Solutions of half-linear differential equations in the classes Gamma and Pi

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Abstract

Abstract We study asymptotic behavior of (all) positive solutions of the nonoscillatory half-linear differential equation of the form $(r(t)|y'|^{\alpha-1}\operatorname{sgn} y')' = p(t)|y|^{\alpha-1}\operatorname{sgn} y$, where $\alpha \in (1,\infty)$ and r, p are positive continuous functions on $[a,\infty)$, with the help of the Karamata theory of regularly varying functions and the de Haan theory. We show that increasing resp. decreasing solutions belong to the de Haan class Γ resp. Γ_- under suitable assumptions. Further we study behavior of slowly varying solutions for which asymptotic formulas are established. Some of our results are new even in the linear case $\alpha = 2$.

Keywords: half-linear differential equation; positive solution; asymptotic formula; regular variation; class Gamma; class Pi

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1 Introduction

Consider the half-linear equation

$$(r(t)\Phi(y'))' = p(t)\Phi(y), \tag{1}$$

where r, p are positive continuous functions on $[a,\infty)$ and $\Phi(u) = |u|^{\alpha-1}\operatorname{sgn} u$ with $\alpha > 1$. Denote by β the conjugate number of α, i.e. $\beta = \frac{\alpha}{\alpha-1}$.

Equation (1) is nonoscillatory and all its nontrivial solution are eventually monotone (see [7]). Our aim is to describe asymptotic behavior of (all) positive increasing and decreasing solutions to (1) with the help of the Karamata theory of regularly varying functions and the de Haan theory.

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We establish first conditions guaranteeing that all positive increasing solutions (which tend to ∞) resp. positive decreasing solutions (which tend to zero) of (1) belong to the de Haan class Γ resp. Γ_- with auxiliary functions expressed in terms of the coefficients; these classes form proper subsets of rapidly varying functions. The ideas of the proofs of these statements will be utilized to discuss also regular variation of solutions to (1).

The second part of the paper is devoted to solutions of (1) in the de Haan class Π which forms a proper subset of slowly varying functions. We will distinguish two cases with respect to a certain behavior of the coefficients, where in one case, all positive decreasing solutions are shown to be in Π and to satisfy certain asymptotic formulas, while in the other case such kind of results is established for increasing solutions.

Theory of regular variation has been shown very useful in the study of asymptotic properties of differential equations, see in particular the monograph [16] which summarizes the research up to 2000. Half-linear differential equations in the framework of regular variation have been considered in [13, 14, 15, 22, 23], see also [24, Chapter 3] and [7, Subsection 4.3].

Our results can be understood in various ways. We give complementary information to a standard asymptotic classification of nonoscillatory solutions (see e.g. [2, 3, 5, 17], [7, Chapter 4]), and to some results on behavior of solutions to (1) made in the framework of regular variation ([13, 14, 23, 24]). Further, our theory can be seen as an extension of results for the linear equation $y'' = p(t)y$ (see [8, 9, 16, 18, 19, 21, 24]). We however append several observations and further extensions which are new even in the linear case.

The paper is organized as follows. In the next section we recall several useful facts about half-linear differential equations, regularly varying functions, and de Haan classes. Section 3 consists of three subsections. In the first resp. the second one we establish conditions which guarantee that increasing resp. decreasing solutions of (1) are in Γ resp. Γ_-. In both cases we give examples and simplified versions of sufficient conditions. Subsection 3.3 discusses related results which are either by-products of the proofs (as a Hartman-Wintner type result) or use modified ideas of the proofs (sufficient conditions for regular variation of solutions). Section 4 consists also of three subsections. The first two are logically distinguished with respect to the behavior of the coefficients. In Subsection 4.1 we examine decreasing slowly varying solutions (which are in Π) and derive asymptotic formulas. The setting of the second subsection requires to seek for slowly varying solutions among increasing solutions and, again, asymptotic formulas for such solutions are established. In both cases we give examples and additional observations, see Subsection 4.3. The paper is concluded with a short section which indicates some directions for a future research.

2 Preliminaries

It is well known (see [7, Chapter 4]) that (1) with positive r, p is nonoscillatory, i.e. all its solutions are eventually of constant sign. Thus, without loss of generality, we work just with positive solutions, i.e. with the class

$$\mathbb{M} = \{y : y(t) \text{ is a positive solution of (1) for large } t\}.$$

Basic classification of nonoscillatory solutions and existence results can be found in [2, 3, 5, 17]. For a survey see [7, Chapter 4]). Because of the sign conditions on the coefficients, all positive solutions of (1) are eventually monotone, therefore they belong to one of the following disjoint classes:

$$\begin{aligned}\mathbb{M}^+ &= \{y \in \mathbb{M} : y'(t) > 0 \text{ for large } t\},\\ \mathbb{M}^- &= \{y \in \mathbb{M} : y'(t) < 0 \text{ for large } t\}.\end{aligned}$$

It can be shown that both these classes are nonempty (see ([7, Lemma 4.1.2]). The classes $\mathbb{M}^+, \mathbb{M}^-$ can be further divided into four mutually disjoint subclasses:

$$\begin{aligned}\mathbb{M}^+_\infty &= \{y \in \mathbb{M}^+ : \lim_{t\to\infty} y(t) = \infty\},\\ \mathbb{M}^+_B &= \{y \in \mathbb{M}^+ : \lim_{t\to\infty} y(t) = l \in \mathbb{R}\},\\ \mathbb{M}^-_B &= \{y \in \mathbb{M}^- : \lim_{t\to\infty} y(t) = l > 0\},\\ \mathbb{M}^-_0 &= \{y \in \mathbb{M}^- : \lim_{t\to\infty} y(t) = 0\}.\end{aligned}$$

We set

$$J_1 = \lim_{T\to\infty} \int_a^T r(t)^{1-\beta} \left(\int_a^t p(s) ds \right)^{\beta-1} dt$$

and

$$J_2 = \lim_{T\to\infty} \int_a^T r(t)^{1-\beta} \left(\int_t^T p(s) ds \right)^{\beta-1} dt.$$

The convergence or divergence of the above integrals fully characterize the above subclasses. In particular, according to [7, Theorem 4.1.4], $\mathbb{M}^+ = \mathbb{M}^+_\infty$ if and only if $J_1 = \infty$, while $\mathbb{M}^+ = \mathbb{M}^+_B$ if and only if $J_1 < \infty$. Moreover, $\mathbb{M}^- = \mathbb{M}^-_B$ if and only if $J_1 = \infty$ and $J_2 < \infty$, while $\mathbb{M}^- = \mathbb{M}^-_0$ if and only if $J_2 = \infty$. Finally, if $J_1 < \infty$ and $J_2 < \infty$, then $\mathbb{M}^-_0 \neq \emptyset \neq \mathbb{M}^-_B$.

Let $y \in \mathbb{M}$ and take $f \in C^1$ with $f(t) \neq 0$ for every t. Denoted $w = fr\Phi(\frac{y'}{y})$, it satisfies the generalized Riccati equation

$$w' - \frac{f'}{f} w - fp + (\alpha - 1) \frac{r^{1-\beta}}{\Phi^{-1}(f)} |w|^\beta = 0, \tag{2}$$

where Φ^{-1} stands for the inverse of Φ, i.e., $\Phi^{-1}(u) = |u|^{\beta-1} \operatorname{sgn} u$. If $f(t) \equiv 1$, then (2) reduces to the usual generalized Riccati equation

$$w' - p(t) + (\alpha - 1) r^{1-\beta}(t) |w|^\beta = 0. \tag{3}$$

Dividing (2) by fp, we get

$$\frac{w'}{fp} = \frac{f'w}{f^2 p} + 1 - (\alpha - 1) \frac{r^{1-\beta}|w|^\beta}{|f|^\beta p}. \tag{4}$$

A solution y of (1) is said to be principal solution if for every solution u of (1) such that $u \neq \lambda y$, $\lambda \in \mathbb{R}$, it holds $y'(t)/y(t) < x'(t)/x(t)$ for large t, see [7, Section 4.2]. According to [4, Corollary 1], the set of (eventually positive) principal solutions is either $\mathbb{M}^-_B$ if $J_1 = \infty$ and $J_2 < \infty$ or $\mathbb{M}^-_0$ otherwise. A solution of the associated generalized Riccati equation which is generated by a principal solution is called an eventually minimal solution. According to [7, Theorem 4.2.2.], if $P(t) \leq p(t)$ for large t, then the eventually minimal solutions $w = r\Phi(y'/y)$ and $z = r\Phi(x'/x)$ of the generalized Riccati equations respectively associated to (1) and $(r(t)\Phi(x'))' = P(t)\Phi(x)$ satisfy

$$w(t) \le z(t) \tag{5}$$

for large t.

In the second part of this section we recall some basic informations on the Karamata theory of regularly varying functions and the de Haan theory; for a deeper study of this topic see the monographs [1, 9]. Given $\delta \in \mathbb{R} \cup \{\pm\infty\}$ and a measurable function $f : [a, \infty) \to (0, \infty)$ such that

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\delta$$

for every $\lambda > 0$, we say that

- f is *regularly varying of index* δ, we write $f \in RV(\delta)$, if $\delta \in \mathbb{R}\backslash\{0\}$;
- f is *slowly varying*, we write $f \in SV$, if $\delta = 0$;
- f is *rapidly varying*, we write $f \in RPV(\infty)$ resp. $f \in RPV(-\infty)$, if $\delta = \infty$ resp. $\delta = -\infty$.

Here we use the convention $\lambda^\infty = 0$, $\lambda^{-\infty} = \infty$ for $\lambda \in (0, 1)$, and $\lambda^\infty = \infty$, $\lambda^{-\infty} = 0$ for $\lambda > 0$.

It follows that $f \in RV(\delta)$ if and only if there exists a function $L \in SV$ such that $f(t) = t^\delta L(t)$ for every t. The slowly varying component of $f \in RV(\delta)$ will be denoted by L_f, i.e., $L_f(t) = t^{-\delta} f(t)$.

The Representation theorem (see e.g. [1]) says that $f \in RV(\delta)$ if and only if

$$f(t) = \varphi(t) t^\delta \exp\left\{ \int_a^t \frac{\psi(s)}{s} ds \right\}, \tag{6}$$

$t \ge a$, for some $a > 0$, where φ, ψ are measurable with $\lim_{t\to\infty} \varphi(t) = C \in (0, \infty)$ and $\lim_{t\to\infty} \psi(t) = 0$. A regularly varying function f is said to be *normalized regularly varying*, we write $f \in NRV(\delta)$, if $\varphi(t) \equiv C$ in (6). If (6) holds with $\delta = 0$ and $\varphi(t) \equiv C$, we say that f is *normalized slowly varying*, we write $f \in NSV$. Clearly, if f is a C^1 function and $\lim_{t\to\infty} \frac{tf'(t)}{f(t)} = \delta$, then $f \in NRV(\delta)$.

The following Karamata theorem (direct-half) will be very useful in the sequel. As usual, $f(t) \sim g(t)$ (as $t \to \infty$) means $\lim_{t\to\infty} f(t)/g(t) = 1$.

Theorem 1. *([1, 9]) If $L \in SV$ then*

$$\int_t^\infty s^\delta L(s) ds \sim \frac{1}{-\delta - 1} t^{\delta+1} L(t) \quad \text{if } \delta < -1,$$
$$\int_a^t s^\delta L(s) ds \sim \frac{1}{\delta + 1} t^{\delta+1} L(t) \quad \text{if } \delta > -1$$

as $t \to \infty$. Moreover, if $\int_a^\infty L(s)/s\, ds$ converges, then $\tilde{L}(t) = \int_t^\infty L(s)/s\, ds$ is a SV function; if $\int_a^\infty L(s)/s\, ds$ diverges, then $\tilde{L}(t) = \int_a^t L(s)/s\, ds$ is a SV function; in both cases, $L(t)/\tilde{L}(t) \to 0$ as $t \to \infty$.

Here are some simple properties of RV functions which we frequently use: If $L \in SV$ and $\vartheta > 0$, then $t^{\vartheta} L(t) \to \infty$ and $t^{-\vartheta} L(t) \to 0$ as $t \to \infty$. Let $f_i \in RV(\vartheta_i)$, $i = 1, 2$, $\vartheta_1, \vartheta_2, \gamma \in \mathbb{R}$. Then $f_1^{\gamma} \in RV(\gamma\vartheta_1)$, $f_1 + f_2 \in RV(\max\{\vartheta_1, \vartheta_2\})$, $f_1 f_2 \in RV(\vartheta_1 + \vartheta_2)$. Several further properties are spread in the text, at the places where we need them.

We now recall useful subclasses of slowly and rapidly varying functions, which were introduced by de Haan, see [1, 6, 9]. A nondecreasing function $f : \mathbb{R} \to (0, \infty)$ is said to belong to the class Γ if there exists a function $v : \mathbb{R} \to (0, \infty)$ such that for all $\lambda \in \mathbb{R}$

$$\lim_{t\to\infty} \frac{f(t + \lambda v(t))}{f(t)} = e^{\lambda};$$

we write $f \in \Gamma$ or $f \in \Gamma(v)$. The function v is called an *auxiliary function* for f. Further, $f \in \Gamma_-(v)$ if $1/f \in \Gamma(v)$. Sometimes we write Γ as Γ_+.

A measurable function $f : [a, \infty) \to \mathbb{R}$ is said to belong to the class Π if there exists a function $w : (0, \infty) \to (0, \infty)$ such that for $\lambda > 0$

$$\lim_{t\to\infty} \frac{f(\lambda t) - f(t)}{w(t)} = \ln \lambda;$$

we write $f \in \Pi$ or $f \in \Pi(w)$. The function w is called an *auxiliary function* for f.

It is known that functions in class Γ are rapidly varying of index ∞ (see [6, Theorem 1.5.1]). From $\Gamma(v) \subset RPV(\infty)$, it follows that $\Gamma_-(v) \subset RPV(-\infty)$. Moreover, for a function f in class Π, there exists $l = \lim_{t\to\infty} f(t)$ and, provided f is positive, f is slowly varying (see [9, Corollary 1.18]).

A measurable function $f : \mathbb{R} \to (0, \infty)$ is *Beurling slowly varying* if

$$\lim_{t\to\infty} \frac{f(t + \lambda f(t))}{f(t)} = 1 \quad \text{for all } \lambda \in \mathbb{R}; \tag{7}$$

we write $f \in BSV$. If (7) holds locally uniformly in λ, then f is called *self-neglecting*; we write $f \in SN$. It is known that if $f \in BSV$ is continuous, then f is self-neglecting (see [9, Theorem 1.34]). Moreover, $f \in SN$ if and only there exists ϕ integrable such that $\lim_{t\to\infty} \phi(t) = 0$ and $f(t) \sim \int_a^t \phi(s) ds$ as $t \to \infty$ (see [9, Theorem 1.35]). Trivially this implies that $f \in C^1$ is BSV if and only if f' is integrable and $\lim_{t\to\infty} f'(t) = 0$.

3 Solutions in classes Γ and Γ_-

3.1 Increasing solutions in the class Γ

In this section we deal with solutions of (1) belonging to the de Haan class Γ where an auxiliary function is expressed in terms of the coefficients of the equation.

Theorem 2. *Suppose that* $J_1 = \infty$ *and* $\left(\frac{r}{p}\right)^{\frac{1}{\alpha}} \in BSV$*. If there exists a function* $\tilde{f} \in C^1$ *satisfying*

$$\tilde{f}(t) \sim f(t) := \left(\frac{p(t)}{r(t)}\right)^{-\frac{1}{\beta}} \frac{1}{r(t)} \quad \textit{and} \quad \lim_{t\to\infty} \frac{\tilde{f}'(t)}{\tilde{f}^2(t) p(t)} = 0, \tag{8}$$

then $\mathbb{M}^+ \subset \Gamma\left(\left[\frac{r(\alpha-1)}{p}\right]^{\frac{1}{\alpha}}\right)$.

Proof. Take a positive solution $y \in \mathbb{M}^+$. By definition, $\Phi(y) = y^{\alpha-1}$ and $\Phi(y') = (y')^{\alpha-1}$ for t sufficiently large. We now show that $\frac{y'(t)}{y(t)} \sim [\frac{p(t)}{r(t)(\alpha-1)}]^{\frac{1}{\alpha}}$ as $t \to \infty$.

Let us first assume that $f \in C^1$ and that $\lim_{t\to\infty} \frac{f'(t)}{f(t)^2 p(t)} = 0$ and define $w = fr(\frac{y'}{y})^{\alpha-1}$. Then w satisfies the generalized Riccati equation (4) and, since f is positive and $r^{1-\beta} f^{1-\beta} = fp$, we obtain

$$\frac{w'}{fp} = 1 + w\left[\frac{f'}{f^2 p} - (\alpha-1)w^{\beta-1}\right]. \tag{9}$$

We claim that $\lim_{t\to\infty} w(t) = (\alpha-1)^{-\frac{1}{\beta}}$. Consider first the case when $w'(t) > 0$ for t large. Then $\frac{w'}{fp}$ is definitely positive, w is increasing and tends to $l \in (0, \infty]$ when $t \to \infty$. If $l = \infty$, according to (9) and $\frac{f'}{f^2 p} \to 0$, we would have that $\lim_{t\to\infty} \frac{w'(t)}{f(t)p(t)} = -\infty$, which leads to a contradiction. Thus $l < \infty$. If $l \neq (\alpha-1)^{-\frac{1}{\beta}}$, again by (9) and $\frac{f'}{f^2 p} \to 0$, we would get that $\lim_{t\to\infty} \frac{w'(t)}{f(t)p(t)} = 1 - (\alpha-1)l^\beta \neq 0$, yielding the existence of two positive constants M_0, M_1 such that

$$w^{\beta-1}(t) \leq M_0 \tag{10}$$

and

$$\frac{w'(t)}{f(t)p(t)} \geq M_1$$

i.e., recalling that $fp = (\frac{p}{r})^{\frac{1}{\alpha}}$,

$$w(t) \geq w(t_0) + M_1 \int_{t_0}^{t} \left(\frac{p(t)}{r(t)}\right)^{\frac{1}{\alpha}} dt$$

for $t > t_0$ with t_0 sufficiently large. By definition,

$$w^{\beta-1} = f^{\beta-1} r^{\beta-1} \left(\frac{y'}{y}\right)^{(\alpha-1)(\beta-1)} = \left(\frac{p}{r}\right)^{-1+\frac{1}{\beta}} \left(\frac{y'}{y}\right) = \left(\frac{p}{r}\right)^{-\frac{1}{\alpha}} \left(\frac{y'}{y}\right),$$

hence (10) implies that

$$\left(\frac{p(t)}{r(t)}\right)^{\frac{1}{\alpha}} \geq \frac{y'(t)}{M_0 y(t)}$$

for $t > t_0$. Integrating between t_0 and t and recalling that $J_1 = \infty$ implies $y \to \infty$, we then would get

$$w(t) \geq w(t_0) + M_1 \int_{t_0}^{t} \left(\frac{p(t)}{r(t)}\right)^{\frac{1}{\alpha}} dt \geq w(t_0) + \frac{M_1}{M_0} \ln \frac{y(t)}{y(t_0)} \to \infty,$$

a contradiction. Reasoning similarly, we obtain a contradiction also in the case when $w'(t) < 0$ for t sufficiently large. In fact, in this case w is a positive decreasing function and tends to $l \in [0, \infty)$. Finally, suppose the existence of a sequence $\{t_n\}_n$

with $\lim_{n\to\infty} t_n = \infty, \lim_{n\to\infty} w(t_n) = l \in \mathbb{R}\cup\{-\infty,\infty\}$ and $w'(t_n) = 0$ for every n. Then, according to (9), it follows that

$$1 + w(t_n)\left[\frac{f'(t_n)}{f(t_n)^2 p(t_n)} - (\alpha-1)w(t_n)^{\beta-1}\right] = 0$$

for every n and, passing to the limit, we obtain $1-(\alpha-1)l^\beta = 0$, because $\frac{f'(t)}{f^2(t)p(t)} \to 0$ as $t\to\infty$. In conclusion, $\lim_{t\to\infty} w(t) = (\alpha-1)^{-\frac{1}{\beta}}$, i.e. $(\frac{y'(t)}{y(t)})^{\alpha-1} \sim (\frac{p(t)}{r(t)})^{\frac{1}{\beta}}(\alpha-1)^{-\frac{1}{\beta}}$ as $t\to\infty$. Raising by $\beta-1$ we then obtain $\frac{y'(t)}{y(t)} \sim [\frac{p(t)}{r(t)(\alpha-1)}]^{\frac{1}{\alpha}}$.

We return now to the general case. Let $\tilde f \in C^1$ be such that (8) is fulfilled. Take $\tilde p = (r\tilde f)^{-\beta} r$. Then $\tilde p(t) \sim r(t)^{1-\beta}\frac{p(t)}{r(t)}(\frac{1}{r(t)})^{-\beta} = p$, therefore for every $\varepsilon\in(0,1)$ there exists t_ε such that $(1-\varepsilon)\tilde p(t) \le p(t) \le (1+\varepsilon)\tilde p(t)$ for all $t > t_\varepsilon$. Since $\varepsilon\in(0,1)$, according to [7, Lemma 4.1.2], there exist eventually positive increasing solutions v, u, respectively, of problems

$$\begin{cases} (r(t)\Phi(v'))' = (1-\varepsilon)\tilde p(t)\Phi(v) \\ v(t_\varepsilon) = v_0, v'(t_\varepsilon) = v_0', \end{cases} \tag{11}$$

and

$$\begin{cases} (r(t)\Phi(u'))' = (1+\varepsilon)\tilde p(t)\Phi(u) \\ u(t_\varepsilon) = u_0, u'(t_\varepsilon) = u_0' \end{cases} \tag{12}$$

with v_0, v_0', u_0, u_0' positive and such that $(\frac{v_0'}{v_0})^{\alpha-1} \le (\frac{y'(t_\varepsilon)}{y(t_\varepsilon)})^{\alpha-1} \le (\frac{u_0'}{u_0})^{\alpha-1}$. Define $w_v = r(\frac{v'}{v})^{\alpha-1}, w_y = r(\frac{y'}{y})^{\alpha-1}$, and $w_u = r(\frac{u'}{u})^{\alpha-1}$. These functions satisfy, respectively, the generalized Riccati equations

$$w_v' = (1-\varepsilon)\tilde p - (\alpha-1)r^{1-\beta}w_v^\beta,$$

$$w_y' = p - (\alpha-1)r^{1-\beta}w_y^\beta,$$

and

$$w_u' = (1+\varepsilon)\tilde p - (\alpha-1)r^{1-\beta}w_u^\beta.$$

By the theory on differential inequalities (see [11, Chapter III, Section 4]), since $w_v(t_\varepsilon) \le w_y(t_\varepsilon) \le w_u(t_\varepsilon)$ and $(1-\varepsilon)\tilde p(t) \le p(t) \le (1+\varepsilon)\tilde p(t)$ for every $t > t_\varepsilon$, it follows that $w_v(t) \le w_y(t) \le w_u(t)$, yielding

$$\left(\frac{v'(t)}{v(t)}\right)^{\alpha-1}\left(\frac{p(t)}{r(t)}\right)^{-\frac{1}{\beta}} \le \left(\frac{y'(t)}{y(t)}\right)^{\alpha-1}\left(\frac{p(t)}{r(t)}\right)^{-\frac{1}{\beta}} \le \left(\frac{u'(t)}{u(t)}\right)^{\alpha-1}\left(\frac{p(t)}{r(t)}\right)^{-\frac{1}{\beta}} \tag{13}$$

for every $t > t_\varepsilon$. Define $\tilde w_v = \tilde f r(\frac{v'}{v})^{\alpha-1}$. Then

$$\frac{\tilde w_v'}{\tilde f\tilde p} = 1-\varepsilon+\tilde w_v\left[\frac{\tilde f'}{\tilde f^2\tilde p} - (\alpha-1)\tilde w_v^{\beta-1}\right],$$

therefore, reasoning as above, we get that $\lim_{t\to\infty}\tilde w_v(t) = (\frac{1-\varepsilon}{\alpha-1})^{\frac{1}{\beta}}$ which implies

$$\left(\frac{v'(t)}{v(t)}\right)^{\alpha-1}\left(\frac{p(t)}{r(t)}\right)^{-\frac{1}{\beta}} \sim \left(\frac{v'(t)}{v(t)}\right)^{\alpha-1}\left(\frac{\tilde p(t)}{r(t)}\right)^{-\frac{1}{\beta}} \sim \left(\frac{1-\varepsilon}{\alpha-1}\right)^{\frac{1}{\beta}} \tag{14}$$

as $t \to \infty$, because $\tilde{p}(t) \sim p(t)$. From (13) and (14) we obtain that

$$\liminf_{t\to\infty} \left(\frac{y'(t)}{y(t)}\right)^{\alpha-1} \left(\frac{p(t)}{r(t)}\right)^{-\frac{1}{\beta}} \geq \left(\frac{1-\varepsilon}{\alpha-1}\right)^{\frac{1}{\beta}}$$

for every $\varepsilon > 0$, i.e. that

$$\liminf_{t\to\infty} \left(\frac{y'(t)}{y(t)}\right)^{\alpha-1} \left(\frac{p(t)}{r(t)}\right)^{-\frac{1}{\beta}} \geq \left(\frac{1}{\alpha-1}\right)^{\frac{1}{\beta}}.$$

Similarly, from (13) we also get that

$$\limsup_{t\to\infty} \left(\frac{y'(t)}{y(t)}\right)^{\alpha-1} \left(\frac{p(t)}{r(t)}\right)^{-\frac{1}{\beta}} \leq \left(\frac{1}{\alpha-1}\right)^{\frac{1}{\beta}},$$

hence $(\frac{y'(t)}{y(t)})^{\alpha-1} \sim (\frac{p(t)}{r(t)})^{\frac{1}{\beta}}(\frac{1}{\alpha-1})^{\frac{1}{\beta}}$, which implies again $\frac{y'(t)}{y(t)} \sim [\frac{p(t)}{r(t)(\alpha-1)}]^{\frac{1}{\alpha}}$ as $t \to \infty$. Denoted $[\frac{r(\alpha-1)}{p}]^{\frac{1}{\alpha}} = Q$, we have that $\frac{y'(t)}{y(t)} \sim \frac{1}{Q(t)}$, hence for every $\varepsilon \in (0,1)$ there exists t_ε such that $\frac{1-\varepsilon}{Q(t)} \leq \frac{y'(t)}{y(t)} \leq \frac{1+\varepsilon}{Q(t)}$ for $t \geq t_\varepsilon$. For every $\lambda > 0$, integrating by substitution, we obtain that

$$\int_t^{t+\lambda Q(t)} \frac{1}{Q(s)} ds = \int_0^\lambda \frac{Q(t)}{Q(t+\xi Q(t))} d\xi \to \lambda$$

as $t \to \infty$, because $Q \in BSV$ continuous implies $Q \in SN$. Thus

$$(1-\varepsilon)\lambda \leq \liminf_{t\to\infty} \int_t^{t+\lambda Q(t)} \frac{y'(s)}{y(s)} ds \leq \limsup_{t\to\infty} \int_t^{t+\lambda Q(t)} \frac{y'(s)}{y(s)} ds \leq (1+\varepsilon)\lambda.$$

Hence, in view of the arbitrariness of ε, for every $\lambda > 0$,

$$\lambda = \lim_{t\to\infty} \int_t^{t+\lambda Q(t)} \frac{y'(s)}{y(s)} ds = \lim_{t\to\infty} \ln \frac{y(t+\lambda Q(t))}{y(t)}.$$

By definition, $y \in \Gamma([\frac{r(\alpha-1)}{p}]^{\frac{1}{\alpha}})$ and the theorem is proved. □

Remark 1. (i) The previous result was obtained in [23, Theorem 3] in the special case when $r(t) \equiv 1$ and then extended in the same paper to the case $\int_a^\infty r(t)^{1-\beta} dt = \infty$ via a suitable transformation of dependent variable. Using different techniques, we are able to extend the theorem to the case when the integral can also converge. Moreover, we do not need to distinguish whether the integral converges or diverges. Note that, in contrast to the linear case, the transformation of dependent variable (which can transform a 'convergent' case into a 'divergent' one) is not at disposal for equation (1). Recall that the sufficient condition from [23, Theorem 3] reads as $\int_a^\infty r^{1-\beta}(s)\, ds = \infty$ and

$$\left(\frac{p}{r^{1-\beta}}\right)^{-\frac{1}{\alpha}} \circ R^{-1} \in BSV, \quad \left(\frac{p}{r}\right)^{-\frac{1}{\alpha}} \in BSV, \tag{15}$$

where R^{-1} is the inverse of $R(t) = \int_a^t r^{1-\beta}(s)\, ds$. As noted in [23], for $p, r \in C^1$, (15) is guaranteed by

$$\left(\left(\frac{p(t)}{r(t)}\right)^{-\frac{1}{\alpha}}\right)' \to 0 \ \text{ and } \ \left(\frac{p(t)}{r(t)}\right)^{-\frac{1}{\alpha}} \frac{r'(t)}{r(t)} \to 0 \tag{16}$$

as $t \to \infty$. Thanks to identity (20), condition (16) implies also the condition $\frac{f'(t)}{f^2(t)p(t)} \to 0$ as $t \to \infty$, with $p, r \in C^1$, cf. (8).

(ii) A comparison with linear results is described in Remark 3, where also solutions in Γ_- are discussed.

The next corollary gives various sufficient conditions which guarantee the statement of Theorem 2.

Corollary 1. *Assume that* $J_1 = \infty$, $\left(\frac{r}{p}\right)^{\frac{1}{\alpha}} \in BSV$. *Any of the following assumptions guarantee* $\mathbb{M}^+ \subset \Gamma\left(\left[\frac{r(\alpha-1)}{p}\right]^{\frac{1}{\alpha}}\right)$, $\tilde{f}$ *being defined in* (8)*:*

(i) $\frac{1}{p}$ *is bounded and there exists a function* $\tilde{f} \in C^1$ *such that* $\tilde{f}(t) \sim f(t)$ *as* $t \to \infty$ *and* $\frac{1}{\tilde{f}} \in BSV$;

(ii) $\left(\frac{r}{p}\right)^{\frac{1}{\alpha}}$ *is bounded and there exists a function* $\tilde{f} \in C^1$ *such that* $\tilde{f}(t) \sim f(t)$ *as* $t \to \infty$ *and* $\ln \tilde{f} \in BSV$;

(iii) $\left(\frac{p}{r}\right)^{\frac{1}{\alpha}-\frac{1}{\beta}} r$ *is bounded and there exists a function* $\tilde{f} \in C^1$ *such that* $\tilde{f}(t) \sim f(t)$ *as* $t \to \infty$ *and* $\tilde{f} \in BSV$,

Proof. (i) Since $\tilde{f} \in C^1$, then $\frac{1}{\tilde{f}} \in BSV$ is equivalent to $\lim_{t\to\infty} \frac{\tilde{f}'(t)}{\tilde{f}(t)^2} = 0$, hence $\lim_{t\to\infty} \frac{\tilde{f}'(t)}{\tilde{f}(t)^2 p(t)} = 0$ from the boundedness of $\frac{1}{p}$.

Similarly, it is possible to prove the cases (ii) and (iii). □

Example 1. Suppose that $p \in RV(\delta)$ and $r \in RV(\sigma)$. Then, $f \in RV(-\frac{\delta}{\beta} - \frac{\sigma}{\alpha})$ and $(\frac{r}{p})^{\frac{1}{\alpha}} \in RV(\frac{\sigma-\delta}{\alpha})$. According to [9, Proposition 1.7], if $\delta\alpha + \sigma\beta \neq 0$ and $\delta \neq \sigma$, then f is asymptotically equivalent to $\tilde{f} \in C^1$, with $|\tilde{f}'| \in RV(-\frac{\delta}{\beta} - \frac{\sigma}{\alpha} - 1)$, while $(\frac{r}{p})^{\frac{1}{\alpha}}$ is asymptotically equivalent to $h \in C^1$ with $|h'| \in RV(\frac{\sigma-\delta}{\alpha} - 1)$. Then $\frac{|\tilde{f}'|}{\tilde{f}^2 p} \in RV(\frac{\sigma-\delta}{\alpha} - 1)$. Thus $\sigma - \delta < \alpha$ implies $\frac{|\tilde{f}'|}{\tilde{f}^2 p} \to 0$ and consequently $\frac{\tilde{f}'}{\tilde{f}^2 p} \to 0$. Moreover, since $(\frac{r(t)}{p(t)})^{\frac{1}{\alpha}} \sim \int_a^t h'(s)ds$, the same condition implies $h' \to 0$, i.e. $(\frac{r}{p})^{\frac{1}{\alpha}} \in BSV$.

Now, if $\delta < -1$, then $\int_a^t p(s)ds \to l \in \mathbb{R}$. Hence, $J_1 = \infty$ if and only if $\int_a^\infty r(t)^{1-\beta}dt = \infty$ (see [7, page 136]). Thus $\sigma < \alpha - 1$, i.e. $\sigma(1-\beta) > -1$, implies $J_1 = \infty$. Notice that $\sigma - \delta < \alpha$ implies $\sigma < \alpha + \delta < \alpha - 1$.

On the other hand, if $\delta > -1$, by Karamata's theorem we get

$$r(t)^{1-\beta}\left(\int_a^t p(s)ds\right)^{\beta-1} = t^{\sigma(1-\beta)} L_r(t)^{1-\beta}\left(\int_a^t s^\delta L_p(s)ds\right)^{\beta-1} \sim t^{(\delta+1-\sigma)(\beta-1)}\left(\frac{L_p(t)}{L_r(t)(\delta+1)}\right)^{\beta-1},$$

and $(\delta+1-\sigma)(\beta-1)>-1$, i.e. $\sigma-\delta<\alpha$, implies again $J_1=\infty$.

In conclusion, the assumptions of Theorem 2 are satisfied if $\sigma-\delta<\alpha$ and $\delta\alpha+\sigma\beta\neq 0$ with $\delta\neq -1$. The result holds also when $\delta=0$ or $\sigma=0$, i.e. when p or r are SV.

3.2 Decreasing solutions in the class Γ_-

In this section we deal with solutions of (1) belonging to the de Haan class Γ_-. As far as we know, the only result related with a solution in this class was obtained in [19, Corollary 3.2] for the linear equation, i.e. when $\alpha=2$, in the special case when $r(t)\equiv 1$. Using a quite different approach we extend the statement to a quite wider class of equations and, moreover, we deal with an entire subclass of decreasing solutions. In some aspects, the following result is new even in the linear case, see Remark 3.

Theorem 3. *Suppose that $J_1<\infty$ or $J_2=\infty$ and $\left(\frac{p}{r}\right)^{-\frac{1}{\alpha}}\in BSV$. If there exist functions $\tilde{p},\tilde{r}\in C^1$ such that $\tilde{p}\sim p,\tilde{r}\sim r$ and $\tilde{f}:=\left(\frac{\tilde{p}}{\tilde{r}}\right)^{-\frac{1}{\beta}}\frac{1}{\tilde{r}}$ satisfies (8), then $\mathbb{M}_0^-\subset\Gamma_-\left[\left(\frac{r(\alpha-1)}{p}\right)^{\frac{1}{\alpha}}\right]$.*

Proof. Take $y\in\mathbb{M}_0^-$. By definition, $\Phi(y)=y^{\alpha-1}$ and $\Phi(y')=-(-y')^{\alpha-1}$ for t sufficiently large. The main ideas of the proof of the theorem are similar to the ones of Theorem 2, but there are substantial differences in some steps. Therefore we give just a sketch of this proof, pointing out the differences.

We first show the relation $(-\frac{y'(t)}{y(t)})\sim[\frac{p(t)}{r(t)(\alpha-1)}]^{\frac{1}{\alpha}}$ as $t\to\infty$. Assuming that $p,r\in C^1$ and that $\lim_{t\to\infty}\frac{f'(t)}{f(t)^2p(t)}=0$, we define $w=-fr(-\frac{y'}{y})^{\alpha-1}$. Then w is negative and satisfies the equation

$$\frac{w'}{fp}=1+(-w)\left[-\frac{f'}{f^2p}-(\alpha-1)(-w)^{\beta-1}\right]. \tag{17}$$

Let us now show that $\lim_{t\to\infty}w(t)=-(\alpha-1)^{-\frac{1}{\beta}}$. Consider first the case when $w'(t)<0$ for t sufficiently large. Then $\frac{w'}{fp}$ is definitely negative, w is decreasing and tends to $l\in[-\infty,0)$ when $t\to\infty$. Suppose by contradiction that $l=-\infty$. Then $|w|\to\infty$, therefore according to the definition of f, it holds $(\frac{p}{r})^{-\frac{1}{\beta}}[\frac{(-y')}{y}]^{\alpha-1}\to\infty$. Hence, raising by β, we get $(\frac{p}{r})^{-1}[\frac{(-y')}{y}]^{(\alpha-1)\beta}=(\frac{p}{r})^{-1}[\frac{(-y')}{y}]^{\alpha}\to\infty$, which implies

$$\frac{p}{r}\left[\frac{y}{(-y')}\right]^{\alpha}\to 0. \tag{18}$$

Since $p,r\in C^1$, condition $(\frac{p}{r})^{-\frac{1}{\alpha}}\in BSV$ is equivalent to $[(\frac{p}{r})^{-\frac{1}{\alpha}}]'\to 0$, i.e.

$$\left(\frac{p}{r}\right)^{-\frac{1}{\alpha}-1}\left(\frac{p}{r}\right)'\to 0. \tag{19}$$

Making now computations, we obtain

$$\frac{f'}{f^2p} = \frac{-\frac{1}{\beta}\left(\frac{p}{r}\right)^{-\frac{1}{\beta}-1}\left(\frac{p}{r}\right)'\frac{1}{r} - \left(\frac{p}{r}\right)^{-\frac{1}{\beta}}\frac{r'}{r^2}}{\left(\frac{p}{r}\right)^{-\frac{2}{\beta}}\frac{p}{r^2}} = -\frac{1}{\beta}\left(\frac{p}{r}\right)^{-\frac{1}{\alpha}-1}\left(\frac{p}{r}\right)' - \left(\frac{p}{r}\right)^{-\frac{1}{\alpha}}\frac{r'}{r}$$
$$= (\alpha-1)\left(\left(\left(\frac{p}{r}\right)^{-\frac{1}{\alpha}}\right)' - \left(\frac{p}{r}\right)^{-\frac{1}{\alpha}}\frac{r'}{r}\right), \tag{20}$$

therefore (19) and $\frac{f'}{f^2p} \to 0$ imply

$$\left(\frac{p}{r}\right)^{-\frac{1}{\alpha}}\frac{r'}{r} \to 0 \tag{21}$$

thus

$$\frac{r'}{r}\cdot\frac{y}{y'} = \left(\frac{p}{r}\right)^{-\frac{1}{\alpha}}\frac{r'}{r}\left(\frac{p}{r}\right)^{\frac{1}{\alpha}}\frac{y}{y'} \to 0$$

by (18) and (21). The conditions $p, r \in C^1$ yield $y \in C^2$ and, from (1),

$$-r'(-y')^{\alpha-1} + (\alpha-1)r(-y')^{\alpha-2}y'' = py^{\alpha-1},$$

thus, according to (18) and (21),

$$\frac{y''y}{(y')^2} = \frac{py^\alpha + r'(-y')^{\alpha-1}y}{(\alpha-1)r(-y')^\alpha} = \frac{py^\alpha}{(\alpha-1)r(-y')^\alpha} - \frac{r'y}{(\alpha-1)ry'} \to 0.$$

It follows that

$$\left(\frac{y}{y'}\right)' = \frac{(y')^2 - yy''}{(y')^2} = 1 - \frac{y''y}{(y')^2} \to 1,$$

hence $\frac{y}{y'} \to \infty$ which implies $y' > 0$, a contradiction with $y \in \mathbb{M}^-$. Thus $l > -\infty$. If $l \neq -(\alpha-1)^{-\frac{1}{\beta}}$, again by (17) and $\frac{f'}{f^2p} \to 0$, we would get that $\lim_{t\to\infty}\frac{w'(t)}{f(t)p(t)} = 1-(\alpha-1)(-l)^\beta \neq 0$, yielding the existence of two positive constants M_0, M_1 such that

$$\left(\frac{p(t)}{r(t)}\right)^{\frac{1}{\alpha}} \geq -\frac{y'(t)}{M_0 y(t)} \tag{22}$$

and

$$w(t) \leq w(t_0) - M_1\int_{t_0}^t \left(\frac{p(t)}{r(t)}\right)^{\frac{1}{\alpha}} dt \tag{23}$$

for $t > t_0$ sufficiently large. From (22) and (23), recalling that $y \to 0$, we then would get

$$w(t) \leq w(t_0) + \frac{M_1}{M_0}\ln\frac{y(t)}{y(t_0)} \to -\infty,$$

a contradiction. Reasoning similarly, we obtain a contradiction also in the case when $w'(t) < 0$ for t sufficiently large or there exists a sequence $\{t_n\}_n$ with $\lim_{n\to\infty} t_n = \infty, \lim_{n\to\infty} w(t_n) = l \in \mathbb{R}\cup\{-\infty,\infty\}$ and $w'(t_n) = 0$ for every n. We have so proved that $\lim_{t\to\infty} w(t) = -(\alpha-1)^{-\frac{1}{\beta}}$, i.e. $(-\frac{y'(t)}{y(t)})^{\alpha-1} \sim (\frac{p(t)}{r(t)})^{\frac{1}{\beta}}(\alpha-1)^{-\frac{1}{\beta}}$. Raising by $\beta-1$ we then obtain $-\frac{y'(t)}{y(t)} \sim [\frac{p(t)}{r(t)(\alpha-1)}]^{\frac{1}{\alpha}}$ as $t\to\infty$.

In the general case, given $\tilde{p}, \tilde{r} \in C^1$ as in the assumptions, let v and u be solutions respectively of problems (11) and (12) in class $\mathbb{M}_0^-$ and consider the solutions w_v, w_y and w_u of the respectively associated Riccati equations. Since v, y and u are principal solutions and $(1-\epsilon)\tilde{p}(t) \le p(t) \le (1+\epsilon)\tilde{p}(t)$ for $t > t_\epsilon$, then $w_v(t) \le w_y(t) \le w_u(t)$ for large t by (5). Hence it is possible to reason similarly as in Theorem 2 to prove that $(-\frac{y'(t)}{y(t)})^{\alpha-1} \sim (\frac{p(t)}{r(t)})^{\frac{1}{\beta}}(\frac{1}{\alpha-1})^{\frac{1}{\beta}}$, which implies again $-\frac{y'(t)}{y(t)} \sim [\frac{p(t)}{r(t)(\alpha-1)}]^{\frac{1}{\alpha}}$.

Denoted again $[\frac{r(\alpha-1)}{p}]^{\frac{1}{\alpha}} = Q$, from $-\frac{y'(t)}{y(t)} \sim \frac{1}{Q(t)}$ and $Q \in SN$ we obtain, for every $\lambda > 0$,

$$\lambda = \lim_{t\to\infty} \int_t^{t+\lambda Q(t)} \frac{-y'(s)}{y(s)} ds = \lim_{t\to\infty} -\ln \frac{y(t+\lambda Q(t))}{y(t)} = \lim_{t\to\infty} \ln \frac{y(t)}{y(t+\lambda Q(t))}.$$

By definition, $\frac{1}{y} \in \Gamma([\frac{r(\alpha-1)}{p}]^{\frac{1}{\alpha}})$, i.e. $y \in \Gamma_-([\frac{r(\alpha-1)}{p}]^{\frac{1}{\alpha}})$ and the theorem is proved. $\square$

The next corollary gives conditions which guarantee the statement of Theorem 3. The proof is similar to the one of Corollary 1.

Corollary 2. *Assume that $J_1 < \infty$ or $J_2 = \infty$. If any of the conditions (i) or (ii) or (iii) of Corollary 1 holds for some $\tilde{p}, \tilde{r} \in C^1$ with $\tilde{p} \sim p, \tilde{r} \sim r$, then $\mathbb{M}_0^- \subset \Gamma_-\left(\left[\frac{r(\alpha-1)}{p}\right]^{\frac{1}{\alpha}}\right)$.*

Example 2. Suppose that $p \in RV(\delta)$ and $r \in RV(\sigma)$. According to [9, Proposition 1.7], if $\delta, \sigma \neq 0, p$ and r are asymptotically equivalent respectively to $\tilde{p}$ and $\tilde{r} \in C^1$, with $|\tilde{p}'| \in RV(\delta-1)$ and $|\tilde{r}'| \in RV(\sigma-1)$. Reasoning as in Example 1, it follows that $\frac{f'}{f^2 p} \to 0$ and $(\frac{r}{p})^{\frac{1}{\alpha}} \in BSV$. Now, if $\delta < -1$, then $\int_a^\infty p(s)ds < \infty$. According to Karamata's theorem we have $\int_t^\infty p(s)ds \sim \frac{1}{-\delta-1} t^{\delta+1} L_p(t)$, thus

$$r(t)^{1-\beta}\left(\int_t^\infty p(s)ds\right)^{\beta-1} \sim t^{(\delta+1-\sigma)(\beta-1)}\left(\frac{L_p(t)}{L_r(t)(-\delta-1)}\right)^{\beta-1}.$$

Hence, if we further assume $\sigma < \alpha+\delta$, i.e. $(\delta+1-\sigma)(\beta-1) > -1$, we obtain $J_2 = \infty$. Note that while we are able to guarantee $J_2 = \infty$, we cannot assure $J_1 < \infty$ when $\delta < -1$, since it would yield $-\sigma \le 1-\alpha < -\delta-\alpha$, which is in a contradiction with $\sigma < \alpha+\delta$; the later inequality being required for $(\frac{r}{p})^{\frac{1}{\alpha}} \in BSV$. If $\delta > -1$, then $\int_a^t p(s)ds \to \infty$, which implies $J_2 = \infty$.

In conclusion, the assumptions of Theorem 2 are satisfied if $\delta, \sigma \neq 0$ and $\sigma - \delta < \alpha$ with $\delta \neq -1$.

3.3 Related observations and regularly varying solutions

The following corollary gives a precise classification of all solutions of (1) in terms of de Haan classes. Its proof is a direct consequence of Theorem 2 and Theorem 3.

Corollary 3. *Assume that* $J_1 = J_2 = \infty$ *and that* $\left(\frac{p}{r}\right)^{-\frac{1}{\alpha}} \in BSV$*. If there exist functions* $\tilde{p}, \tilde{r}$ *such that* $\tilde{p} \sim p, \tilde{r} \sim r$ *and* $\tilde{f} := \left(\frac{\tilde{p}}{\tilde{r}}\right)^{-\frac{1}{\beta}} \frac{1}{\tilde{r}}$ *satisfies* (8)*, then*

$$\mathbb{M}^+ \subset \Gamma\left(\left[\frac{r(\alpha-1)}{p}\right]^{\frac{1}{\alpha}}\right) \quad \text{and} \quad \mathbb{M}^- \subset \Gamma_-\left(\left[\frac{r(\alpha-1)}{p}\right]^{\frac{1}{\alpha}}\right).$$

Remark 2. As a by-product of Corollary 3, we get conditions guaranteeing that

$$\mathbb{M}^+ = \mathbb{M}^+_\infty \subset RPV(\infty) \text{ and } \mathbb{M}^- = \mathbb{M}^-_0 \subset RPV(-\infty).$$

Remark 3. A closer examination of the proofs of Theorem 2 and Theorem 3 shows that under the conditions of Corollary 3 we have guaranteed the existence of solutions y_i of (1) such that

$$y_i'(t) \sim \pm\left(\frac{p(t)}{r(t)(\alpha-1)}\right)^{\frac{1}{\alpha}} y_i(t) \tag{24}$$

as $t \to \infty$, $i = 1, 2$. If $\alpha = 2$ and $r(t) = 1$, then (24) reduces to $y_i'(t) \sim \pm\sqrt{p(t)}y_i(t)$, y_i being solutions of $y'' - p(t)y = 0$. The same formulas in the linear case were obtained in [12] by Hartman and Wintner under the assumptions $\int_a^\infty \sqrt{p(s)}\, ds = \infty$ and $p'(t)/p^{\frac{3}{2}}(t) \to 0$ as $t \to \infty$. Omey in [18] rediscovered this statement for an increasing solution y_1 and showed that $y_1 \in \Gamma(p^{-\frac{1}{2}})$ under the assumption $p^{-\frac{1}{2}} \in BSV$, see also [9, 19, 21]. A decreasing solution which is in Γ_- was found by Omey in [19] with the help of reduction of order formula, having at disposal an increasing solution in Γ. Note that this tool cannot be used in the half-linear case. We emphasize that in Corollary 3 we work with all (possible) decreasing solutions and with general r, which makes this statement new also in the linear case.

The ideas of the proofs of Theorems 2, 3 can be used to establish conditions which guarantee regular variation of solutions to (1).

Theorem 4. *Assume that there exist* $\tilde{p}, \tilde{r} \in C^1$ *such that*

$$\lim_{t\to\infty} \frac{\tilde{f}'(t)}{\tilde{f}^2(t)\tilde{p}(t)} = D \in \mathbb{R} \quad \text{and} \quad \lim_{t\to\infty}\left(\left(\frac{\tilde{p}(t)}{\tilde{r}(t)}\right)^{-\frac{1}{\alpha}}\right)' = C \in (0, \infty), \tag{25}$$

where $\tilde{f} = \left(\frac{\tilde{p}}{\tilde{r}}\right)^{-\frac{1}{\beta}} \frac{1}{\tilde{r}}$*, and* $\tilde{p}(t) \sim p(t), \tilde{r}(t) \sim r(t)$ *as* $t \to \infty$*. Let* $\varrho_1 > 0 > \varrho_2$ *denote the roots of the equation*

$$|\varrho|^\beta - \frac{D}{\alpha-1}\varrho - \frac{1}{\alpha-1} = 0. \tag{26}$$

(i) If $J_1 = \infty$*, then* $\mathbb{M}^+ \subset NRV\left(\frac{\Phi^{-1}(\varrho_1)}{C}\right)$*.*

(ii) If $J_1 < \infty$ *or* $J_2 = \infty$*, then* $\mathbb{M}^-_0 \subset NRV\left(\frac{\Phi^{-1}(\varrho_2)}{C}\right)$*.*

Proof. (i) Take $y \in \mathbb{M}^+$. Similarly as in the proof of Theorem 2 we get the relation $f(t)r(t)\left(\frac{y'(t)}{y(t)}\right)^{\alpha-1} \sim \varrho_1$ as $t \to \infty$ with $f = \left(\frac{p}{r}\right)^{-\frac{1}{\beta}}\frac{1}{r}$. Since $C > 0$, according to the l'Hospital rule, $\frac{1}{t}(\frac{\tilde{p}(t)}{\tilde{r}(t)})^{-\frac{1}{\alpha}} \sim [(\frac{\tilde{p}(t)}{\tilde{r}(t)})^{-\frac{1}{\alpha}}]' \sim C$, hence from the second condition in (25) we get $\frac{p(t)}{r(t)} \sim \frac{\tilde{p}(t)}{\tilde{r}(t)} \sim \frac{1}{C^\alpha t^\alpha}$ as $t \to \infty$. Thus,

$$\varrho_1 \sim \left(\frac{p(t)}{r(t)}\right)^{-\frac{1}{\beta}} \left(\frac{y'(t)}{y(t)}\right)^{\alpha-1} \sim (Ct)^{\frac{\alpha}{\beta}} \left(\frac{y'(t)}{y(t)}\right)^{\alpha-1} = C^{\alpha-1}\left(\frac{ty'(t)}{y(t)}\right)^{\alpha-1}$$

as $t \to \infty$. This implies $\lim_{t\to\infty} \frac{ty'(t)}{y(t)} = \frac{\varrho_1^{\beta-1}}{C}$, or $y \in NRV\left(\frac{\Phi^{-1}(\varrho_1)}{C}\right)$.

(ii) Take $y \in \mathbb{M}_0^-$. Similarly as in the proof of Theorem 3 we get the relation $f(t)r(t)\left(\frac{-y'(t)}{y(t)}\right)^{\alpha-1} \sim -\varrho_2$ as $t \to \infty$. We only note that to show that $w(t) \to -\infty$ is excluded for a negative solution w of the associated Riccati type equation we can use the same arguments as in the proof Theorem 3, since $\left(\frac{\tilde{p}}{\tilde{r}}\right)^{-\frac{1}{\alpha}}\frac{\tilde{r}'}{\tilde{r}}$ is bounded. The boundedness follows from the identity

$$\frac{\tilde{f}'(t)}{\tilde{f}^2(t)\tilde{p}(t)} = (\alpha-1)\left(\left(\left(\frac{\tilde{p}(t)}{\tilde{r}(t)}\right)^{-\frac{1}{\alpha}}\right)' - \left(\frac{\tilde{p}(t)}{\tilde{r}(t)}\right)^{-\frac{1}{\alpha}}\frac{\tilde{r}'(t)}{\tilde{r}(t)}\right) \tag{27}$$

and condition (25). Similarly as in (i), we obtain $\lim_{t\to\infty} \frac{ty'(t)}{y(t)} = \frac{\Phi^{-1}(\varrho_2)}{C}$. □

Remark 4. Jaroš, Kusano, and Tanigawa in [14], assuming $\int_a^\infty r^{1-\beta}(s)\,ds = \infty$, showed that equation (1) (with no sign condition on p) possesses a pair of solutions $y_i \in NRV_R(\Phi^{-1}(\lambda_i))$, $i = 1, 2$, if and only if

$$\lim_{t\to\infty} R^{\alpha-1}(t)\int_t^\infty p(s)\,ds = A \in \left(-\frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}, \infty\right), \tag{28}$$

where $R(t) = \int_a^t r^{1-\beta}(s)\,ds$ and λ_1, λ_2 are the real roots of the equation

$$|\lambda|^\beta - \lambda - A = 0. \tag{29}$$

The notation $f \in NRV_R(\vartheta)$ means $f \circ R^{-1} \in NRV(\vartheta)$; we speak about generalized regular variation with respect to R. A similar statement is proved for the case $\int_a^\infty r^{1-\beta}(s)\,ds < \infty$. To make a comparison with our results simpler, let $r(t) = 1$. Then the limit in condition (28) reduces to $\lim_{t\to\infty} t^{\alpha-1}\int_t^\infty p(s)\,ds$ and generalized regular variation becomes usual regular variation, thus $y_i \in NRV(\Phi^{-1}(\lambda_i))$, $i = 1, 2$. The second condition in (25) then implies $t^\alpha p(t) \sim t^\alpha \tilde{p}(t) \sim C^{-\alpha}$ as $t \to \infty$ and so, by the L'Hospital rule, $\lim_{t\to\infty} t^{\alpha-1}\int_t^\infty p(s)\,ds = \frac{1}{(\alpha-1)C^\alpha}$, which yields $(\alpha-1)C^\alpha = 1/A$. Further, from the identity (27) and the first condition in (25), we have $D = (\alpha-1)C$. The relation between the corresponding real roots λ of (29) and ϱ of (26) reads as $\lambda = \varrho(A(\alpha-1))^{\frac{1}{\beta}}$. Hence, $\Phi^{-1}(\lambda_i) = \Phi^{-1}(\varrho_i)/C$, $i = 1, 2$, and so, as expected, the corresponding indices of regular variation in both results are the same. Note that the integral condition from [14] is more general than our conditions in Theorem 4 and, moreover, it is shown to be necessary. On the other hand, the fixed point approach used in [14] guarantees the existence of at least one positive increasing (decreasing) RV solution, while Theorem 4 says that all positive increasing (decreasing) solutions are regularly varying.

Remark 5. It is well known that $J_1 < \infty$ implies $\mathbb{M}^+ = \mathbb{M}_B^+ \subset SV$. Consequently, assuming (25), no matter what additional integral condition holds, $\mathbb{M}^+ \subset RV \cup SV$. The same conclusion holds for $\mathbb{M}^-$, recalling that, if $J_1 = \infty$ and $J_2 < \infty$, then $\mathbb{M}^- = \mathbb{M}_B^- \subset SV$.

Remark 6. Note that, with $r \in C^1$, having obtained

$$\left(\frac{p(t)}{r(t)}\right)^{-\frac{1}{\beta}} = f(t)r(t)\Phi\left(\frac{y'(t)}{y(t)}\right) \sim \varrho \tag{30}$$

as $t \to \infty$, where y is a solution of (1) and ϱ is a root of (26) (which cannot be zero), D being a real number, the statements of the proofs of Theorems 2, 3, and 4 can be proved in an alternative and unified way; compare with the latter parts of their proofs. The number C in (25) is assumed to be in $[0, \infty)$. We show that there exists the limit

$$\lim_{t\to\infty} \frac{y''(t)y(t)}{y'^2(t)} = K \in \mathbb{R}. \tag{31}$$

We have

$$\frac{y''y}{y'^2} = \frac{p}{(\alpha-1)r} \cdot \frac{y^\alpha}{|y'|^\alpha} - \frac{r'}{(\alpha-1)r} \cdot \frac{y}{y'}.$$

Relation (30) implies $\frac{p(t)}{r(t)} \frac{y^\alpha(t)}{|y'(t)|^\alpha} \sim \frac{1}{|\varrho|^\beta}$. From identity (27), in view of $\tilde{r}(t) = r(t)$, (30), and $\tilde{p}(t) \sim p(t)$ as $t \to \infty$, we get

$$\frac{r'(t)}{r(t)} \cdot \frac{y(t)}{y'(t)} = \frac{r'(t)}{r(t)} \left(\frac{r(t)}{p(t)}\right)^{\frac{1}{\alpha}} \left(\frac{p(t)}{r(t)}\right)^{\frac{1}{\alpha}} \frac{y(t)}{y'(t)} \sim \big((\alpha-1)C - D\big)\frac{1}{\Phi^{-1}(\varrho)}$$

as $t \to \infty$. Since ϱ is a root of (26), for K from (31) it holds

$$K = \frac{1}{|\varrho|^\beta}\left(\frac{1}{\alpha-1} - C\varrho + \frac{1}{\alpha-1}D\varrho\right) = \frac{1}{|\varrho|^\beta}(|\varrho|^\beta - C\varrho) = 1 - \frac{C}{\Phi^{-1}(\varrho)}.$$

If $C = 0$, then $K = 1$, which implies $y \in \Gamma_\pm(|y'/y|)$ depending on whether y is increasing or decreasing, see [20, Proposition 2.1]. Since for the auxiliary function it holds $\frac{|y'(t)|}{y(t)} \sim \left(\frac{p(t)}{(\alpha-1)r(t)}\right)^{\frac{1}{\alpha}}$ as $t \to \infty$, we get the statements of Theorems 2 and 3. If $C > 0$ (as in Theorem 4), then $K \neq 1$ and $y \in NRV(1/(1-K)) = NRV(\Phi^{-1}(\varrho)/C)$ by [20, Proposition 2.1]. Note that in [20], the statement which we just applied is formulated for regularly varying functions, but a closer examination of the proof in that paper shows that regular variation is normalized.

Corollary 4. *Assume that there exist $\tilde{p}, \tilde{r} \in C^1$ such that*

$$\tilde{r} \in NRV(\gamma), \gamma \in \mathbb{R}, \quad \textit{and} \quad \lim_{t\to\infty}\left(\left(\frac{\tilde{p}(t)}{\tilde{r}(t)}\right)^{-\frac{1}{\alpha}}\right)' = C \in (0, \infty). \tag{32}$$

Then the statement of Theorem 4 holds, where D in (26) *satisfies $D = C(\alpha - 1 - \gamma)$.*

Proof. The second condition in (32) implies $\frac{\tilde{r}(t)}{\tilde{p}(t)} \sim C^\alpha t^\alpha$ as $t \to \infty$. Hence, in view of the first condition in (32), we have

$$\left(\frac{\tilde{p}(t)}{\tilde{r}(t)}\right)^{-\frac{1}{\alpha}} \frac{\tilde{r}'(t)}{\tilde{r}(t)} \sim C\frac{t\tilde{r}'(t)}{\tilde{r}(t)} \sim C\gamma$$

as $\to \infty$. For D from condition (25), in view of (27), we get $D = (\alpha - 1)C - C\gamma$. The result now follows from Theorem 4. □

4 Solutions in the class Π

4.1 Decreasing solutions in the case $\delta < -1$

We start with showing that any positive decreasing solution of (1) is normalized slowly varying (and in the de Haan class Π) and satisfies an asymptotic formula.

Theorem 5. *Let $p \in RV(\delta)$ and $r \in RV(\delta+\alpha)$ with $\delta < -1$. If $\dfrac{L_p(t)}{L_r(t)} \to 0$ as $t \to \infty$, then $-y \in \Pi(-ty'(t))$ for every $y \in \mathbb{M}^-$. Moreover, for every $y \in \mathbb{M}^-$,*

(i) if $\displaystyle\int_a^\infty \left(\frac{sp(s)}{r(s)}\right)^{\frac{1}{\alpha-1}} ds = \infty$, then there exists $\varepsilon(t)$ with $\varepsilon(t) \to 0$ as $t \to \infty$ such that

$$y(t) = \exp\left\{-\int_a^t (1+\varepsilon(s))\left(-\frac{sp(s)}{(\delta+1)r(s)}\right)^{\frac{1}{\alpha-1}} ds\right\} \tag{33}$$

with $y \in \mathbb{M}_0^-$.

(ii) if $\displaystyle\int_a^\infty \left(\frac{sp(s)}{r(s)}\right)^{\frac{1}{\alpha-1}} ds < \infty$, then there exists $\varepsilon(t)$ with $\varepsilon(t) \to 0$ as $t \to \infty$ such that

$$y(t) = l\exp\left\{\int_t^\infty (1+\varepsilon(s))\left(-\frac{sp(s)}{(\delta+1)r(s)}\right)^{\frac{1}{\alpha-1}} ds\right\}, \tag{34}$$

where $l = \lim_{t\to\infty} y(t) \in (0,\infty)$, with $y \in \mathbb{M}_B^-$.

Proof. First note that $\int_a^\infty p(s)ds < \infty$ thanks to $\delta < -1$. Further, $r^{1-\beta} \in RV((1-\beta)(\delta+\alpha))$. Since $(1-\beta)(\delta+\alpha) = (1-\beta)(\delta+1+\alpha-1) = (1-\beta)(\delta+1)-1$, $\beta > 1$, and $\delta < -1$, we have $(1-\beta)(\delta+\alpha) > -1$, and so $\int_a^\infty r^{1-\beta}(s)\, ds = \infty$. Take $y \in \mathbb{M}^-$. We first prove that $y \in NSV$. By definition $r\Phi(y') = -r(-y')^{\alpha-1}$ for t sufficiently large and it is negative increasing . Therefore there exists

$$\lim_{t\to\infty} -r(t)(-y'(t))^{\alpha-1} = -M \in (-\infty, 0].$$

If $-M < 0$, then, according to the monotonicity, $-r(t)(-y'(t))^{\alpha-1} \le -M$ for every t, thus $y'(t) \le -M^{\beta-1}r(t)^{1-\beta}$. Integrating now from a to t we would get $y(t) \le y(a) - M^{\beta-1}\int_a^t r(s)^{1-\beta}ds \to -\infty$ as $t \to \infty$, a contradiction. Hence $M = 0$. Integrating now (1) from t to ∞, we get

$$r(t)(-y'(t))^{\alpha-1} = \int_t^\infty p(s)y(s)^{\alpha-1}ds, \tag{35}$$

thus, dividing by $r(t)$ and recalling that y is decreasing,

$$(-y'(t))^{\alpha-1} = \frac{1}{r(t)}\int_t^\infty p(s)y(s)^{\alpha-1}ds \le \frac{y(t)^{\alpha-1}}{r(t)}\int_t^\infty p(s)ds.$$

Therefore, dividing by $y(t)^{\alpha-1}$ and multiplying by $t^{\alpha-1}$, we obtain

$$0 < \left(-\frac{ty'(t)}{y(t)}\right)^{\alpha-1} \le \frac{t^{\alpha-1}}{r(t)}\int_t^\infty p(s)ds \tag{36}$$

for large t. By Karamata's theorem,

$$\frac{t^{\alpha-1}}{r(t)}\int_t^\infty p(s)ds \sim \frac{t^{\alpha-1}}{t^{\delta+\alpha}L_r(t)}\cdot\frac{1}{-\delta-1}t^{\delta+1}L_p(t)=\frac{L_p(t)}{L_r(t)}\cdot\frac{1}{-\delta-1}.$$

Hence, according to the assumption, $\frac{t^{\alpha-1}}{r(t)}\int_t^\infty p(s)ds \to 0$, and from (36) we get $ty'(t)/y(t)\to 0$, which implies $y\in NSV$.

Now, since $p\in RV(\delta)$, it follows that $(-r(-y')^{\alpha-1})'=py^{\alpha-1}\in RV(\delta)$. From (35) and Karamata's theorem, we obtain $r(-y')^{\alpha-1}\in RV(\delta+1)$. Since $r\in RV(\delta+\alpha)$, we get $(-y')^{\alpha-1}\in RV(1-\alpha)$, i.e. $-y'\in RV(-1)$. Finally, integrating by substitution, we get, for every $\lambda>0$,

$$\frac{-y(\lambda t)+y(t)}{-ty'(t)}=\int_t^{\lambda t}\frac{-y'(u)}{-ty'(t)}du=\int_1^{\lambda}\frac{-y'(st)}{-y'(t)}ds\to\int_1^{\lambda}\frac{1}{s}ds=\ln\lambda, \qquad (37)$$

as $t\to\infty$, i.e. $-y\in\Pi(-ty'(t))$, because $\frac{-y'(st)}{-y'(t)}\to\frac{1}{s}$ uniformly in the interval $[\min\{1,\lambda\},\max\{1,\lambda\}]$.

Set

$$h(t)=-t^{-\delta-1}r(t)(-y'(t))^{\alpha-1}-(\delta+1)\int_a^t s^{-\delta-2}r(s)(-y'(s))^{\alpha-1}ds.$$

Let us show that $h\in\Pi(th'(t))$ and $h\in\Pi(-(\delta+1)t^{-\delta-1}r(t)(-y'(t))^{\alpha-1})$. Indeed,

$$\begin{aligned}h'(t)= &-(-\delta-1)t^{-\delta-2}r(t)(-y'(t))^{\alpha-1}+t^{-\delta-1}p(t)y(t)^{\alpha-1}\\ &-(\delta+1)t^{-\delta-2}r(t)(-y'(t))^{\alpha-1}\\ =\ & t^{-\delta-1}p(t)y(t)^{\alpha-1},\end{aligned}$$

hence, recalling that $p\in RV(\delta)$ and $y\in SV$, we have $h'\in RV(-\delta-1+\delta+0)=RV(-1)$ and, reasoning like in (37), we obtain that $h\in\Pi(th'(t))$. Moreover, integrating by substitution, we get

$$\begin{aligned}&\frac{h(\lambda t)-h(t)}{-(\delta+1)t^{-\delta-1}r(t)(-y'(t))^{\alpha-1}}\\ &=-\frac{\lambda^{-\delta-1}t^{-\delta-1}r(\lambda t)(-y'(\lambda t))^{\alpha-1}}{-(\delta+1)t^{-\delta-1}r(t)(-y'(t))^{\alpha-1}}+\frac{t^{-\delta-1}r(t)(-y'(t))^{\alpha-1}}{-(\delta+1)t^{-\delta-1}r(t)(-y'(t))^{\alpha-1}}\\ &\quad-\frac{(\delta+1)\int_t^{\lambda t}s^{-\delta-2}r(s)(-y'(s))^{\alpha-1}ds}{-(\delta+1)t^{-\delta-1}r(t)(-y'(t))^{\alpha-1}}\\ &=-\frac{\lambda^{-\delta-1}r(\lambda t)(-y'(\lambda t))^{\alpha-1}}{-(\delta+1)r(t)(-y'(t))^{\alpha-1}}-\frac{1}{\delta+1}-\int_1^\lambda\frac{t^{-\delta-2}u^{-\delta-2}r(tu)(-y'(tu))^{\alpha-1}}{-t^{-\delta-1}r(t)(-y'(t))^{\alpha-1}}t\,du\\ &=-\frac{\lambda^{-\delta-1}r(\lambda t)(-y'(\lambda t))^{\alpha-1}}{-(\delta+1)r(t)(-y'(t))^{\alpha-1}}-\frac{1}{\delta+1}-\int_1^\lambda\frac{u^{-\delta-2}r(tu)(-y'(tu))^{\alpha-1}}{-r(t)(-y'(t))^{\alpha-1}}\,du.\end{aligned} \qquad (38)$$

Since $-r(-y')^{\alpha-1}\in RV(\delta+1)$, it follows that

$$\lim_{t\to\infty}-\frac{\lambda^{-\delta-1}r(\lambda t)(-y'(\lambda t))^{\alpha-1}}{-(\delta+1)r(t)(-y'(t))^{\alpha-1}}=\frac{\lambda^{-\delta-1}}{\delta+1}\lambda^{\delta+1}=\frac{1}{\delta+1}$$

and the uniform convergence of $\frac{-y'(tu)}{-y'(t)}$ to $\frac{1}{u}$ in $[\min\{1,\lambda\},\max\{1,\lambda\}]$ implies

$$\lim_{t\to\infty}\Big[-\int_1^\lambda\frac{u^{-\delta-2}r(tu)(-y'(tu))^{\alpha-1}}{-r(t)(-y'(t))^{\alpha-1}}du\Big]=\int_1^\lambda u^{-\delta-2}u^{\delta+1}du=\ln\lambda,$$

thus

$$\lim_{t\to\infty}\frac{h(\lambda t)-h(t)}{-(\delta+1)t^{-\delta-1}r(t)(-y'(t))^{\alpha-1}}=\ln\lambda,$$

i.e. $h\in\Pi(-(\delta+1)t^{-\delta-1}r(t)(-y'(t))^{\alpha-1})$.

Because of the uniqueness of the auxiliary function up to asymptotic equivalence, $-(\delta+1)t^{-\delta-1}r(t)(-y'(t))^{\alpha-1}\sim th'(t)=t^{-\delta}p(t)y(t)^{\alpha-1}$, which implies $[-\frac{y'(t)}{y(t)}]^{\alpha-1}\sim -\frac{tp(t)}{(\delta+1)r(t)}$, i.e. $\frac{y'(t)}{y(t)}\sim -[-\frac{tp(t)}{(\delta+1)r(t)}]^{\frac{1}{\alpha-1}}$. Therefore there exists a function $\varepsilon(t)$, with $\lim_{t\to\infty}\varepsilon(t)=0$, such that

$$\frac{y'(t)}{y(t)}=-(1+\varepsilon(t))\left[-\frac{tp(t)}{(\delta+1)r(t)}\right]^{\frac{1}{\alpha-1}} \tag{39}$$

Assume now that $\int_a^\infty[-\frac{sp(s)}{r(s)}]^{\frac{1}{\alpha-1}}ds=\infty$. Integrating (39) from a to t, it follows

$$\ln y(t)-\ln y(a)=\int_a^t\frac{y'(s)}{y(s)}ds=-\int_a^t(1+\varepsilon(s))\left[-\frac{sp(s)}{(\delta+1)r(s)}\right]^{\frac{1}{\alpha-1}}ds,$$

which yields (33). In fact, it easily follows that there exists $\tilde\varepsilon(t)\to 0$ such that $\ln y(a)-\int_a^t(1+\varepsilon(s))[-\frac{sp(s)}{(\delta+1)r(s)}]^{\frac{1}{\alpha-1}}ds=-\int_a^t(1+\tilde\varepsilon(s))[-\frac{sp(s)}{(\delta+1)r(s)}]^{\frac{1}{\alpha-1}}ds$. It is clear that $y(t)\to 0$ as $t\to\infty$. On the contrary, assuming $\int_a^\infty[-\frac{sp(s)}{r(s)}]^{\frac{1}{\alpha-1}}ds<\infty$ and integrating (39) from t to ∞, it follows

$$\ln l-\ln y(t)=\int_t^\infty\frac{y'(s)}{y(s)}ds=-\int_t^\infty(1+\varepsilon(s))\left[-\frac{sp(s)}{(\delta+1)r(s)}\right]^{\frac{1}{\alpha-1}}ds,$$

with $l=\lim_{t\to\infty}y(t)$, which implies (34). It is clear that l must be positive. □

The next remark reveals that the condition guaranteeing normalized slow variation of decreasing solutions can be relaxed. Moreover, this condition is shown to be necessary for the existence of a decreasing slowly varying solution of (1). In addition, in Remark 8, we prove that slowly varying solutions necessarily decrease.

Remark 7. (i) From the proof of Theorem 5 it can be deduced that $\mathbb{M}^-\subset NSV$ follows from the weaker conditions

$$\int_a^\infty p(t)dt<\infty,\qquad \int_a^\infty r(t)^{1-\beta}dt=\infty$$

and

$$\lim_{t\to\infty}\frac{t^{\alpha-1}}{r(t)}\int_t^\infty p(s)ds=0. \tag{40}$$

We point out that, in this case, it is not necessary to assume the regular variation of p or r.

(ii) We now show the necessity of (40). More precisely, we prove the following statement. Assume $r\in RV(\delta+\alpha)$ with $\delta<-1$. If there exists $y\in\mathbb{M}^-\cap NSV$, then (40) holds. Indeed, set $w=r\Phi(y'/y)=-r(-y'/y)^{\alpha-1}$. Then w satisfies the generalized Riccati equation (3) for large t, and $0<-\frac{t^{\alpha-1}}{r(t)}w(t)=\left(\frac{-ty'(t)}{y(t)}\right)^{\alpha-1}\to 0$ as

$t \to \infty$, because $y \in NSV$. Hence, there exists $M > 0$ such that $|w(t)| \le Mr(t)t^{1-\alpha} \in RV(\delta+1)$, and so $w(t) \to 0$ as $t \to 0$, because $\delta < -1$. Further, since $y \in NSV$, there exists $N > 0$ such that $r^{1-\beta}(t)|w(t)|^\beta \le Nr(t)/t^\alpha \in RV(\delta)$, which implies $\int_a^\infty r^{1-\beta}(s)|w(s)|^\beta ds < \infty$. Integrating (3) from t to ∞ and multiplying by $t^{\alpha-1}/r(t)$ we obtain

$$-\frac{t^{\alpha-1}}{r(t)}w(t) = \frac{t^{\alpha-1}}{r(t)}\int_t^\infty p(s)\,ds + (\alpha-1)z(t), \quad z(t) := \frac{t^{\alpha-1}}{r(t)}\int_t^\infty r^{1-\beta}(s)|w(s)|^\beta ds. \tag{41}$$

We claim that $z(t) \to 0$ as $t \to \infty$. Without loss of generality we may assume $r \in NRV(\delta+\alpha) \cap C^1$. Indeed, if r is not normalized or is not in C^1, then we can take $\tilde{r} \in NRV(\delta+\alpha) \cap C^1$ with $\tilde{r}(t) \sim r(t)$ when $t \to \infty$, and we have

$$z(t) \sim \frac{t^{\alpha-1}}{\tilde{r}(t)}\int_t^\infty \tilde{r}^{1-\beta}(s)|\tilde{r}(s)\Phi(y'(s)/y(s))|^\beta ds.$$

By the L'Hospital rule,

$$\begin{aligned}
\lim_{t\to\infty} z(t) &= \lim_{t\to\infty} \frac{-r^{1-\beta}(t)|w(s)|^\beta}{r'(t)t^{1-\alpha} + (1-\alpha)r(t)t^{-\alpha}} \\
&= \lim_{t\to\infty} \frac{r^{-\beta}(t)|w(t)|^\beta t^\alpha}{-tr'(t)/r(t) + (\alpha-1)} \\
&= \lim_{t\to\infty} \frac{|ty'(t)/y(t)|^\alpha}{-tr'(t)/r(t) + (\alpha-1)} = \frac{0}{-\delta-\alpha+\alpha-1} = 0.
\end{aligned}$$

Condition (40) then follows from (41).

If, in addition $p \in RV(\delta)$, then the necessary condition may read as $L_p(t)/L_r(t) \to 0$ as $t \to \infty$. A closer examination of the proofs shows that the condition $r \in RV(\delta+\alpha)$ can be relaxed to the existence of $r_i \in RV(\delta_i+\alpha)$, $i = 1, 2$, with $r_1(t) \le r(t) \le r_2(t)$ for large t, and $\delta_1, \delta_2 < -1$.

The next remark shows that SV solutions cannot increase. Hence, in Theorem 5 we are dealing with all SV solutions of (1).

Remark 8. Assume $p \in RV(\delta), r \in RV(\delta+\alpha)$, with $\delta < -1$, and take $y \in \mathbb{M}^+$. Then $\Phi(y') = (y')^{\alpha-1}$ and $\Phi(y) = y^{\alpha-1}$. Since y is positive, then $r(y')^{\alpha-1}$ is positive increasing, hence there exists a positive constant M such that $r(t)y'(t)^{\alpha-1} \ge M$ for t sufficiently large. Dividing by $r(t)$ and raising by $\frac{1}{\alpha-1}$, it follows that $y'(t) \ge (\frac{M}{r(t)})^{\frac{1}{\alpha-1}}$, which implies

$$y(t) \ge y(a) + M^{\frac{1}{\alpha-1}}\int_a^t \left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha-1}} ds. \tag{42}$$

Since $r \in RV(\delta+\alpha)$, it holds $(\frac{1}{r})^{\frac{1}{\alpha-1}} \in RV(\frac{-\delta-\alpha}{\alpha-1})$. From hypothesis, $\delta < -1$, thus $-\delta-\alpha > 1-\alpha$, i.e. $\frac{-\delta-\alpha}{\alpha-1} > -1$. Applying Karamata's theorem, we then get that

$$\int_a^t \left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha-1}} ds \in RV\left(\frac{-\delta-\alpha}{\alpha-1}+1\right) = RV\left(\frac{-\delta-1}{\alpha-1}\right).$$

Since $\delta < -1$, it follows that $\frac{-\delta-1}{\alpha-1} > 0$, therefore (42) implies that y is greater than or equal to a $RV(\frac{-\delta-1}{\alpha-1})$ function, and therefore cannot be SV. We have so proved that if $\delta < -1$, then $\mathbb{M} \cap SV \subseteq \mathbb{M}^-$. Observe that regular variation of p actually was not used.

Remark 9. (i) If $\alpha = 2$ and $r(t) = 1$, then Theorem 5 reduces to [8, Theorem 0.1-A].

(ii) Under the conditions of Theorem 5-(i), it does not follow that

$$y(t) \sim \exp\left\{-\int_a^t \left(\frac{sp(s)}{-(\delta+1)r(s)}\right)^{\frac{1}{\alpha-1}} ds\right\}$$

as $t \to \infty$. This fact was observed already in the linear case (and with $r(t) = 1$), see [8, Remark 2].

4.2 Increasing solutions in the case $\delta > -1$

The next theorem deals with SV solutions in the complementary case $\delta > -1$. As it follows from subsequent Remark 11, we must sought for SV solutions among elements of $\mathbb{M}^+$. The result is new also in the linear case.

Theorem 6. *Let $p \in RV(\delta)$ and $r \in RV(\delta + \alpha)$ with $\delta > -1$. If $\dfrac{L_p(t)}{L_r(t)} \to 0$ as $t \to \infty$, then $y \in \Pi(ty'(t))$ for every $y \in \mathbb{M}^+$. Moreover, for every $y \in \mathbb{M}^+$,*

(i) if $\displaystyle\int_a^\infty \left(\frac{sp(s)}{r(s)}\right)^{\frac{1}{\alpha-1}} ds = \infty$, then there exists $\varepsilon(t)$ with $\varepsilon(t) \to 0$ as $t \to \infty$ such that

$$y(t) = \exp\left\{\int_a^t (1+\varepsilon(s))\left(\frac{sp(s)}{(\delta+1)r(s)}\right)^{\frac{1}{\alpha-1}} ds\right\} \tag{43}$$

with $y \in \mathbb{M}^+_\infty$.

(ii) if $\displaystyle\int_a^\infty \left(\frac{sp(s)}{r(s)}\right)^{\frac{1}{\alpha-1}} ds < \infty$, then there exists $\varepsilon(t)$ with $\varepsilon(t) \to 0$ as $t \to \infty$ such that

$$y(t) = \exp\left\{-\int_t^\infty (1+\varepsilon(s))\left(\frac{sp(s)}{(\delta+1)r(s)}\right)^{\frac{1}{\alpha-1}} ds\right\}, \tag{44}$$

where $l = \lim_{t\to\infty} y(t) \in (0, \infty)$, with $y \in \mathbb{M}^+_B$.

Proof. First note that, since $\delta > -1$, we have $\int_a^\infty p(s)ds = \infty$. Take $y \in \mathbb{M}^+$. We first prove that $y \in NSV$. By definition, $\Phi(y') = (y')^{\alpha-1}$ for $t > t_0$ sufficiently large and hence, integrating (1) from t_0 to t and recalling that y is increasing we get

$$\begin{aligned} r(t)y'(t)^{\alpha-1} &= r(t_0)y'(t_0)^{\alpha-1} + \textstyle\int_{t_0}^t p(s)y(s)^{\alpha-1}ds \\ &\geq r(t_0)y'(t_0)^{\alpha-1} + y(t_0)^{\alpha-1}\textstyle\int_{t_0}^t p(s)ds \to \infty \end{aligned} \tag{45}$$

as $t \to \infty$. Moreover, it is possible to find a positive constant A such that

$$r(t)y'(t)^{\alpha-1} \leq A\int_{t_0}^t p(s)y(s)^{\alpha-1}ds$$

for large t. Now, dividing the last inequality by $r(t)$ and recalling that y is increasing, we get

$$y'(t)^{\alpha-1} \leq \frac{Ay(t)^{\alpha-1}}{r(t)}\int_{t_0}^t p(s)ds.$$

Thus, dividing by $y(t)^{\alpha-1}$ and multiplying by $t^{\alpha-1}$, we obtain

$$0 < \left(\frac{ty'(t)}{y(t)}\right)^{\alpha-1} \leq A\frac{t^{\alpha-1}}{r(t)}\int_a^t p(s)ds. \tag{46}$$

By Karamata's theorem,

$$\frac{t^{\alpha-1}}{r(t)}\int_a^t p(s)ds \sim \frac{L_p(t)}{L_r(t)}\frac{1}{\delta+1}$$

as $t\to\infty$. Hence, from (46) and the hypothesis, we get $ty'(t)/y(t)\to 0$, which implies $y\in NSV$.

From $py^{\alpha-1}\in RV(\delta)$ and (45), we obtain $ry'^{\alpha-1}\in RV(\max\{0,\delta+1\}) = RV(\delta+1)$, because $\delta>-1$. Consequently, $(y')^{\alpha-1}\in RV(1-\alpha)$, i.e. $y'\in RV(-1)$, and concluding as in (37) we obtain $y\in \Pi(ty')$.

Since $py^{\alpha-1}\in RV(\delta)$, with $\delta>-1$, from (45) and Karamata's theorem we get

$$\begin{aligned} r(t)y'(t)^{\alpha-1} &\sim \textstyle\int_{t_0}^t p(s)y(s)^{\alpha-1}ds \sim \int_a^t s^\delta L_p(s)L_y^{\alpha-1}(s)ds \\ &\sim \textstyle\frac{1}{\delta+1}t^{\delta+1}L_p(t)L_y^{\alpha-1}(t) = \frac{1}{\delta+1}tp(t)y(t)^{\alpha-1}. \end{aligned}$$

Thus, dividing by $r(t)y(t)^{\alpha-1}$, we get $(\frac{y'(t)}{y(t)})^{\alpha-1}\sim\frac{tp(t)}{(\delta+1)r(t)}$, hence, raising by $\frac{1}{\alpha-1}$,

$$\frac{y'(t)}{y(t)}\sim\left(\frac{tp(t)}{(\delta+1)r(t)}\right)^{\frac{1}{\alpha-1}}.$$

Therefore there exists a function $\varepsilon(t)$, with $\lim_{t\to\infty}\varepsilon(t)=0$, such that

$$\frac{y'(t)}{y(t)} = (1+\varepsilon(t))\left(\frac{tp(t)}{(\delta+1)r(t)}\right)^{\frac{1}{\alpha-1}}. \tag{47}$$

Assume now that $\int_a^\infty(\frac{tp(t)}{(\delta+1)r(t)})^{\frac{1}{\alpha-1}}dt=\infty$. Integrating (47) from a to t, we get

$$\ln y(t)-\ln y(a) = \int_a^t (1+\varepsilon(s))\left(\frac{sp(s)}{(\delta+1)r(s)}\right)^{\frac{1}{\alpha-1}}ds,$$

and (43) follows with $y(t)\to\infty$ as $t\to\infty$. Otherwise, if $\int_a^\infty(\frac{tp(t)}{(\delta+1)r(t)})^{\frac{1}{\alpha-1}}dt<\infty$, then, integrating (47) from t to ∞, we obtain

$$\ln y(t) - l = -\int_t^\infty (1+\varepsilon(s))\left(\frac{sp(s)}{(\delta+1)r(s)}\right)^{\frac{1}{\alpha-1}}ds,$$

with $l=\lim_{t\to\infty}y(t)$, which yields (44). Clearly, $l\in(0,\infty)$. □

The next remark claims that sufficient conditions guaranteeing $\mathbb{M}^+\subset NSV$ can be relaxed. Moreover, the condition (48) which can be understood as a counterpart to (40) is necessary for the existence of an increasing SV solution.

Remark 10. The proof of Theorem 6 easily implies $\mathbb{M}^+ \subset NSV$ when assuming, instead of the regular variation of p and r, the weaker conditions

$$\int_a^\infty p(t)dt = \infty$$

and

$$\lim_{t\to\infty} \frac{t^{\alpha-1}}{r(t)} \int_a^t p(s)ds = 0. \tag{48}$$

Similarly as in Remark 7, it is possible to prove the following statement. Assume $r \in RV(\delta + \alpha)$ with $\delta > -1$. If there exists $y \in \mathbb{M}^+ \cap NSV$, then (48) holds. Note that in this case, instead of (41), we work with the Riccati type integral equation of the form

$$\frac{t^{\alpha-1}w(t)}{r(t)} - \frac{t^{\alpha-1}}{r(t)} = \frac{t^{\alpha-1}}{r(t)} \int_a^t p(s)\, ds - \frac{t^{\alpha-1}}{r(t)}(\alpha-1) \int_a^t r^{1-\beta}(s)|w(s)|^\beta ds.$$

Here, $t^{\alpha-1}/r(t) \to 0$ and $t^{\alpha-1}w(t)/r(t) \to 0$ as $t \to \infty$.

If, in addition $p \in RV(\delta)$, then a necessary condition reads as $L_p(t)/L_r(t) \to 0$ as $t \to \infty$. Moreover, the condition $r \in RV(\delta + \alpha)$ can be relaxed to the existence of $r_i \in RV(\delta_i + \alpha)$, $i = 1, 2$, with $r_1(t) \le r(t) \le r_2(t)$ for large t, and $\delta_1, \delta_2 > -1$.

As we will see next, slowly varying solutions cannot decrease in our current setting. Thus, in Theorem 6 we are dealing with all SV solutions.

Remark 11. Assume $p \in RV(\delta), r \in RV(\delta + \alpha)$, with $\delta > -1$, and let $y \in \mathbb{M}^-$. Then $\Phi(y') = -(-y')^{\alpha-1}$ and $\Phi(y) = y^{\alpha-1}$. Integrating (1) from a to t, we get

$$-r(t)(-y'(t))^{\alpha-1} = -r(a)(-y'(a))^{\alpha-1} + \int_a^t p(s)y(s)^{\alpha-1}ds. \tag{49}$$

From the fact that $-r(-y')^{\alpha-1}$ is negative increasing, we obtain that there exists $\lim_{t\to\infty} -r(t)(-y'(t))^{\alpha-1} = M \in (-\infty, 0]$. Suppose now that $y \in SV$. Then $py^{\alpha-1} \in RV(\delta)$ thus $\int_a^t p(s)y(s)^{\alpha-1}ds \to \infty$, because $\delta > -1$, a contradiction. We have so proved that if $\delta > -1$, then $\mathbb{M} \cap SV \subseteq \mathbb{M}^+$. Observe that regular variation of r actually was not used.

4.3 Related observations and example

We start with justifying the fact that the relation about the indices of regular variation of the coefficients is quite natural when looking for slowly varying solutions.

Remark 12. Assume that $r \in RV(\gamma), p \in RV(\delta)$, and that $y \in SV$ is a solution of equation (1). Then $(r\Phi(y'))' = py^{\alpha-1} \in RV(\delta)$. Suppose first $\delta > -1$. Then integrating (1) we have $r(t)\Phi(y'(t)) = r(a)\Phi(y'(a)) + \int_a^t p(s)y^{\alpha-1}(s)\, ds \to +\infty$, thus, without loss of generality, we may assume $y'(t) > 0$ for $t \ge a$. Hence, $r(t)(y'(t))^{\alpha-1} = r(a)(y'(a))^{\alpha-1} + \int_a^t p(s)y^{\alpha-1}(s)\, ds \in RV(\max\{0, \delta+1\}) = RV(\delta+1)$ and so $(y')^{\alpha-1} \in RV(\delta+1-\gamma)$, because $r \in RV(\gamma)$, which implies $y' \in RV(\frac{\delta+1-\gamma}{\alpha-1})$. Now, if $\frac{\delta+1-\gamma}{\alpha-1} \neq -1$, similarly as before it is possible to prove that $y \in RV(\frac{\delta+1-\gamma}{\alpha-1} + 1)$, therefore, since $y \in SV$, we get $\frac{\delta+1-\gamma}{\alpha-1} + 1 = 0$, a contradiction. Thus $\frac{\delta+1-\gamma}{\alpha-1} = -1$, i.e. $\gamma = \delta + \alpha$. It turns out that $\delta > -1$ is equivalent to $\gamma > \alpha - 1$, so now let $\gamma < \alpha - 1$. This implies $\gamma(1-\beta) > -1$, consequently $\int_a^\infty r^{1-\beta}(s)\,ds = \infty$. Reasoning like in Remark 8 we obtain that $y \in \mathbb{M}^-$, then $r(t)(-y'(t))^{\alpha-1} = \int_t^\infty p(s)y(s)^{\alpha-1}ds \in RV(\delta+1)$, according to (35) and Karamata's theorem. Similarly as before we get $\gamma = \delta + \alpha$.

The results from Theorems 5 and 6 can be unified to obtain the following corollary.

Corollary 5. *Assume that $p \in RV(\delta)$ and $r \in RV(\delta + \alpha)$, with $\delta \neq -1$ and $L_p(t)/L_r(t) \to 0$ as $t \to \infty$. Then, for every $y \in \mathbb{M}$ there exists $\varepsilon(t)$ with $\varepsilon(t) \to 0$ as $t \to \infty$ such that*

(i) if $\int_a^\infty (\frac{sp(s)}{r(s)})^{\frac{1}{\alpha-1}} ds = \infty$, then

$$y(t) = \exp\left\{\operatorname{sgn}(\delta+1)\int_a^t (1+\varepsilon(s))\left(\frac{sp(s)}{|\delta+1|r(s)}\right)^{\frac{1}{\alpha-1}} ds\right\}$$

(ii) if $\int_a^\infty (\frac{sp(s)}{r(s)})^{\frac{1}{\alpha-1}} ds < \infty$, then

$$y(t) = l\exp\left\{\operatorname{sgn}(\delta+1)\int_t^\infty (1+\varepsilon(s))\left(\frac{sp(s)}{|\delta+1|r(s)}\right)^{\frac{1}{\alpha-1}} ds\right\},$$

where $l = \lim_{t\to\infty} y(t) \in (0, \infty)$.

Remarks 8 and 11 in combination with Theorems 5 and 6 yield the following corollary.

Corollary 6. *Assume that $p \in RV(\delta), r \in RV(\delta+\alpha)$ with $\frac{L_p(t)}{L_r(t)} \to 0$,*

(i) Let $\delta < -1$. Then

(a) if $\int_a^\infty (\frac{sp(s)}{r(s)})^{\frac{1}{\alpha-1}} ds = \infty$ then $\mathbb{M} \cap SV = \mathbb{M}^- = \mathbb{M}_0^-$;

(b) if $\int_a^\infty (\frac{sp(s)}{r(s)})^{\frac{1}{\alpha-1}} ds < \infty$ then $\mathbb{M} \cap SV = \mathbb{M}^- = \mathbb{M}_B^-$.

(ii) Let $\delta > -1$. Then

(a) if $\int_a^\infty (\frac{sp(s)}{r(s)})^{\frac{1}{\alpha-1}} ds = \infty$ then $\mathbb{M} \cap SV = \mathbb{M}^+ = \mathbb{M}_\infty^+$;

(b) if $\int_a^\infty (\frac{sp(s)}{r(s)})^{\frac{1}{\alpha-1}} ds < \infty$ then $\mathbb{M} \cap SV = \mathbb{M}^+ = \mathbb{M}_B^+$.

In the next remark we discuss relations between the conditions from Theorems 5 and 6 which involve the integral $\int_a^\infty (\frac{sp(s)}{r(s)})^{\frac{1}{\alpha-1}} ds$ and the integral conditions involving J_1 and J_2 from the general existence theory (see Preliminaries) under our setting.

Remark 13. Assume that $p \in RV(\delta)$ and $r \in RV(\gamma)$.

We recall, first of all, that $\mathbb{M}^+ = \mathbb{M}_B^+$ if and only if $J_1 < \infty$, while $\mathbb{M}^+ = \mathbb{M}_\infty^+$ if and only if $J_1 = \infty$. Notice that, if $\delta > -1$, then $\int_a^\infty p(s)ds = \infty$, and, according to Karamata's theorem, $(\frac{\int_a^t p(s)ds}{r(t)})^{\beta-1} \sim t^{(\delta+1-\gamma)(\beta-1)}(\frac{L_p(t)}{(\delta+1)L_r(t)})^{\beta-1}$. Under our natural assumption $\gamma = \delta + \alpha$, we obtain that $J_1 = \infty$ if and only if $\int_a^\infty \frac{1}{t}(\frac{L_p(t)}{L_r(t)})^{\frac{1}{\alpha-1}} dt = \infty$, which precisely corresponds to our conditions in setting (ii) of Corollary 6.

Recall now that $\mathbb{M}^- = \mathbb{M}_B^+$ if and only if $J_1 = \infty$ and $J_2 < \infty$, while $\mathbb{M}^- = \mathbb{M}_0^-$ if and only if $J_2 = \infty$. Suppose $\delta < -1$. Then $\int_a^\infty p(s)ds < \infty$, thus $J_1 = \infty$ if $\gamma < \frac{1}{\beta-1} = \alpha - 1$, which is fulfilled when $\gamma = \alpha + \delta$. Moreover, again from Karamata's theorem,

$$\left(\frac{\int_t^\infty p(s)ds}{r(t)}\right)^{\beta-1} \sim -t^{(\delta+1-\gamma)(\beta-1)}\left(\frac{L_p(t)}{(\delta+1)L_r(t)}\right)^{\beta-1} = \frac{1}{t}\left(\frac{L_p(t)}{(\delta+1)L_r(t)}\right)^{\frac{1}{\alpha-1}}$$

recalling that we assume $\gamma = \delta + \alpha$. Therefore, also in this case, we obtain that the integral conditions in setting (i) of Corollary 6 are equivalent to the known general existence conditions involving J_1 and J_2.

Example 3. Consider equation (1) with $p(t) = t^\delta L_p(t)$ and $r(t) = t^{\delta+\alpha}$, where $L_p(t) = (\ln t)^{\gamma_1} + h_1(t), L_r(t) = (\ln t)^{\gamma_2} + h_2(t)$, with $|h_i(t)| = o((\ln t)^{\gamma_i}), i = 1, 2$, for some $\gamma_1 < \gamma_2$. For example, both $h_i(t) = \cos t$ or $h_i(t) = \ln(\ln(t))$ satisfy the previous condition. Trivially, for every $\lambda > 0$, we have that

$$\begin{aligned}\lim_{t\to\infty}\frac{(\ln\lambda t)^{\gamma_i} + h_i(\lambda t)}{(\ln t)^{\gamma_i} + h_i(t)} &= \lim_{t\to\infty}\frac{(\frac{\ln(\lambda t)}{\ln t})^{\gamma_i} + \frac{h_i(\lambda t)}{(\ln t)^{\gamma_i}}}{1 + \frac{h_i(t)}{(\ln t)^{\gamma_i}}}\\ &= \lim_{t\to\infty}\frac{(\frac{\ln(\lambda t)}{\ln t})^{\gamma_i} + \frac{h_i(\lambda t)}{(\ln\lambda t)^{\gamma_i}}(\frac{\ln\lambda t}{\ln t})^{\gamma_i}}{1 + \frac{h_i(t)}{(\ln t)^{\gamma_i}}} = 1,\end{aligned}$$

i.e. $L_p, L_r \in SV$. Now,

$$\begin{aligned}\lim_{t\to\infty}\frac{L_p(t)}{L_r(t)} &= \lim_{t\to\infty}\frac{(\ln t)^{\gamma_1} + h_1(t)}{(\ln t)^{\gamma_2} + h_2(t)} = \lim_{t\to\infty}\frac{1 + \frac{h_1(t)}{(\ln t)^{\gamma_1}}}{(\ln t)^{\gamma_2-\gamma_1} + \frac{h_2(t)}{(\ln t)^{\gamma_1}}}\\ &= \lim_{t\to\infty}\frac{1 + \frac{h_1(t)}{(\ln t)^{\gamma_1}}}{(\ln t)^{\gamma_2-\gamma_1}[1 + \frac{h_2(t)}{(\ln t)^{\gamma_2}}]} = 0,\end{aligned} \tag{50}$$

because $\gamma_2 > \gamma_1$. Finally, according to (50), it follows that

$$\left(\frac{tp(t)}{r(t)}\right)^{\frac{1}{\alpha-1}} = \left(\frac{t^{\delta+1}L_p(t)}{t^{\delta+\alpha}L_r(t)}\right)^{\frac{1}{\alpha-1}} = \frac{1}{t}\left\{\frac{1 + \frac{h_1(t)}{(\ln t)^{\gamma_1}}}{(\ln t)^{\gamma_2-\gamma_1}[1 + \frac{h_2(t)}{(\ln t)^{\gamma_2}}]}\right\}^{\frac{1}{\alpha-1}} \sim \frac{1}{t}(\ln t)^{\frac{\gamma_1-\gamma_2}{\alpha-1}}$$

as $t \to \infty$. Thus, since $\int_a^t \frac{1}{s}(\ln s)^\lambda ds < \infty$ if and only if $\lambda < -1$, we have that $\int_a^t (\frac{sp(s)}{r(s)})^{\frac{1}{\alpha-1}} ds < \infty$ if and only if $\xi := \frac{\gamma_1-\gamma_2}{\alpha-1} < -1$. In this case, according to Corollary 5, every solution has a finite non zero limit l and

$$y(t) = l\exp\left\{\operatorname{sgn}(\delta+1)(1+o(1))(\ln t)^{\xi+1}\frac{-1}{\xi+1}\cdot\frac{1}{|\delta+1|^{\frac{1}{\alpha-1}}}\right\}$$

as $t \to \infty$. Further, if $\xi > -1$, then increasing solutions are unbounded, while decreasing solutions have zero limit, and

$$y(t) = \exp\left\{\operatorname{sgn}(\delta+1)(1+o(1))(\ln t)^{\xi+1}\frac{1}{\xi+1}\cdot\frac{1}{|\delta+1|^{\frac{1}{\alpha-1}}}\right\}$$

as $t \to \infty$. Finally, if $\xi = -1$, then

$$y(t) = (\ln t)^{\operatorname{sgn}(\delta+1)(1+o(1))|\delta+1|^{-\frac{1}{\alpha-1}}}$$

as $t \to \infty$.

Remark 14. For related results (sufficient and necessary conditions for the existence of RV solutions of (1)) see [14] where generalized regular variation and a fixed point theorem play important roles. Compare also with Remark 4 and the discussion on conditions (40) and (48).

5 Some open problems

In this last paragraph we indicate some directions for a possible future research related to the above topics:

- To establish a "second order" result associated to Theorems 2 and 3, i.e., to take a closer look at the behavior of $y'(t) \mp \left(\frac{p(t)}{r(t)(\alpha-1)}\right)^{\frac{1}{\alpha}} y(t)$, y being a solutions of (1), cf. (24), in the sense of [19, Section 3.2] and [21, Section 5.1], where the linear equation $y'' = p(t)y$ is considered.
- To give an improvement of Theorems 5 and 6 in the sense of [8, Theorem 0.1-B], where the linear equation $y'' = p(t)y$ is considered and the class $\Pi R_2(u, v)$ is utilized.
- To deal with $RV(\vartheta)$ solutions (ϑ being a certain positive number) in the situation of Theorem 5, in particular, to show $\mathbb{M}^+ \subset RV(\vartheta)$ and derive an asymptotic formula. Similarly for Theorem 6. Note that in contrast to the linear case, the reduction of order formula or some usual transformation tricks are not at our disposal.
- To establish asymptotic formulas for $RV(\varrho)$ solutions with ϱ different from 0 and ϑ (here we mean the ϑ from the previous item), and possibly under a more general integral condition, in the sense of [10, Theorem 1.2], where the linear equation $y'' = p(t)y$ is considered.
- To examine the borderline case for δ (in Theorems 5 and 6), namely $\delta = -1$.
- To extend (some of) the above results to the so-called nearly half-linear equation, i.e., $(r(t)G(y'))' = p(t)F(y)$, where F, G are regularly varying functions with the same positive index. Some observations along this line (more precisely, extension of Theorem 5 to the equation where $F(|\cdot|), G(|\cdot|)$ are regularly varying at zero of index 1) can be found in [25].

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