$$\frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^-} \times \frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^+} \le 0.$$

It is easy to see that

$$\frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^+} \le 0 \le \frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^-} \iff \lambda_k(\theta(t)) \le \lambda_k(\theta) + o(t) \text{ as t} \to 0$$

and

$$\frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^-} \le 0 \le \frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^+} \iff \lambda_k(\theta(t)) \ge \lambda_k(\theta) + o(t) \text{ as t} \to 0.$$

Of course, if θ is a local maximizer or a local minimizer of λ_k, then θ is critical in the sense of the previous definition. We set

$$\mathcal{A}_0(M,\theta) = \left\{ f \in C^\infty(M) \ : \ \int_M f\, \psi_\theta = 0 \right\}$$

and recall the definition of the quadratic form $Q_{f,k} : E_k(\theta) \to \mathbb{R}$ associated to a pair $(f,k) \in C^\infty(M) \times \mathbb{N}^*$ (see Corollary 3.1):

$$Q_{f,k}(v) = -\int_M \left(\lambda_k(\theta)v^2 + \frac{n}{2}\Delta_b v^2\right) f\, \psi_\theta.$$

Theorem 5.1. *Let M be a compact strictly pseudoconvex CR manifold. Let $\theta \in \mathcal{P}_{+,0}(M)$ be a pseudo-Hermitian structure and $k \in \mathbb{N}^*$.*

1) If θ is a critical pseudo-Hermitian structure of the functional λ_k restricted to $\mathcal{P}_{+,0}(M)$, then, $\forall f \in \mathcal{A}_0(M,\theta)$, the quadratic form $Q_{f,k}$ is indefinite on $E_k(\theta)$.

2) Assume that $\lambda_k(\theta) > \lambda_{k-1}(\theta)$ or $\lambda_k(\theta) < \lambda_{k+1}(\theta)$. The pseudo-Hermitian structure θ is critical for the functional λ_k restricted to $\mathcal{P}_{+,0}(M)$ if and only if, $\forall f \in \mathcal{A}_0(M,\theta)$, the quadratic form $Q_{f,k}$ is indefinite on $E_k(\theta)$.

Proof. Let $f \in \mathcal{A}_0(M,\theta)$ and let $\theta(t)$ be the analytic deformation of θ given by

$$\theta(t) = \left(\frac{\mathrm{vol}(\theta)}{\mathrm{vol}(e^{tf}\theta)}\right)^{\frac{1}{n+1}} e^{tf}\theta = e^{u_t}\theta,\ t \in \mathbb{R},$$

with $u_t = tf - \frac{1}{n+1}\ln(\mathrm{vol}(e^{tf}\theta))$. Since

$$\psi_{e^{tf}\theta} = e^{(n+1)tf}\psi_\theta,$$

it is easy to check that $\mathrm{vol}(\theta(\mathrm{t})) = 1$, that is $\theta(t)$ belongs to $\mathcal{P}_{+,0}(M)$ for every $t \in \mathbb{R}$. One has

$$\frac{d}{dt}\mathrm{vol}(e^{tf}\theta(t))\Big|_{t=0} = \frac{d}{dt}\int_M e^{(n+1)tf}\psi_\theta\Big|_{t=0} = (n+1)\int_M f\psi_\theta = 0.$$

Therefore,

$$\frac{d}{dt}u_t\Big|_{t=0} = \cdots = f$$

and, then,

$$\Delta_b' v = \frac{d}{dt}\Delta_{b,t}\Big|_{t=0} = -f\Delta_b v - nG_\theta(\nabla^H v, \nabla^H f).$$

Thus, we have (see (3.3)),

$$Q_{f,k}(v) = \int_M v\Delta_b' v\ \psi_\theta. \tag{5.1}$$

Now, assuming that θ is a critical pseudo-Hermitian structure of λ_k restricted to $\mathcal{P}_{+,0}(M)$, we obtain, using the definition of criticality and Theorem 3.3 (2), that the operator $\Pi_k \circ \Delta_b'$ admits both nonnegative and nonpositive eigenvalues in $E_k(\theta)$, which means (thanks to (5.1) that the quadratic form $Q_{f,k}$ is indefinite on $E_k(\theta)$. This proves the first part of the theorem.

The last part of the theorem follows from Theorem 3.3 (3) and (4), and (5.1).

□

Proposition 5.1. *Let M be a compact strictly pseudoconvex CR manifold and let $\theta \in \mathcal{P}_{+,0}(M)$ be a pseudo-Hermitian structure. For any positive integer k, the two following conditions are equivalent:*

1. *For all $f \in \mathcal{A}_0(M,\theta)$, the quadratic form $Q_{f,k}$ is indefinite on $E_k(\theta)$.*
2. *There exists a finite family $\{v_1, \cdots, v_d\} \subset E_k(\theta)$ of eigenfunctions associated with $\lambda_k(\theta)$ such that $\sum_{i=1}^d v_i^2 = 1$.*

Proof. Assume first that there exist $v_1, \cdots, v_d \in E_k(\theta)$ such that $\sum_i^d v_i^2 = 1$. Therefore, $\forall f \in \mathcal{A}_0(M,\theta)$,

$$\sum_{i=1}^{d} Q_{f,k}(v_i) = \cdots = -\lambda_k(\theta)\int_M f\psi_\theta = 0$$

which implies that $Q_{f,k}$ is indefinite on $E_k(\theta)$.

Conversely, assume that $Q_{f,k}$ is indefinite on $E_k(\theta)$ for all $f \in \mathcal{A}_0(M,\theta)$ and consider the convex set

$$K = \left\{\sum_{i\in J}\left[\lambda_k(\theta)v_i^2 + \frac{n}{2}\Delta_b v_i^2\right]; v_i \in E_k(\theta),\ J \subset \mathbb{N},\ J \text{ finite}\right\} \subset L^2(M).$$

Let us prove that the constant function 1 belongs to K. Indeed, if $1 \notin K$, then, applying classical separation theorem in the finite dimensional subspace of $L^2(M,\theta)$ generated by K and 1, we deduce the existence of $h \in L^2(M)$ such that $(h,1)_{L^2(M)} = \int_M h\psi_\theta > 0$ and, $\forall\ w \in K$, $(h,w)_{L^2} = \int_M hw\psi_\theta \le 0$. Let $f = h - \frac{1}{\mathrm{vol}(\theta)}\int_M h\psi_\theta \in \mathcal{A}_0(M,\theta)$. Then , $\forall v \in E_k(\theta)$

$$\begin{aligned} Q_{f,k}(v) &= -\int_M \left(\lambda_k(\theta)v^2 + \frac{n}{2}\Delta_b v^2\right) f\psi_\theta \\ &= -\int_M \left(\lambda_k(\theta)v^2 + \frac{n}{2}\Delta_b v^2\right) h\psi_\theta + \frac{\int_M h\psi_\theta}{\mathrm{vol}(\theta)}\lambda_k(\theta)\int_M v^2\psi_\theta \end{aligned}$$

since $\int_M \Delta_b v^2\psi_\theta = 0$. Moreover, $\forall v \in E_k(\theta)$, the function $\left(\lambda_k(\theta)v^2 + \frac{n}{2}\Delta_b v^2\right)$ belongs to K which implies that $\int_M \left(\lambda_k(\theta)v^2 + \frac{n}{2}\Delta_b v^2\right) h\psi_\theta \le 0$ and, then

$$Q_{f,k}(v) \ge \frac{\lambda_k(\theta)\int_M h\psi_\theta}{\mathrm{vol}(\theta)}\int_M v^2\psi_\theta.$$

Therefore, the quadratic form $Q_{f,k}$ is positive definite on $E_k(\theta)$ which contradicts the assumptions.

Now, since $1 \in K$, there exist $v_1, \cdots, v_d \in E_k(\theta)$ such that

$$\sum_{i=1}^{d}\left(\lambda_k(\theta)v_i^2 + \frac{n}{2}\Delta_b v_i^2\right) = \lambda_k(\theta) \tag{5.2}$$

which leads to

$$\Delta_b\left(\sum_{i\le d} v_i^2 - 1\right) = -\frac{2}{n}\lambda_k(\theta)\left(\sum_{i\le d} v_i^2 - 1\right)$$

This implies that $\sum_{i\le d} v_i^2 - 1 = 0$ since the sub-Laplacian admits no negative eigenvalues.

□

Theorem 5.1 and Proposition 5.1 lead to the following

Corollary 5.1. *Let M be a compact strictly pseudoconvex CR manifold. Let $\theta \in \mathcal{P}_{+,0}(M)$ be a pseudo-Hermitian structure and $k \in \mathbb{N}^*$.*

(1) If θ is a critical pseudo-Hermitian structure of the functional λ_k restricted to $\mathcal{P}_{+,0}(M)$, then there exists a finite family $\{v_1, \cdots, v_d\} \subset E_k(\theta)$ of eigenfunctions associated with $\lambda_k(\theta)$ such that $\sum_i^d v_i^2 = 1$.

(2) Assume that $\lambda_k(\theta) > \lambda_{k-1}(\theta)$ or $\lambda_k(\theta) < \lambda_{k+1}(\theta)$. Then, θ is critical for the functional λ_k restricted to $\mathcal{P}_{+,0}(M)$ if and only if there exists a finite family $\{v_1, \cdots, v_d\} \subset E_k(\theta)$ of eigenfunctions associated with $\lambda_k(\theta)$ such that $\sum_i^d v_i^2 = 1$.

According to [30, Proposition 4.4], the first positive eigenvalue of the standard CR sphere $\mathbb{S}^{2n+1}$ is equal to $2n$ and the corresponding eigenspace is generated by the restriction of coordinate functions, the sum of whose squares is 1 on $\mathbb{S}^{2n+1}$. Hence, the standard contact form of $\mathbb{S}^{2n+1}$ is a critical pseudo-Hermitian structure of λ_1 restricted to $\mathcal{P}_{+,0}(\mathbb{S}^{2n+1})$. On the other hand, the condition that there exists a finite family $\{v_1, \cdots, v_d\} \subset E_k(\theta)$ such that $\sum_i^d v_i^2 = 1$, is equivalent to the existence of a pseudo-harmonic map from (M, θ) to the sphere $\mathbb{S}^{d-1}$ (see [6, Lemma 6.1] and [12]).

An immediate consequence of Corollary 5.1 is the following:

Corollary 5.2. *Let M be a compact strictly pseudoconvex CR manifold. If $\theta \in \mathcal{P}_{+,0}(M)$ is a critical metric of the functional λ_k restricted to $\mathcal{P}_{+,0}(M)$, then $\lambda_k(\theta)$ is a degenerate eigenvalue, that is*

$$\dim E_k(\theta) \geq 2.$$

In the case when θ is a local maximizer or a local minimizer, we have the following more precise result

Proposition 5.2. *Let M be a compact strictly pseudoconvex CR manifold.*

(1) If $\theta \in \mathcal{P}_{+,0}(M)$ is a local minimizer of the functional λ_k restricted to $\mathcal{P}_{+,0}(M)$, then $\lambda_k(\theta) = \lambda_{k-1}(\theta)$.

(2) If $\theta \in \mathcal{P}_{+,0}(M)$ is a local maximizer of the functional λ_k restricted to $\mathcal{P}_{+,0}(M)$, then $\lambda_k(\theta) = \lambda_{k+1}(\theta)$.

Proof. Let $\theta \in \mathcal{P}_{+,0}(M)$ be a local minimizer of λ_k, that is $\lambda_k(\hat{\theta}) \geq \lambda_k(\theta)$ for every $\hat{\theta}$ in a neighborhood of θ in $\mathcal{P}_{+,0}(M)$. Assume for a contradiction that $\lambda_k(\theta) > \lambda_{k-1}(\theta)$. Let $f \in \mathcal{A}_0(M, \theta)$ and let $\theta(t) = e^{u_t}\theta \in \mathcal{P}_{+,0}(M)$ with $u_t = tf - \frac{1}{n+1} \ln(\mathrm{vol}(e^{tf}\theta))$. Then $\theta(t)$ is a volume-preserving analytic deformation of θ such that $\frac{d}{dt}u_t\Big|_{t=0} = f$ (see the proof of Theorem 5.1). Denote by $\Lambda_1(t), \cdots, \Lambda_m(t)$ the associated family of eigenvalues of $\Delta_{b,t}$, depending analytically on t and such that $\Lambda_1(0) = \cdots = \Lambda_m(0) = \lambda_k(\theta)$ with $m = \dim E_k(\theta)$ (see Theorem 3.3). For continuity reasons, we have, for sufficiently small t and all $i \leq m$,

$$\Lambda_i(t) > \lambda_{k-1}(\theta(t)).$$

Hence, $\forall i \leq m$ and $\forall t$ sufficiently small,

$$\Lambda_i(t) \geq \lambda_k(\theta(t)) \geq \lambda_k(\theta) = \Lambda_i(0).$$

Consequently, $\Lambda_i'(0) = 0$ for all $i \leq m$. Since (Theorem 3.3) $\Lambda_1'(0), \cdots, \Lambda_m'(0)$ are eigenvalues of the operator $\Pi_k \circ \Delta_b' : E_k(\theta) \to E_k(\theta)$, it follows that $\Pi_k \circ \Delta_b'$ is identically zero on $E_k(\theta)$. Consequently, thanks to (3.3), for any $f \in \mathcal{A}_0(M, \theta)$, the quadratic form $Q_{f,k}$ is identically zero on $E_k(\theta)$ which implies that, $\forall v \in E_k(\theta)$,

$$\lambda_k(\theta) v^2 + \frac{n}{2} \Delta_b v^2 = c$$

for some constant $c \in \mathbb{R}$. Therefore,

$$\Delta_b \left(v^2 - \frac{c}{\lambda_k(\theta)} \right) = -\frac{2}{n} \lambda_k(\theta) \left(v^2 - \frac{c}{\lambda_k(\theta)} \right)$$

which leads to a contradiction since the sub-Laplacian admits no negative eigenvalues.

A similar proof works for (2). □

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