HARNACK INEQUALITY FOR HYPOELLIPTIC SECOND ORDER PARTIAL DIFFERENTIAL OPERATORS

ALESSIA E. KOGOJ AND SERGIO POLIDORO

Abstract. We consider nonnegative solutions $u : \Omega \longrightarrow \mathbb{R}$ of second order hypoelliptic equations

$$\mathscr{L}u(x) = \sum_{i,j=1}^{n} \partial_{x_i} \left(a_{ij}(x) \partial_{x_j} u(x) \right) + \sum_{i=1}^{n} b_i(x) \partial_{x_i} u(x) = 0,$$

where Ω is a bounded open subset of $\mathbb{R}^n$ and x denotes the point of Ω. For any fixed $x_0 \in \Omega$, we prove a Harnack inequality of this type

$$\sup_K u \le C_K u(x_0) \qquad \forall\, u \ \text{ s.t. } \ \mathscr{L}u = 0, u \ge 0,$$

where K is any compact subset of the interior of the *$\mathscr{L}$-propagation set of* x_0 and the constant C_K does not depend on u.

1. Introduction

We consider second order partial differential operators $\mathscr{L}$ acting on functions $u \in C^2(\Omega)$ as follows

$$\mathscr{L}u(x) := \sum_{i,j=1}^{n} \partial_{x_i} \left(a_{ij}(x) \partial_{x_j} u(x) \right) + \sum_{i=1}^{n} b_i(x) \partial_{x_i} u(x) \tag{1.1}$$

for x belonging to any open *bounded* subset Ω of $\mathbb{R}^n$. The coefficients a_{ij}, b_i are real functions and belong to $C^\infty(\overline{\Omega})$ for $1 \le i, j \le n$. Moreover, $A := (a_{ij})$ is a $n \times n$ symmetric and non-negative matrix. We also assume the following hypotheses:

- (H1) $\mathscr{L} - \beta$ *and* $\mathscr{L}^*$ *are hypoelliptic* for every constant $\beta \ge 0$;
- (H2) $\inf_\Omega \ a_{11} > 0$.

We recall that $\mathscr{L}$ is said hypoelliptic if every distribution u in Ω such that $\mathscr{L}u \in C^\infty(\Omega)$ is a smooth function. We note that condition (H2) ensures that for every $x \in \Omega$ there exists $\xi \in \mathbb{R}^n$ such that $\langle A(x)\xi, \xi \rangle > 0$ that is $\mathscr{L}$ is non-totally degenerate, in accordance with Definition 5.1 in [Bon69]. We can drop condition (H2) if the operator $\widetilde{\mathscr{L}} = \partial^2_{x_{n+1}} + \mathscr{L}$ acting on $\mathbb{R}^{n+1}$ satisfies (H1) (see Corollary 2).

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The main result of this paper is the following Harnack inequality for the non-negative solutions of the equation $\mathscr{L} u = 0$. It obviously applies to the Laplacian and to the heat operators and in these cases it restores the classical elliptic and parabolic Harnack inequalities.

Theorem 1. *Assume that $\mathscr{L}$ statisfies (H1) and (H2). Let x_0 in Ω and let K be any compact set contained in the interior of $\overline{\mathscr{P}(x_0, \Omega)}$, then there exists a positive constant $C = C(x_0, K, \Omega, \mathscr{L})$ such that*

$$\sup_K u \le C u(x_0),$$

for every non-negative solution u of $\mathscr{L} u = 0$ in Ω.

We introduce here the definition of $\mathscr{L}$-*propagation set* $\mathscr{P}(x_0, \Omega)$ appearing it the above statement. It is the set of all points x reachable from x_0 by a *propagation path*:

$$\mathscr{P}(x_0, \Omega) := \{x \in \Omega \mid \exists\, \gamma\ \mathscr{L}\text{-propagation path}, \gamma(0) = x_0, \gamma(T) = x\}.$$

A $\mathscr{L}$-*propagation path* is any absolutely continuous path $\gamma : [0, T] \longrightarrow \Omega$ such that

$$\gamma'(t) = \sum_{j=1}^{n} \lambda_j(t) X_j(\gamma(t)) + \mu(t) Y(\gamma(t)) \qquad \text{a.e. in } [0, T]$$

for suitable piecewise constant real functions $\lambda_1, \dots, \lambda_n$, and μ, $\mu \ge 0$.
$X_1(x), \dots, X_n(x), Y(x)$ are the vector fields defined in the following way:

$$X_j(x) := \sum_{i=1}^{n} a_{ji}(x) \partial_{x_i}, \quad j = 1, \dots, n, \qquad Y(x) := \sum_{i=1}^{n} b_i(x) \partial_{x_i}. \tag{1.2}$$

As we said at the beginning of the Introduction, our main result can holds also under somehow weaker assumptions on $\mathscr{L}$. Only in the following Corollary the hypotheses (H1) and (H2) on $\mathscr{L}$ are replaced by the assumption that the operator $\widetilde{\mathscr{L}} = \partial^2_{x_{n+1}} + \mathscr{L}$ in $\mathbb{R}^{n+1}$ satisfies (H1). Of course if $\widetilde{\mathscr{L}}$ satisfies assumption (H1) then also $\mathscr{L}$ does. A simple example of operator satisfying the hypotheses of this Corollary but not the ones of Theorem 1 is $\partial_{x_1} = {x_1}^2 \partial^2_{x_2}$ in $\mathbb{R}^2$.

Corollary 2. *If the operator $\widetilde{\mathscr{L}} = \partial^2_{x_{n+1}} + \mathscr{L}$ acting on $\Omega \times \mathbb{R}$ satisfies (H1) then for every x_0 in Ω and for every compact set K contained in the interior of the $\mathscr{L}$-propagation set $\overline{\mathscr{P}(x_0, \Omega)}$, there exists a positive constant $C = C(x_0, K, \Omega, \mathscr{L})$ such that*

$$\sup_K u \le C u(x_0),$$

for every non-negative solution u of $\mathscr{L} u = 0$ in Ω.

The notion of propagation set $\mathscr{P}(x_0, \Omega)$ has been introduced by Amano in his work on maximum principle (see [Ama79, Theorem 2]). In our case it reads as follows.

Assume that u is a (smooth) solution of $\mathscr{L}u = 0$ in Ω. If u attains its maximum at a point x_0 in Ω, then $u \equiv u(x_0)$ in $\overline{\mathscr{P}(x_0, \Omega)}$.

The Amano maximum principle is a crucial tool to prove Theorem 1 using the abstract Harnack inequality of the Potential Theory ([CC72, Proposition 6.1.5]). We observe that Harnack inequalities based on results of Potential Theory were proved in [BLU02, Theorem 4.2] for heat equations on Carnot groups and in [CNP10, Theorem 1.1] and in [CMP15, Theorem 5.2] for more general evolution equations. The class of hypoelliptic operators considered in [CNP10, CMP15] is

$$\sum_{j=1}^{m} \widetilde{X}_j^2(y) + \widetilde{X}_0(y) - \partial_t, \tag{1.3}$$

where $\widetilde{X}_j$ are smooth vector fields on $\mathbb{R}^N$ and (y, t) denoted the point of any subset of $\mathbb{R}^{N+1}$. We explicitly note that operator (1.3) is a particular example of the operators (1.1), with respect to the variable $x = (y, t)$. In both papers the operators are assumed left translation invariant w.r.t. a Lie group in $\mathbb{R}^{N+1}$ and endowed with a *global* fundamental solution. We point out that the use of the fundamental solution is a key step in verifying the *separation axiom* of the axiomatic Potential Theory.

The approach used in this note allows us to prove the validity of the separation axiom in Section 2.4 without requiring the existence of any global fundamental solution on every bounded open set. We rely only on hypoellipticity, on non total degeneracy of $\mathscr{L}$ and the following maximum principle due to Picone that we recall for the sake of completeness.

Let V be any open (bounded) subset of Ω. Assume that exists a function $w : V \to \mathbb{R}$ such that $\mathscr{L}w < 0$ in V and $\inf_V w > 0$. Then for every $u \in C^2(V)$ such that

$$\mathscr{L}u \geq 0 \quad \text{in } V, \quad \limsup_{x \to \xi} u(x) \leq 0 \quad \forall \xi \in \partial V,$$

we have $u \leq 0$ in V.

In our case the existence of a function w follows from (H2) and from the smoothness of the coefficients. Indeed, under these assumptions, we can choose two positive real constants M and λ such that the function

$$w(x) = w(x_1, \ldots, x_N) = M - e^{\lambda x_1} \tag{1.4}$$

has the required properties.

This paper is organized as follows. In Section 2 all the notions and results from Potential Theory that we need are briefly recalled. In Section 3 we show that the set of the solutions u of $\mathscr{L}u = 0$ in Ω satisfies the axioms of the Doob Potential Theory. In Section 4 we prove that the $\mathscr{L}$-propagation set of x_0 is a subset of the smallest absorbent set containing x_0. In this way we derive the Harnack inequality for the non-negative solutions u of $\mathscr{L}u = 0$. In Section 5 the propagation sets of some meaningful operators are studied. In particular we focus on the following operators: $\partial_{x_1}^2 + x_1\partial_{x_2}$ in $\mathbb{R}^2$ and $\partial_{x_1}^2 + \sin(x_1)\partial_{x_2} + \cos(x_1)\partial_{x_3}$ in $\mathbb{R}^3$, and we show that the geometry of the relevant Harnack inequality may appear either of *parabolic* or *elliptic*-type, depending on the choice of Ω, even if both operators are *parabolic*.

2. Some recalls from Potential Theory

We recall some definitions and results of the Potential Theory that we need to prove our Harnack inequality. For a detailed description of the general theory of *harmonic spaces* we refer to [BLU07, chapter 6], [CC72] and to [Bau66].

2.1. Sheafs of functions and harmonic sheafs in Ω.

Let V be any open subset of Ω. We denote by $\overline{\mathbb{R}}$ the set $\mathbb{R} \cup \{\infty, -\infty\}$ and by $\overline{\mathbb{R}}^V$ the set of functions $u : V \longrightarrow \overline{\mathbb{R}}$. Moreover $C(V, \mathbb{R})$ is the vector space of real continuous functions defined on V.

A map

$$\mathscr{F} : V \mapsto \mathscr{F}(V) \subseteq \overline{\mathbb{R}}^V$$

is a *sheaf of functions* in Ω if

(*i*) $V_1, V_2 \subseteq \Omega$, $V_1 \subseteq V_2$, $u \in \mathscr{F}(V_2) \implies u|_{V_1} \in \mathscr{F}(V_1)$;

(*ii*) $V_\alpha \subseteq \Omega\ \forall \alpha \in \mathcal{A}, u : \bigcup_{\alpha\in A} V_\alpha \longrightarrow \overline{\mathbb{R}}$, $u|_{V_\alpha} \in \mathscr{F}(V_\alpha) \implies u \in \mathscr{F}(\bigcup_{\alpha\in\mathcal{A}} V_\alpha)$.

When $\mathscr{F}(V)$ is a linear subspace of $C(V, \mathbb{R})$ for every $V \subseteq \Omega$, we say that the sheaf of functions $\mathscr{F}$ on V is *harmonic* and we denote it $\mathcal{H}(\Omega)$.

2.2. Regular open sets, harmonic measures and absorbent sets.

Let $\mathcal{H}$ be a harmonic sheaf on Ω. We say that an open set $V \subseteq \Omega$ is *regular* if:

(*i*) $\overline{V} \subseteq \Omega$ is compact and $\partial V \neq \emptyset$;

(*ii*) for every continuous function $\varphi : \partial V \longrightarrow \mathbb{R}$, there exists a unique function in $\mathcal{H}(V)$, that we denote by h_φ^V, such that $h_\varphi^V(x) \underset{x\to\xi}{\longrightarrow} \varphi(\xi)$ for every $\xi \in \partial V$;

(*iii*) if $\varphi \geq 0$ then $h_\varphi^V \geq 0$.

From (*ii*) and (*iii*) it follows that, for every regular set V and for every $x \in V$, the map

$$C(\partial V) \ni \varphi \longmapsto h_\varphi^V(x) \in \mathbb{R}$$

is linear and positive. Thus, the Riesz representation theorem (see e.g. [Rud87]), implies that, for every regular set V and for every $x \in V$, there exists a *regular Borel measure*, that we denote by μ_x^V, supported in ∂V, such that

$$h_\varphi^V(x) = \int_{\partial V} \varphi(y)\ d\mu_x^V(y) \qquad \forall\ \varphi \in C(\partial V).$$

The measure μ_x^V is called the *harmonic measure* related to V and x.

Now, let A be a closed subset of Ω . We say that A is *absorbent* if it contains the supports of all the harmonic measures related to its points. More precisely,

$$\text{for every } x \in A \text{ and every regular set } V \text{ containing } x, \text{ supp } \mu_x^V \subseteq A.$$

If $x_0 \in \Omega$, we define Ω_{x_0} as the smallest absorbent set containing x_0:

$$\Omega_{x_0} := \bigcap_{\substack{A \text{ absorbent} \\ A \ni x_0}} A.$$

2.3. **Superharmonic functions.**

A function $u : \Omega \longrightarrow]-\infty, \infty]$ is called *superharmonic in* Ω if

(i) u is lower semi-continuous;

(ii) for every regular set V, $\overline{V} \subseteq \Omega$, and for every $\varphi \in C(\partial V, \mathbb{R})$, $\varphi \leq u|_{\partial\Omega}$, it follows $u \geq h_\varphi^V$ in V;

(iii) the set $\{x \in \Omega \mid u(x) < \infty\}$ is dense in Ω.

We denote by $\mathscr{S}(\Omega)$ the family of the superharmonic functions on Ω.

By the maximum principle, we have that every function $u \in C^2(\Omega)$ such that $\mathscr{L}u \leq 0$ in Ω is superharmonic (see [BLU07, Proposition 7.2.5]).

2.4. **Doob harmonic spaces and Harnack inequality.**

We say that a harmonic sheaf $\mathcal{H}(\Omega)$ is a *Doob harmonic space* if the following axioms are satisfied.

(A1) *Positivity axiom:*
For every $x \in \Omega$, there exists an open set $V \ni z$ and a function $u \in \mathcal{H}(V)$ such that $u(x) > 0$.

(A2) *Doob convergence axiom:*
Let $(u_n)_{n\in\mathbb{N}}$ be a monotone increasing sequence in $\mathcal{H}(\Omega)$ and let $u := \sup_{n\in\mathbb{N}} u_n$. If the set $\{x \in \Omega \mid u(x) < \infty\}$ is dense in Ω, then $u \in \mathcal{H}(\Omega)$.

(A3) *Regularity axiom:*
There is a basis of the euclidean topology of Ω formed by regular sets.

(A4) *Separation axiom:*
$\mathscr{S}(\Omega)$ separates the points of Ω in this sense: for every y and z in Ω,

$y \neq z$, there exist two non-negative functions u and v in $\mathscr{S}(\Omega)$ such that $u(y)v(z) \neq u(z)v(y)$.

We close the section recalling that in this setting the *abstract Harnack inequality* from the Parabolic Potential Theory holds [CC72, Proposition 6.1.5].

Theorem A. *Let $(\Omega, \mathcal{H})$ be a Doob harmonic space, $x_0 \in \Omega$ and let K be a compact set contained in the interior of Ω_{x_0}, the smallest absorbent set containing x_0. Then there exists a positive constant $C = C(x_0, K, \Omega)$ such that*

$$\sup_K u \leq C u(x_0) \qquad \forall u \in \mathcal{H}(\Omega), u \geq 0.$$

3. The harmonic space of the solutions of $\mathscr{L}u = 0$

We show that the set of the solutions of the equation $\mathscr{L}u = 0$ is a Doob harmonic space in Ω. For every $V \subseteq \Omega$ we consider the harmonic sheaf

$$\mathbb{R}^n \supseteq V \longmapsto \mathcal{H}(V)$$

where

$$\mathcal{H}(V) = \{u \in C^\infty(V) \mid \mathscr{L}u = 0\}$$

and $\mathscr{L}$ is the operator (1.1).

The *positivity axiom* (A1) is plainly verified. Indeed every constant function belongs to $\mathcal{H}(\Omega)$.

(A2) is a consequence of a weak Harnack inequality due to Bony (see [Bon69, Theoreme 7.1]); see also [KL04, Proposition 7.4]).

(A3), i.e. the existence of a basis of the euclidean topology of Ω formed by regular sets, can be proved as in [Bon69, Corollarie 5.2], see also [BLU07, Proposition 7.1.5]. We stress that the tools used in its proof are only the hypoellipticity, the non totally degeneracy of the operator $\mathscr{L}$ and the classical Picone Maximum Principle.

Now we are left to verify the *separation axiom* (A4). As in our setting the constant are superharmonic functions, we need to prove that

$$\forall\, y, z \in \Omega, y \neq z, \exists\, u \in \mathscr{S}(\Omega),\ u \geq 0,\ \text{such that } u(y) \neq u(z). \tag{3.1}$$

Now, let $y = (y_1, \dots, y_n)$ and $z = (z_1, \dots, z_n)$ be two different points in Ω.

We observe that the function $w(x) = w(x_1, \dots, x_N) = M - e^{\lambda x_1}$, as in (1.4), for suitable real positive constants λ and M, is non-negative and $\mathscr{L}w(x) < 0$ for every $x \in \Omega$, hence $w \in \mathscr{S}(\Omega)$.

If $y_1 \neq z_1$, we can choose $u(x) = w(x)$ to separate y and z and we are done.

If $y_1 = z_1$, we set $u(x) = |x - y|^2 + w(x)$. Also in this case, for suitable λ and M, u is non-negative, $u \in C^2(\Omega)$ and $\mathscr{L}u(x) = \mathscr{L}(|x - y|^2) + \mathscr{L}(w(x)) < 0$ in Ω. Moreover $u(y) - u(z) = |z - y|^2$, so (3.1) is satisfied.

4. Propagation sets and Harnack inequality

Let $X_1(x), \dots, X_n(x), Y(x)$ be the vector fields defined in the following way:

$$X_i(x) = \sum_{j=1}^{n} a_{ij}(x)\partial_{x_j}, \qquad 1 \leq i \leq n,$$

$$Y(x) = \sum_{i=1}^{n} b_i(x)\partial_{x_i}.$$

We recall that a *$\mathscr{L}$-propagation path* is any absolutely continuous path $\gamma : [0, T] \longrightarrow \Omega$ such that

$$\gamma'(t) = \sum_{j=1}^{n} \lambda_j(t) X_j(\gamma(t)) + \mu(t) Y(\gamma(t)) \qquad \text{a.e. in } [0, T]$$

for suitable piecewise constant real functions $\lambda_1, \dots, \lambda_n$, and μ with $\mu \geq 0$.

For a point x_0 in Ω, we define the *$\mathscr{L}$-propagation set* as the set of all points x such that x and x_0 can be connected by a propagation path, running from x_0 to x:

$$\mathscr{P}(x_0, \Omega) := \{x \in \Omega \mid \exists\, \gamma : [0, T] \to \Omega, \gamma\ \mathscr{L}\text{-propagation path}, \gamma(0) = x_0, \gamma(T) = x\}.$$

Proceeding as in [CMP15, Lemma 5.8], we prove now that the $\mathscr{L}$-propagation set of x_0 is a subset of every absorbent set containing x_0. This Lemma, based on the maximum propagation principle, is a key lemma in order to get our Harnack inequality so we prefer to give here its detailed proof.

Lemma 2. *For every x_0 in Ω, $\mathscr{P}(x_0, \Omega) \subseteq \Omega_{x_0}$.*

Proof. By contradiction, suppose $x \in \mathscr{P}(x_0, \Omega)$ and $x \notin \Omega_{x_0}$. There exists an absolutely continuous path γ connecting x_0 and x:

$$\gamma : [0, T] \longrightarrow \Omega, \qquad \gamma(0) = x_0, \qquad \gamma(T) = x.$$

As Ω_{x_0} is a subset closed in Ω and γ is continuous, there will be a time t_1 such that $\gamma(t_1) = x_1 \in \Omega_{x_0}$ and $\gamma(t) \notin \Omega_{x_0}$ when t is in $]t_1, T]$.

Let's take a regular open set V containing x_1. There will be $t_2 \in]t_1, T]$ such that $x_2 = \gamma(t_2) \in \partial V$. From what we wrote before, x_2 does not belong to Ω_{x_0}.

Take now a neighborhood of x_2, U such that $U \cap \partial V \subseteq \Omega \backslash \Omega_{x_0}$ and consider a function φ defined on ∂V such that φ is strictly positive in $U \cap \partial V$ and 0 otherwise.

$$h_{\varphi}^{V}(x_1) = \int_{\partial V} \varphi(y)\ d\mu_{x_1}^{V}(y) = \int_{U\cap\partial V} \varphi(\zeta)\ d\mu_{x_1}^{V}(\zeta) = 0,$$

because x_1 is in Ω_{x_0} and supp $\mu_{x_1}^{V} \subseteq \Omega_{x_0}$ for every regular set V. But h_{φ}^{V} is nonnegative and it would attain its minimum at x_1. From Amano minimum propagation principle [Ama79, Theorem 2], it would follow that

$$h_{\varphi}^{V}(\gamma(t)) = 0 \qquad \forall\ t \in]t_1, t_2[.$$

In conclusion, we would have that

$$h_{\varphi}^{V}(x) \xrightarrow[x\to x_2]{} \varphi(x_2) > 0,$$

and

$$h_{\varphi}^{V}(\gamma(t)) \xrightarrow[t\to t_2^-]{} h_{\varphi}^{V}(\gamma(t_2)) = 0$$

that is a contraddiction. □

We are now ready to give the proofs of our main results.

Proof of Theorem 1. Let x_0 in Ω and let K be a compact set contained in the interior of $\overline{\mathscr{P}(x_0,\Omega)}$. As Ω_{x_0} is a closed subset of Ω, Lemma 2 implies that $\overline{\mathscr{P}(x_0,\Omega)} \subseteq \Omega_{x_0}$. On the other hand, as we showed in Section 3, the set of the solutions of the equation $\mathscr{L}u = 0$ is a Doob harmonic space in Ω. Then, by Theorem A, there exists a positive constant $C = C(x_0, K, \Omega, \mathscr{L})$ such that

$$\sup_K u \le Cu(x_0)$$

for every non-negative solution u of $\mathscr{L}u = 0$ in Ω. □

Proof of Corollary 2. We set $\widetilde{x} := (x, x_{n+1})$, $\widetilde{\Omega} := \Omega\times]-1,1[$, $\widetilde{K} := K \times [-\frac{1}{2}, \frac{1}{2}]$ and $\widetilde{u}(\widetilde{x}) := u(x)$ for every $\widetilde{x} \in \widetilde{\Omega}$. We observe that the $\widetilde{\mathscr{L}}$-propagation set of $(x_0, 0)$, $\widetilde{\mathscr{P}}_{(x_0,0)}(\widetilde{\Omega})$, equals $\mathscr{P}_{x_0}(\Omega)\times]-1,1[$. Then $K \subseteq \text{int}\mathscr{P}_{x_0}(\Omega)$ if and only if $\widetilde{K} \subseteq \text{int}\widetilde{\mathscr{P}}_{(x_0,0)}(\widetilde{\Omega})$. By Theorem 1

$$\sup_{\widetilde{K}} \widetilde{u} \le C\ \widetilde{u}(x_0, 0)$$

and the conclusion follows immediately.

□

5. Examples

In this Section we give two examples of operators for which we give Harnack-type inequalities that, to our knowledge, are new. In general, the main step in the application of our Theorem 1 is the characterization of the propagation set

$\mathscr{P}(x_0,\Omega)$ of the operator $\mathscr{L}$. We recall that the Control Theory provides us with several tools useful for this problem. We refer, for example, to the book [AS04, Chapter 8] by Agrachev and Sachkov.

5.1. **A Harnack inequality for the stationary Mumford operator.**

We consider the operator $\mathscr{L} = \partial^2_{x_1} + \sin(x_1)\partial_{x_2} + \cos(x_1)\partial_{x_3}$ in the set:

$$\Omega =]-a,a[\times B(0,r) \subseteq \mathbb{R}\times\mathbb{R}^2. \tag{5.1}$$

$x_1 \in]-a,a[$ where $a>\pi$, and $(x_2,x_3)\in B(0,r)$, the euclidean ball centered at 0 with radius $r>0$. This operator has been introduced by Mumford [Mum94] in the study of computer vision problems. The relevant Harnack inequality of Theorem 1 takes the following form:

Theorem 3. *Let Ω be the set introduced in* (5.1)*, with $a>\pi$. For every compact set $K\subset\Omega$ there exists a positive constant $C=C(K,\Omega,\mathscr{L})$ such that*

$$\sup_K u \le Cu(0),$$

for every non-negative solution u of

$$\partial^2_{x_1}u + \sin(x_1)\partial_{x_2}u + \cos(x_1)\partial_{x_3}u = 0 \quad in \quad \Omega.$$

Proof. In view of Theorem 1, we need only to prove that in this case the *propagation set* $\mathscr{P}(0,\Omega)$ agrees with Ω. With this aim, we fix any point $z=(z_1,z_2,z_3)$ in Ω, and we construct a *$\mathscr{L}$-propagation path* steering 0 to z. Note that, in our case, the vector fields defined in (1.2) are

$$X = \partial_{x_1} \qquad \text{and} \qquad Y = \sin(x_1)\partial_{x_2} + \cos(x_1)\partial_{x_3}.$$

We connect 0 and z by a path $\gamma:[0,T]\to\Omega$ such that $\gamma'(t) = \pm X(\gamma(t))$ in the first interval $[0,t_1]$, then $\gamma'(t) = Y(\gamma(t))$ in the second interval $[t_1,t_2]$, and $\gamma'(t) = \pm X(\gamma(t))$ in the third interval $[t_2,T]$, for t_1,t_2,T such that $0\le t_1\le t_2\le T$ chosen as follows.

We set $t^* = \arg(z_2,z_3) \in]-\pi,\pi] \subset]-a,a[$, and we choose $t_1 := |t^*|$. If $t^*>0$, the function $\gamma(t) = (t,0,0)$ is a solution of $\gamma'(t) = X$, for $t\in[0,t_1]$, $\gamma(0)=0$. If $t^*<0$ we consider $\gamma(t) = (-t,0,0)$. In both cases, we have that $\gamma'(t) = \pm X(\gamma(t))$. If $t^*=0$ we simply skip this step.

We next set $t_2 = t_1 + \sqrt{z_2^2+z_3^2}$, and we choose γ such that $\gamma'(t) = Y(\gamma(t))$ for $t_1<t<t_2$. Also in this case, if $(z_2,z_3)=(0,0)$, we skip this step. We conclude the construction of γ by choosing $s^* = z_1 - t^*$, $T = t_2 + |s^*|$ and following the same method used in the first step. The path γ then writes as follows.

$$\gamma(t) = \begin{cases} (\pm t, 0, 0) & \text{if} \quad 0 \le t \le t_1, \\ (t^*, (t-t_1)\cos t^*, (t-t_1)\sin t^*) & \text{if} \quad t_1 \le t \le t_2, \\ (t^* \pm (t-t_2), z_2, z_3) & \text{if} \quad t_2 \le t \le T. \end{cases}$$

□

Remark 4. The above construction can be reproduced to translated cylinders

$$\Omega_y =]y_1 - a, y_1 + a[\times B((y_2, y_3), r) \subseteq \mathbb{R} \times \mathbb{R}^2,$$

for every $y = (y_1, y_2, y_3) \in \mathbb{R}^3$. We find $\mathscr{P}((y_1, y_2, y_3), \Omega_y) = \Omega_y$.

We point out that, on the other hand, the geometry of the propagation set $\mathscr{P}(0, \Omega)$ changes completely as the width of the interval $]-a, a[$ is smaller than 2π. For instance, if we consider the set

$$\widetilde{\Omega} =]-\pi/2, \pi/2[\times B(0, r) \subseteq \mathbb{R} \times \mathbb{R}^2,$$

we easily see that $\mathscr{P}(0, \widetilde{\Omega}) = \widetilde{\Omega} \cap \{x_3 > 0\}$. This fact is in accordance with the invariance of the operator $\mathscr{L}$ with respect to the following left translation introduced in [BL12]. Denote $x = (t, z), y = (s, w) \in \mathbb{R} \times \mathbb{C}$. then

$$x \circ y := (t + s, z + we^{it}).$$

5.2. A Harnack inequality for a degenerate Ornstein Uhlenbeck operator.

We consider the operator $\mathscr{L} = \partial_{x_1}^2 + x_1 \partial_{x_2}$ in the set

$$\Omega =]-a, a[\times]-b, b[\tag{5.2}$$

for some positive a and b.

As in the case of Mumford operator, Theorem 1 gives an *elliptic* Harnack inequality.

Theorem 5. *Let Ω be the set introduced in* (5.2). *For every compact set $K \subset \Omega$ there exists a positive constant $C = C(K, \Omega, \mathscr{L})$ such that*

$$\sup_K u \le C u(0),$$

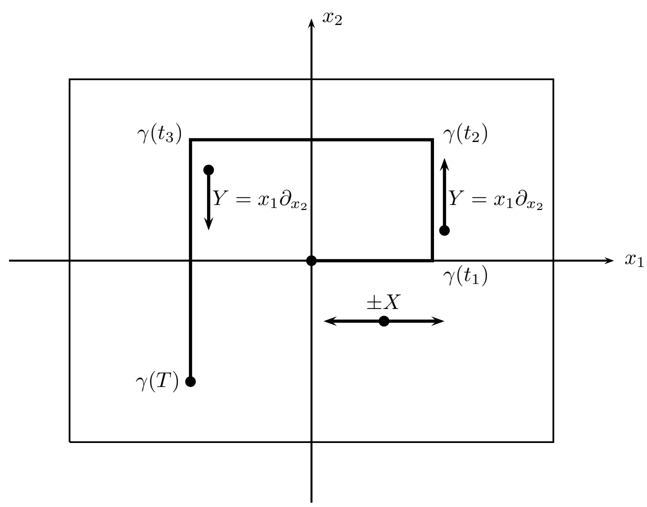
for every non-negative solution u of

$$\partial_{x_1}^2 u + x_1 \partial_{x_2} u = 0 \quad in \quad \Omega.$$

Proof. We prove that, also in this case, the *propagation set* $\mathscr{P}(0,\Omega)$ agrees with Ω. The vector fields defined in (1.2) are

$$X = \partial_{x_1} \qquad \text{and} \qquad Y = x_1 \partial_{x_2}.$$

We choose an integral curve γ such that $\gamma'(t) = \pm X(\gamma(t))$ in some intervals. A curve like that writes as $\gamma(t) = (\widetilde{x}_1 \pm t, \widetilde{x}_2)$. In particular, we will use the field X to increase or decrease the first coordinate x_1. In some other intervals we choose $\gamma'(t) = Y(\gamma(t))$. Such a curve writes as $\gamma(t) = (\widetilde{x}_1, \widetilde{x}_2 + \widetilde{x}_1 t)$. In this case, we rely on the sign of $\widetilde{x}_1$ to increase or decrease the second component x_2. We prefer not to give the details of the construction and to refer to the following figure.



□

Remark 6. The above result fails as

$$\Omega =]a_1, a_2[\times] - b, b[\subseteq \mathbb{R}^2,$$

and a_1 and a_2 have the same sign. In particular, if a_1 and a_2 are both positive, and we consider $x_0 = \left(\frac{a_1+a_2}{2}, 0\right)$, we have $\mathscr{P}(x_0,\Omega) = \Omega \cap \{x_2 > 0\}$. On the contrary, if a_1 and a_2 are both negative, we have $\mathscr{P}(x_0,\Omega) = \Omega \cap \{x_2 < 0\}$.

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Dipartimento di Ingegneria dell'Informazione, Ingegneria Elettrica e Matematica Applicata, Università degli Studi di Salerno, Via Giovanni Paolo II, 132, IT-84084 Fisciano (SA) - Italy

E-mail address: alessia.kogoj@gmail.com

Dipartimento di Scienze Fisiche, Informatiche e Matematiche - Università di Modena e Reggio Emilia - via Campi 213/b, IT-41125 Modena - Italy

E-mail address: sergio.polidoro@unimore.it