A note on Nordhaus-Gaddum-type inequalities for the automorphic $\mathcal{H}$-chromatic index of graphs *

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Abstract

The automorphic $\mathcal{H}$-chromatic index of a graph G is the minimum integer m for which G has a proper edge-coloring with m colors which is preserved by a given automorphism group $\mathcal{H}$ of G. We consider the sum and the product of the automorphic $\mathcal{H}$-chromatic index of a graph and its complement. We prove upper and lower bounds in terms of the order of the graph when $\mathcal{H}$ is chosen to be either a cyclic group of prime order or a group of order four.

Keywords Nordhaus-Gaddum-type inequalities, automorphism, automorphic chromatic index

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1 Introduction

All graphs under consideration are simple. For graph terminology and notation we refer to [6]. Let $G = (V, E)$ be a graph of order n with vertex set V and edge set E. The complement $\bar{G}$ of a graph G is the graph whose vertex set is that of G and in which two vertices are adjacent if and only if they are not adjacent in G. Let $k \geq 2$ be an integer. Following [7] we define a k-decomposition of a graph G_0 as a family $(G_1, G_2, \ldots, G_k)$ of spanning subgraphs of G_0 such that each edge of G_0 is contained in exactly one member of $(G_1, G_2, \ldots, G_k)$, see also [3]. We shall occasionally refer to the subgraphs $G_1, G_2, \ldots, G_k$ as being the "blocks" of the k-decomposition.

The following two problems can be formulated for an arbitrary graph parameter P:

(1) finding upper and lower bounds of the set

$$\{P(G_1) + \cdots + P(G_k) : (G_1, G_2, \ldots, G_k) \text{ is a } k\text{-decomposition of } G_0\};$$

(2) finding upper and lower bounds of the set

$$\{P(G_1) \cdot P(G_2) \cdots P(G_k) : (G_1, G_2, \ldots, G_k) \text{ is a } k\text{-decomposition of } G_0\}.$$

The study of the above problems started in 1956 with the paper by Nordhaus and Gaddum [10] in the particular case $k = 2$, G_0 the complete graph K_n of order n and $P = \chi$ the chromatic number. Nordhaus and Gaddum gave answers to problems (1) and (2) in terms of the order n of $G_0 = K_n$. Only 10 years later Vizing in [11] solved the same problems for another graph parameter, namely the chromatic index $P = \chi'$.

Theorem 1.1. *[11] For an arbitrary graph G of order n the following inequalities hold*

$$2\left\lfloor \frac{n+1}{2} \right\rfloor - 1 \leq \chi'(G) + \chi'(\bar{G}) \leq n + 2\left\lfloor \frac{n-2}{2} \right\rfloor$$

$$0 \leq \chi'(G)\chi'(\bar{G}) \leq (p-1)\left(2\left\lfloor \frac{n}{2} \right\rfloor - 1\right)$$

Theorem 1.1 was independently proved by Alavi and Behzard in [1] and by Capobianco and Molluzzo in [5]. This type of result, called Nordhaus-Gaddum-type inequalities, have been studied for several different graph parameters. We

refer to [2] for a recent survey on Nordhaus-Gaddum-type inequalities. In particular in [2, Sec.3.6] Nordhaus-Gaddum-type inequalities can be found for several chromatic graph parameters, for example, the total chromatic number. In this paper we consider Nordhaus-Gaddum-type inequalities for a particular chromatic graph parameter: the automorphic chromatic index. Let $\phi : E \to \mathcal{C}$ be an edge-coloring of a graph G with color set $\mathcal{C}$. An automorphism σ of G preserves ϕ if there exists a permutation a of the color-set $\mathcal{C}$ such that the relation $\phi\sigma(e) = a\phi(e)$ holds for each $e \in E$. Denote by $Aut(G)$ the full automorphism group of a graph G and by $\mathcal{H}$ a given subgroup of $Aut(G)$. The *automorphic $\mathcal{H}$-chromatic index* of G, as defined in [8] and denoted by $\chi'_{\mathcal{H}}(G)$, is the minimum integer m for which G has a proper edge-coloring with m colors preserved by each automorphism of the subgroup $\mathcal{H}$.
Upper bounds for $\chi'_{\mathcal{H}}(G)$ in terms of the chromatic index $\chi'(G)$ are established in [8] and [9] when $\mathcal{H}$ is either a cyclic group of prime order or a group of order four. We recall these results which shall be used in Section 3.

Proposition 1.2. *[8] Let G be a graph with chromatic index $\chi'(G)$ and assume that $\mathcal{H}$ is cyclic of order 2. The inequality holds*

$$\chi'_{\mathcal{H}}(G) \leq \chi'(G) + 2\left\lceil \frac{\chi'(G)}{2} \right\rceil.$$

Let σ be an automorphism of G of odd prime order p. A σ-cycle is a cycle of G of length p which is preserved by σ while none of its vertices is fixed by σ.

Proposition 1.3. *[8] Let G be a graph with chromatic index $\chi'(G)$ and assume that $\mathcal{H}$ is cyclic of odd prime order p and generated by σ. Then the inequality holds*

$$\chi'_{\mathcal{H}}(G) \leq \chi'(G) + p\left\lceil \frac{\chi'(G)}{p} \right\rceil$$

provided that G has either no σ-cycles or maximum degree not divisible by p.

Proposition 1.4. *[9] Let G be a graph with chromatic index $\chi'(G)$ and assume that $\mathcal{H}$ is the Klein group. Then the inequality holds*

$$\chi'_{\mathcal{H}}(G) \leq \chi'(G) + 6\left\lceil \frac{\chi'(G)}{2} \right\rceil + 4\left\lceil \frac{\chi'(G)}{4} \right\rceil.$$

Proposition 1.5. *[9] Let G be a graph with chromatic index $\chi'(G)$ and assume that $\mathcal{H}$ is cyclic of order four. Then the inequality holds*

$$\chi'_{\mathcal{H}}(G) \leq \chi'(G) + 2\left\lceil \frac{\chi'(G)}{2} \right\rceil + 4\left\lceil \frac{\chi'(G)}{2} \right\rceil.$$

All the above described bounds, with the exception of one, are best possible (see [8] and [9]). In this note the main purpose is to find Nordhaus-Gaddum-type inequalities for the $\mathcal{H}$-automorphic chromatic index of a graph G with $\mathcal{H}$ either a cyclic group of prime order or a cyclic group of order four.

2 Some general bounds

In this section a result is shown in analogy to [3, Theorem 4.9 p. 27].

Definition 2.1. *Let $(G_1, G_2, \dots, G_k)$ be a k-decomposition of a graph G_0 and $\mathcal{H}$ a subgroup of $Aut(G_0)$. The k-decomposition $(G_1, G_2, \dots, G_k)$ is said to be blockwise fixed by $\mathcal{H}$ if $G_i^h = G_i$ holds for $h \in \mathcal{H}$ and $i = 1, 2, \dots, k$.*

In what follows we shall omit the word blockwise and we denote by $\mathcal{H}_i$ the automorphism group of G_i induced by $\mathcal{H}$ on G_i for $i = 1, 2, \dots, k$.

Lemma 2.2. *Let $(G_1, G_2, \dots, G_k)$ be a k-decomposition of G_0 which is fixed by $\mathcal{H}$, $\mathcal{H} \leqslant Aut(G_0)$. Then,*

$$\chi'_{\mathcal{H}}(G_0) \leq \chi'_{\mathcal{H}_1}(G_1) + \chi'_{\mathcal{H}_2}(G_2) + \cdots + \chi'_{\mathcal{H}_k}(G_k).$$

Proof. For $i = 1, 2, \dots, k$, let $m_i = \chi'_{\mathcal{H}_i}(G_i)$ be and let ϕ_i be an edge-coloring of G_i with colors $\mathcal{C}_i$ preserved by $\mathcal{H}_i$ such that $|\mathcal{C}_i| = m_i$ and $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$ for $i \neq j$. Each edge of G_0 is contained in exactly one member of $(G_1, G_2, \dots, G_k)$. If an edge e of G_0 belongs to G_i then we color it with $\phi_i(e)$. Therefore an edge-coloring of G with $(m_1 + m_2 + \cdots + m_k)$ colors is obtained which is preserved by $\mathcal{H}$ by construction. Obviously we have the following inequality

$$\chi'_{\mathcal{H}}(G_0) \leq m_1 + m_2 + \cdots + m_k = \chi'_{\mathcal{H}_1}(G_1) + \chi'_{\mathcal{H}_2}(G_2) + \cdots + \chi'_{\mathcal{H}_k}(G_k),$$

and the statement follows. □

Proposition 2.3. *Let n and k be positive integers and $(G_1, G_2, \dots, G_k)$ be a k-decomposition of K_n which is fixed by $\mathcal{H}$, $\mathcal{H} \leqslant Aut(K_n)$. Then,*

$$2\left\lfloor \frac{n+1}{2} \right\rfloor - 1 \leq \chi'_{\mathcal{H}_1}(G_1) + \chi'_{\mathcal{H}_2}(G_2) + \cdots + \chi'_{\mathcal{H}_k}(G_k).$$

Proof. Since $\chi'(K_n) = 2\lfloor \frac{n+1}{2} \rfloor - 1$ and $\chi'(K_n) \leq \chi'_{\mathcal{H}}(K_n)$, Lemma 2.2 implies the statement. This lower bound is the best possible: if $\mathcal{H}$ is the identity group and $G_1 = K_n$, then $\chi'_{\mathcal{H}}(G_i)$ coincides with $\chi'(G_i)$. Hence, we get $\chi'_{\mathcal{H}}(G_1) = 2\lfloor \frac{n+1}{2} \rfloor - 1$ and $\chi'_{\mathcal{H}}(G_i) = 0$ for $i \neq 1$. □

3 Some Nordhaus-Gaddum-type inequalities for the $\mathcal{H}$-automorphic chromatic index

In this section G_0 will be the complete graph K_n of order n and a 2-decomposition of K_n will be denoted by $(G, \bar{G})$ where $\bar{G}$ is the complement of G. The automorphism group of a graph G coincides with the automorphism group of the complement of G, [4, Theorem 1.1 p. 139], therefore, if $\mathcal{H} \leq Aut(G)$ both $\chi'_{\mathcal{H}}(G)$ and $\chi'_{\mathcal{H}}(\bar{G})$ can be studied simultaneously. Proposition 2.3 in the particular case $k = 2$ implies the following:

Lemma 3.1. *For an arbitrary graph G of order n with $\mathcal{H} \leq Aut(G)$ the following inequality holds*

$$2\left\lfloor \frac{n+1}{2} \right\rfloor - 1 \leq \chi'_{\mathcal{H}}(G) + \chi'_{\mathcal{H}}(\bar{G}).$$

The above bound is best possible as shown in Proposition 2.3.

Lemma 3.2. *Let G be a graph of order n. Then,*

$$\left\lceil \frac{\chi'(G)}{r} \right\rceil + \left\lceil \frac{\chi'(\bar{G})}{r} \right\rceil \leq \frac{1}{r}\left(n + 2\left\lfloor \frac{(n-2)}{2} \right\rfloor\right) + 2$$

where r is an integer greater than or equal to 1.

Proof. The statement follows from the following inequalities

$$\left\lceil \frac{\chi'(G)}{r} \right\rceil + \left\lceil \frac{\chi'(\bar{G})}{r} \right\rceil \leq \frac{1}{r}\left(\chi'(G) + \chi'(\bar{G})\right) + 2 \leq \frac{1}{r}\left(n + 2\left\lfloor \frac{(n-2)}{2} \right\rfloor\right) + 2$$

where the last one is obtained from Theorem 1.1. □

Proposition 3.3. *Let G be a graph of order n with $\mathcal{H} \leq Aut(G)$ and assume that $\mathcal{H}$ is cyclic of order 2. Then the following inequality holds*

$$\chi'_{\mathcal{H}}(G) + \chi'_{\mathcal{H}}(\bar{G}) \leq 2\left(n + 2\left\lfloor \frac{(n-2)}{2} \right\rfloor\right) + 4$$

Proof. We get

$$\chi'_{\mathcal{H}}(G) + \chi'_{\mathcal{H}}(\bar{G}) \leq \chi'(G) + \chi'(\bar{G}) + 2\left\lceil \frac{\chi'(G)}{2} \right\rceil + 2\left\lceil \frac{\chi'(\bar{G})}{2} \right\rceil$$

$$\leq n + 2\left\lfloor \frac{(n-2)}{2} \right\rfloor + 2\left(\frac{1}{2}\left(n + 2\left\lfloor \frac{(n-2)}{2} \right\rfloor\right) + 2\right) \leq 2\left(n + 2\left\lfloor \frac{(n-2)}{2} \right\rfloor\right) + 4$$

where the first of the above relations is obtained from Proposition 1.2, while the second one from Theorem 1.1 and Lemma 3.2. □

Proposition 3.4. *Let G be a graph of order n, with $\mathcal{H} \leq Aut(G)$. Assume that $\mathcal{H}$ is cyclic of odd prime order p and generated by σ. The following inequalities hold*

$$p \leq \chi'_{\mathcal{H}}(G) + \chi'_{\mathcal{H}}(\bar{G}) \leq 2\left(n + 2\left\lfloor \frac{(n-2)}{2} \right\rfloor\right) + 2p$$

provided that G does not contain σ-cycles and $\bar{G}$ has maximum degree not divisible by p.

Proof. From Proposition 1.3, Theorem 1.1 and Lemma 3.2 we have

$$\chi'_{\mathcal{H}}(G) + \chi'_{\mathcal{H}}(\bar{G}) \leq \chi'(G) + \chi'(\bar{G}) + p\left\lceil \frac{\chi'(G)}{p} \right\rceil + p\left\lceil \frac{\chi'(\bar{G})}{} \right\rceil$$

$$\leq n + 2\left\lfloor \frac{(n-2)}{2} \right\rfloor + p\left(\frac{1}{p}\left(n + 2\left\lfloor \frac{(n-2)}{2} \right\rfloor\right) + 2\right)$$

$$\leq 2\left(n + 2\left\lfloor \frac{(n-2)}{2} \right\rfloor\right) + 2p.$$

Hence, the upper bound is proved.

In G there exist p vertices $v_0, v_1, \ldots, v_{p-1}$ such that $\sigma(v_i) = v_{i+1}$, where the indices are taken modulo p. Since G has no σ-cycles, then $C = v_0 v_1 \ldots v_{p-1} v_0$ is a cycle in $\bar{G}$. In order to have an edge-coloring of $\bar{G}$ preserved by σ, the edges of the cycle C must be colored with p different colors . Therefore $\chi'_{\mathcal{H}}(\bar{G}) \geq p$ and the lower bound is shown.

We now prove that the lower bound is best possible. Using the cyclic group $\mathbb{Z}_p$ of rotations of the p-gon, let $\bar{G} = K_p$ be with $V(\bar{G}) = \mathbb{Z}_p = \{0, 1, \ldots, p-1\}$, $\sigma(i) = i + 1$, (the indices are taken modulo p) and G the null graph of order p. The "standard" edge-coloring of K_p with p colors and with all the edges of the near 1-factor $F_i = \{\{i + j, i - j\} : j \in \mathbb{Z}_p \setminus \{0\}\}$ colored by the color i ($i = 0, 1, \ldots, p-1$), is obviously preserved by σ. Hence, $p = \chi'(\bar{G}) \leq \chi'_{\mathcal{H}}(\bar{G}) \leq p$. Therefore, $\chi'_{\mathcal{H}}(\bar{G}) = p$ and the lower bound is attained. □

In analogy to the above propositions, Proposition 1.4 and Proposition 1.5 imply the following, respectively:

Proposition 3.5. *Let G be a graph of order n with $\mathcal{H} \leq Aut(G)$. Assume that $\mathcal{H}$ is the Klein group, then the following inequality holds*

$$\chi'_{\mathcal{H}}(G) + \chi'_{\mathcal{H}}(\bar{G}) \leq 5\left(n + 2\left\lfloor \frac{(n-2)}{2} \right\rfloor\right) + 20$$

Proposition 3.6. *Let G be a graph of order n with $\mathcal{H} \leq Aut(G)$. Assume that $\mathcal{H}$ is cyclic of order 4, then the following inequality holds*

$$\chi'_{\mathcal{H}}(G) + \chi'_{\mathcal{H}}(\bar{G}) \leq 4\left(n + 2\left\lfloor \frac{n-2}{2} \right\rfloor\right) + 12$$

Note that if $G = K_n$ then $\bar{G}$ is the null graph with $\chi'_{\mathcal{H}}(\bar{G}) = 0$, hence $0 \leq \chi'_{\mathcal{H}}(G)\chi'_{\mathcal{H}}(\bar{G})$. Since

$$\chi'_{\mathcal{H}}(G)\chi'_{\mathcal{H}}(\bar{G}) = \left(\sqrt{\chi'_{\mathcal{H}}(G)\chi'_{\mathcal{H}}(\bar{G})}\right)^2 \leq \left(\frac{(\chi'_{\mathcal{H}}(G) + \chi'_{\mathcal{H}}(\bar{G}))}{2}\right)^2$$

then by Propositions 3.3, 3.4, 3.5 and 3.6 we obtain the following:

Proposition 3.7. *Let G be a graph of order n with $\mathcal{H} \leq Aut(G)$. Assume $\mathcal{H}$ cyclic of order 2, then the following inequalities hold*

$$0 \leq \chi'_{\mathcal{H}}(G)\chi'_{\mathcal{H}}(\bar{G}) \leq \left(n + 2\left\lfloor \frac{n-2}{2} \right\rfloor + 2\right)^2$$

Proposition 3.8. *Let G be a graph of order n with $\mathcal{H} \leq Aut(G)$. Assume that $\mathcal{H}$ is cyclic of odd prime order p and generated by σ. Then the following inequalities hold*

$$0 \leq \chi'_{\mathcal{H}}(G)\chi'_{\mathcal{H}}(\bar{G}) \leq \left(n + 2\left\lfloor \frac{(n-2)}{2} \right\rfloor + p\right)^2$$

provided that G does not contain σ-cycles and the maximum degree of $\bar{G}$ is not divisible by p.

Proposition 3.9. *Let G be a graph of order n with $\mathcal{H} \leq Aut(G)$. Assume that $\mathcal{H}$ is the Klein group, then the following inequalities hold*

$$0 \leq \chi'_{\mathcal{H}}(G)\chi'_{\mathcal{H}}(\bar{G}) \leq \left(\frac{5}{2}\left(n + 2\left\lfloor \frac{(n-2)}{2} \right\rfloor\right) + 10\right)^2$$

Proposition 3.10. *Let G be a graph of order n with $\mathcal{H} \leq Aut(G)$. Assume that $\mathcal{H}$ is cyclic of order 4, then the following inequalities hold*

$$0 \leq \chi'_{\mathcal{H}}(G)\chi'_{\mathcal{H}}(\bar{G}) \leq \left(2n + 4\left\lfloor \frac{(n-2)}{2} \right\rfloor + 6\right)^2$$

It remains an open problem to verify if some of the above bounds are sharp.

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