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ABSTRACT. We prove that the first positive eigenvalue, normalized by the volume, of the sub-Laplacian associated with a strictly pseudoconvex pseudo-Hermitian structure θ on the CR sphere $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$, achieves its maximum when θ is the standard contact form.

1. Introduction and statement of the main result

According to a classical result of Hersch [15], given any Riemannian metric g on the 2-dimensional sphere $\mathbb{S}^2$, the first positive eigenvalue $\lambda_1(g)$ of the Laplace-Beltrami operator Δ_g satisfies the estimate

$$\lambda_1(g)A(g) \leq \lambda_1(g_0)A(g_0) \tag{1}$$

where g_0 is the standard metric of $\mathbb{S}^2$ and $A(g)$ is the area of $\mathbb{S}^2$ with respect to g. Moreover, the equality holds in (1) if and only if g is isometric to g_0. This result was extended to higher dimensional spheres by Ilias and the second author as follows (see [12, proposition 3.1]) : If a Riemannian metric g on the n-dimensional sphere $\mathbb{S}^n$ is conformal to the standard metric g_0, then

$$\lambda_1(g)V(g)^{\frac{2}{n}} \leq \lambda_1(g_0)V(g_0)^{\frac{2}{n}} \tag{2}$$

where $V(g)$ denotes the Riemannian volume of the sphere with respect to g. Again, the equality holds in (2) if and only if g is isometric to g_0.

The aim of the present paper is to establish a version of the estimate (2) for the first positive eigenvalue of the sub-Laplacian on the CR sphere $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$. Indeed, let

$$\theta_0 = \frac{i}{2}\sum_{j=1}^{n+1}\left(\zeta_j\, d\bar{\zeta}_j - \bar{\zeta}_j\, d\zeta_j\right)$$

be the standard contact form on $\mathbb{S}^{2n+1}$ whose kernel coincides with the Levi distribution $H(\mathbb{S}^{2n+1}) = T\mathbb{S}^{2n+1} \cap JT\mathbb{S}^{2n+1}$, where J is the complex structure of $\mathbb{C}^{n+1}$. The set $\mathcal{P}_+(\mathbb{S}^{2n+1}) = \{f\theta_0\ ;\ f \in C^\infty(\mathbb{S}^{2n+1}) \text{ and } f > 0\}$ contains all pseudo-Hermitian structures on $\mathbb{S}^{2n+1}$ whose Levi form is positive definite. Given a pseudo-Hermitian structure $\theta \in \mathcal{P}_+(\mathbb{S}^{2n+1})$, we denote by $\lambda_1(\theta)$ the first positive eigenvalue of the corresponding sub-Laplacian Δ_θ, and by $V(\theta)$ the volume of $\mathbb{S}^{2n+1}$ with respect to the volume form $\psi_\theta = \frac{1}{n!2^n}\,\theta \wedge (d\theta)^n$ (see the next section for precise definitions). The main result of this paper is

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Theorem 1.1. *For every pseudo-Hermitian structure* $\theta \in \mathcal{P}_+(\mathbb{S}^{2n+1})$ *we have*

$$\lambda_1(\theta)V(\theta)^{\frac{1}{n+1}} \leq \lambda_1(\theta_0)V(\theta_0)^{\frac{1}{n+1}}. \tag{3}$$

The equality holds in (3) *if and only if there exists a CR-automorphism* γ *of* $\mathbb{S}^{2n+1}$ *such that* $\theta = c\,\gamma^*\theta_0$ *for some constant* $c > 0$, *or if and only if there exist* $p \in \mathbb{S}^{2n+1}$ *and* $t \geq 0$ *such that*

$$\theta = \frac{c}{|\cosh t + \sinh t\,(\zeta, p)|^2}\,\theta_0,$$

where $(\ ,\ )$ *denotes the standard Hermitian product of* $\mathbb{C}^{n+1}$.

This result can be seen as a contribution to the program aiming to recovering the main results of spectral geometry, established for the eigenvalues of the Laplace-Beltrami operator on a compact Riemannian manifold, in the realm of CR and pseudo-Hermitian geometry. This program has motivated a lot of research in recent years and we can find significant contributions in [1, 2, 3, 4, 6, 5, 7, 8, 9, 10, 13, 14, 17, 16, 18, 19, 20, 21, 22].

2. Proof of Theorem 1.1

Let M be a connected differentiable manifold of dimension $2n+1 \geq 3$. A CR structure on M is a couple $(H(M), J)$ where $H(M)$ is a $2n$-dimensional subbundle of the tangent bundle TM, the so-called Levi distribution, endowed with a pseudo-complex operator J satisfying the following integrability condition : $\forall X, Y \in \Gamma(H(M))$,

$$[X,Y] - [JX, JY] \ \in \ \Gamma(H(M))$$

and

$$J\left([X,Y] - [JX,JY]\right) = [JX,Y] + [X,JY].$$

Real hypersurfaces of $\mathbb{C}^{n+1}$ are the most natural examples of CR manifolds. Indeed, if $M \subset \mathbb{C}^{n+1}$ is such a hypersurface, then $H(M) := TM \cap J(TM)$ endowed with the restriction of the standard complex structure J of $\mathbb{C}^{n+1}$, is a CR structure on M.

If $(M, H(M), J)$ is an orientable CR manifold, then there exists a nontrivial 1-form $\theta \in \Gamma(T^*M)$ such that $Ker\theta = H(M)$. Such a 1-form, called pseudo-Hermitian structure, is of course not unique. The set of pseudo-Hermitian structures consists in all the forms $f\theta$, where f is a smooth nowhere zero function on M. To each pseudo-Hermitian structure θ we associate its Levi form L_θ defined on $H(M)$ by

$$L_\theta(X,Y) = -d\theta(JX,Y) = \theta([JX,Y]).$$

The integrability of J implies that L_θ is symmetric and J-invariant. The CR manifold M is called strictly pseudoconvex if L_θ is definite. Of course, this condition does not depend on the choice of θ (since $L_{f\theta} = fL_\theta$). In the sequel, we denote by $\mathcal{P}_+(M)$ the set of all pseudo-Hermitian structures with positive definite Levi form. Every $\theta \in \mathcal{P}_+(M)$ is in fact a contact form which induces on M the following volume form

$$\psi_\theta = \frac{1}{2^n\ n!}\,\theta \wedge (d\theta)^n.$$

The associated divergence div_θ is defined, for every smooth vector field Z on M, by

$$\mathcal{L}_Z\psi_\theta = \mathrm{div}_\theta(Z)\,\psi_\theta.$$

The sub-Laplacian Δ_θ is then defined for all $u \in C^\infty(M)$, by

$$\Delta_\theta u = \mathrm{div}_\theta(\nabla^H u)$$

where $\nabla^H u \in \Gamma(H(M))$ is the horizontal vectorfield such that, $\forall X \in H(M)$, $du(X) = L_\theta(\nabla^H u, X)$. The following integration by parts formula holds for any $u, v \in C_0^\infty(M)$:

$$\int_M (\Delta_\theta u)\, v\, \psi_\theta = -\int_M L_\theta(\nabla^H u, \nabla^H v)\, \psi_\theta.$$

Given $\theta \in \mathcal{P}_+(M)$, there is a unique vectorfield ξ, often called Reeb vectorfield, that satisfies $\theta(\xi) = 1$ and $\xi \rfloor d\theta = 0$. The Levi form L_θ extends to a Riemannian metric on M (the Webster metric) given by

$$g_\theta(X,Y) = L_\theta(X^H, Y^H) + \theta(X)\theta(Y)$$

with $X^H = X - \theta(X)\xi$. The corresponding Laplace-Beltrami operator Δ_{g_θ} is related to Δ_θ by the following formula (see [13])

$$\Delta_\theta = \Delta_{g_\theta} - \xi^2. \tag{4}$$

The sub-Laplacian Δ_θ is a sub-elliptic operator of order 1/2. When M is compact, it admits a self-adjoint extension to an unbounded operator of $L^2(M, \psi_\theta)$ whose resolvent is compact (see for instance [2, Lemma 2.2]). Hence, the spectrum of $-\Delta_\theta$ is discrete and consists of a sequence of nonnegative eigenvalues of finite multiplicity $\{\lambda_k(\theta)\}_{k\geq 0}$ with $\lambda_0(\theta) = 0$. The min-max variational principle gives

$$\lambda_1(\theta) = \inf_{\int_M u\,\psi_\theta = 0} \frac{\int_M |\nabla^H u|_\theta^2\, \psi_\theta}{\int_M u^2\, \psi_\theta} \tag{5}$$

where $|\nabla^H u|_\theta^2 = L_\theta(\nabla^H u, \nabla^H u)$.

2.1. **The CR Sphere.** Let $\mathbb{S}^{2n+1}$ be the unit Sphere in $\mathbb{C}^{n+1}$

$$\mathbb{S}^{2n+1} = \left\{ \zeta = (\zeta_1, ..., \zeta_{n+1}) \in \mathbb{C}^{n+1}\,;\ \sum_{j\leq n+1} |\zeta_j|^2 = 1 \right\}$$

endowed with its standard CR-structure. The restriction to $\mathbb{S}^{2n+1}$ of the contact form

$$\theta_0 = -\frac{i}{2} \sum_{j=1}^{n+1} \left(\bar{\zeta}_j\, d\zeta_j - \zeta_j\, d\bar{\zeta}_j \right)$$

is a pseudohermitian structure whose Reeb field $\xi = i \sum_{j=1}^{n+1} \left(\zeta_j \frac{\partial}{\partial \zeta_j} - \bar{\zeta}_j \frac{\partial}{\partial \bar{\zeta}_j} \right)$ generates the natural action of $\mathbb{S}^1$ on $\mathbb{S}^{2n+1}$. Since $d\theta_0$ is the standard Kähler form of $\mathbb{C}^{n+1}$, the induced Levi form on $H(\mathbb{S}^{2n+1})$ coincides with the standard metric of the sphere.

If $V^{p,q}$ is the space of harmonic polynomials of bi-degree (p,q) in $\mathbb{C}^{n+1}$, then, ξ acts on $V^{p,q}$ as the multiplication by $i(p-q)$ and it can be deduced, using (4), that $V^{p,q}$ is an eigenspace of Δ_{θ_0} on $\mathbb{S}^{2n+1}$ with eigenvalue $2n(p+q) + 4pq$ (see [10, Theorem 4.1] and [22, Proposition 4.4] for details). Therefore,

$$\lambda_1(\theta_0) = 2n. \tag{6}$$

2.2. **One-parameter groups of CR-automorphisms of the sphere.** A differentiable map $\varphi : M \to \widetilde{M}$ between two CR manifolds is a CR map if for any $x \in M$,

$$d_x\varphi(H_x(M)) \subset H_{\varphi(x)}(\widetilde{M}) \qquad \text{and} \qquad d_x\varphi \circ J_x^M = J_{\varphi(x)}^{\widetilde{M}} \circ d_x\varphi. \tag{7}$$

A CR-automorphism of a CR manifold M is a diffeomorphism of M which is a CR map.

Let $e_{n+1} = (0, \cdots, 0, 1) \in \mathbb{S}^{2n+1}$. The punctured sphere $\mathbb{S}^{2n+1} \setminus \{e_{n+1}\}$ can be identified with the boundary of the so-called Siegel domain $\Omega_{n+1} = \{(z, w) \in \mathbb{C}^n \times \mathbb{C} \ : \ \operatorname{Im} w > |z|^2\} \subset \mathbb{C}^{n+1}$ through the CR diffeomorphism $\Phi : \mathbb{S}^{2n+1} \setminus \{e_{n+1}\} \to \partial\Omega_{n+1}$ given by

$$\Phi(\zeta) = \frac{1}{1-\zeta_{n+1}}\left(\zeta_1, \cdots, \zeta_n, i(1+\zeta_{n+1})\right)$$

with

$$\Phi^{-1}(z, w) = \frac{1}{w+i}(2iz_1, \cdots, 2iz_n, w-i)$$

For every $t \in \mathbb{R}$, the "dilation"

$$\begin{array}{rccc} H_t & : & \partial\Omega_{n+1} & \to & \partial\Omega_{n+1} \\ & & (z,w) & \mapsto & (e^t z, e^{2t} w) \end{array}$$

is a CR-automorphism of $\partial\Omega_{n+1}$. We define $\gamma_t : \mathbb{S}^{2n+1} \to \mathbb{S}^{2n+1}$ by $\gamma_t(e_{n+1}) = e_{n+1}$ and, $\forall \zeta \in \mathbb{S}^{2n+1} \setminus \{e_{n+1}\}$,

$$\gamma_t(\zeta) = \Phi^{-1} \circ H_t \circ \Phi(\zeta) = \frac{1}{\cosh t + \sinh t\, \zeta_{n+1}}\left(\zeta_1, \cdots, \zeta_n, \sinh t + \cosh t\ \zeta_{n+1}\right)$$

or

$$\gamma_t(\zeta) = \frac{1}{\cosh t + \sinh t\, \zeta_{n+1}}\left(\zeta + (\sinh t + (\cosh t - 1)\ \zeta_{n+1})\, e_{n+1}\right).$$

Lemma 2.1. *For every t, the map γ_t is a CR-automorphism of $\mathbb{S}^{2n+1}$ which satisfies*

$$(\gamma_t)^*\theta_0 = \frac{1}{|\cosh t + \sinh t\, \zeta_{n+1}|^2}\, \theta_0.$$

Proof. Let $f_j(\zeta) = \frac{\zeta_j}{\cosh t + \sinh t\, \zeta_{n+1}}$, $j \leq n$, and $f_{n+1}(\zeta) = \frac{\sinh t + \cosh t\, \zeta_{n+1}}{\cosh t + \sinh t\, \zeta_{n+1}}$. Then

$$df_j = \frac{d\zeta_j}{\cosh t + \sinh t\, \zeta_{n+1}} - \frac{\sinh t\, \zeta_j\, d\zeta_{n+1}}{(\cosh t + \sinh t\, \zeta_{n+1})^2} \quad \text{and} \quad df_{n+1} = \frac{d\zeta_{n+1}}{(\cosh t + \sinh t\, \zeta_{n+1})^2}.$$

Therefore

$$f_j\, d\bar{f}_j = \frac{\zeta_j\, d\bar{\zeta}_j}{|\cosh t + \sinh t\, \zeta_{n+1}|^2} - \frac{|\zeta_j|^2 \sinh t\, d\bar{\zeta}_{n+1}}{|\cosh t + \sinh t\, \zeta_{n+1}|^2\, (\cosh t + \sinh t\, \bar{\zeta}_{n+1})},$$

$$\bar{f}_j\, df_j = \frac{\bar{\zeta}_j\, d\zeta_j}{|\cosh t + \sinh t\, \zeta_{n+1}|^2} - \frac{|\zeta_j|^2 \sinh t\, d\zeta_{n+1}}{|\cosh t + \sinh t\, \zeta_{n+1}|^2\, (\cosh t + \sinh t\, \zeta_{n+1})}$$

which gives with $\sum_{j=1}^n |\zeta_j|^2 = 1 - |\zeta_{n+1}|^2$,

$$\sum_{j=1}^n \left(f_j d\bar{f}_j - \bar{f}_j df_j\right) = \frac{1}{|\cosh t + \sinh t\, \zeta_{n+1}|^2} \sum_{j=1}^n \left(\zeta_j\, d\bar{\zeta}_j - \bar{\zeta}_j\, d\zeta_j\right) +$$

$$\frac{\left(1-|\zeta_{n+1}|^2\right)\sinh t}{|\cosh t+\sinh t\,\zeta_{n+1}|^2}\left(\frac{d\zeta_{n+1}}{\cosh t+\sinh t\,\zeta_{n+1}}-\frac{d\bar\zeta_{n+1}}{\cosh t+\sinh t\,\bar\zeta_{n+1}}\right).$$

On the other hand,

$$f_{n+1}\,d\bar f_{n+1}-\bar f_{n+1}\,df_{n+1}=$$
$$\frac{1}{|\cosh t+\sinh t\,\zeta_{n+1}|^2}\left(\frac{\sinh t+\cosh t\;\zeta_{n+1}}{\cosh t+\sinh t\,\bar\zeta_{n+1}}\;d\bar\zeta_{n+1}-\frac{(\sinh t+\cosh t\,\bar\zeta_{n+1})}{\cosh t+\sinh t\,\zeta_{n+1}}\,d\zeta_{n+1}\right)$$

Now (with $|\zeta_{n+1}|^2=\zeta_{n+1}\bar\zeta_{n+1}$),

$$\frac{\left(1-|\zeta_{n+1}|^2\right)\sinh t-\left(\sinh t+\cosh t\,\bar\zeta_{n+1}\right)}{\cosh t+\sinh t\,\zeta_{n+1}}=-\bar\zeta_{n+1}$$

and

$$\frac{-\left(1-|\zeta_{n+1}|^2\right)\sinh t+(\sinh t+\cosh t\,\zeta_{n+1})}{\cosh t+\sinh t\,\bar\zeta_{n+1}}=\zeta_{n+1}$$

Thus,

$$\sum_{j=1}^{n+1}\left(f_j d\bar f_j-\bar f_j df_j\right)=\frac{1}{|\cosh t+\sinh t\,\zeta_{n+1}|^2}\sum_{j=1}^{n+1}\left(\zeta_j\,d\bar\zeta_j-\bar\zeta_j\,d\zeta_j\right)$$

that is,

$$(\gamma_t)^*\theta_0=\frac{i}{2}\sum_{j=1}^{n+1}\left(f_j d\bar f_j-\bar f_j df_j\right)=\frac{1}{|\cosh t+\sinh t\,\zeta_{n+1}|^2}\,\theta_0.$$

□

Let $p\in\mathbb{S}^{2n+1}$ be any point of the sphere and let $\alpha_p\in U(n+1)$ be such that $\alpha_p(p)=e_{n+1}$. The family $\gamma_t^p=\alpha_p^{-1}\circ\gamma_t\circ\alpha_p$ is a 1-parameter group of CR-automorphisms of the sphere $\mathbb{S}^{2n+1}$ with

$$\gamma_t^p(\zeta)=\frac{1}{\cosh t+\sinh t\,(\zeta,p)}\left\{\zeta+(\sinh t+(\cosh t-1)\,(\zeta,p))\,p\right\}\tag{8}$$

and

$$(\gamma_t^p)^*\theta_0=\frac{1}{|\cosh t+\sinh t\,(\zeta,p)|^2}\,\theta_0.\tag{9}$$

2.3. **Preparatory lemmas.**

Lemma 2.2. *Let M be a strictly pseudoconvex CR manifold of dimension $2n+1$ and let θ, $\hat\theta\in\mathcal{P}_+(M)$ be two pseudo-Hermitian structures with $\hat\theta=f\,\theta$, $f\in C^\infty(M)$. Then*

$$\psi_{\hat\theta}=f^{n+1}\psi_\theta\tag{10}$$

Proof. From $d\hat\theta=f\,d\theta+df\wedge\theta$ we deduce, by induction,

$$(d\hat\theta)^n=f^n(d\theta)^n+\alpha_n\wedge\theta$$

where α_n is a differential form of degree $2n-1$. Thus,

$$\hat\theta\wedge(d\hat\theta)^n=f\theta\wedge(f^n(d\theta)^n+\alpha_n\wedge\theta)=f^{n+1}\theta\wedge(d\theta)^n.$$

□

Lemma 2.3. *Let M be a strictly pseudoconvex CR manifold of dimension $2m+1$ and let $\phi : M \to (\mathbb{S}^{2n+1}, \theta_0)$ be a CR map. Then, for every $\theta \in \mathcal{P}_+(M)$,*

$$\phi^*\theta_0 = \frac{1}{2m}\left(\sum_{i=1}^{2n+2} \left|\nabla^H \phi_i\right|_\theta^2\right)\theta \tag{11}$$

where $\phi_1, \dots, \phi_{2n+2}$ are the Euclidean components of ϕ.

Proof. Since ϕ is a CR map, the 1-form $\phi^*\theta_0$ vanishes on $H(M)$ which implies that there exists $f \in C^\infty(M)$ such that

$$\phi^*\theta_0 = f\theta. \tag{12}$$

Differentiating, we get

$$\phi^* d\theta_0 = df \wedge \theta + f d\theta.$$

Hence, for every $X, Y \in H_x(M)$, one has $\phi^* d\theta_0(X, Y) = f d\theta(X, Y)$ and, using (7),

$$\begin{aligned} L_{\theta_0}(d\phi(X), d\phi(X)) &= d\theta_0(d\phi(X), J^{\mathbb{S}^{2n+1}} d\phi(X)) = d\theta_0(d\phi(X), d\phi(J^M X)) \\ &= \phi^* d\theta_0(X, J^M X) = f d\theta(X, J^M X) = f L_\theta(X, X). \end{aligned}$$

On the other hand, since L_{θ_0} coincides with the standard inner product on $H_{\phi(x)}(\mathbb{S}^{2n+1})$,

$$L_{\theta_0}(d\phi(X), d\phi(X)) = \sum_{i=1}^{2n+2} (d\phi_i(X))^2 = \sum_{i=1}^{2n+2} L_\theta(\nabla^H \phi_i, X)^2.$$

Thus,

$$f L_\theta(X, X) = \sum_{i=1}^{2n+2} L_\theta(\nabla^H \phi_i, X)^2.$$

Taking an L_θ-orthonormal basis $\{e_1, \dots, e_{2m}\}$ of $H_x(M)$, we get

$$2mf = \sum_{j=1}^{2m}\sum_{i=1}^{2n+2} L_\theta(\nabla^H \phi_i, e_j)^2 = \sum_{i=1}^{2n+2} L_\theta(\nabla^H \phi_i, \nabla^H \phi_i) = \sum_{i=1}^{2n+2} \left|\nabla^H \phi_i\right|_\theta^2$$

which implies (11), thanks to (12). □

If $\phi : M \to \mathbb{R}^N$ is a map and μ is a measure on M, we denote by $\int_M \phi\, d\mu$ the vector $\left(\int_M \phi_1 d\mu, \dots, \int_M \phi_N d\mu\right) \in \mathbb{R}^N$, where $\phi = (\phi_1, \cdots, \phi_N)$.

Lemma 2.4. *Let M be a compact manifold and let μ be a measure on M such that no open set has measure zero. If $\phi : M \to \mathbb{S}^{2n+1}$ is a non constant continuous map, then there exists a pair $(p, t) \in \mathbb{S}^{2n+1} \times [0, +\infty)$ such that*

$$\int_M \gamma_t^p \circ \phi\, d\mu = 0.$$

Proof. The proof uses standard arguments (see [11, 15]). We consider the map

$$\begin{array}{rcccl} F & : & \mathbb{S}^{2n+1} \times (0, +\infty) & \to & \mathbb{B}^{2n+2} \subset \mathbb{R}^{2n+2} \\ & & (p, t) & \mapsto & \frac{1}{V}\int_M \gamma_t^p \circ \phi\, d\mu \end{array}$$

where $V = \int_M d\mu$ and $\mathbb{B}^{2n+2}$ is the unit Euclidean ball. Observe that (see (8)), $\forall p \in \mathbb{S}^{2n+1}$, γ_0^p is the identity map while, $\forall \zeta \neq -p$, $\gamma_t^p(\zeta)$ tends to p as $t \to +\infty$. Consequently, $F(\cdot, 0) = \frac{1}{V}\int_M \varphi\, d\mu$ is a constant map and $F(\cdot, t)$ tends to the identity of $\mathbb{S}^{2n+1}$ as

$t \to +\infty$. Such a map F is necessarily onto which implies that the origin of $\mathbb{R}^{2n+2}$ belongs to its image. □

2.4. **Proof Theorem 1.1.** Let $\theta \in \mathcal{P}_+(\mathbb{S}^{2n+1})$ be a strictly pseudoconvex pseudo-Hermitian structure. We apply Lemma 2.4 to the identity map of $\mathbb{S}^{2n+1}$ and the measure induced by ψ_θ to obtain the existence of a pair $(p,t) \in \mathbb{S}^{2n+1} \times [0,+\infty)$ such that

$$\int_{\mathbb{S}^{2n+1}} \gamma_t^p \, \psi_\theta = 0.$$

For simplicity, we write γ for γ_t^p. The Euclidean components $\gamma_1, \dots \gamma_{2n+2}$ of γ satisfy $\int_{\mathbb{S}^{2n+1}} \gamma_j \, \psi_\theta = 0$. Hence, applying the min-max principle (5), we get for every $j \le 2n+2$,

$$\lambda_1(\theta) \int_{\mathbb{S}^{2n+1}} \gamma_j^2 \, \psi_\theta \le \int_{\mathbb{S}^{2n+1}} \left|\nabla^H \gamma_j\right|_\theta^2 \psi_\theta.$$

Summing up we obtain, with $\sum_{j \le 2n+2} \gamma_j^2 = 1$,

$$\lambda_1(\theta) V(\theta) \le \sum_{j \le 2n+2} \int_{\mathbb{S}^{2n+1}} \left|\nabla^H \gamma_j\right|_\theta^2 \psi_\theta \tag{13}$$

$$\le 2n \left(\int_{\mathbb{S}^{2n+1}} \left(\frac{1}{2n} \left|\nabla^H \gamma_j\right|_\theta^2 \right)^{n+1} \psi_\theta \right)^{\frac{1}{n+1}} V(\theta)^{1-\frac{1}{n+1}}. \tag{14}$$

Using Lemma 2.3, we see that, since γ is a CR map from $(\mathbb{S}^{2n+1}, \theta)$ to $(\mathbb{S}^{2n+1}, \theta_0)$,

$$\gamma^* \theta_0 = \left(\frac{1}{2n} \sum_{j \le 2n+2} \left|\nabla^H \gamma_j\right|_\theta^2 \right) \theta \tag{15}$$

which gives, thanks to Lemma 2.2

$$\psi_{\gamma^* \theta_0} = \left(\frac{1}{2n} \sum_{j \le 2n+2} \left|\nabla^H \gamma_j\right|_\theta^2 \right)^{n+1} \psi_\theta.$$

Thus

$$\lambda_1(\theta) V(\theta) \le 2n V(\gamma^* \theta_0)^{\frac{1}{n+1}} V(\theta)^{1-\frac{1}{n+1}}.$$

Since $V(\gamma^* \theta_0) = V(\theta_0)$, we finally get

$$\lambda_1(\theta) V(\theta)^{\frac{1}{n+1}} \le 2n V(\theta_0)^{\frac{1}{n+1}}$$

which proves the inequality of the theorem thanks to (6).

Now, if the equality holds in (3), this means that we have equality in the Cauchy-Schwarz inequality used in (14). Thus, $\sum_{j \le 2n+2} \left|\nabla^H \gamma_j\right|_\theta^2$ is constant and, thanks to (15), θ is proportional to $\gamma^* \theta_0$ which takes the form given by (9).

Conversely, it is clear that when γ is a CR-automorphism of the sphere then $\lambda_1(\gamma^* \theta_0) = \lambda_1(\theta_0) = 2n$ and $V(\gamma^* \theta_0) = V(\theta_0)$. Hence, the equality holds in (3).

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