PHYSICAL REVIEW D **67**, 104009 (2003)

Dimensional regularization for $\mathcal{N}=1$ supersymmetric sigma models and the worldline formalism

Fiorenzo Bastianelli,* Olindo Corradini,† and Andrea Zirotti‡

Dipartimento di Fisica, Università di Bologna and INFN, Sezione di Bologna via Irnerio 46, I-40126 Bologna, Italy

(Received 28 February 2003; published 19 May 2003)

We generalize the worldline formalism to include spin 1/2 fields coupled to gravity. To this purpose we first extend dimensional regularization to supersymmetric nonlinear sigma models in one dimension. We consider a finite propagation time and find that dimensional regularization is a manifestly supersymmetric regularization scheme, since the classically supersymmetric action does not need any counterterm to preserve worldline supersymmetry. We apply this regularization scheme to the worldline description of Dirac fermions coupled to gravity. We first compute the trace anomaly of a Dirac fermion in 4 dimensions, providing an additional check on the regularization with finite propagation time. Then we come to the main topic and consider the one-loop effective action for a Dirac field in a gravitational background. We describe how to represent this effective action as a worldline path integral and compute explicitly the one- and two-point correlation functions, i.e. the spin 1/2 particle contribution to the graviton tadpole and graviton self-energy. These results are presented for the general case of a massive fermion. It is interesting to note that in the worldline formalism the coupling to gravity can be described entirely in terms of the metric, avoiding the introduction of a vielbein. Consequently, the fermion-graviton vertices are always linear in the graviton, just like the standard coupling of fermions to gauge fields.

DOI: 10.1103/PhysRevD.67.104009

PACS number(s): 04.62.+v

I. INTRODUCTION

One dimensional supersymmetric nonlinear sigma models are useful to describe in first quantization the propagation of Fermionic particles in a curved background. In fact, it is well known that $\mathcal{N}=1$ supersymmetric sigma models describe the worldline dynamics of a spinning particle [1]. Mastering the path integral quantization of such models provides a useful tool for treating spin 1/2 particles coupled to gravity. The purpose of this paper is twofold. We first extend dimensional regularization to supersymmetric sigma models with finite propagation (proper) time. Then, with this regularization scheme at hand, we generalize the worldline formalism to include spin 1/2 fields coupled to gravity. This extends the scalar particle case treated in Ref. [2]. The resulting Feynman rules are simpler than the standard ones obtained from the second quantized action. In particular, the fermion-graviton vertices can always be taken linear in the graviton field, a fact which seems to point once more to unexpected perturbative relations between gravity and gauge theories, as reviewed in Ref. [3].

Path integrals for supersymmetric sigma models in one dimension were originally used for deriving formulas for index theorems and chiral anomalies [4–6]. However, for obtaining those results the details of how to properly define and regulate the path integrals at higher loops were not necessary. Due to the worldline supersymmetry the chiral anomalies are seen as a topological quantity, the Witten index [7], which is independent of β, the propagation time in the sigma model. Thus a semiclassical approximation (which consists in calculating a few determinants) already gives the complete results. The quantum mechanical calculation of chiral anomalies can be extended to trace anomalies [8,9]. However, in the latter case the details of how to define the path integral is essential since one-loop (in target space) trace anomalies correspond to higher-loop calculations on the worldline, namely the one-loop trace anomaly in D dimensions is given by a $D/2+1$ loop calculation on the worldline. Several regularization schemes have been developed for this purpose: mode regularization (MR) [8–10], time slicing (TS) [11,12], and dimensional regularization (DR) [13,14]. The DR regularization was developed after the results of Ref. [15] which dealt with the nonlinear sigma model in the infinite propagation time limit.[1] The first objective of this paper is to extend dimensional regularization to include fermionic fields on the worldline and treat supersymmetric nonlinear sigma models. Worldline fermions coupled to gravity give rise to new (superficially) divergent Feynman diagrams, other than those associated to the coupling of gravity with the bosonic coordinates. Hence one may *a priori* expect additional counterterms to arise. In fact, in time slicing, the inclusion of the fermionic fields brings in additional noncovariant counterterms of order β^2: they are proportional to $g^{\mu\nu}\Gamma^{\lambda}_{\mu\rho}\Gamma^{\rho}_{\nu\lambda}$ if one uses fermions with curved target space indices, or $g^{\mu\nu}\omega_{\mu}{}^{ab}\omega_{\nu ab}$ if one uses fermions with flat space indices [12]. Note that such counterterms only arise at two loops, and thus they do not affect the calculation of the chiral anomalies, but should be included if one wants to check with TS that there are no higher order corrections in β [17]. We are going to show that in dimensional regularization no extra counterterms arise. This implies that dimensional regulariza-

*Email address: bastianelli@bo.infn.it

†Email address: corradini@bo.infn.it

‡Email address: zirotti@bo.infn.it

[1]Recently, Kleinert and Chervyakov [16] have also analyzed nonlinear sigma models for finite propagation time, discussing how DR defines products of distributions, and finding results for the Feynman rules which agree with those obtained in Ref. [13].

tion manifestly preserves supersymmetry. In fact the bosonic part produces a coupling to the scalar curvature with the precise coefficient required by supersymmetry. We describe how to use both flat target space indices and curved ones, for the fermionic fields. Using curved indices will bring in a new set of bosonic ''ghost'' fields, in the same fashion of Refs. [8,9].

Having at hand a simple and reliable regularization scheme for supersymmetric sigma models, we turn to the worldline formalism. As a warm up, we first compute the trace anomaly for a Dirac fermion in 4 dimensions. We obtain the expected result, providing a further test on our application of the DR scheme. Then we come to the core of our paper: the generalization of the worldline formalism to include spin 1/2 particles coupled to gravity. Many simplifications are known to occur in the worldline path integral formulation of quantum field theory (QFT), which for this very reason provides an efficient and alternative method for computing Feynman diagrams. This method has quite a long history, rooted in Ref. [18]. Later it was developed further by viewing it as the particle limit of string theory [19], and then discussed directly as the first quantization of point particles [20,21] (see Ref. [22] for a review and a list of references). The inclusion of background gravity was presented in Ref. [2] for the case of a scalar particle. Results obtained using string inspired rules with gravity were presented in Refs. [23,24,3].

Here we consider the case of the one-loop effective action for a Dirac fermion in a gravitational background. We describe how to represent it as a worldline path integral. We compute explicitly the one- and two-point correlation functions, i.e. the spin 1/2 particle contribution to the graviton tadpole and graviton self-energy. These results are presented for the general case of a massive fermion. In our calculations we use the DR scheme constructed in the previous sections. The other known scheme explicitly developed to include worldline fermions (time slicing [12]) can be used as well, but lack of manifest covariance makes its use more complicated. It is interesting to note that in the worldline formalism the coupling to gravity can be described entirely in terms of the metric, avoiding the introduction of a vielbein. The fermion-graviton vertices are always linear in the metric field, just like the standard coupling of fermions to gauge fields are linear in the gauge potential. This fact seems to point once more to the unexpected perturbative relations between gravity and gauge theories encoded in the so-called Kawai-Lewellen-Tye (KLT) relations [25], as reviewed in Ref. [3].

The paper is organized as follows. In Sec. II we introduce dimensional regularization applied to the worldline Majorana fermions and to supersymmetric sigma models. We mainly consider antiperiodic boundary conditions (which break supersymmetry), but also briefly discuss periodic boundary conditions. In Sec. III we apply DR to compute the trace anomaly of a Dirac field in 4 dimensions with quantum mechanics. Then in Sec. IV we describe the worldline formalism with Dirac fields coupled to gravity and compute explicitly the spin 1/2 particle contribution to the graviton tadpole and graviton self-energy. This is the main section, and the reader uninterested in the details of the DR regularization scheme may jump directly to it. Section V contains our conclusions. Conventions and useful formulas are collected in the Appendix. We work with a Euclidean time both on the worldline and in target space. The latter is assumed to have even dimensions D.

II. DIMENSIONAL REGULARIZATION WITH FERMIONS

In this section we describe the dimensional regularization of fermionic path integrals obtained by extending the method presented in Ref. [13] for bosonic models. We shall discuss explicitly path integrals for Majorana fermions on a circle with antiperiodic boundary conditions (ABC's), as these are the only boundary conditions that will be directly needed in the applications to trace anomalies and effective action calculations. Our strategy will be as follows: we first set up the rules of dimensional regularization for fermions following Ref. [13], then we require that a two-loop computation with DR reproduces known results, and precisely those obtained by a path integral with time slicing [12] (or equivalently by heat kernel methods [26]). This requirement plays the role of a standard (in QFT) renormalization condition, and fixes once and for all the DR two-loop counterterm due to fermions. Since counterterms are due to ultraviolet effects, the infrared vacuum structure and the related boundary conditions on the fields should not matter in their evaluation. Therefore one expects that the same counterterm should apply to fermionic path integral with periodic boundary conditions (PBC's) as well. No higher-loop contributions to the counterterm are expected as the model is super-renormalizable, just as in the purely bosonic case.

Let us consider the path integral quantization of the $\mathcal{N}=1$ supersymmetric model written in terms of Majorana fermions with flat target space indices,

$$Z=\int DxDaDbDcD\psi e^{-S}, \tag{1}$$

$$\begin{aligned} S=\frac{1}{\beta}\int_0^1 d\tau\bigg(&\frac{1}{2}g_{\mu\nu}(x)(\dot{x}^\mu\dot{x}^\nu+a^\mu a^\nu+b^\mu c^\nu)\\ &+\frac{1}{2}\psi_a[\dot{\psi}^a+\dot{x}^\mu\omega_\mu{}^a{}_b(x)\psi^b]+\beta^2[V(x)\\ &+V_{CT}(x)+V'_{CT}(x)]\bigg) \end{aligned} \tag{2}$$

where as usual we have scaled the propagation time β out of the action. The propagation time β will be considered as the expansion parameter for a perturbative evaluation (i.e. the loop counting parameter). In the action we have included: (i) the bosonic a^μ and fermionic b^μ,c^μ ghost fields which exponentiate the nontrivial path integral measure; (ii) the counterterm V_{CT} which arises in the chosen regularization scheme from the bosonic sector and which is fixed in order to produce a quantum Hamiltonian without nonminimal coupling to the scalar curvature R; (iii) the additional counterterm V'_{CT} which may arise from the fermionic sector; and (iv) the potential $V=\frac{1}{8}R$ which is required to have a supersymmetric quantum Hamiltonian as given by the square of the supersymmetry charge.

The action is classically supersymmetric if all the potential terms multiplied by β^2 are set to zero (the ghosts can be trivially eliminated by using their algebraic equations of motion). Supersymmetry may be broken by boundary conditions, e.g. periodic for the bosons and antiperiodic for the fermions. Here we assume antiperiodic boundary conditions (ABC's) for the Majorana fermions $\psi^a(1)=-\psi^a(0)$. Majorana fermions realize the Dirac gamma matrices in a path integral context, and ABC's compute the trace over the Dirac matrices. For simplicity we consider a target space with even dimensions D, and thus the curved indices $\mu,\nu,\ldots$ and the flat space indices $a,b,\ldots$ both run from 1 to D.

One may explicitly compute by time slicing the transition amplitude for going from the background point x_0 at time $t=0$ back to the same point x_0 at a later time $t=\beta$ using ABC's for the Majorana fermions. In the two-loop approximation this calculation gives

$$Z\equiv \mathrm{tr}\langle x_0|e^{-\beta\hat{H}}|x_0\rangle=\frac{2^{D/2}}{(2\pi\beta)^{D/2}}\left(1-\frac{\beta}{24}R+O(\beta^2)\right) \quad (3)$$

where the trace on the left-hand side is only over the Dirac matrices, and where

$$\hat{H}=-\frac{1}{2}\nabla\!\!\!\!/\,\nabla\!\!\!\!/=-\frac{1}{2}\nabla^2+\frac{1}{8}R \quad (4)$$

is the supersymmetric Hamiltonian of the $\mathcal{N}=1$ model [one can normalize the supersymmetric charge as $\hat{Q}=(i/\sqrt{2})\nabla\!\!\!\!/$, so that $\hat{H}=\hat{Q}^2$]. Note that there is an explicit coupling to the scalar curvature in Eq. (4), thus one needs to use a potential $V=\frac{1}{8}R$ in the action together with the time slicing counterterms $V_{TS}=-\frac{1}{8}R+\frac{1}{8}g^{\mu\nu}\Gamma^{\lambda}_{\mu\rho}\Gamma^{\rho}_{\nu\lambda}$ and $V'_{TS}=\frac{1}{16}g^{\mu\nu}\omega_{\mu}{}^{ab}\omega_{\nu ab}$ (see Ref. [12]; later on we will derive once more this value of V'_{TS} as well). Our conventions for the curvature tensors can be found in Sec. 1 of the Appendix.

Now we want to reproduce Eq. (3) in dimensional regularization with a path integral over Majorana fermions. This will unambiguously fix the additional counterterm V'_{DR} due to the fermions. Note that in dimensional regularization the potential $V=\frac{1}{8}R$ cancels exactly with the counterterm $V_{DR}=-\frac{1}{8}R$ coming from the bosons [13].

We focus directly on the regularization of the Feynman graphs arising in perturbation theory. To recognize how to dimensionally continue the various Feynman graphs we extend the action in Eq. (2) from 1 to $d+1$ dimensions as follows:

$$S=\frac{1}{\beta}\int_{\Omega}d^{d+1}t\bigg(\frac{1}{2}g_{\mu\nu}(\partial^{\alpha}x^{\mu}\partial_{\alpha}x^{\nu}+a^{\mu}a^{\nu}+b^{\mu}c^{\nu}) + \frac{1}{2}\overline{\psi}_a\gamma^{\alpha}(\partial_{\alpha}\psi^a+\partial_{\alpha}x^{\mu}\omega_{\mu}{}^{a}{}_{b}\psi^b)+\beta^2V'_{DR}\bigg) \quad (5)$$

where $\Omega=I\times R^d$ is the region of integration containing the finite interval $I=[0,1]$, γ^{α} are the gamma matrices in $d+1$ dimensions satisfying $\{\gamma^{\alpha},\gamma^{\beta}\}=2\delta^{\alpha\beta}$, and $t^{\alpha}\equiv(\tau,\mathbf{t})$ with $\alpha=0,1,\ldots,d$ and with boldface indicating vectors in the extra d dimensions. Here we assume that we can first continue to those Euclidean integer dimensions where Majorana fermions can be defined. The Majorana conjugate is defined by $\overline{\psi}_a=\psi^T_aC_{\pm}$ with a suitable charge conjugation matrix $C_{\pm}$ such that $\overline{\psi^a}\gamma^{\alpha}\psi^b=-\overline{\psi^b}\gamma^{\alpha}\psi^a$. This can be achieved for example in 2 dimensions.[2] It realizes the basic requirement for the Majorana fermions of the $\mathcal{N}=1$ supersymmetric model which must have a nonvanishing coupling $\omega_{\mu ab}\psi^a\psi^b=-\omega_{\mu ab}\psi^b\psi^a$. The actual details of how to represent $C_{\pm}$ and the gamma matrices in $d+1$ dimensions are not important, as the most important thing for the rules which define the DR scheme for fermions is to keep track of how derivatives are going to be contracted in higher dimensions. Apart from the above requirements, no additional Dirac algebra on the gamma matrices γ^{α} in $d+1$ dimensions is needed. With these rules one can recognize from the action (5) the propagators and vertices in $d+1$ dimensions, and thus rewrite those Feynman diagrams which are ambiguous in one dimension directly in $d+1$ dimensions.

The bosonic and ghost propagators are as usual and reported in Sec. 2 of the Appendix. The fermionic fields with ABC's on the worldline, $\psi^a(1)=-\psi^a(0)$, can be expanded in half integer modes

$$\psi^a(\tau)=\sum_{r\in Z+1/2}\psi^a_re^{2\pi ir\tau} \quad (6)$$

and have the following unregulated propagator:

$$\langle\psi^a(\tau)\psi^a(\sigma)\rangle=\beta\delta^{ab}\Delta_{AF}(\tau-\sigma),$$

$$\Delta_{AF}(\tau-\sigma)=\sum_{r\in Z+1/2}\frac{1}{2\pi ir}e^{2\pi ir(\tau-\sigma)}. \quad (7)$$

Note that the Fourier sum defining the function Δ_{AF} for the antiperiodic fermions is conditionally convergent for $\tau\neq\sigma$, and yields

$$\Delta_{AF}(\tau-\sigma)=\frac{1}{2}\epsilon(\tau-\sigma) \quad (8)$$

where $\epsilon(x)=\theta(x)-\theta(-x)$ is the sign function [with the value $\epsilon(0)=0$, obtained by symmetrically summing the Fourier series]. The function Δ_{AF} satisfies

$$\partial_{\tau}\Delta_{AF}(\tau-\sigma)=\delta_A(\tau-\sigma) \quad (9)$$

where $\delta_A(\tau-\sigma)$ is the Dirac's delta on functions with antiperiodic boundary conditions

[2] In Euclidean 2 dimensions one can choose $\gamma^1=\sigma^3$, $\gamma^2=\sigma^1$ and $C_+=1$. Recall that $C_{\pm}$ are defined by $C_{\pm}\gamma^{\mu}C_{\pm}^{-1}=\pm\gamma^{\mu T}$.

$$\delta_A(\tau-\sigma)=\sum_{r\in Z+1/2} e^{2\pi i r(\tau-\sigma)}. \tag{10}$$

The dimensionally regulated propagator obtained by adding a number d of extra infinite coordinates is derived from Eq. (5) and reads

$$\langle \psi^a(t)\bar{\psi}^b(s)\rangle=\beta\delta^{ab}\Delta_{AF}(t,s) \tag{11}$$

where the function

$$\Delta_{AF}(t,s)=-i\int\frac{d^d\mathbf{k}}{(2\pi)^d}\times\sum_{r\in Z+1/2}\frac{2\pi r\gamma^0+\mathbf{k}\cdot\vec{\gamma}}{(2\pi r)^2+\mathbf{k}^2}e^{2\pi i r(\tau-\sigma)}e^{i\mathbf{k}\cdot(\mathbf{t}-\mathbf{s})} \tag{12}$$

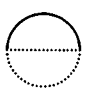
satisfies

$$\gamma^\alpha\frac{\partial}{\partial t^\alpha}\Delta_{AF}(t,s)=-\frac{\partial}{\partial s^\beta}\Delta_{AF}(t,s)\gamma^\beta=\delta_A(\tau-\sigma)\delta^d(\mathbf{t}-\mathbf{s})\equiv\delta_A^{d+1}(t-s). \tag{13}$$

The latter are the basic relations which will be used in the application of DR to fermions. They keep track of which derivative can be contracted to which vertex to produce the $d+1$ delta function. This delta function is only to be used in $d+1$ dimensions, as we assume that only in such a situation the regularization due to the extra dimensions is taking place.[3] By using partial integration one casts the various loop integrals in a form which can be computed by sending first $d\to 0$. At this stage one can use $\gamma^0=1$, and no extra factors arise from the Dirac algebra in $d+1$ dimensions. This procedure will be exemplified in the subsequent calculations. Having specified how to compute the ambiguous Feynman graphs by continuation to $d+1$ dimensions the DR scheme is now complete.

Now we are ready to perform the two-loop calculation in the $\mathcal{N}=1$ nonlinear sigma model using DR. The bosonic vertices together with the ghosts, V and V_{DR} give the standard contribution, as for example in Ref. [2]. The overall normalization of the fermionic path integral gives the extra factor $2^{D/2}$ which equals the number of components of a Dirac fermion in a target space of even dimensions D. This already produces the full expected result in Eq. (3).

Thus the sum of the additional fermion graphs arising from the cubic vertex contained in $\Delta S=\int_0^1 d\tau(1/2\beta)\dot{x}^\mu\omega_{\mu ab}\psi^a\psi^b$ and the contribution from the extra counterterm V'_{DR} must vanish at two loops. The cubic vertex arise by evaluating the spin connection at the background point x_0 and reads $\Delta S_3=(1/2\beta)\omega_{\mu ab}\int_0^1 d\tau\dot{y}^\mu\psi^a\psi^b$, where y^μ denotes the quantum fluctuations around the background point x_0^μ with vanishing boundary conditions at $\tau=0,1$. Using Wick contractions (see Appendix Sec. 2 for the explicit form of the bosonic propagators with vanishing Dirichlet boundary conditions) we identify the following nontrivial contribution to $\langle e^{-S^{int}}\rangle$ (other graphs vanish trivially)

$$\frac{1}{2}\langle(\Delta S_3)^2\rangle=$$  $$=\frac{1}{2}(-2)\left(\frac{1}{2\beta}\omega_{\mu ab}\right)^2(-\beta^3)\int_0^1 d\tau\int_0^1 d\sigma\ {}^\bullet\Delta^\bullet(\tau,\sigma)[\Delta_{AF}(\tau,\sigma)]^2 \tag{14}$$

where dotted lines represent fermions. As usual, we denote with a left/right dot the derivative with respect to the first/second variable. Using DR this contribution is regulated by

$$\int_0^1 d\tau\int_0^1 d\sigma\ {}^\bullet\Delta^\bullet(\tau,\sigma)[\Delta_{AF}(\tau,\sigma)]^2 \to -\int\int {}_\alpha\Delta_\beta(t,s)\mathrm{tr}[\gamma^\alpha\Delta_{AF}(t,s)\gamma^\beta\Delta_{AF}(s,t)] \tag{15}$$

where ${}_\alpha\Delta_\beta(t,s)\equiv(\partial/\partial t^\alpha)(\partial/\partial s^\beta)\Delta(t,s)$ (note the minus sign obtained in exchanging t and s in the last propagator; it is the usual minus sign arising for fermionic loops). We can partially integrate ∂_α without picking boundary terms and obtain

$$\begin{aligned}
&2\int\int\Delta_\beta(t,s)\mathrm{tr}\{[\gamma^\alpha\partial_\alpha\Delta_{AF}(t,s)]\gamma^\beta\Delta_{AF}(s,t)\}\\
&=2\int\int\Delta_\beta(t,s)\mathrm{tr}[\delta_A^{d+1}(t,s)\gamma^\beta\Delta_{AF}(s,t)]\\
&=2\int\Delta_\beta(t,t)\mathrm{tr}[\gamma^\beta\Delta_{AF}(t,t)]\\
&\to 2\int_0^1 d\tau\Delta^\bullet(\tau,\tau)\Delta_{AF}(0)=0
\end{aligned} \tag{16}$$

[3] We are not able to show this in full generality, and at this stage this rule is taken as an assumption. One way to prove it explicitly would be to compute all integrals arising in perturbation theory at arbitrary d and check the location of the poles.

because $\Delta_{AF}(0)=\frac{1}{2}\epsilon(0)=0$ (and $\gamma^0=1$ at $d=0$). As this example shows, the Dirac gamma matrices in $d+1$ dimensions are just a bookkeeping device to keep track where one can use the Green equation (13). Actually, the vanishing of this graph is achieved already before removing the regularization $d\to 0$ by using symmetric integration in the momentum space representation of $\Delta_{AF}(t,t)$.

Thus no contributions arise from the fermions at order β^2, and this fixes

$$V'_{DR}=0. \tag{17}$$

This is exactly what one expects to preserve supersymmetry, as the counterterm V_{DR} is exactly canceled by the extra potential term $V=\frac{1}{8}R$ needed to have the correct coupling to the scalar curvature in the Hamiltonian (4). Thus dimensional regularization without any counterterm preserves the supersymmetry of the classical $\mathcal{N}=1$ action

$$S=\frac{1}{\beta}\int_0^1 d\tau\left(\frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu+\frac{1}{2}\psi_a(\dot{\psi}^a+\dot{x}^\mu\omega_\mu{}^a{}_b\psi^b)\right) \tag{18}$$

since the amount of the curvature coupling brought in by DR is of the exact amount to render the quantum Hamiltonian $\hat{H}$ supersymmetric.

To compare with TS, we can compute again the Feynman graph (14), but now using the TS rules. According to Ref. [12] we must use that ${}^\bullet\Delta^\bullet(\tau,\sigma)=1-\delta(\tau,\sigma)$, and integrate the delta function even if it acts on discontinuous functions. The delta function is ineffective as $\epsilon(0)=0$, but the rest gives

$$\begin{aligned}\frac{1}{2}\langle(\Delta S_3)^2\rangle(TS)&=\frac{1}{2}(-2)\left(\frac{1}{2\beta}\omega_{\mu ab}\right)^2(-\beta^3)\int_0^1 d\tau\int_0^1 d\sigma\frac{1}{4}\\&=\frac{\beta}{16}(\omega_{\mu ab})^2.\end{aligned} \tag{19}$$

This is canceled by using an extra counterterm $V'_{TS}=\frac{1}{16}(\omega_{\mu ab})^2$ which at this order contributes with a term $-\beta V'_{TS}$ evaluated at the background point x_0. Thus as expected we recover the counterterm V'_{TS} found in Ref. [12].

To summarize, we have proven that DR extended to fermions does not require additional counterterms on top of those described in Ref. [13]. In addition, supersymmetry requires that no counterterms should be added at all to the classical sigma model action.

A. Periodic boundary conditions

We present here some comments on the case of Majorana fermions with PBC's. The mode expansion of $\psi^a(\tau)$ has now only integer modes,

$$\psi^a(\tau)=\sum_{n\in Z}\psi_n^a e^{2\pi i n\tau}. \tag{20}$$

The zero modes ψ_0^a of the free kinetic operator (∂_τ) are treated separately, and the unregulated propagator in the sector of periodic functions orthogonal to the zero mode reads

$$\langle\psi'^a(\tau)\psi'^b(\sigma)\rangle=\beta\delta^{ab}\Delta_{PF}(\tau-\sigma),$$

$$\Delta_{PF}(\tau-\sigma)=\sum_{n\neq 0}\frac{1}{2\pi i n}e^{2\pi i n(\tau-\sigma)}, \tag{21}$$

where $\psi'^a(\tau)=\psi^a(\tau)-\psi_0^a$. The function Δ_{PF} satisfies

$$\partial_\tau\Delta_{PF}(\tau-\sigma)=\delta_P(\tau-\sigma)-1 \tag{22}$$

with $\delta_P(\tau-\sigma)$ the Dirac delta on periodic functions. Its continuum limit can be obtained by summing up the Fourier series and reads (for $(\tau-\sigma)\in[-1,1]$)

$$\Delta_{PF}(\tau-\sigma)=\frac{1}{2}\epsilon(\tau-\sigma)-(\tau-\sigma). \tag{23}$$

The dimensionally regulated propagator is instead

$$\langle\psi'^a(t)\overline{\psi'}^b(s)\rangle=\beta\delta^{ab}\Delta_{PF}(t,s) \tag{24}$$

where the function

$$\begin{aligned}\Delta_{PF}(t,s)=&-i\int\frac{d^d\mathbf{k}}{(2\pi)^d}\\&\times\sum_{n\neq 0}\frac{2\pi n\gamma^0+\mathbf{k}\cdot\vec{\boldsymbol{\gamma}}}{(2\pi n)^2+\mathbf{k}^2}e^{2\pi i n(\tau-\sigma)}e^{i\mathbf{k}\cdot(\mathbf{t}-\mathbf{s})}\end{aligned} \tag{25}$$

satisfies

$$\begin{aligned}\gamma^\alpha\frac{\partial}{\partial t^\alpha}\Delta_{PF}(t,s)&=-\frac{\partial}{\partial s^\beta}\Delta_{PF}(t,s)\gamma^\beta\\&=[\delta_P(\tau-\sigma)-1]\delta^d(\mathbf{t}-\mathbf{s}).\end{aligned} \tag{26}$$

Even if one uses PBC's, one does not expect additional counterterms in DR, as mentioned earlier. It could be interesting to check in DR the expected β independence of the supertrace which computes the Witten index, i.e. the chiral anomaly. This is given by the path integral with periodic boundary conditions for both bosons and fermions [4–6]. The treatment of the bosonic zero modes is known to be somewhat delicate as a total derivative term may appear at higher loops [2,27]. However it should be possible to do a manifestly supersymmetric computation using superfields, and one could thus check if these total derivative terms survive in the supersymmetric case and, in case they do, study their meaning.[4]

[4]In a very recent paper, Kleinert and Chervyakov [28] have discussed how to avoid these total derivative terms which appear using naively the bosonic string inspired propagators.

B. Curved indices

It is interesting to consider as well the case of fermions with curved target space indices. This should be equivalent to the case of fermions with flat target space indices: it is just a change of integration variables in the path integral. However it is an useful exercise to work out, as some formulas will become simpler. The classical $\mathcal{N}=1$ supersymmetric sigma model is now written as

$$S=\frac{1}{\beta}\int_0^1 d\tau\frac{1}{2}g_{\mu\nu}(x)\{\dot{x}^\mu\dot{x}^\nu+\psi^\mu[\dot{\psi}^\nu+\dot{x}^\lambda\Gamma^\nu_{\lambda\rho}(x)\psi^\rho]\}. \tag{27}$$

The fermionic term could also be written more compactly using the covariant derivative $(D/d\tau)\psi^\nu=\dot{\psi}^\nu+\dot{x}^\lambda\Gamma^\nu_{\lambda\rho}(x)\psi^\rho$. Note that the action is now expressed in terms of the metric and Christoffel connection only, and there is no need of introducing the vielbein and spin connection.

The treatment of the bosonic part goes on unchanged. For the fermionic part we can derive the correct path integral measure by taking into account the Jacobian for the change of variables from the free measure with flat indices

$$\begin{aligned}D\psi^a&=D[e^a{}_\mu(x)\psi^\mu]=\mathrm{Det}^{-1}[e^a{}_\mu(x)]D\psi^\mu\\&=\left(\prod_{0\leqslant\tau<1}\frac{1}{\sqrt{\det g_{\mu\nu}[x(\tau)]}}\right)D\psi^\mu.\end{aligned} \tag{28}$$

Note the inverse functional determinant appearing because of the Grassmann nature of the integration variables. This extra factor arising in the measure can be exponentiated using bosonic ghosts $\alpha^\mu(\tau)$ with the same boundary condition of the fermions (ABC's or PBC's) and it leads to the following extra term in the ghost action:

$$S_{gh}^{extra}=\frac{1}{\beta}\int_0^1 d\tau\frac{1}{2}g_{\mu\nu}(x)\alpha^\mu\alpha^\nu. \tag{29}$$

One can check that the counterterms of dimensional regularization are left unchanged. The full quantum action for the $\mathcal{N}=1$ supersymmetric sigma model now reads

$$\begin{aligned}S=\frac{1}{\beta}\int_0^1 d\tau\frac{1}{2}g_{\mu\nu}(x)\{&\dot{x}^\mu\dot{x}^\nu+a^\mu a^\nu+b^\mu c^\nu\\&+\psi^\mu[\dot{\psi}^\nu+\dot{x}^\lambda\Gamma^\nu_{\lambda\rho}(x)\psi^\rho]+\alpha^\mu\alpha^\nu\}\end{aligned} \tag{30}$$

and appears in the path integral as

$$Z=\int DxDaDbDcD\psi D\alpha e^{-S}. \tag{31}$$

It is clear that supersymmetry is not broken by the boundary conditions if one uses PBC's. Then the effects of the ghosts cancel by themselves: the ghosts have the same boundary conditions and can be eliminated altogether from the path integral

$$\left(\prod_{0\leqslant\tau<1}\sqrt{\det g_{\mu\nu}[x(\tau)]}\right)\left(\prod_{0\leqslant\tau<1}\frac{1}{\sqrt{\det g_{\mu\nu}[x(\tau)]}}\right)=1. \tag{32}$$

One can recognize that the potential divergences arising in the bosonic $\dot{x}\dot{x}$ contractions are canceled by the fermionic $\psi\dot{\psi}$ contractions, while the remaining UV ambiguities are treated by dimensional regularization as usual. In this scheme it should be simpler for example to test that the Witten index (i.e. the gravitational contribution to the chiral anomaly for a spin 1/2 field) does not get higher order contributions in worldline loops, and is thus β independent.

If one uses ABC's the ghosts have different boundary conditions. Hence their cancellation is not complete, and they must be kept in the action.

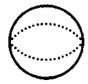
III. TRACE ANOMALY FOR A SPIN 1/2 FIELD IN 4 DIMENSIONS

As a further test on the DR scheme applied to fermions, we compute the trace anomaly for a spin 1/2 fields in 4 dimensions. This anomaly is given by: (i) extending the formula (3) to include the three-loop correction (order β^2 inside the round bracket); (ii) setting $D=4$; (iii) picking up the order β^0 term [8]; and (iv) including an overall minus sign which takes care of the fermionic nature of the target space loop. The bosonic part has been computed already in DR using Riemann normal coordinates (see Ref. [14], use $\xi=\frac{1}{4}$, and recall our conventions on the scalar curvature reported in Appendix Sec. 1). Multiplied by $2^{D/2}$ (the additional normalization due the worldline ABC Majorana fermions) it reads

$$\begin{aligned}Z_{bos}&=\mathrm{tr}\langle x_0|e^{-\beta\hat{H}}|x_0\rangle_{bos}\\&=\frac{2^{D/2}}{(2\pi\beta)^{D/2}}\left(1-\frac{\beta}{24}R+\frac{\beta^2}{1152}R^2+\frac{\beta^2}{720}(R^2_{\mu\nu\lambda\rho}\right.\\&\qquad\left.-R^2_{\mu\nu})-\frac{\beta^2}{480}\nabla^2R+O(\beta^3)\right).\end{aligned} \tag{33}$$

We have now to include the fermionic contributions. On top of Riemann normal coordinates we may use a Fock-Schwinger gauge for the spin connection $\omega_{\mu ab}(x_0+y)=\frac{1}{2}y^\nu R_{\nu\mu ab}(x_0)+\ldots$ with y^μ the Riemann normal coordinates around the background point x_0^μ. Then the leading quartic vertex $S_{4,f}=(1/4\beta)R_{\mu\nu ab}\int_0^1 d\tau y^\mu\dot{y}^\nu\psi^a\psi^b$ which originates from the spin connection produces the following 3-loop diagram:

$$\left\langle \frac{1}{2}(S_{4,f})^2 \right\rangle =$$



$$= -\frac{\beta^2}{16} R^2_{\mu\nu ab} \int_0^1 d\tau \int_0^1 d\sigma \; ({}^{\bullet}\Delta^{\bullet}\,\Delta - {}^{\bullet}\Delta\Delta^{\bullet})\,\Delta^2_{AF} \tag{34}$$

where all functions Δ and Δ_{AF} are functions of τ and σ in this precise order [recall that Δ_{AF} is antisymmetric, $\Delta_{AF}(\tau,\sigma)=-\Delta_{AF}(\sigma,\tau)$].

Now we regulate this graph in DR as follows [the second contribution in Eq. (34) does not need regularization and could be directly computed at $d=0$, but we carry it along anyway]

$$\begin{aligned}
&\int_0^1 d\tau \int_0^1 d\sigma ({}^{\bullet}\Delta^{\bullet}\Delta - {}^{\bullet}\Delta\Delta^{\bullet})\Delta^2_{AF} \\
&\to \int\int ({}_{\alpha}\Delta_{\beta}\,\Delta - {}_{\alpha}\Delta\Delta_{\beta})\mathrm{tr}[-\gamma^{\alpha}\Delta_{AF}(t,s) \\
&\quad \times \gamma^{\beta}\Delta_{AF}(s,t)] \\
&= \int\int \Delta_{\beta}\Delta\, \mathrm{tr}\Bigg[2\underbrace{\left(\gamma^{\alpha}\frac{\partial}{\partial t^{\alpha}}\Delta_{AF}(t,s)\right)}_{\delta_A^{d+1}(t-s)}\gamma^{\beta}\Delta_{AF}(s,t)\Bigg] \\
&\quad -2\int\int {}_{\alpha}\Delta\Delta_{\beta}\mathrm{tr}[-\gamma^{\alpha}\Delta_{AF}(t,s)\gamma^{\beta}\Delta_{AF}(s,t)] \\
&= 0 - 2\int\int {}_{\alpha}\Delta\Delta_{\beta}\mathrm{tr}[-\gamma^{\alpha}\Delta_{AF}(t,s)\gamma^{\beta}\Delta_{AF}(s,t)] \\
&\to -2\int_0^1 d\tau \int_0^1 d\sigma\, {}^{\bullet}\Delta\Delta^{\bullet}\Delta^2_{AF} \\
&= -2\int_0^1 d\tau \int_0^1 d\sigma\, {}^{\bullet}\Delta\Delta^{\bullet}\frac{1}{4} \\
&= -2\left(-\frac{1}{12}\right)\left(\frac{1}{4}\right) = \frac{1}{24}
\end{aligned} \tag{35}$$

where we have integrated by parts the α derivative in ${}_{\alpha}\Delta_{\beta}$, which then produces a delta function when acting on fermions ("equations of motion terms"). This delta function is integrated in $d+1$ dimensions and gives a vanishing contribution since $\Delta_{AF}(0)=0$. The remaining terms are then computed at $d\to 0$. Thus

$$\left\langle \frac{1}{2}(S_{4,f})^2 \right\rangle = -\frac{\beta^2}{384}R^2_{\mu\nu ab}. \tag{36}$$

This fermionic contribution must now be added to the terms inside the round bracket of Eq. (33). Setting $D=4$ one recognizes the following anomaly:

$$\begin{aligned}
Z|_{\beta^0-term} = \frac{1}{4\pi^2}\bigg(&\frac{1}{288}R^2 - \frac{7}{1440}R^2_{\mu\nu\lambda\rho} - \frac{1}{180}R^2_{\mu\nu} \\
&- \frac{1}{120}\nabla^2 R\bigg).
\end{aligned} \tag{37}$$

This is the correct trace anomaly for a Dirac fermion in 4 dimensions once we include the minus sign due to the target space fermionic loop.

IV. ONE-LOOP EFFECTIVE ACTION FOR A DIRAC FIELD IN A GRAVITATIONAL BACKGROUND

It is known that, for a wide class of field theories, the one-loop effective action and the relative N-point vertex functions can be computed using one-dimensional path integrals [21,22]. Two of us have presented in Ref. [2] the extension of this formalism to include a gravitational background, considering the simplest case of a scalar field. The extension of DR to worldline fermions allows us to do the same for a Dirac field. We will get an expression for the effective action from which we derive explicitly the one- and two-point correlation functions, namely the contribution to the tadpole and self-energy of the graviton. We perform this program considering both flat and curved indices for the worldline Majorana fermions. The use of flat indices produces an effective action $\bar{\Gamma}[e_{a\mu}]$ which is naturally a functional of the vielbein. The use of curved indices produces instead an effective action $\Gamma[g_{\mu\nu}]$ which is naturally a functional of the metric. Local Lorentz invariance guarantees that $\bar{\Gamma}[e_{a\mu}]=\Gamma[g_{\mu\nu}(e_{a\mu})]$. In the following we shall discuss both cases. As we shall see, the simplest set up is to use curved indices: in this case the sigma model couples linearly to the metric fluctuations $h_{\mu\nu}=g_{\mu\nu}-\delta_{\mu\nu}$, and the effective N-point vertices for the metric are obtained by integrating over the proper time the quantum average of N graviton vertex operators.

A. The worldline formalism

Let us consider the one-loop effective action obtained by quantizing a Dirac field Ψ coupled to gravity through the vielbein $e_{a\mu}$,

$$S[\Psi,\bar{\Psi},e_{a\mu}] = \int d^D x e \bar{\Psi}(\nabla\!\!\!/ + m)\Psi \tag{38}$$

where $e=\det e^a{}_{\mu}$, $\omega_{\mu ab}$ is the spin connection, and

$$\nabla\!\!\!/ = \gamma^a e_a{}^{\mu}\nabla_{\mu}, \quad \nabla_{\mu} = \partial_{\mu} + \frac{1}{4}\omega_{\mu ab}\gamma^a\gamma^b. \tag{39}$$

The effective action depends on the background vielbein field $e_{a\mu}$ and formally reads as $[e^{-\bar{\Gamma}[e_{a\mu}]} \equiv \int \mathcal{D}\Psi\mathcal{D}\bar{\Psi}e^{-S[\Psi,\bar{\Psi},e_{a\mu}]} = \mathrm{Det}(\nabla\!\!\!/ + m)]$

$$\bar{\Gamma}[e_{a\mu}] = -\log \mathrm{Det}(\nabla\!\!\!/ + m). \tag{40}$$

For a Dirac field one does not expect anomalies (the Euclidean effective action is real) and one can exploit standard arguments to write

$$\overline{\Gamma}[e_{a\mu}] = -\log[\mathrm{Det}(\nabla\!\!\!/ + m)\mathrm{Det}(-\nabla\!\!\!/ + m)]^{1/2}$$
$$= -\frac{1}{2}\mathrm{Tr}\log(-\nabla\!\!\!/^2 + m^2)$$
$$= -\frac{1}{2}\mathrm{Tr}\log\left(-\nabla^2 + \frac{1}{4}R + m^2\right). \quad (41)$$

In this formula we recognize the logarithm of an operator which up the mass term is proportional to the supersymmetric Hamiltonian (4). Thus we can immediately write down a path integral representation for the effective action in terms of a proper time as

$$\overline{\Gamma}[e_{a\mu}] = \frac{1}{2}\int_0^\infty \frac{dT}{T} \oint_{PBC} \mathcal{D}x^\mu \oint_{ABC} D\psi^a e^{-S[x^\mu,\psi^a;e_{a\mu}]} \quad (42)$$

where[5]

$$S[x^\mu,\psi^a;e_{a\mu}] = \int_0^1 d\tau\left(\frac{1}{4T}g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu + \frac{1}{4T}\psi_a\dot{\psi}^a + \frac{1}{4T}\dot{x}^\mu\omega_{\mu ab}(x)\psi^a\psi^b + Tm^2\right). \quad (43)$$

The subscripts PBC and ABC remind of the boundary condition at $\tau=0,1$, periodic for the bosonic coordinates $x^\mu(\tau)$ and antiperiodic for the fermionic ones $\psi^a(\tau)$: these boundary conditions have to be imposed to obtain the trace in Eq. (41). We have used a rescaled proper time $T=\beta/2$ with respect to the previous sections to agree with standard normalizations used in the worldline formalism [22]. We have not added any counterterm since we are going to use dimensional regularization to compute the path integral.[6] Of course, the covariant measure in Eq. (42) contains the ghost fields

$$\mathcal{D}x^\mu = Dx^\mu \prod_{0\leq\tau<1}\sqrt{\det g_{\mu\nu}[x(\tau)]}$$
$$= Dx^\mu \oint_{PBC} Da^\mu Db^\mu Dc^\mu e^{-S_{gh}[x,a,b,c]} \quad (44)$$

where

$$S_{gh}[x,a,b,c] = \int_0^1 d\tau \frac{1}{4T}g_{\mu\nu}(x)(a^\mu a^\nu + b^\mu c^\nu). \quad (45)$$

One may also compute the effective action directly as a functional of the metric. This is achieved by using the sigma model written in terms of the Majorana fermions with curved indices. The corresponding formula is

$$\Gamma[g_{\mu\nu}] = \frac{1}{2}\int_0^\infty \frac{dT}{T} \oint_{PBC} \mathcal{D}x^\mu \oint_{ABC} \mathcal{D}\psi^\mu e^{-S[x^\mu,\psi^\mu;g_{\mu\nu}]} \quad (46)$$

with

$$S[x^\mu,\psi^\mu;g_{\mu\nu}] = \int_0^1 d\tau\left(\frac{1}{4T}g_{\mu\nu}(x)(\dot{x}^\mu\dot{x}^\nu + \psi^\mu\dot{\psi}^\nu + \psi^\mu\Gamma^\nu_{\lambda\rho}(x)\dot{x}^\lambda\psi^\rho) + Tm^2\right). \quad (47)$$

Note that the covariant fermionic measure now contains the new bosonic ghost α^μ

$$\mathcal{D}\psi^\mu = D\psi^\mu \prod_{0\leq\tau<1}\frac{1}{\sqrt{\det g_{\mu\nu}[x(\tau)]}}$$
$$= D\psi^\mu \oint_{ABC} D\alpha^\mu e^{-S_{gh}^{extra}[x,\alpha]} \quad (48)$$

with

$$S_{gh}^{extra}[x,\alpha] = \int_0^1 d\tau \frac{1}{4T}g_{\mu\nu}(x)\alpha^\mu\alpha^\nu. \quad (49)$$

The fermionic term in the action (47) may be written using the covariant derivative as $g_{\mu\nu}\psi^\mu(D/d\tau)\psi^\nu$, making manifest its geometrical meaning. However, one can write the Christoffel connection directly in terms of the metric and, because of the Grassmannian nature of the fields ψ^μ, the action simplifies to

$$S[x^\mu,\psi^\mu;g_{\mu\nu}] = \int_0^1 d\tau\left(\frac{1}{4T}g_{\mu\nu}(x)(\dot{x}^\mu\dot{x}^\nu + \psi^\mu\dot{\psi}^\nu) - \frac{1}{4T}\partial_\mu g_{\nu\lambda}(x)\psi^\mu\psi^\nu\dot{x}^\lambda + Tm^2\right) \quad (50)$$

which shows that there is only a linear coupling to the background $g_{\mu\nu}(x)$. To summarize, we have two options for representing the effective action in the worldline formalism, and we will consider both of them.

The next step is to discuss how to treat the boundary conditions. Due to the translational invariance of the resulting propagators, we adopt the "string inspired" option: one expands the coordinate fields with periodic boundary conditions into Fourier modes and then separates the zero mode $x_0^\mu = \int_0^1 d\tau x^\mu(\tau)$ from the quantum fluctuations $y^\mu(\tau) = x^\mu(\tau) - x_0^\mu$. The latter have an invertible kinetic term and the integration over the constants zero mode x_0^μ is performed

[5]Presumably, this final action can be obtained also by gauge fixing the locally supersymmetric formulation of the spinning particle action [29,30], at least in the massless case, as the corresponding ghosts decouple from the background geometry and can be ignored.

[6]Let us recall that other regularization schemes (such as time slicing [12]) require additional noncovariant counterterms.

separately. For the alternative option of using Dirichlet boundary conditions, see a discussion in Ref. [2]. Other options for treating the zero modes can be found in Refs. [31,22].

These subtleties do not arise for the anticommuting variables ψ^a as the boundary conditions are now antiperiodic and the kinetic term has no zero mode. All these propagators are collected in Sec. 3 of the Appendix.

For later convenience, it is useful to introduce the following notations:

$$\overline{\Gamma}^{a_1\mu_1\cdots a_N\mu_N}_{(x_1,\ldots,x_N)} = \frac{\delta^N \overline{\Gamma}}{\delta e_{a_1\mu_1}(x_1)\cdots \delta e_{a_N\mu_N}(x_N)}\bigg|_{e_{a\mu}=\delta_{a\mu}}, \tag{51}$$

$$\Gamma^{\mu_1\nu_1\cdots\mu_N\nu_N}_{(x_1,\ldots,x_N)} = \frac{\delta^N \Gamma}{\delta g_{\mu_1\nu_1}(x_1)\cdots \delta g_{\mu_N\nu_N(x_N)}}\bigg|_{g_{\mu\nu}=\delta_{\mu\nu}} \tag{52}$$

and the corresponding Fourier transform for the vielbein vertex functions:

$$\begin{aligned}\tilde{\overline{\Gamma}}^{a_1\mu_1\cdots a_N\mu_N}_{(p_1,\ldots,p_N)} &= (2\pi)^D \delta^D(p_1+\ \ldots\ +p_N)\overline{\Gamma}^{a_1\mu_1\cdots a_N\mu_N}_{(p_1,\ldots,p_N)}\\ &= \int dx_1\ldots dx_N e^{ip_1x_1+\ \ldots\ +ip_Nx_N}\overline{\Gamma}^{a_1\mu_1\cdots a_N\mu_N}_{(x_1,\ldots,x_N)}\end{aligned} \tag{53}$$

plus a similar one for the metric vertex functions. The correlation functions obtained by varying the vielbein are symmetric under the interchange of indices belonging to the same couple as a consequence of local Lorentz invariance. Notice also that after restricting these vertices to flat space there is no intrinsic difference between curved and flat indices. For $N=1,2$, and using $\overline{\Gamma}[e_{a\rho}]=\Gamma[g_{\mu\nu}(e_{a\rho})]$ together with the relation $g_{\mu\nu}=\delta_{ab}e^a{}_\mu e^b{}_\nu$, one finds

$$\Gamma^{\mu\nu}_{(x)} = \frac{1}{2}\overline{\Gamma}^{\mu\nu}_{(x)}, \tag{54}$$

$$\Gamma^{\mu_1\nu_1\mu_2\nu_2}_{(x,y)} = \frac{1}{4}\overline{\Gamma}^{\mu_1\nu_1\mu_2\nu_2}_{(x,y)} - \frac{1}{4}\overline{\Gamma}^{\underline{\mu_1}\overline{\mu_2}}_{\underline{(x)}}\delta^{\underline{\nu_1}\overline{\nu_2}}\delta^D(x-y) \tag{55}$$

where we indicate with underline and overline a normalized symmetrization.

Following a standard technique, one can obtain the vertex functions directly in momentum space [22]. Let us describe it for the effective action $\overline{\Gamma}[e_{a\mu}]$. One considers $\overline{\Gamma}[e_{a\mu}]$ as a power series in $c_{a\mu}\equiv e_{a\mu}-\delta_{a\mu}$ (note that this definition induces a relative expression for the metric: $g_{\mu\nu}=\delta_{\mu\nu}+c_{\mu\nu}+c_{\nu\mu}+c_{a\mu}c^a_\nu$, where $c_{\mu\nu}=c_{a\nu}\delta^a_\mu$), takes the c^N term as a sum of N plane waves of given polarizations (our polarization tensors include the gravitational coupling constant)

$$c_{a\mu}(x)=\sum_{i=1}^{N}\ \varepsilon^{(i)}_{a\mu}e^{ip_i\cdot x}, \tag{56}$$

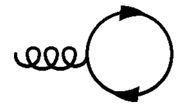


FIG. 1. Graviton tadpole.

and then picks up the terms linear in each $\varepsilon^{(i)}_{a\mu}$: this gives directly the N-point function in momentum space,

$$\tilde{\overline{\Gamma}}^{\varepsilon_1\cdots\varepsilon_N}_{(p_1,\ldots,p_N)} \equiv \varepsilon^{(1)}_{a_1\mu_1}\cdots\varepsilon^{(N)}_{a_N\mu_N}\tilde{\overline{\Gamma}}^{a_1\mu_1\cdots a_N\mu_N}_{(p_1,\ldots,p_N)} \tag{57}$$

[the tilde symbol can be dropped by removing the momentum delta functions as in Eq. (53)].

In the following sections we are going to compute the one- and two-point correlation functions. We will employ the worldline "string inspired" propagators together with dimensional regularization on the worldline (and in target space).

B. One- and two-point functions from $\overline{\Gamma}[e_{a\mu}]$

The one-point vertex function can be depicted by the Feynman diagram of Fig. 1 where the external line refers to the vielbein. It gives the Dirac particle contribution to the cosmological constant. The recipe just outlined tells that the term in the effective action linear in $c_{a\mu}$, and with $c_{a\mu}$ expressed as a single plane wave, produces

$$\begin{aligned}\tilde{\overline{\Gamma}}^{\varepsilon}_{(p)} = \frac{1}{2}\int_0^\infty \frac{dT}{T}e^{-m^2T}\frac{2^{D/2}}{(4\pi T)^{D/2}}\int d^Dx_0\left(-\frac{1}{4T}\right)\\ \times\int_0^1 d\tau\{\langle 2\varepsilon_{(\mu\nu)}(\dot{y}^\mu\dot{y}^\nu+a^\mu a^\nu+b^\mu c^\nu)e^{ip\cdot(x_0+y)}\rangle\\ +\langle \dot{y}^\mu\omega^{(1)}_{\mu ab}(x_0+y)\rangle\langle\psi^a\psi^b\rangle\},\end{aligned} \tag{58}$$

where the superscript on $\omega^{(1)}_{\mu ab}$ denotes the part linear in $\varepsilon_{a\mu}$, and round brackets around indices denote symmetrization normalized to 1.

It can be immediately noted that the contribution of the spin connection term vanishes, being proportional to $\omega_{\mu ab}\delta^{ab}\Delta_{AF}(0)=0$. Therefore everything proceeds as in the scalar field case [2], and the one-point function reads

$$\overline{\Gamma}^{\mu\nu}_{(0)} = 2^{D/2}\frac{\delta^{\mu\nu}}{2}\frac{(m^2)^{D/2}}{(4\pi)^{D/2}}\Gamma\left(-\frac{D}{2}\right). \tag{59}$$

Clearly it diverges for even target space dimension D and renormalization is needed.

Let us now discuss the two-point vertex function. We set

$$c_{a\mu}(x)=\varepsilon^{(1)}_{a\mu}e^{ip_1\cdot x}+\varepsilon^{(2)}_{a\mu}e^{ip_2\cdot x}. \tag{60}$$

One sees that there are two kinds of contributions, illustrated by the Feynman graphs in Figs. 2 and 3, which we denote as $\Delta_1\overline{\Gamma}^{\mu\nu\alpha\beta}$ and $\Delta_2\overline{\Gamma}^{\mu\nu\alpha\beta}$, respectively.

In the first one there is just one vertex. It is simple to compute it, being quite similar to the tadpole. It reads

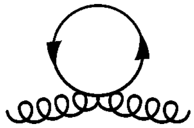


FIG. 2. One-vertex graph for graviton self-energy.

$$\Delta_1 \tilde{\tilde{\Gamma}}^{\varepsilon_1\varepsilon_2}_{(p_1,p_2)} = \frac{1}{2}\int_0^\infty \frac{dT}{T} e^{-m^2T}\frac{2^{D/2}}{(4\pi T)^{D/2}}\int d^D x_0 \left(-\frac{1}{4T}\right) \times \int_0^1 d\tau \{\langle c_{a\mu}c^a_\nu(\dot{y}^\mu\dot{y}^\nu + a^\mu a^\nu + b^\mu c^\nu)\rangle + \langle \dot{y}^\mu \omega^{(2)}_{\mu ab}(x_0+y)\rangle\langle\psi^a\psi^b\rangle\}|_{\mathrm{m.l.}} \tag{61}$$

where $\omega^{(2)}_{\mu ab}$ is the part of the spin connection quadratic in the $c_{a\mu}$ field; the prescription m.l. (multilinear) refers to the two different polarization tensors. The contribution from the spin connection term vanishes for the same reason as before. We are then left with the bosonic contribution, which gives

$$\Delta_1 \bar{\Gamma}^{\mu\nu\alpha\beta}_{(p,-p)} = \frac{2^{D/2}}{4}(\delta^{\mu\alpha}\delta^{\nu\beta}+\delta^{\mu\beta}\delta^{\nu\alpha})\frac{(m^2)^{D/2}}{(4\pi)^{D/2}}\Gamma\left(-\frac{D}{2}\right) \tag{62}$$

where, according to the notation (53), we have factored out $(2\pi)^D\delta^D(p_1+p_2)$ and used $p=p_1=-p_2$.

The two-vertex graph of Fig. 3 produces

$$\Delta_2 \tilde{\tilde{\Gamma}}^{\varepsilon_1\varepsilon_2}_{(p_1,p_2)} = \frac{1}{2}\int_0^\infty \frac{dT}{T} e^{-m^2T}\frac{2^{D/2}}{(4\pi T)^{D/2}} \times \int d^D x_0 \left\langle \frac{1}{2}\left[\int_0^1 d\tau \left(\frac{1}{2T} c_{(\mu\nu)}(\dot{y}^\mu\dot{y}^\nu + a^\mu a^\nu + b^\mu c^\nu) + \frac{1}{4T}\dot{y}^\mu\omega^{(1)}_{\mu ab}\psi^a\psi^b\right)\right]^2\right\rangle\Bigg|_{\mathrm{m.l.}}.$$

Three kinds of contributions are included in the previous expression: (i) the square of the bosonic part which yields a term proportional to the contribution of a scalar field (nonminimally coupled with $\xi=1/4$, already computed in Ref. [2]); (ii) the mixed terms of the product which are zero, again being proportional to $\omega_{\mu ab}\delta^{ab}\Delta_{AF}(0)$; (iii) the square of the fermionic term which contains

$$\omega^{(1)}_{\mu ab}(x_0+y) = \sum_{i=1}^{2}(-ip_\mu\varepsilon^{(i)}_{[ab]} + i\varepsilon^{(i)}_{\mu[a}p_{b]} - ip_{[a}\varepsilon^{(i)}_{b]\mu})e^{ip_i\cdot(x_0+y)} \tag{63}$$

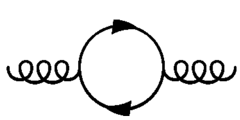


FIG. 3. Two-vertex graph for graviton self-energy.

where square brackets around indices denote antisymmetrization normalized to 1. This third term produces the following contribution:

$$\int_0^\infty \frac{dT}{32T^3}e^{-m^2T}\frac{2^{D/2}}{(4\pi T)^{D/2}}(-ip_\mu\varepsilon^{(1)}_{[ab]} + i\varepsilon^{(1)}_{\mu[a}p_{b]} - ip_{[a}\varepsilon^{(1)}_{b]\mu})(-ip_\rho\varepsilon^{(2)}_{[cd]} + i\varepsilon^{(2)}_{\rho[c}p_{d]} - ip_{[c}\varepsilon^{(2)}_{d]\rho})\int_0^1 d\tau \times \int_0^1 d\sigma \langle e^{ip\cdot y}\dot{y}^\mu\psi^a\psi^b(\tau)e^{-ip\cdot y}\dot{y}^\rho\psi^c\psi^d(\sigma)\rangle. \tag{64}$$

After performing Wick contractions, the second line of this expression becomes

$$(\delta^{ac}\delta^{bd}-\delta^{ad}\delta^{bc})(2T)^3\int_0^1 d\tau\int_0^1 d\sigma\{\delta^{\mu\rho}\,{}^\bullet\Delta^\bullet(\tau-\sigma) + 2Tp^\mu p^\rho\,{}^\bullet\Delta^2(\tau-\sigma)\}e^{-2Tp^2\Delta_0(\tau-\sigma)}\Delta^2_{AF}(\tau-\sigma), \tag{65}$$

where $\Delta_0(\tau-\sigma)=\Delta(\tau-\sigma)-\Delta(0)$, and needs worldline regularization. Following the rules of dimensional regularization we write the last line of the above expression as we would have done starting from the action in $1+d$ dimensions,

$$\int\int\{\delta^{\mu\rho}{}_\alpha\Delta_\beta(t-s) + 2Tp^\mu p^\rho{}_\alpha\Delta(t-s)\,{}_\beta\Delta(t-s)\} \times e^{-2Tp^2\Delta_0(t-s)}\mathrm{tr}[\gamma^\alpha\Delta_{AF}(t-s)\gamma^\beta\Delta_{AF}(t-s)] \tag{66}$$

and perform an integration by parts on the α index of the first term, as already explained in Eqs. (16) and (35), to get the following result:

$$-\frac{1}{2}Tp^2\left(\delta^{\mu\rho}-\frac{p^\mu p^\rho}{p^2}\right)\int_0^1 d\tau\left(\tau-\frac{1}{2}\right)^2 e^{-Tp^2(\tau-\tau^2)}. \tag{67}$$

The remaining worldline integral can be computed as described in Sec. 4 of the Appendix. Using the result into Eq. (64) gives

$$\varepsilon^{(1)}_{\mu\nu}\varepsilon^{(2)}_{\alpha\beta}\frac{1}{8}\frac{2^{D/2}}{(4\pi)^{D/2}}\Gamma\left(1-\frac{D}{2}\right)p^2[(P^2)^{D/2-1} - (m^2)^{D/2-1}]S_2^{\mu\nu\alpha\beta} \tag{68}$$

where P^2 and S_2 are defined below. Collecting all terms, we find for the two-vertex part of the self-energy

$$\Delta_2\bar{\Gamma}_{(p,-p)} = \frac{2^{D/2}}{2(4\pi)^{D/2}}\left\{\Gamma\left(-\frac{D}{2}\right)\{(m^2)^{D/2}(R_1-R_2) + [(P^2)^{D/2}-(m^2)^{D/2}](S_1+S_2)\} + \frac{1}{4}\Gamma\left(1-\frac{D}{2}\right)p^2(P^2)^{D/2-1}S_2\right\}. \tag{69}$$

Consequently, the full graviton self-energy obtained by summing Eqs. (62) and (69) reads

$$\overline{\Gamma}_{(p,-p)}=\frac{2^{D/2}}{2(4\pi)^{D/2}}\Bigg\{\Gamma\left(-\frac{D}{2}\right)\Bigg[(m^2)^{D/2}\left(R_1-\frac{1}{2}R_2-S_1-S_2\right)+(P^2)^{D/2}(S_1+S_2)\Bigg]+\frac{1}{4}\Gamma\left(1-\frac{D}{2}\right)\times p^2(P^2)^{D/2-1}S_2\Bigg\} \quad (70)$$

where, as in Ref. [2], we have suppressed tensor indices, and used the following basis of dimensionless tensors with the required symmetry properties:

$$R_1^{\mu\nu\alpha\beta}=\delta^{\mu\nu}\delta^{\alpha\beta},$$

$$R_2^{\mu\nu\alpha\beta}=\delta^{\mu\alpha}\delta^{\nu\beta}+\delta^{\mu\beta}\delta^{\nu\alpha},$$

$$R_3^{\mu\nu\alpha\beta}=\frac{1}{p^2}(\delta^{\mu\alpha}p^\nu p^\beta+\delta^{\nu\alpha}p^\mu p^\beta+\delta^{\mu\beta}p^\nu p^\alpha+\delta^{\nu\beta}p^\mu p^\alpha),$$

$$R_4^{\mu\nu\alpha\beta}=\frac{1}{p^2}(\delta^{\mu\nu}p^\alpha p^\beta+\delta^{\alpha\beta}p^\mu p^\nu),$$

$$R_5^{\mu\nu\alpha\beta}=\frac{1}{p^4}p^\mu p^\nu p^\alpha p^\beta. \quad (71)$$

For simplicity we have introduced the manifestly transverse combinations

$$S_1^{\mu\nu\alpha\beta}=R_1-R_4+R_5=\left(\delta^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right)\left(\delta^{\alpha\beta}-\frac{p^\alpha p^\beta}{p^2}\right), \quad (72)$$

$$S_2^{\mu\nu\alpha\beta}=R_2-R_3+2R_5=2\left(\delta^{\mu\overline{\alpha}}-\frac{p^\mu p^{\overline{\alpha}}}{p^2}\right)\left(\delta^{\underline{\nu}\overline{\beta}}-\frac{p^{\underline{\nu}}p^{\overline{\beta}}}{p^2}\right) \quad (73)$$

and defined

$$(P^2)^x=\int_0^1 d\tau[m^2+p^2(\tau-\tau^2)]^x. \quad (74)$$

Further details may be found in Sec. 4 of the Appendix.

The final results for the one- and two-point functions, Eqs. (59) and (70), satisfy the gravitational Ward identities (see Appendix Sec. 5). Of course, one may now extract the divergent part and renormalize these functions in the chosen spacetime dimensions.[7]

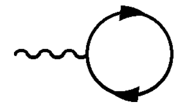


FIG. 4. Graviton tadpole.

C. One- and two-point functions from $\Gamma[g_{\mu\nu}]$

In this section we describe the calculation of the one- and two-point functions employing curved indices for the worldline Majorana fermions. As already explained in Secs. II B and IV A, the total action including the ghost fields is given by

$$S=\int_0^1 d\tau\ \Bigg\{\frac{1}{4T}g_{\mu\nu}(x)(\dot{x}^\mu\dot{x}^\nu+a^\mu a^\nu+b^\mu c^\nu+\psi^\mu\dot{\psi}^\nu+\alpha^\mu\alpha^\nu)-\frac{1}{4T}\partial_\mu g_{\nu\lambda}(x)\psi^\mu\psi^\nu\dot{x}^\lambda+Tm^2\Bigg\}. \quad (75)$$

Clearly, there are no vertices with two or more gravitons in this picture. Using

$$h_{\mu\nu}(x)\equiv g_{\mu\nu}(x)-\delta_{\mu\nu}=\sum_{i=1}^N \epsilon_{\mu\nu}^{(i)}e^{ip_i\cdot x} \quad (76)$$

with the gravitational coupling constant included into the polarization tensors, one gets the following general expression for the N-point effective vertices:

$$\frac{1}{2}\int_0^\infty\frac{dT}{T}e^{-m^2T}\frac{2^{D/2}}{(4\pi T)^{D/2}}\left(-\frac{1}{4T}\right)^N\left\langle\prod_{i=1}^N\int_0^1 d\tau_i V^{(i)}(\tau_i)\right\rangle \quad (77)$$

where the graviton vertex operator $V^{(i)}(\tau_i)$ is given by

$$V^{(i)}(\tau_i)=\epsilon_{\mu\nu}^{(i)}(\dot{x}^\mu\dot{x}^\nu+a^\mu a^\nu+b^\mu c^\nu+\psi^\mu\dot{\psi}^\nu+\alpha^\mu\alpha^\nu-ip_i^\lambda\psi_\lambda\dot{x}^\mu\psi^\nu)(\tau_i)e^{ip_i\cdot x(\tau_i)}. \quad (78)$$

The explicit calculations of the one- and two-point vertex functions—depicted in Figs. 4 and 5, respectively—give the same results previously obtained from the coupling to the vielbein [after using relations (54) and (55)]. Note that external lines in Figs. 4 and 5 now refer to metric fluctuations. Let us describe briefly these calculations.

In the one-point function the connection term [i.e. the last term inside the round brackets of the vertex operator (78)] does not contribute, and the remaining terms lead to the same worldline integral obtained previously,

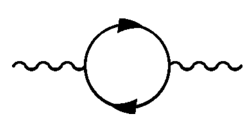


FIG. 5. Graviton self-energy.

[7]If one is interested in odd dimensions, then there is no divergence at one loop, but the formulas should be modified by substituting $2^{D/2}\rightarrow 2^{[D/2]}$ to account for the correct number of components of a Dirac spinor. Here $[D/2]$ denotes the integer part of $D/2$.

$$\int_0^1 d\tau({}^\bullet\Delta^\bullet+\Delta_{gh}-\Delta^\bullet_{AF}-\Delta_{Agh})(0)$$

$$=\int_0^1 d\tau({}^\bullet\Delta^\bullet+\Delta_{gh})(0)=1. \qquad (79)$$

In fact, the propagator of the extra ghost field

$$\langle\alpha^\mu(\tau)\alpha^\nu(\sigma)\rangle=2T\delta^{\mu\nu}\Delta_{Agh}(\tau-\sigma) \qquad (80)$$

where

$$\Delta_{Agh}(\tau-\sigma)={}^\bullet\Delta_{AF}(\tau-\sigma)=\delta_A(\tau-\sigma) \qquad (81)$$

cancels with $\Delta^\bullet_{AF}(\tau-\sigma)=-{}^\bullet\Delta_{AF}(\tau-\sigma)$ due to the fermionic propagator. This exemplifies the effect of the new ghost α^μ which cancels a contraction arising from the $\psi^\mu\dot\psi^\nu$ term. The final answer is

$$\Gamma^{\mu\nu}_{(0)}=2^{D/2}\frac{\delta^{\mu\nu}}{4}\frac{(m^2)^{D/2}}{(4\pi)^{D/2}}\Gamma\left(-\frac{D}{2}\right). \qquad (82)$$

As one might have expected, this result is $-2^{D/2}$ times the contribution of a scalar field [2]: the minus sign is the usual one due to a fermionic loop, while $2^{D/2}$ is the number of degrees of freedom of a Dirac fermion in even D dimensions.

Let us now look at the two-point function. It corresponds to the single diagram of Fig. 5, as in this scheme all vertices are linear in the graviton field. Again one may note that the α^μ ghosts cancel all Wick contractions arising from the $\psi\dot\psi$ term of the vertex operators (notice that $\Delta_{AF}\Delta^\bullet_{AF}=-\Delta_{AF}\delta_A=0$). Thus only two nonvanishing contributions survive: one from the square of the kinetic term of the bosonic sector [i.e. $\sim(\dot x^2+a^2+bc)^2$]; the other, transverse, from the square of the connection term. The final result is

$$\Gamma_{(p,-p)}=\frac{2^{D/2}}{8(4\pi)^{D/2}}\left\{\Gamma\left(-\frac{D}{2}\right)[(m^2)^{D/2}(R_1-R_2-S_1-S_2)+(P^2)^{D/2}(S_1+S_2)]+\frac{1}{4}\Gamma\left(1-\frac{D}{2}\right)\times p^2(P^2)^{D/2-1}S_2\right\}. \qquad (83)$$

This expression satisfies the expected gravitational Ward identities (for details see Sec. 5 of the Appendix). The graviton self-energy due to a massless fermion has been already computed in Ref. [32], and agrees with the massless limit of this general result.[8]

[8]A similar result for a massive scalar field with minimal coupling to the scalar curvature can be found in Ref. [33] and agrees with the worldline result [2]. A calculation with standard Feynman rules for the scalar has been recently reported again in Ref. [34]. It may be noticed how the worldline computation produces simpler and more compact expressions.

V. CONCLUSIONS

We have extended the worldline formalism to include fermionic fields coupled to gravity. To achieve this task we have found it useful to study dimensional regularization on supersymmetric worldline sigma models. We have shown that dimensional regularization preserves worldline supersymmetry in that no counterterms need to be added to the classical action to maintain supersymmetry. This is in contrast to the time slicing regularization scheme, previously used for supersymmetric sigma models, which required specific counterterms to restore supersymmetry. Of course, final physical results are independent of the regularization scheme adopted. We have applied this set up to describe quantum properties of a Dirac fermion coupled to gravity. Then, we have described the one-loop effective action for a Dirac fermion coupled to gravity in the worldline formalism, and computed the corresponding one- and two-point functions, namely the one-loop fermionic contribution to the cosmological constant and graviton self-energy. We have seen that one can use a formulation either in terms of the vielbein or in terms of the metric, the latter being much simpler as the coupling to gravity is linear (and it avoids the introduction of the local Lorentz symmetry related to a choice of the vielbein). The computations are rather simple and demonstrate the efficiency of the worldline formalism in computing Feynman graphs even in the presence of gravitational fields. Our conclusion is that one can be confident and address more complicated processes using the worldline method. In particular, mixed photon-graviton amplitudes are under study [35].

ACKNOWLEDGMENTS

We would like to thank Christian Schubert for useful discussions and comments. We also thank the EC Commission for financial support via the FP5 Grant HPRN-CT-2002-00325.

APPENDIX

1. Covariant derivatives and curvature tensors

The covariant derivative for a vector with curved indices is $\nabla_\mu V^\nu=\partial_\mu V^\nu+\Gamma^\nu_{\mu\lambda}V^\lambda$, where $\Gamma^\nu_{\mu\lambda}=\frac{1}{2}g^{\nu\rho}(\partial_\mu g_{\lambda\rho}+\partial_\lambda g_{\mu\rho}-\partial_\rho g_{\mu\lambda})$ is the usual Christoffel connection. The corresponding curvatures are defined by

$$[\nabla_\mu,\nabla_\nu]V^\lambda=R_{\mu\nu}{}^\lambda{}_\rho(\Gamma)V^\rho, \quad R_{\mu\nu}=R_{\lambda\mu}{}^\lambda{}_\nu(\Gamma),$$

$$R=R^\mu{}_\mu>0 \quad \text{on spheres.} \qquad (A1)$$

The covariant derivative of a vector with flat indices is $\nabla_\mu V^a=\partial_\mu V^a+\omega_\mu{}^a{}_b V^b$, where $\omega_\mu{}^a{}_b$ is the spin connection satisfying the "vielbein postulate" $\nabla_\mu(\Gamma,\omega)e^a{}_\nu=0$. The corresponding curvature is

$$[\nabla_\mu,\nabla_\nu]V^a=R_{\mu\nu}{}^a{}_b(\omega)V^b. \qquad (A2)$$

These curvatures are related by

$$R_{\mu\nu}{}^\lambda{}_\rho(\Gamma)=R_{\mu\nu}{}^a{}_b(\omega)e^\lambda{}_a e^b{}_\rho. \qquad (A3)$$

2. Propagators for bosons and related ghosts with vanishing Dirichlet boundary conditions

For quantum fields that vanish at $\tau=0,1$, we have the following propagators:

$$\begin{aligned}\langle y^{\mu}(\tau)y^{\nu}(\sigma)\rangle &= -\beta\delta^{\mu\nu}\Delta(\tau,\sigma),\\ \langle a^{\mu}(\tau)a^{\nu}(\sigma)\rangle &= \beta\delta^{\mu\nu}\Delta_{gh}(\tau,\sigma),\\ \langle b^{\mu}(\tau)c^{\nu}(\sigma)\rangle &= -2\beta\delta^{\mu\nu}\Delta_{gh}(\tau,\sigma)\end{aligned}\tag{A4}$$

with Green functions Δ and Δ_{gh} satisfying vanishing Dirichlet boundary conditions

$$\begin{aligned}\Delta(\tau,\sigma) &= \sum_{m=1}^{\infty}\left(-\frac{2}{\pi^2m^2}\sin(\pi m\tau)\sin(\pi m\sigma)\right)\\ &= (\tau-1)\sigma\theta(\tau-\sigma)+(\sigma-1)\tau\theta(\sigma-\tau),\\ \Delta_{gh}(\tau,\sigma) &= \sum_{m=1}^{\infty} 2\sin(\pi m\tau)\sin(\pi m\sigma)\\ &= {}^{\bullet\bullet}\Delta(\tau,\sigma)=\delta(\tau,\sigma)\end{aligned}\tag{A5}$$

where $\theta(\tau-\sigma)$ is the standard step function and $\delta(\tau,\sigma)$ is the Dirac's delta function which vanishes at the boundaries $\tau,\sigma=0,1$. These functions are not translationally invariant.

Their extensions to $d+1$ dimensions read

$$\begin{aligned}\Delta(t,s) = \int \frac{d^d\mathbf{k}}{(2\pi)^d} & \\ \times\sum_{m=1}^{\infty} & \frac{-2}{(\pi m)^2+\mathbf{k}^2}\sin(\pi m\tau)\sin(\pi m\sigma)e^{i\mathbf{k}\cdot(\mathbf{t}-\mathbf{s})},\end{aligned}\tag{A6}$$

$$\begin{aligned}\Delta_{gh}(t,s) &= \int \frac{d^d\mathbf{k}}{(2\pi)^d}\sum_{m=1}^{\infty} 2\sin(\pi m\tau)\sin(\pi m\sigma)e^{i\mathbf{k}\cdot(\mathbf{t}-\mathbf{s})}\\ &= \delta(\tau,\sigma)\delta^d(\mathbf{t}-\mathbf{s})=\delta^{d+1}(t,s).\end{aligned}\tag{A7}$$

Note that the function $\Delta(t,s)$ satisfies the relation (Green's equation)

$$\partial^{\alpha}\partial_{\alpha}\Delta(t,s)=\Delta_{gh}(t,s)=\delta^{d+1}(t,s).\tag{A8}$$

The $d\to 0$ limits of these propagators reproduce the unregulated expressions.

3. The "string inspired" propagators

The propagators we used in the worldline formalism are the "string inspired" ones. More specifically, on the circle the free kinetic term for x^{μ} is proportional to ∂_τ^2 and has a zero mode. Thus one splits

$$\begin{aligned}x^{\mu}(\tau) &= x_0^{\mu}+y^{\mu}(\tau),\\ x_0^{\mu} &= \int_0^1 d\tau x^{\mu}(\tau),\\ y^{\mu}(\tau) &= \sum_{n\neq 0} y_n^{\mu}e^{2\pi i n\tau}\end{aligned}\tag{A9}$$

and the path integration measure becomes

$$Dx=\frac{1}{(4\pi T)^{D/2}}d^Dx_0Dy.\tag{A10}$$

The kinetic term for the quantum bosonic fields y^{μ} is invertible and the corresponding free path integral is normalized to unity,

$$\int Dye^{-\int_0^1 d\tau(1/4T)\dot{y}^2}=1.\tag{A11}$$

The value of the free fermionic path integral defines implicitly its measure. Using flat indices it reads

$$\int_{ABC} D\psi^a e^{-\int_0^1 d\tau(1/2T)\psi_a\dot{\psi}^a}=\mathrm{tr}(1)=2^{D/2}.\tag{A12}$$

The propagators for the free fields are

$$\begin{aligned}\langle y^{\mu}(\tau)y^{\nu}(\sigma)\rangle &= -2T\delta^{\mu\nu}\Delta(\tau-\sigma),\\ \langle a^{\mu}(\tau)a^{\nu}(\sigma)\rangle &= 2T\delta^{\mu\nu}\Delta_{gh}(\tau-\sigma),\\ \langle b^{\mu}(\tau)c^{\nu}(\sigma)\rangle &= -4T\delta^{\mu\nu}\Delta_{gh}(\tau-\sigma),\\ \langle \psi^{a}(\tau)\psi^{b}(\sigma)\rangle &= 2T\delta^{ab}\Delta_{AF}(\tau-\sigma),\end{aligned}\tag{A13}$$

where Δ, Δ_{gh} and Δ_{AF} are given by

$$\begin{aligned}\Delta(\tau-\sigma) &= -\sum_{n\neq 0}\frac{1}{4\pi^2n^2}e^{2\pi i n(\tau-\sigma)}\\ &= \frac{1}{2}|\tau-\sigma|-\frac{1}{2}(\tau-\sigma)^2-\frac{1}{12},\\ \Delta_{gh}(\tau-\sigma) &= \sum_{n=-\infty}^{\infty}e^{2\pi i n(\tau-\sigma)}\\ &= \delta_P(\tau-\sigma),\\ \Delta_{AF}(\tau-\sigma) &= \sum_{r\in Z+1/2}\frac{1}{2\pi i r}e^{2\pi i r(\tau-\sigma)}\\ &= \frac{1}{2}\epsilon(\tau-\sigma)\end{aligned}\tag{A14}$$

and satisfy ${}^{\bullet\bullet}\Delta=\Delta_{gh}-1=\delta_P-1, {}^{\bullet}\Delta_{AF}=\delta_A$, where δ_P and δ_A are the Dirac delta functions on the space of periodic and antiperiodic functions on $[0,1]$, respectively. All these free propagators are translationally invariant and have a well defined parity under $(\tau-\sigma)\to(\sigma-\tau)$.

When using curved indices for the Majorana fermions ψ^μ there appears an extra set of bosonic ghosts α^μ. Their propagators with ABC's are

$$\langle\psi^\mu(\tau)\psi^\nu(\sigma)\rangle=2T\delta^{\mu\nu}\Delta_{AF}(\tau-\sigma),$$
$$\langle\alpha^\mu(\tau)\alpha^\nu(\sigma)\rangle=2T\delta^{\mu\nu}\Delta_{Agh}(\tau-\sigma) \tag{A15}$$

where

$$\Delta_{Agh}(\tau-\sigma)=\sum_{r\in Z+1/2}e^{2\pi ir(\tau-\sigma)}=\delta_A(\tau-\sigma). \tag{A16}$$

Clearly ${}^\bullet\Delta_{AF}(\tau-\sigma)=\Delta_{Agh}(\tau-\sigma)=\delta_A(\tau-\sigma)$.

4. Recursive formula for some worldline integrals

In the calculation of one-particle irreducible (1PI) correlation functions via the worldline formalism described in Sec. IV A one needs integrals of the form

$$A_n=\int_0^1 d\tau\left(\tau-\frac{1}{2}\right)^n e^{-Tp^2(\tau-\tau^2)}=\int_{-1/2}^{1/2}dx x^n e^{Tp^2(x^2-1/4)}. \tag{A17}$$

It is not difficult to prove the following recursive relations:

$$A_{2n+1}=0,$$
$$A_{2n}=\frac{1}{2Tp^2}\left[\left(\frac{1}{2}\right)^{2(n-1)}-(2n-1)A_{2(n-1)}\right] \tag{A18}$$

and express all integrals in terms of A_0.

Recalling the definition of the gamma function

$$\Gamma(x)=\int_0^\infty dT T^{x-1}e^{-T} \tag{A19}$$

one can obtain the following result for the proper time integration of A_0

$$\int_0^\infty dT T^{-x-1}e^{-m^2T}A_0=\Gamma(-x)(P^2)^x \tag{A20}$$

where we have defined

$$(P^2)^x\equiv\int_0^1 d\tau[m^2+p^2(\tau-\tau^2)]^x$$
$$=(m^2)^x{}_2F_1\left(-x,1;\frac{3}{2};-\frac{p^2}{4m^2}\right). \tag{A21}$$

It is useful for the comparison with Ref. [32] to note that

$$\lim_{m^2\to 0}(P^2)^x=(p^2)^x B(x+1,x+1). \tag{A22}$$

Here we have used the hypergeometric function ${}_2F_1$ and the Euler beta function B.

5. Ward identities

A test for our results on one- and two-point functions is provided by the Ward identities due to general coordinate and local Lorentz invariances. Local Lorentz symmetry $\delta e^a{}_\mu=\Lambda^a{}_b(x)e^b{}_\mu$ with arbitrary antisymmetric local $\Lambda_{ab}(x)$ implies that

$$\frac{\delta\bar{\Gamma}[e]}{\delta e^a{}_\mu(x)}\Lambda^a{}_b(x)e^b{}_\mu(x)=0 \tag{A23}$$

which shows that the induces stress tensor $T_{\mu\nu}=(1/e)(\delta\bar{\Gamma}[e]/\delta e_a{}^\mu)e_{a\nu}$ is symmetric. General coordinate invariance leads instead to the conservation law for the induced energy-momentum tensor,

$$\nabla_\mu^{(x)}\frac{1}{e(x)}\frac{\delta\bar{\Gamma}[e]}{\delta e^a_\mu(x)}=0. \tag{A24}$$

Taking functional derivatives of this last expression produces Ward identities that must be satisfied by the one- and two-point functions,

$$p_\mu\bar{\Gamma}^{\mu\nu}_{(p)}=0, \tag{A25}$$

$$p_\mu\bar{\Gamma}^{\mu\nu,\alpha\beta}_{(p,-p)}+\frac{1}{2}p_\mu(\delta^{\nu\beta}\bar{\Gamma}^{\mu\alpha}_{(0)}+\delta^{\nu\alpha}\bar{\Gamma}^{\mu\beta}_{(0)})-p^\nu\bar{\Gamma}^{\alpha\beta}_{(0)}=0. \tag{A26}$$

It is easy to check that Eqs. (59) and (70) do indeed satisfy the latter, while the former is rather straightforward ($p^\mu=0$ due to momentum conservations).

Alternatively, one can derive equivalent Ward identities for the effective action Γ expressed as a functional of the metric and obtain [equivalently, one may use relations (54) and (55)]

$$p_\mu\tilde{\Gamma}^{\mu\nu}_{(p)}=0, \tag{A27}$$

$$p_\mu\Gamma^{\mu\nu,\alpha\beta}_{(p,-p)}+\frac{1}{2}p_\mu(\delta^{\nu\beta}\Gamma^{\mu\alpha}_{(0)}+\delta^{\nu\alpha}\Gamma^{\mu\beta}_{(0)})-\frac{1}{2}p^\nu\Gamma^{\alpha\beta}_{(0)}=0. \tag{A28}$$

Also in this case it is simple to verify that Eqs. (82) and (83) satisfy these Ward identities.

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