IMPROVED CONVERGENCE THEOREMS FOR BUBBLE CLUSTERS. I. THE PLANAR CASE

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Abstract. We develop an *improved convergence theorem* for a case study variational problem with singularities, namely, the isoperimetric problem on planar bubble clusters. We exploit this theorem in the description of isoperimetric clusters, possibly perturbed by a potential. Our methods are not specific to bubble clusters, and should provide a starting point to address similar issues in other variational problems where minimizers are known to possibly develop singularities. Further applications and extensions are discussed in companion papers.

Contents

1. Introduction

1.1. **Overview.** The aim of this two-parts paper is developing a basic technique in the Calculus of Variations, that we call *improved convergence*, in a case study where minimizers can exhibit singularities. We focus on the isoperimetric problem for bubble clusters, whose study was initiated by Almgren [Alm76] and Taylor [Tay76]. As reviewed in section 1.2, the technique of improved convergence has found several applications in recent years to variational problems involving minimization on sets: these applications include sharp stability inequalities, qualitative and quantitative descriptions (and even characterizations) of minimizers, and the relation between strict stability (in the sense of positive second variation) and local minimality. The crucial assumption to exploit improved convergence in these problems is the smoothness of the limit set, which is always the case in ambient space dimension $n \le 7$. In section 1.3 we explain why our very limited understanding of near-to-singularities behavior of minimizing sets prevents the possibility of obtaining improved converge theorems to singular limit sets. For this reason we move to the context of clusters. In this more general context, singularities arise even in dimension $n = 2$. When $n = 2, 3$ we have a full understanding of near-to-singularities behavior of clusters, and thus we can try to obtain improved convergence theorems. In section 1.4 we introduce the theory of bubble clusters and formulate our improved convergence theorem for planar clusters, while in section 1.5 we present some of its applications. As explained in section 1.6, further applications to planar clusters are discussed in [CLMb, CLMc], while an improved convergence theorem in dimension $n = 3$ is obtained in the second part of this paper [CLMa].

1.2. **Improved convergence to a regular limit and applications.** A basic fact about sequences of perimeter almost-minimizing sets, which comes as a direct consequence of the classical De Giorgi's regularity theory [DG60], is that L^1-convergence improves to C^1-convergence whenever the limiting set has smooth boundary, that is to say

$$\begin{cases} \{E_k\}_{k\in\mathbb{N}} \text{ are perimeter almost-minimizing sets} \\ E_k \to E \text{ in } L^1 \text{ with } \partial E \text{ smooth} \end{cases} \qquad \Rightarrow \qquad \partial E_k \to \partial E \text{ in } C^1. \tag{1.1}$$

Let us recall that given $\Lambda \geq 0$, $r_0 > 0$, and an open set $A \subset \mathbb{R}^n$ ($n \geq 2$), a set E of locally finite perimeter in A is a *perimeter* (Λ, r_0)*-minimizing set in* A if

$$P(E; B_{x,r}) \leq P(F; B_{x,r}) + \Lambda\, |E\Delta F|\,, \tag{1.2}$$

whenever $E\Delta F \subset\subset B_{x,r} = \{y \in \mathbb{R}^n : |y - x| < r\} \subset\subset A$ and $r < r_0$; see section 3.1 for the standard notation and terminology used here. In this way, (1.1) means that if $\{E_k\}_{k\in\mathbb{N}}$ is a sequence of perimeter (Λ, r_0)-minimizing sets in $\mathbb{R}^n$ with $|E_k\Delta E| \to 0$ as $k \to \infty$ and if ∂E is a smooth hypersurface, then there exist $\alpha \in (0, 1]$ and $\{\psi_k\}_{k\in\mathbb{N}} \subset C^{1,\alpha}(\partial E)$ such that, for k large enough, and denoting by ν_E the outer unit normal to E

$$\partial E_k = (\mathrm{Id} + \psi_k \nu_E)(\partial E)\,, \qquad \sup_{k\in\mathbb{N}} \|\psi_k\|_{C^{1,\alpha}(\partial E)} < \infty\,, \qquad \lim_{k\to\infty} \|\psi_k\|_{C^1(\partial E)} = 0\,. \tag{1.3}$$

(Here we have set $(\mathrm{Id} + \psi_k\nu_E)(\partial E) = \{x + \psi_k(x)\nu_E(x) : x \in \partial E\}$.) A local version of this improved convergence result is found in [Mir67] in the case $\Lambda = 0$, but actually holds true even for more general notions of almost-minimality than the one considered here; see [Tam84, Theorem 1.9]. It immediately implies a regularizing property of the sets E_k, in the sense that ∂E_k must be an $C^{1,\alpha}$-hypersurface as a consequence of (1.3). Improved convergence finds numerous applications to geometric variational problems. These include:

(A) Sharp quantitative inequalities: In [CL12], (1.1) was used (with $E = B$, where $B = B_{0,1}$ is the unit ball of $\mathbb{R}^n$ with center at the origin) in combination with a selection principle and a result by Fuglede on nearly spherical sets [Fug89] to give an alternative proof of the sharp quantitative isoperimetric inequality of [FMP08], namely

$$P(E) \geq P(B) \Big\{1 + c(n) \min_{x\in\mathbb{R}^n} |E\Delta(x + B)|^2\Big\}\,, \qquad \forall E \subset \mathbb{R}^n\,, |E| = |B|\,.$$

This strategy of proof has been subsequently adopted to prove many other geometric inequalities in sharp quantitative form. Examples are the Gaussian isoperimetric inequality [BBJ14] (see also [CFMP11]), the Euclidean isoperimetric inequality in higher codimension [BDF12], the isoperimetric inequalities on spheres and hyperbolic spaces [BDF12, BDF13], isoperimetric inequalities for eigenvalues [BDPV13] (see also [FMP09]), minimality inequalities of area minimizing hypersurfaces [DPM14], and non-local isoperimetric inequalities [FFM$^+$]; moreover, in [FJ14] the same strategy is used to control by $P(E) - P(B)$ a more precise distance from the family of balls (see also [Neu14] for the case of the Wulff inequality).

(B) Qualitative properties (and characterization) of minimizers: Given a potential $g : \mathbb{R}^n \to \mathbb{R}$ with $g(x) \to +\infty$ as $|x| \to \infty$ and a one-homogeneous and convex integrand $\Phi : \mathbb{R}^n \to [0, \infty)$, in [FM11] the variational problems (parameterized by $m > 0$)

$$\inf \Big\{ \int_{\partial^* E} \Phi(\nu_E)\, d\mathcal{H}^{n-1} + \int_{\mathbb{R}^n} g(x)\, dx : |E| = m \Big\}\,, \tag{1.4}$$

are considered in the small volume regime $m \to 0^+$. Denoting by E_m a minimizer with volume m, one expects $m^{-1/n}\, E_m$ to converge to K, the unit volume Wulff shape of Φ. One of the main results proved in [FM11] is that if Φ is a smooth elliptic integrand and g is smooth, then $m^{-1/n}\, E_m \to K$ as $m \to 0^+$ in every $C^{k,\alpha}$, with explicit rates of convergence in terms of m. The improved convergence theorem (1.1), applied with $E = K$ and on (Φ, Λ, r_0)-minimizing sets, plays of course a basic role in this kind of analysis. The same circle of ideas has been exploited in the qualitative description of minimizers of the Ohta-Kawasaki energy for diblock copolymers [CS13], and to characterize balls as minimizers in isoperimetric problems with competing nonlocal terms [KM13, KMar, BC13, FFM$^+$], and in isoperimetric problems with log-convex densities [FM13].

(C) Stability and L^1-local minimality: A classical problem in the Calculus of Variations is that of understanding whether stable critical points of a given functional are also local minimizers. This question was addressed in the case of the Plateau's problem by White [Whi94], who has proved that a *smooth* surface that is a stable critical point of the area functional is automatically locally area minimizing in L^∞ (see [MR10, DPM14] for the L^1-case). A key step in his argument is again an improved convergence theorem (for area almost-minimizing currents) towards a smooth limit. Similarly, in the case of the Otha-Kawasaki energy, volume-constrained stable critical points with smooth boundary turn out to be volume-constrained L^1-local minimizers, see [AFM13]. Once again, (1.1) is the starting point of the analysis.

1.3. **Improved convergence to a singular limit.** We now try to address the question of the precise meaning one should give to an assertion like

$$\begin{cases} \{E_k\}_{k\in\mathbb{N}} \text{ are perimeter almost-minimizing sets} \\ E_k \to E \text{ in } L^1 \end{cases} \qquad \Rightarrow \qquad \partial E_k \to \partial E \text{ in } C^1\,, \tag{1.5}$$

when ∂E is possibly singular. To this end we split ∂E into its regular and singular parts: precisely, recalling that the reduced boundary $\partial^* E$ of a perimeter (Λ, r_0)-minimizing set in $\mathbb{R}^n$ is a $C^{1,\alpha}$-hypersurface for every $\alpha < 1$, we define the singular part $\Sigma(E)$ of ∂E as

$$\Sigma(E) = \partial E \setminus \partial^* E\,.$$

It turns out that $\Sigma(E)$ is always closed: moreover, it is empty if $2 \le n \le 7$, discrete if $n = 8$, and $\mathcal{H}^s$-negligible for every $s > n-8$ if $n \ge 9$; see, for example, [Mag12, Theorem 21.8,Theorem 28.1]. The regularity theory behind these results also leads to obtain a weak form of (1.3), which in turn reduces to (1.3) when $\Sigma(E) = \emptyset$. More precisely, given a sequence $\{E_k\}_{k\in\mathbb{N}}$ of perimeter (Λ, r_0)-minimizing sets with $E_k \to E$ in L^1, denoting by $I_\rho(S)$ the ρ-neighborhood of $S \subset \mathbb{R}^n$, and setting

$$[\partial E]_\rho = \partial E \setminus I_\rho(\Sigma(E)) \subset \partial^* E\,, \qquad \rho > 0\,, \tag{1.6}$$

one finds that, for every ρ small enough and for $k \ge k(\rho)$, there exists $\{\psi_k\}_{k\ge k(\rho)} \subset C^{1,\alpha}([\partial E]_\rho)$ such that

$$\partial E_k \setminus I_{2\rho}(\Sigma(E)) \subset (\mathrm{Id} + \psi_k \nu_E)([\partial E]_\rho) \subset \partial^* E_k\,, \qquad \forall k \ge k(\rho)\,, \tag{1.7}$$

$$\sup_{k\ge k(\rho)} \|\psi_k\|_{C^{1,\alpha}([\partial E]_\rho)} \le C\,, \qquad \lim_{k\to\infty} \|\psi_k\|_{C^1([\partial E]_\rho)} = 0\,. \tag{1.8}$$

Of course, if $\Sigma(E) = \emptyset$, then (1.7) and (1.8) coincide with (1.3). Moreover, we notice that to replace $\partial E_k \setminus I_{2\rho}(\Sigma(E))$ with, say, $[\partial E_k]_{3\rho}$ in the first inclusion in (1.7), one would need to prove Hausdorff convergence of $\Sigma(E_k)$ to $\Sigma(E)$; however, in this generality, one just knows that $\Sigma(E_k) \subset I_\rho(\Sigma(E))$ provided $k \ge k(\rho)$.

Even though (1.7) and (1.8) seem to contain all the information we can extract from the "standard" regularity theory, this is however not sufficient, for several reasons, to address any of the above mentioned applications. The first evident gap is that we do not parameterize the full boundaries ∂E_k on ∂E. Of course, in presence of singularities we cannot expect to represent the whole ∂E_k as a normal deformation of ∂E; see Figure 1. Therefore, the best we can hope for is to find a sequence $\{f_k\}_{k\in\mathbb{N}}$ of $C^{1,\alpha}$-diffeomorphisms between ∂E and ∂E_k such that

$$\sup_{k\in\mathbb{N}} \|f_k\|_{C^{1,\alpha}(\partial E)} < \infty\,, \qquad \lim_{k\to\infty} \|f_k - \mathrm{Id}\|_{C^1(\partial E)} = 0\,. \tag{1.9}$$

A difficulty here is to specify what is meant by a $C^{1,\alpha}$-diffeomorphism between ∂E and ∂E_k, since these are singular hypersurfaces. Moreover, in passing from (1.7)–(1.8) to (1.9) we may

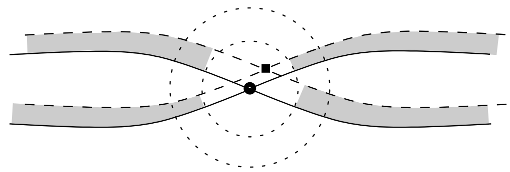

FIGURE 1. The limit boundary ∂E is depicted with continuous lines, the approximating boundaries ∂E_k by dashed lines, the singular set $\Sigma(E)$ by a black circle, and its ρ and 2ρ-neighborhoods $I_\rho(\Sigma(E))$ and $I_{2\rho}(\Sigma(E))$ by concentric balls: $I_\rho(\Sigma(E))$ contains the singular set of ∂E_k (depicted by a black square), while (1.7) says that $\partial E_k \setminus I_{2\rho}(\Sigma(E))$ can be covered by a normal deformation of $[\partial E]_\rho = \partial E \setminus I_\rho(\Sigma(E))$ (depicted as a grey region) which is C^1-close to the identity thanks to (1.8). Of course, we cannot describe ∂E_k by a normal deformation of the four components of $\partial^* E$ unless $\Sigma(E_k) = \Sigma(E)$.

lose the useful information that ∂E_k is actually a C^1-small normal deformation of ∂E away from the singular sets. It is therefore natural to require that, if $k \geq k(\rho)$, then

$$f_k = \mathrm{Id} + \psi_k\, \nu_E \qquad \text{on } [\partial E]_\rho\,, \tag{1.10}$$

with ψ_k as in (1.7)–(1.8). The maps f_k must have a nontrivial tangential displacement

$$u_k = (f_k - \mathrm{Id}) - \Big((f_k - \mathrm{Id}) \cdot \nu_E\Big)\, \nu_E\,,$$

on $[\partial E]_\rho$ if $\Sigma(E_k) \neq \Sigma(E)$: and, actually, in order the maps f_k to be usable in addressing problem (C), it seems crucial to have a control of the C^1-norm of u_k in terms of the distance between $\Sigma(E_k)$ and $\Sigma(E)$. A possibility is requiring that $f_k(\Sigma(E)) = \Sigma(E_k)$, with $f_k = \mathrm{Id}$ on $\Sigma(E)$ if $\Sigma(E_k) = \Sigma(E)$, and, for some constant C depending on ∂E,

$$\|u_k\|_{C^1(\partial E)} \leq C\, \|f_k - \mathrm{Id}\|_{C^1(\Sigma(E))}\,. \tag{1.11}$$

Due to our limited understanding of singular sets, proving (1.7)–(1.11) seems a goal out of reach, and so the possibility of understanding improved convergence to singular limit sets. The theory of bubble clusters (partitions of the space into sets of finite perimeter) provides us with a (more complex) setting where singularities appear even in dimension $n = 2$. However, at least when $n = 2, 3$, these singularities have been classified and understood. This fact opens to the possibility of studying improved convergence in this setting, which is the content of this paper concerning the case $n = 2$, and of [CLMa] when $n = 3$.

1.4. **Perimeter minimizing clusters.** We now briefly introduce the basics of the theory of perimeter minimizing clusters, following [Mag12, Part IV] (which, in turn, is based on [Alm76]). Given $n, N \in \mathbb{N}$ with $n, N \geq 2$, one says that $\mathcal{E} = \{\mathcal{E}(h)\}_{h=1}^N$ is an *N-cluster* if each $\mathcal{E}(h)$ is a set of locally finite perimeter in $\mathbb{R}^n$ with

$$0 < |\mathcal{E}(h)| < \infty\,, \qquad 1 \leq h \leq N\,, \tag{1.12}$$

$$|\mathcal{E}(h) \cap \mathcal{E}(k)| = 0\,, \qquad 1 \leq h < k \leq N\,. \tag{1.13}$$

The sets $\mathcal{E}(h)$, $1 \leq h \leq N$, are called the *chambers* of $\mathcal{E}$, while $\mathcal{E}(0) = \mathbb{R}^n \setminus \bigcup_{h=1}^N \mathcal{E}(h)$ is called the *exterior chamber* of $\mathcal{E}$ (so that $|\mathcal{E}(0)| = \infty$). The *perimeter* of $\mathcal{E}$ relative to some $F \subset \mathbb{R}^n$ is defined by setting

$$P(\mathcal{E}; F) = \frac{1}{2} \sum_{h=0}^N P(\mathcal{E}(h); F) = \sum_{0 \leq h < k \leq N} \mathcal{H}^{n-1}\Big(F \cap \mathcal{E}(h,k)\Big)\,, \tag{1.14}$$

where in the last identity we have set $\mathcal{E}(h,k) = \partial^*\mathcal{E}(h) \cap \partial^*\mathcal{E}(k)$ for the (h,k)th *interface* of $\mathcal{E}$. Setting $P(\mathcal{E}) = P(\mathcal{E}; \mathbb{R}^n)$, the basic variational problem motivating the introduction of clusters is the isoperimetric problem

$$\inf\Big\{P(\mathcal{E}):\mathrm{vol}\,(\mathcal{E})=m\Big\}\,,\qquad m\in\mathbb{R}^N_+\ \text{given}\,,\tag{1.15}$$

where $\mathbb{R}^N_+=\{m\in\mathbb{R}^N:m_h>0\,\forall h=1,...,N\}$, and where $\mathrm{vol}\,(\mathcal{E})$ stands for the vector in $\mathbb{R}^N$ whose hth entry is equal to $|\mathcal{E}(h)|$. A minimizer in (1.15) is called an *isoperimetric cluster*. It is of course natural to study partitioning problems in the presence of a potential energy term, like

$$\inf\Big\{P(\mathcal{E})+\sum_{h=1}^N\int_{\mathcal{E}(h)}g(x)\,dx:\mathrm{vol}\,(\mathcal{E})=m\Big\}\,,\tag{1.16}$$

where, say, $g:\mathbb{R}^n\to\mathbb{R}$ with $g(x)\to+\infty$ as $|x|\to\infty$. The existence of minimizers in these two problems can be proved by a careful restoration of compactness argument due to Almgren, see [Mag12, Chapter 29]. It turns out that if $\mathcal{E}$ is a minimizer either in (1.15) or in (1.16), then there exist positive constants Λ and r_0 such that $\mathcal{E}$ is *a perimeter* (Λ,r_0)*-minimizing cluster* in $\mathbb{R}^n$, that is (in analogy with (1.2))

$$P(\mathcal{E};B_{x,r})\le P(\mathcal{F};B_{x,r})+\Lambda\,\mathrm{d}(\mathcal{E},\mathcal{F})\,,\tag{1.17}$$

whenever $x\in\mathbb{R}^n$, $r<r_0$ and $\mathcal{E}(h)\Delta\mathcal{F}(h)\subset\subset B_{x,r}$ for every $h=1,...,N$. Here we have set

$$\mathrm{d}(\mathcal{E},\mathcal{F})=\frac{1}{2}\sum_{h=0}^N\Big|\,\mathcal{E}(h)\Delta\mathcal{F}(h)\Big|\,,\tag{1.18}$$

for the L^1-distance between $\mathcal{E}$ and $\mathcal{F}$. A partial regularity theorem holds for (Λ,r_0)-minimizing clusters. Precisely, let us set

$$\partial\mathcal{E}=\bigcup_{h=1}^N\partial\mathcal{E}(h)\,,\qquad\partial^*\mathcal{E}=\bigcup_{0\le h<k\le N}\mathcal{E}(h,k)\,,\tag{1.19}$$

where, by our convention on sets of finite perimeter, see section 3.1, $\partial\mathcal{E}(h)=\mathrm{cl}\,(\partial^*\mathcal{E}(h))$ for every $h=1,...,N$. Then (see [Mag12, Chapter 30] for the case $\Lambda=0$, and section 3 below otherwise) $\partial^*\mathcal{E}$ is a $C^{1,\alpha}$-hypersurface (for every $\alpha\in(0,1)$), $\partial^*\mathcal{E}$ is relatively open into $\partial\mathcal{E}$, and $\mathcal{H}^{n-1}(\Sigma(\mathcal{E}))=0$ where $\Sigma(\mathcal{E})$ is the singular set

$$\Sigma(\mathcal{E})=\partial\mathcal{E}\setminus\partial^*\mathcal{E}\,.$$

One does not expect this almost-everywhere regularity result to be optimal in any dimension n, although the situation is clear only when $n=2$ (by elementary arguments) and when $n=3$ by [Tay76].

Let us now review the structure of singular sets when $n=2$, and then exploit this description to formulate an improved convergence result for planar clusters. With the notation introduced in section 2.1, if $\mathcal{E}$ is a perimeter (Λ,r_0)-minimizing cluster in $\mathbb{R}^2$, then one has

$$\left\{\begin{array}{l}\partial\mathcal{E}=\bigcup_{i\in I}\gamma_i\,,\\ \partial^*\mathcal{E}=\bigcup_{i\in I}\mathrm{int}\,(\gamma_i)\,,\end{array}\right.\qquad\begin{array}{l}\text{where } I \text{ is at most countable}\,,\\ \gamma_i \text{ is a closed connected } C^{1,1}\text{-curve with boundary}\,,\\ \{\gamma_i\}_{i\in I}\text{ is locally finite}\,,\end{array}\tag{1.20}$$

(see [Ble87], [Mor94], or [Mag12, Section 30.3] in the case $\Lambda=0$, and Theorem 3.16 below in the general case – which is a simple variant of the $\Lambda=0$ case). Moreover,

$$\Sigma(\mathcal{E})=\bigcup_{j\in J}\{p_j\}=\bigcup_{i\in I}\mathrm{bd}\,(\gamma_i)\,,\qquad\begin{array}{l}\text{where } J \text{ is at most countable}\,,\\ \{p_j\}_{j\in J}\text{ is locally finite}\,,\end{array}\tag{1.21}$$

and each $p_j\in\Sigma(\mathcal{E})$ is a common end-point to three different curves from $\{\gamma_i\}_{i\in I}$, which form three 120 degree angles at p_j.

Remark 1.1. As already noticed, if $\mathcal{E}$ is an isoperimetric cluster in $\mathbb{R}^2$, or if $\mathcal{E}$ is a minimizer in (1.16) with $n = 2$ and g is smooth, then $\mathcal{E}$ is a perimeter (Λ, r_0)-minimizing cluster in $\mathbb{R}^2$ for some Λ and r_0, with the additional property of being bounded, so that I and J are finite. Moreover, if $\mathcal{E}$ is an isoperimetric cluster, then each γ_i is either a circular arc or a segment; if $\mathcal{E}$ is a minimizer in (1.16), then γ_i is a closed connected smooth curve with boundary, whose curvature is equal to (the restriction to γ_i of) g up to an additive constant. Motivated by these examples, we give the following definition.

Definition 1.2. Let $\mathcal{E}$ be a cluster in $\mathbb{R}^2$. One says that $\mathcal{E}$ is a $C^{k,\alpha}$*-cluster in* $\mathbb{R}^2$ if there exists a family of $C^{k,\alpha}$-curves with boundary $\{\gamma_i\}_{i\in I}$ such that (1.20) and (1.21) hold.

We premise two additional definitions to the statement of Theorem 1.5.

Definition 1.3. Let $\mathcal{E}$ be a $C^{1,\alpha}$-cluster in $\mathbb{R}^2$. Given a map $f : \partial\mathcal{E} \to \mathbb{R}^2$ one says that $f \in C^{1,\alpha}(\partial\mathcal{E};\mathbb{R}^2)$ if f is continuous on $\partial\mathcal{E}$, $f \in C^{1,\alpha}(\gamma_i;\mathbb{R}^2)$ for every $i \in I$, and

$$\|f\|_{C^{1,\alpha}(\partial\mathcal{E})} := \sup_{i\in I} \|f\|_{C^{1,\alpha}(\gamma_i)} < \infty\,.$$

If $\mathcal{E}$ and $\mathcal{E}'$ are $C^{1,\alpha}$-clusters in $\mathbb{R}^2$, then one says that f is a $C^{1,\alpha}$-diffeomorphism between $\partial\mathcal{E}$ and $\partial\mathcal{E}'$ provided f is an homeomorphism between $\partial\mathcal{E}$ and $\partial\mathcal{E}'$ with $f \in C^{1,\alpha}(\partial\mathcal{E};\mathbb{R}^2)$, $f^{-1} \in C^{1,\alpha}(\partial\mathcal{E}';\mathbb{R}^2)$, and $f(\Sigma(\mathcal{E})) = \Sigma(\mathcal{E}')$.

Definition 1.4. Given a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ and a cluster $\mathcal{E}$ in $\mathbb{R}^2$, the tangential component of f with respect to $\mathcal{E}$ is the map $\boldsymbol{\tau}_{\mathcal{E}} f : \partial^*\mathcal{E} \to \mathbb{R}^2$ defined by

$$\boldsymbol{\tau}_{\mathcal{E}} f(x) = f(x) - (f(x)\cdot\nu_{\mathcal{E}}(x))\nu_{\mathcal{E}}(x)\,, \qquad x \in \partial^*\mathcal{E}\,,$$

where $\nu_{\mathcal{E}} : \partial^*\mathcal{E} \to \mathbb{S}^1$ is any Borel function such that either $\nu(x) = \nu_{\mathcal{E}(h)}(x)$ or $\nu(x) = \nu_{\mathcal{E}(k)}(x)$ for every $x \in \mathcal{E}(h,k)$, $h \neq k$.

Theorem 1.5 (Improved convergence for planar almost-minimizing clusters)**.** *Given* $\Lambda \geq 0$, $r_0 > 0$ *and a bounded* $C^{2,1}$*-cluster* $\mathcal{E}$ *in* $\mathbb{R}^2$*, there exist positive constants* μ_0 *and* C_0 *(depending on* Λ *and* $\mathcal{E}$*) with the following property.*

If $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ *is a sequence of perimeter* (Λ, r_0)*–minimizing clusters in* $\mathbb{R}^2$ *such that* $\mathrm{d}(\mathcal{E}_k,\mathcal{E}) \to 0$ *as* $k \to \infty$*, then for every* $\mu < \mu_0$ *there exist* $k(\mu) \in \mathbb{N}$ *and a sequence of maps* $\{f_k\}_{k\geq k(\mu)}$ *such that each* f_k *is a* $C^{1,1}$*-diffeomorphism between* $\partial\mathcal{E}$ *and* $\partial\mathcal{E}_k$ *with*

$$\begin{aligned}
\|f_k\|_{C^{1,1}(\partial\mathcal{E})} &\leq C_0\,,\\
\lim_{k\to\infty} \|f_k - \mathrm{Id}\|_{C^1(\partial\mathcal{E})} &= 0\,,\\
\|\boldsymbol{\tau}_{\mathcal{E}}(f_k - \mathrm{Id})\|_{C^1(\partial^*\mathcal{E})} &\leq \frac{C_0}{\mu}\,\|f_k - \mathrm{Id}\|_{C^0(\Sigma(\mathcal{E}))}\,,\\
\boldsymbol{\tau}_{\mathcal{E}}(f_k - \mathrm{Id}) &= 0\,, \qquad \textit{on } \partial\mathcal{E}\setminus I_\mu(\Sigma(\mathcal{E}))\,.
\end{aligned}$$

Remark 1.6. When $\Sigma(\mathcal{E}) = \Sigma(\mathcal{E}_k)$ then $f_k = \mathrm{Id}$ on $\Sigma(\mathcal{E})$ and f_k is a normal perturbation of the identity on $\partial^*\mathcal{E}$. In general, the C^1-size of the tangential displacement is controlled by the distance between the singular sets. Moreover, for every $x \in \partial\mathcal{E}\setminus I_\mu(\Sigma(\mathcal{E}))$, $f_k(x)$ is just the nearest point to x on $\partial\mathcal{E}_k$, while for every $x \in \Sigma(\mathcal{E})$, $f_k(x)$ is the nearest point to x on $\Sigma(\mathcal{E})$.

Remark 1.7. A natural question is of course whether the maps f_k in Theorem 1.5 can be extended to $C^{1,1}$-diffeomorphisms g_k of $\mathbb{R}^2$ with $\|g_k\|_{C^{1,1}(\mathbb{R}^2)} \leq C_0$ and $\|g_k - \mathrm{Id}\|_{C^1(\mathbb{R}^2)} \to 0$ as $k \to \infty$. The answer is yes, but at the cost of a longer proof and of a heavier use of Whitney's extension theorem. The kind of argument needed here is a particular case of the one used in the construction of almost-normal diffeomorphisms between surfaces with corners addressed in [CLMa]. At the same time, in view of the applications to planar clusters we have in mind, it is not evident that working with such extensions g_k would bring substantial advantages. For these reasons we have decided not to include this stronger form of Theorem 1.5.

1.5. **Some applications of Theorem 1.5.** As explained in section 1.2, a result like Theorem 1.5 opens the way to several applications. The ones given below, see Theorem 1.8 and Theorem 1.9, are inspired by a list of questions concerning partitioning problems proposed by Almgren in [Alm76, VI.1(6)], precisely "to classify in some reasonable way the different minimizing clusters corresponding to different choices of $m \in \mathbb{R}^N_+$". In this direction, let us consider the equivalence relation $\approx$ on the family of planar $C^{1,1}$-clusters such that $\mathcal{E} \approx \mathcal{F}$ if there exists a $C^{1,1}$-diffeomorphism between $\mathcal{E}$ and $\mathcal{F}$. Theorem 1.8 shows that isoperimetric clusters of a given volume (or with volume sufficiently close to a given one) generate only finitely many $\approx$-equivalence classes.

Theorem 1.8. *For every $m_0 \in \mathbb{R}^N_+$ there exists $\delta > 0$ with the following property. If Ω is the family of all the isoperimetric N-clusters $\mathcal{E}$ with $|\mathrm{vol}\,(\mathcal{E}) - m_0| < \delta$, then $\Omega/_{\approx}$ is a finite set.*

By an entirely analogous principle, we can describe qualitatively minimizers in (1.16) when the potential energy is small enough. (In the case of planar double bubbles we can upgrade this description to a quantitative one in the spirit of [FM11], see [CLMb].)

Theorem 1.9. *Let $m_0 \in \mathbb{R}^N_+$ be such that there exists a unique (modulo isometries) isoperimetric cluster $\mathcal{E}_0$ in $\mathbb{R}^2$ with $\mathrm{vol}\,(\mathcal{E}_0) = m_0$, and let $g : \mathbb{R}^2 \to [0,\infty)$ be a continuous function with $g(x) \to \infty$ as $|x| \to \infty$. Then there exists $\delta_0 > 0$ (depending on $\mathcal{E}_0$ and g only) such that for every $\delta < \delta_0$ and $|m - m_0| < \delta_0$ there exist minimizers in*

$$\inf\left\{ P(\mathcal{E}) + \delta \sum_{h=1}^{N} \int_{\mathcal{E}(h)} g(x)\,dx : \mathrm{vol}\,(\mathcal{E}) = m \right\}. \tag{1.22}$$

If $\mathcal{E}$ is a minimizer in (1.22), then $\mathcal{E} \approx \mathcal{E}_0$. Moreover, if $H_{\mathcal{E}(h,k)}$ denotes the scalar curvature of the interface $\mathcal{E}(h,k)$ with respect to $\nu_{\mathcal{E}(h)}$, then $H_{\mathcal{E}(h,k)}$ is continuous on $\mathcal{E}(h,k)$, with

$$\max_{0 \le h < k \le N} \|H_{\mathcal{E}(h,k)} - H_{\mathcal{E}_0(h,k)}\|_{C^0(\mathcal{E}(h,k))} \le C_0\,\delta\,, \tag{1.23}$$

for a constant C_0 depending on $\mathcal{E}_0$ and g only. (Notice that $H_{\mathcal{E}_0(h,k)}$ is a constant for every $0 \le h < k \le N$.)

Of course, in view of Theorem 1.8, if the uniqueness assumption on m_0 in Theorem 1.9 is dropped, then one can still infer that minimizers in (1.22) with $\delta < \delta_0$ and $|m - m_0| < \delta_0$ generate only finitely many $\approx$-equivalence classes.

1.6. **Organization of the paper and overview on companion papers.** In section 2 we construct almost-normal diffeomorphisms between closed curves with boundary. Uniform versions of the inverse and implicit function theorems are needed here, and their proofs are collected in Appendix A for the sake of clarity. Also, it should be noted this part of our paper could possibly be useful in the study of other planar geometric variational problems with singularities, and is totally independent from the theory of clusters. Section 3 addresses the basic regularity properties of (Λ, r_0)-minimizing clusters and provides a weak "improved convergence theorem" in arbitrary dimension, see Theorem 3.12. In section 4 we bring together the results of the previous two sections to prove Theorem 1.5. Here we have a nice application of Whitney's extension theorem, see Proposition B.2. Whitney's theorem, that will play a much more substantial role in [CLMa], is thus quickly reviewed in Appendix B. Finally, in section 5 we give the (closely related proofs of) Theorem 1.8 and Theorem 1.9.

For reasons of space, further applications of Theorem 1.5 are discussed elsewhere. In [CLMb], Theorem 1.5 is the starting point for obtaining a sharp stability inequality for planar double-bubbles. In [CLMc] we exploit Theorem 1.5 to show that every strictly stable (in the sense of positive definite second variation) planar cluster is a local volume-constrained perimeter minimizer in L^1. Again for reasons of space, the extension of Theorem 1.5 to clusters in $\mathbb{R}^3$, which is considerably more delicate from the technical viewpoint, is discussed separately in [CLMa].

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2. Almost-normal diffeomorphisms between curves

2.1. Sets in $\mathbb{R}^n$. Given $x \in \mathbb{R}^n$ and $r > 0$ we set

$$B(x,r) = B_{x,r} = \Big\{ y \in \mathbb{R}^n : |y-x| < r \Big\}, \qquad B(0,r) = B_{0,r} = B_r \,,$$

where $v \cdot w$ is the scalar product of $v, w \in \mathbb{R}^n$ and $|v|^2 = v \cdot v$. We set $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. Given a set $S \subset \mathbb{R}^n$, we shall denote by

$$\mathring{S}\,, \qquad \partial S\,, \qquad \mathrm{cl}\,(S)\,, \qquad \mathrm{int}\,(S)\,,$$

the interior of S, the boundary of S, the closure of S, and the interior of S with respect to the topology of S, respectively. The tubular ε-neighborhood of S in $\mathbb{R}^n$ is denoted by

$$I_\varepsilon(S) = \{x \in \mathbb{R}^n : \mathrm{dist}(x,S) < \varepsilon\}\,, \qquad \varepsilon > 0\,. \tag{2.1}$$

If S is a k-dimensional C^1-manifold in $\mathbb{R}^n$, then the geodesic distance on S is given by

$$\mathrm{dist}_S(x,y) = \inf\Big\{ \int_0^1 |\gamma'(t)|\, dt : \gamma \in C^1([0,1];S)\,, \gamma(0) = x\,, \gamma(1) = y \Big\}\,, \qquad x, y \in S\,.$$

We also define the normal ε-neighborhood of S as

$$N_\varepsilon(S) = \Big\{ x + \sum_{i=1}^{n-k} t_i\, \nu_i(x) : x \in S\,, \sum_{i=1}^{n-k} t_i^2 < \varepsilon^2 \Big\}\,, \tag{2.2}$$

where $\{\nu_i(x)\}_{i=1}^{n-k}$ is an orthonormal basis to $(T_x S)^\perp$ for every $x \in S$. If S is a k-dimensional C^1-manifold with boundary, then we denote by $\mathrm{bd}\,(S)$ the set of boundary points of S, and set

$$[S]_\rho = S \setminus I_\rho(\mathrm{bd}\,(S))\,, \qquad \forall \rho > 0\,.$$

We use the terms *curve* in place of 1-dimensional manifold, and *hypersurface* in place of $(n-1)$-dimensional manifold in $\mathbb{R}^n$. Finally, given two bounded sets S and T in $\mathbb{R}^n$, we denote by $\mathrm{hd}(S,T)$ the Hausdorff distance between S and T,

$$\mathrm{hd}(S,T) = \max\Big\{ \sup\{\mathrm{dist}(x,S) : x \in T\}, \sup\{\mathrm{dist}(x,T) : x \in S\} \Big\}\,.$$

2.2. Uniform inverse and implicit function theorems. If S is a k-dimensional $C^{1,\alpha}$-manifold in $\mathbb{R}^n$ ($\alpha \in (0,1]$), $x \in S$, and $f : S \to \mathbb{R}^n$, then we say that f is differentiable at x with respect to S if we can define a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ by setting

$$\nabla^S f(x) v = \begin{cases} \lim\limits_{t \to 0} \frac{f(\gamma(t)) - f(x)}{t} & \text{if } v \in T_x S, \\ 0 & \text{if } v \in (T_x S)^\perp, \end{cases}$$

where $\gamma \in C^1((-\varepsilon,\varepsilon);S)$ is such that $\gamma(0) = x$ and $\gamma'(0) = v$. We set

$$\|f\|_{C^1(S)} = \sup_{x \in S} |f(x)| + \|\nabla^S f(x)\|\,,$$

where $\|L\|$ denotes the operator norm of a linear map $L : \mathbb{R}^n \to \mathbb{R}^n$. We notice that if f is differentiable in an open neighborhood of S, then $\nabla^S f(x)$ is just the restriction of the differential

$\nabla f(x)$ of f at x to T_xS, extended to take the value 0 on $(T_xS)^\perp$. For $\alpha \in (0,1]$ we set

$$
\begin{aligned}
[\nabla^S f]_{C^{0,\alpha}(S)} &= \sup_{x,y\in S,\, x\neq y} \frac{\|\nabla^S f(x) - \nabla^S f(y)\|}{|x-y|^\alpha}\,,\\
\|\nabla^S f\|_{C^{0,\alpha}(S)} &= \sup_{x\in S} \|\nabla^S f(x)\| + [\nabla^S f]_{C^{0,\alpha}(S)}\,,\\
\|f\|_{C^{1,\alpha}(S)} &= \sup_{x\in S} |f(x)| + \|\nabla^S f\|_{C^{0,\alpha}(S)}\,,
\end{aligned}
$$

and, if $\{\tau_i(x)\}_{i=1}^k$ is an orthonormal basis of T_xS, we define the tangential Jacobian of f as

$$
J^S f(x) = \Big|\bigwedge_{i=1}^k \nabla^S f(x)\tau_i(x)\Big|\,, \qquad x \in S\,.
$$

The next theorem is a uniform version of the inverse function theorem. The proof is included in Appendix A for the sake of clarity.

Theorem 2.1 (Uniform inverse function theorem)**.** *Given $\alpha \in (0,1]$, $L, M > 0$, and S_0 a k-dimensional $C^{1,\alpha}$-manifold in $\mathbb{R}^n$ with $\mathrm{diam}(S_0) \le M$ and*

$$
\begin{aligned}
&\mathrm{dist}_{S_0}(x,y) \le M|x-y|\,, && \forall x,y \in S_0\,, && (2.3)\\
&|y-x| \le 2\,|\pi^0_x(y-x)|\,, && \forall x \in S_0\,, y \in B_{x,1/M}\cap S_0\,, && (2.4)\\
&\|\pi^0_x - \pi^0_y\| \le M\,|x-y|^\alpha\,, && \forall x,y\in S_0\,, && (2.5)
\end{aligned}
$$

(where π^0_x denotes the projection of $\mathbb{R}^n$ onto T_xS_0) there exist positive constants ε_0, ρ_0 and C_0, depending on α, L, M, and k only, with the following properties. If $f \in C^{1,\alpha}(S_0;\mathbb{R}^n)$ is such that

$$
\inf_{S_0} |J^{S_0} f| \ge \frac{1}{L}\,, \qquad \|\nabla^{S_0} f\|_{C^{0,\alpha}(S_0)} \le L\,, \tag{2.6}
$$

then f is injective on $B_{x,\varepsilon_0}\cap S_0$ for every $x\in S_0$. If, moreover,

$$
\|f - \mathrm{Id}\|_{C^0(S_0)} < \rho_0\,, \tag{2.7}
$$

then $S = f(S_0)$ is a k-dimensional $C^{1,\alpha}$-manifold in $\mathbb{R}^n$ and $f : S_0 \to S$ is a $C^{1,\alpha}$-diffeomorphism satisfying $\|f^{-1}\|_{C^{1,\alpha}(S)} \le C_0$.

Remark 2.2. A sufficient condition for the existence of $M > 0$ such that (2.4) and (2.5) hold is that S_0 is compactly contained into a k-dimensional $C^{1,\alpha}$-manifold $\widetilde{S}_0$. Moreover, (2.4) and (2.5) are trivial when $k = n$ and thus S_0 is a (not necessarily bounded) open set in $\mathbb{R}^n$.

Theorem 2.3 (Uniform implicit function theorem)**.** *Let $\alpha \in (0,1]$, $k \ge 1$, and $L, M > 0$. Then there exist positive constants C_0 and η_0 depending on α, k, L and M with the following property. If A is an open set with $\mathrm{dist}_A(x,y) \le M|x-y|$ for every $x, y \in A$, $x_0 \in A$ and $u \in C^{1,\alpha}(A\times(-1,1)^{n-k};\mathbb{R}^{n-k})$ are such that*

$$
u(x_0,\mathbf{0}) = \mathbf{0}\,, \qquad \Big|\bigwedge_{i=1}^{n-k} \frac{\partial u}{\partial t_i}(x_0,\mathbf{0})\Big| \ge \frac{1}{L}\,, \qquad \|\nabla u\|_{C^{0,\alpha}(A\times(-1,1)^{n-k})} \le L\,, \tag{2.8}
$$

where $\mathbf{0} = (0,\dots,0) \in \mathbb{R}^{n-k}$, then there exists a function $\zeta \in C^{1,\alpha}(A\cap B_{x_0,\eta_0};\mathbb{R}^{n-k})$ such that

$$
\zeta(x_0) = \mathbf{0}\,, \qquad u(z,\zeta(z)) = \mathbf{0}\,, \qquad \forall z \in A\cap B_{x_0,\eta_0}\,, \qquad \|\zeta\|_{C^{1,\alpha}(A\cap B(x_0,\eta_0))} \le C_0\,. \tag{2.9}
$$

Proof. This follows from Theorem 2.1 by means of the argument classically used to deduce the implicit function theorem from the inverse function theorem; see, e.g., [Spi65]. $\square$

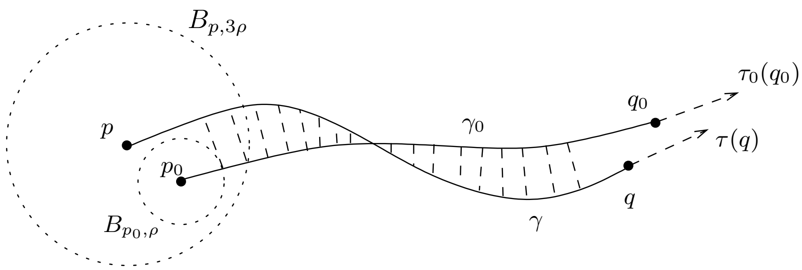

FIGURE 2. In Theorem 2.6 we consider two bounded connected curves with boundary γ_0 and γ such that γ is close in a C^1-sense to γ_0, and is actually a C^1-small normal deformation of γ_0 (at least) up to a distance $3\,\rho$ from its boundary points. The goal of the theorem is extending this normal deformation to a global diffeomorphism, still C^1-close to the identity map, and also using a minimal amount of tangential displacement in order to attach the boundary points.

2.3. **Construction of the diffeomorphisms.** In the main result of this section, Theorem 2.6 below, we are given two compact connected curves with boundary in $\mathbb{R}^n$, denoted by γ_0 and γ respectively, which are close in Hausdorff distance and whose boundaries are also close in Hausdorff distance, see assumption (i). The tangent directions to these curves at their boundary points are close too, see assumption (ii). Finally, the curve γ, up to a certain small distance from $\mathrm{bd}\,(\gamma)$, is a C^1-small normal deformation of the part of γ_0 lying at a certain small distance from $\mathrm{bd}\,(\gamma_0)$; see assumption (iii) and, more generally, Figure 2. Under these assumptions, we want to construct, in a somehow canonical way, a diffeomorphism between γ_0 and γ with a minimum amount of tangential displacement. (This last requirement is crucial in relating stability to local minimality, see also [CLMc].) In order to prove Theorem 1.5 it would suffice to consider curves in $\mathbb{R}^2$ in Theorem 2.6, but the case of curves in $\mathbb{R}^n$ is discussed here in view of the application of Theorem 2.6 in [CLMa] (where $n = 3$). We premise to the statement of Theorem 2.6 the notion of *extension by foliation* of a given curve with boundary.

Definition 2.4. Let γ be a $C^{1,\alpha}$-curve in $\mathbb{R}^n$ with $\{\nu^{(i)}\}_{i=1}^{n-1} \subset C^{0,\alpha}(\gamma;\mathbb{S}^{n-1})$ such that $\{\nu^{(i)}(x)\}_{i=1}^{n-1}$ is an orthonormal basis to $(T_x\gamma)^\perp$ for every $x \in \gamma$. One says that $(\varepsilon_\gamma, d_\gamma)$ is an *extension by foliation* of γ if $d_\gamma \in C^{1,\alpha}(\mathbb{R}^n;\mathbb{R}^{n-1})$ and $\widetilde{\gamma} = I_{\varepsilon_\gamma}(\gamma) \cap \{d_\gamma = (0,...,0)\}$ is a $C^{1,\alpha}$-curve in $\mathbb{R}^n$ with

$$\gamma \subset \widetilde{\gamma}\,, \qquad \nabla d_\gamma(x) = \sum_{i=1}^{n-1} e_j \otimes \nu^{(i)}(x)\,, \qquad \forall x \in \gamma\,. \tag{2.10}$$

Remark 2.5. Thus $\widetilde{\gamma}$ extends γ and is embedded in a foliation of a neighborhood of γ. The function d_γ gives a convenient way of defining the "$C^{1,\alpha}$-norm of γ", and allows one to locate γ in space through the implicit function theorem, see also the sketch of proof given below.

Theorem 2.6. *If $\alpha \in (0,1]$, $L > 0$, and γ_0 is a compact connected $C^{2,1}$-curve with boundary in $\mathbb{R}^n$ with $\mathrm{bd}\,(\gamma_0) \neq \emptyset$, then there exist positive constants $\mu_0 < 1$ and C_0 (depending on α, L, and γ_0 only) with the following property.*

Let γ be a compact connected $C^{1,\alpha}$-curve with boundary in $\mathbb{R}^n$ with $\mathrm{bd}\,(\gamma) \neq \emptyset$, which admits an extension by foliation $(\varepsilon_\gamma, d_\gamma)$ such that

$$\max\left\{\frac{1}{\varepsilon_\gamma}, \|d_\gamma\|_{C^{1,\alpha}(I_{\varepsilon_\gamma}(\gamma))}\right\} \le L\,, \tag{2.11}$$

and which is close to γ_0 in the following sense: for some $\rho \in (0,\mu_0^2)$ one has

(i) $\mathrm{hd}(\gamma,\gamma_0) + \mathrm{hd}(\mathrm{bd}\,(\gamma), \mathrm{bd}\,(\gamma_0)) < \rho$*;*

(ii) there exist unit tangent vector fields $\tau_0 \in C^{1,1}(\gamma_0;\mathbb{S}^{n-1})$ and $\tau \in C^{0,\alpha}(\gamma;\mathbb{S}^{n-1})$, defining start-points p_0 and p, and end-points q_0 and q, to γ_0 and γ respectively, such that

$$|p_0 - p| + |q_0 - q| < \rho\,, \tag{2.12}$$
$$|\tau_0(p_0) - \tau(p)| + |\tau_0(q_0) - \tau(q)| < \rho\,; \tag{2.13}$$

(iii) there exists $\psi \in C^{1,\alpha}([\gamma_0]_\rho;\mathbb{R}^n)$ such that $\psi\cdot\tau_0 = 0$ on $[\gamma_0]_\rho$ and

$$[\gamma]_{3\rho} \subset (\mathrm{Id}+\psi)([\gamma_0]_\rho) \subset \gamma\,, \tag{2.14}$$
$$\|\psi\|_{C^1([\gamma_0]_\rho)} \le \rho\,, \qquad \|\psi\|_{C^{1,\alpha}([\gamma_0]_\rho)} \le L\,. \tag{2.15}$$

Then for every $\mu \in (\sqrt{\rho},\mu_0)$ there exists a $C^{1,\alpha}$-diffeomorphism f between γ_0 and γ such that $f(p_0) = p$, $f(q_0) = q$, and

$$\begin{aligned}
\|f\|_{C^{1,\alpha}(\gamma_0)} &\le C_0\,, && (2.16)\\
\|f-\mathrm{Id}\|_{C^0(\gamma_0)} &\le C_0\,\rho\,, && (2.17)\\
\|f-\mathrm{Id}\|_{C^1(\gamma_0)} &\le \frac{C_0}{\mu}\,\rho\,, && (2.18)\\
\|(f-\mathrm{Id})\cdot\tau_0\|_{C^1(\gamma_0)} &\le \frac{C_0}{\mu}\sup_{\mathrm{bd}\,(\gamma_0)}|(f-\mathrm{Id})\cdot\tau_0|\,, && (2.19)\\
(f-\mathrm{Id})\cdot\tau_0 &= 0 \qquad \text{on } [\gamma_0]_\mu\,. && (2.20)
\end{aligned}$$

Remark 2.7. Note that condition (2.19) guarantees that f is a normal diffeomorphism of γ_0 whenever $p - p_0$ and $q - q_0$ are normal to γ_0 at p_0 and q_0 respectively.

Remark 2.8. The assumption that $\mathrm{bd}\,(\gamma) \ne \emptyset$ is redundant, as it is implicitly contained in $\mathrm{bd}\,(\gamma_0) \ne \emptyset$ and $\mathrm{hd}(\mathrm{bd}\,(\gamma),\mathrm{bd}\,(\gamma_0)) < \infty$. Moreover, if we drop the connectedness assumption on γ, then the map f constructed below is a diffeomorphism between γ_0 and a connected component of γ. In other words, γ could have additional connected components without boundary that (because of assumption (iii)) are close in Hausdorff distance to $\mathrm{bd}\,(\gamma_0)$.

Rough sketch of the proof of Theorem 2.6. One considers an orthonormal basis $\{\nu_0^{(j)}(x)\}_{j=1}^{n-1}$ of $T_x\gamma_0$ and define functions $a_j, b : \gamma_0 \to \mathbb{R}^n$ with

$$\begin{aligned}
a_j(p_0) &= (p-p_0)\cdot\nu_0^{(j)}(p_0)\,, &\qquad b(p_0) &= (p-p_0)\cdot\tau_0(p_0)\,,\\
a_j(q_0) &= (q-q_0)\cdot\nu_0^{(j)}(q_0)\,, &\qquad b(q_0) &= (q-q_0)\cdot\tau_0(q_0)\,.
\end{aligned}$$

In this way, if we define $F : \gamma_0 \times \mathbb{R}^{n-1} \to \mathbb{R}^n$ and $u : \gamma_0 \times \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ as

$$\begin{aligned}
&F(x,\mathbf{t}) = x + b(x)\,\phi_\mu(x)\,\tau_0(x) + \sum_{j=1}^{n-1}(a_j(x)-t_j)\nu_0^{(j)}(x)\,, \qquad \mathbf{t} = (t_1,...,t_{n-1})\,,\\
&u(x,\mathbf{t}) = d_\gamma(F(x,\mathbf{t}))\,,
\end{aligned}$$

where $\mathrm{spt}\phi_\mu \subset\subset I_\mu(\mathrm{bd}\,(\gamma_0)) = B_{p_0,\mu}\cup B_{q_0,\mu}$, then we have $F(p_0,\mathbf{0}) = p$ and $F(q_0,\mathbf{0}) = q$, where $\mathbf{0} = (0,...,0) \in \mathbb{R}^{n-1}$; thus, by $p,q \in \gamma \subset \{d_\gamma = \mathbf{0}\}$, $u(p_0,\mathbf{0}) = u(q_0,\mathbf{0}) = \mathbf{0}$; moreover, by assumption (ii), one checks that $\bigwedge_{j=1}^{n-1}(\partial u/\partial t_j) \ge 1/2$ at $(p_0,\mathbf{0})$ and at $(q_0,\mathbf{0})$. Hence, by the implicit function theorem there exists $\zeta : I_\eta(\mathrm{bd}\,(\gamma_0)) \to \mathbb{R}^{n-1}$ such that $d_\gamma(F(x,\zeta(x)) = \mathbf{0}$ for every $x \in \gamma_0$. The role of ζ is clear: while we dampen the tangential component b (needed to map $\mathrm{bd}\,(\gamma_0)$ into $\mathrm{bd}\,(\gamma)$) by means of the cut-off function ϕ_μ, the function ζ gives us the right amount of normal displacement to find the position in space of γ. The map $f:\gamma_0\to\mathbb{R}^n$ defined by

$$f(x)=F(x,\zeta(x))=x+b(x)\,\phi_\mu(x)\,\tau_0(x)+\sum_{j=1}^{n-1}(a_j(x)-\zeta^{(j)}(x))\nu_0^{(j)}(x)\,,$$

is then a diffeomorphism between γ_0 into γ with $f(p_0)=p$ and $f(q_0)=q$, and a normal deformation of $\gamma_0\setminus I_\mu(\mathrm{bd}\,(\gamma_0))$ thanks to $\mathrm{spt}\phi_\mu\subset\subset I_\mu(\mathrm{bd}\,(\gamma_0))$.

Proof of Theorem 2.6. In the following, we always denote by C a constant which possibly (but, in the case, exclusively) depends on α, L and γ_0.

Extension of γ: Let ε_γ, d_γ and $\widetilde\gamma$ be as in Definition 2.4. The vector fields τ and $\nu^{(i)}$ introduced in assumption (ii) and in Definition 2.4 respectively, are tacitly extended to $\widetilde\gamma$, with $C^{0,\alpha}$-norms depending on L only thanks to (2.11). By (2.10), if $v,v_i\in\mathbb{S}^{n-1}$, $\varepsilon>0$, and $x\in\gamma$, then

$$\begin{aligned}
|\nabla d_\gamma(x)v| &\le C\,\varepsilon\,, &&\text{if } |\tau(x)\cdot v|\ge 1-\varepsilon\,, &(2.21)\\
|\nabla d_\gamma(x)v| &\ge 1-C\,\varepsilon\,, &&\text{if } |\tau(x)\cdot v|\le \varepsilon\,, &(2.22)\\
\Big|\bigwedge_{i=1}^{n-1}\nabla d_\gamma(x)v_i\Big| &\ge 1-C\,\varepsilon\,, &&\text{if } v_i\cdot v_j=\delta_{i,j} \text{ and } |\tau(x)\cdot v_i|\le\varepsilon\,. &(2.23)
\end{aligned}$$

Extension of γ_0: Consider any $C^{2,1}$-curve with boundary $\widetilde\gamma_0$, homeomorphic to γ_0, such that $\gamma_0=\mathrm{cl}\,(\gamma_0)\subset\mathrm{int}\,(\widetilde\gamma_0)\subset\widetilde\gamma_0$. (The various constants appearing in the proof will depend on γ_0 through the particular extension $\widetilde\gamma_0$ we have chosen.) We denote by d_0 the geodesic distance on $\widetilde\gamma_0$, so that

$$\mathrm{d}_0(x,y)\le C\,|x-y|\,,\qquad \forall x,y\in\widetilde\gamma_0\,, \tag{2.24}$$

and define $\tau_0,\nu_0^{(i)}\in C^{1,1}(\widetilde\gamma_0;\mathbb{S}^{n-1})$ in such a way that τ_0 extends to $\widetilde\gamma_0$ the tangent vector field to γ_0 introduced in (ii), and $\{\tau_0(x)\}\cup\{\nu_0^{(i)}\}_{i=1}^{n-1}$ is an orthonormal basis of $\mathbb{R}^n$. In this way,

$$\|\tau_0\|_{C^{1,1}(\widetilde\gamma_0)}+\max_{1\le i\le n-1}\|\nu_0^{(i)}\|_{C^{1,1}(\widetilde\gamma_0)}\le C\,. \tag{2.25}$$

We consider a unit speed parametrization Φ_0 of $\widetilde\gamma_0$, that is $\Phi_0\in C^{2,1}(I;\mathbb{R}^n)$ with

$$\widetilde\gamma_0=\{\Phi_0(s):s\in I\}\,,\qquad \Phi_0'(s)=\tau_0(\Phi_0(s))\,,\qquad \forall s\in I\,, \tag{2.26}$$

where $I\subset\mathbb{R}$ is an interval such that $\mathcal{H}^1(I)=\mathcal{H}^1(\widetilde\gamma_0)$. Clearly, by (2.25),

$$\|\tau_0(\Phi_0)\|_{C^{1,1}(I)}+\max_{1\le i\le n-1}\|\nu_0^{(i)}(\Phi_0)\|_{C^{1,1}(I)}\le C\,. \tag{2.27}$$

If $s_0\in I$ is such that $\Phi_0(s_0)=p_0$, then we set

$$U_{p_0,t}=\Phi_0(I\cap(s_0-t,s_0+t))\subset\widetilde\gamma_0\,.$$

Claim: There exist $\eta_0 = \eta_0(\alpha, L, \gamma_0) > \mu_0$ and a map $f_{p_0} : U_{p_0,\eta_0} \to \widetilde{\gamma}$ with $f_{p_0}(p_0) = p$ and

$$\|f_{p_0}\|_{C^{1,\alpha}(U_{p_0,\eta_0})} \leq C\,, \tag{2.28}$$

$$\|f_{p_0} - \mathrm{Id}\|_{C^0(U_{p_0,\eta_0})} \leq C\,\rho\,, \tag{2.29}$$

$$\|f_{p_0} - \mathrm{Id}\|_{C^1(U_{p_0,\eta_0})} \leq \frac{C}{\mu}\,\rho\,, \tag{2.30}$$

$$\|(f_{p_0} - \mathrm{Id})\cdot\tau_0\|_{C^1(U_{p_0,\eta_0})} \leq \frac{C}{\mu}\,|(p_0 - p)\cdot\tau_0(p_0)|\,, \tag{2.31}$$

$$(f_{p_0} - \mathrm{Id})\cdot\tau_0 = 0\,, \qquad \text{on } U_{p_0,\eta_0}\setminus B_{p_0,\mu}\,, \tag{2.32}$$

$$J^{\widetilde{\gamma}_0} f_{p_0} \geq \frac{1}{2}\,, \qquad \text{on } U_{p_0,\eta_0}\,, \tag{2.33}$$

$$f_{p_0}(\gamma_0 \cap U_{p_0,\eta_0}) \subset \gamma\,. \tag{2.34}$$

We divide the proof of the claim in three steps.

Proof of the claim. Step one: We introduce a one-parameter family of cut-off functions that we use to dampen the tangential displacement used to map p_0 into p: precisely, we fix $\phi \in C^\infty(\mathbb{R}^n \times (0,\infty); [0,1])$ such that, setting $\phi_\mu = \phi(\cdot,\mu)$ for $\mu > 0$, one has

$$\phi_\mu \in C^\infty_c(B_\mu)\,, \qquad \phi_\mu = 1 \text{ on } B_{\mu/2}\,, \tag{2.35}$$

$$|\nabla\phi_\mu(z)| \leq \frac{C}{\mu}\,, \qquad |\nabla^2\phi_\mu(z)| \leq \frac{C}{\mu^2}\,, \qquad \forall (z,\mu) \in \mathbb{R}^n \times (0,\infty)\,. \tag{2.36}$$

We decompose $p - p_0$ in the orthonormal basis $\{\tau_0(p_0)\} \cup \{\nu_0^{(i)}(p_0)\}_{i=1}^{n-1}$ of $\mathbb{R}^n$, and set

$$a_i = (p - p_0)\cdot\nu_0^{(i)}(p_0)\,, \qquad b = (p - p_0)\cdot\tau_0(p_0)\,. \tag{2.37}$$

Of course, by (2.12) we have

$$\sum_{i=1}^{n-1} a_i^2 + b^2 < \rho^2\,. \tag{2.38}$$

We now define $F \in C^{1,1}(I \times \mathbb{R}^{n-1}; \mathbb{R}^n)$ by setting, for $(s,\mathbf{t}) \in I \times \mathbb{R}^{n-1}$,

$$F(s,\mathbf{t}) = \Phi_0(s) + b\,\phi_\mu(\Phi_0(s) - p_0)\,\Phi_0'(s) + \sum_{i=1}^{n-1}(a_i - t_i)\,\nu_0^{(i)}(\Phi_0(s))\,, \tag{2.39}$$

and then exploit $d_\gamma \in C^{1,\alpha}(\mathbb{R}^n; \mathbb{R}^{n-1})$ to define $u \in C^{1,\alpha}(I \times \mathbb{R}^{n-1}; \mathbb{R}^{n-1})$ as

$$u(s,\mathbf{t}) = d_\gamma(F(s,\mathbf{t}))\,, \qquad (s,\mathbf{t}) \in I \times \mathbb{R}^{n-1}\,.$$

By $\Phi_0(s_0) = p_0$, $\Phi_0'(s) = \tau_0(\Phi_0(s))$, $\phi_\mu = 1$ on $B_{\mu/2}$, and (2.37) we find

$$F(s_0,\mathbf{0}) = p\,, \tag{2.40}$$

which combined with $\gamma \subset \{d_\gamma = \mathbf{0}\}$ implies

$$u(s_0,\mathbf{0}) = \mathbf{0}\,. \tag{2.41}$$

We next compute that for every $(s,\mathbf{t}) \in I \times \mathbb{R}^{n-1}$,

$$\frac{\partial F}{\partial s} = \Big(1 + b\,\Big(\nabla\phi_\mu(\Phi_0(s) - p_0)\cdot\Phi_0'(s)\Big)\Big)\,\Phi_0'(s) + b\,\phi_\mu(\Phi_0(s) - p_0)\,\Phi_0''(s) + \sum_{i=1}^{n-1}(a_i - t_i)\,(\nu_0^{(i)}(\Phi_0))'(s)\,, \tag{2.42}$$

$$\frac{\partial F}{\partial t_i} = -\,\nu_0^{(i)}(\Phi_0(s))\,. \tag{2.43}$$

By (2.36) and (2.38) we find

$$\|\nabla F\|_{C^0(I\times(-1,1)^{n-1})}\le C\left(1+\frac{\rho}{\mu}\right),\qquad [\nabla F]_{C^{0,1}(I\times(-1,1)^{n-1})}\le C\left(1+\frac{\rho}{\mu^2}\right),$$

so that $\rho<\mu^2$ gives

$$\|\nabla F\|_{C^{0,1}(I\times(-1,1)^{n-1})}\le C\,. \tag{2.44}$$

By (2.11) and (2.44) we thus find

$$\|\nabla u\|_{C^{0,\alpha}(I\times(-1,1)^{n-1})}\le C\,. \tag{2.45}$$

We claim that if ρ_0 is small enough, then (up to identify $(n-1)$-vectors in $\mathbb{R}^{n-1}$ with real numbers, with the convention that $e_1\wedge\cdots\wedge e_{n-1}=1$),

$$\bigwedge_{i=1}^{n-1}\frac{\partial u}{\partial t_i}(p_0,\mathbf{0})\ge\frac{1}{2}\,. \tag{2.46}$$

Indeed, by (2.40) and (2.43) we find that

$$\bigwedge_{i=1}^{n-1}\frac{\partial u}{\partial t_i}(p_0,\mathbf{0})=\bigwedge_{i=1}^{n-1}\nabla d_\gamma(p)\nu_0^{(i)}(p_0)\,. \tag{2.47}$$

By (2.13), we have

$$\tau_0(p_0)\cdot\tau(p)\ge 1-C\,\rho\,,\qquad \max_{1\le i\le n-1}|\nu_0^{(i)}(p_0)\cdot\tau(p)|\le C\,\rho\,, \tag{2.48}$$

so that (2.23) gives $\bigwedge_{i=1}^{n-1}\frac{\partial u}{\partial t_i}(p_0,\mathbf{0})\ge 1-C\,\rho$, and thus (2.46) if μ_0 is small enough.

Step two: We construct the map f_{p_0} and introduce the parameter η_0 appearing in the claim. By (2.41), (2.45), and (2.46) we can apply Theorem 2.3 to find a positive constant $\eta_0=\eta_0(\alpha,L,\gamma_0)$ and a function $\zeta_0\in C^{1,\alpha}(I\cap(s_0-\eta_0,s_0+\eta_0);\mathbb{R}^{n-1})$ such that

$$u(s,\zeta_0(s))=\mathbf{0}\,,\qquad \forall s\in I\cap(s_0-\eta_0,s_0+\eta_0)\,, \tag{2.49}$$

$$\zeta_0(s_0)=\mathbf{0}\,,\qquad \|\zeta_0\|_{C^{1,\alpha}(I\cap(s_0-\eta_0,s_0+\eta_0))}\le C\,. \tag{2.50}$$

(We notice that we can further decrease the value of μ_0 without affecting the value of η_0. It should be useful to keep in mind that the order of the parameters will be $\mu_0<\eta_0/C$, with $\rho^2<\mu$ in force.) Up to further decrease η_0, we may directly assume that $(s_0-\eta_0,s_0+\eta_0)\subset I$, so that $\Phi_0((s_0-\eta_0,s_0+\eta_0))=U_{p_0,\eta_0}\subset\widetilde{\gamma}_0$, and that

$$\sum_{i=1}^{n-1}|(x-y)\cdot\nu_0^{(i)}(x)|\le C|(x-y)\cdot\tau_0(x)|^2\,,\qquad \forall x,y\in U_{p_0,\eta_0}\,. \tag{2.51}$$

(Indeed, if $x=\Phi_0(t)$ and $y=\Phi_0(s)$ with $(t,s)\subset(s_0-\eta_0,s_0+\eta_0)$ then by $\Phi_0'(t)\cdot\nu_0^{(i)}(\Phi_0(t))=0$ one finds

$$\begin{aligned}
|(y-x)\cdot\nu_0^{(i)}(x)| &= \left|\int_t^s(s-r)\Phi_0''(r)\,dr\cdot\nu_0^{(i)}(\Phi_0(t))\right|\le\|\Phi_0''\|_{C^0(s_0-\eta_0,s_0+\eta_0)}\,(s-t)^2\,,\\
|(y-x)\cdot\tau_0(x)| &= \left|\int_t^s\Phi_0'(r)\,dr\cdot\Phi_0'(t)\right|\ge(s-t)-\int_t^s|\Phi_0'(r)-\Phi_0'(t)|\,dr\\
&\ge \Big(1-2\eta_0\|\Phi_0''\|_{C^0(s_0-\eta_0,s_0+\eta_0)}\Big)(s-t)\ge\frac{s-t}{2}\,,
\end{aligned}$$

provided η_0 is small enough.) Moreover, we notice that, by (2.50),

$$\|\zeta_0\|_{C^0(s_0-\eta_0,s_0+\eta_0)}\le\mathrm{Lip}\,(\zeta_0)\eta_0\le C\,\eta_0\,. \tag{2.52}$$

In particular, we can make quantities of the form $C\,|\zeta_0|$ smaller than other given constants depending on α, L and γ_0 only provided we further decrease the value of η_0. We finally define $f_{p_0}:U_{p_0,\eta_0}\to\mathbb{R}^n$ by setting, for $x\in U_{p_0,\eta_0}$,

$$f_{p_0}(x)=F(\Phi_0^{-1}(x),\zeta_0(\Phi_0^{-1}(x)))\,,\qquad x\in U_{p_0,\eta_0}\,. \tag{2.53}$$

Step three: We check that f_{p_0} satisfies the claimed properties. By construction $f_{p_0}(p_0)=p$ and $f_{p_0}\in C^{1,\alpha}(U_{p_0,\eta_0};\mathbb{R}^n)$ with (2.28) in force thanks to (2.44) and (2.50); moreover,

$$f_{p_0}(U_{p_0,\eta_0})\subset\{d_\gamma=\mathbf{0}\}\,. \tag{2.54}$$

Since $f_{p_0}(p_0)=p$, by (2.28) and up to pick a suitably small value of η_0, we find $f_{p_0}(x)\in B_{p,L^{-1}}$ for every $x\in U_{p_0,\eta_0}$; since $B_{p,L^{-1}}\subset I_{\varepsilon_\gamma}(\gamma)$ by (2.11), we deduce from the definition of $\widetilde\gamma$ and by (2.54) that

$$f_{p_0}(U_{p_0,\eta_0})\subset\widetilde\gamma\,. \tag{2.55}$$

We now define $\zeta:U_{p_0,\eta_0}\to\mathbb{R}^{n-1}$ and $G:U_{p_0,\eta_0}\to\mathbb{R}^n$ as

$$\zeta^{(i)}(x)\;=\;\zeta_0^{(i)}(\Phi_0^{-1}(x))\,, \tag{2.56}$$

$$G(x)\;=\;b\,\phi_\mu(x-p_0)\,\tau_0(x)+\sum_{i=1}^{n-1}a_i\,\nu_0^{(i)}(x)\,, \tag{2.57}$$

so that $\zeta\in C^{1,\alpha}(U_{p_0,\eta_0};\mathbb{R}^{n-1})$ and $G\in C^{1,1}(U_{p_0,\eta_0};\mathbb{R}^n)$, with

$$\|\zeta\|_{C^0(U_{p_0,\eta_0})}\le C\,\eta_0\,,\qquad\|G\|_{C^0(U_{p_0,\eta_0})}\le C\,\rho\qquad\|G\|_{C^1(U_{p_0,\eta_0})}\le C\,\frac{\rho}{\mu}\,. \tag{2.58}$$

$$\|\zeta\|_{C^{1,\alpha}(U_{p_0,\eta_0})}\le C\,,\qquad\|G\|_{C^{1,1}(U_{p_0,\eta_0})}\le C\,, \tag{2.59}$$

thanks to (2.25), (2.37), (2.38), and (2.52). Moreover,

$$f_{p_0}(x)=x+G(x)-\sum_{i=1}^{n-1}\zeta^{(i)}(x)\,\nu_0^{(i)}(x)\,,\qquad\forall x\in U_{p_0,\eta_0}\,. \tag{2.60}$$

By (2.60) and (2.57),

$$(f_{p_0}(x)-x)\cdot\tau_0(x)=b\,\phi_\mu(x-p_0)\,,\qquad\forall x\in U_{p_0,\eta_0}\,, \tag{2.61}$$

so that (2.32) follows by $\operatorname{spt}\phi_\mu\subset\subset B_\mu$. By differentiating (2.61) we find

$$\nabla^{\widetilde\gamma_0}[(f-\mathrm{Id})\cdot\tau_0](x)\tau_0(x)=b\,\Big(\nabla\phi_\mu(x-p_0)\cdot\tau_0(x)\Big)\,\tau_0(x)\,,$$

which implies (2.31) once combined with (2.61) and the definition of b. By differentiating (2.60),

$$\begin{aligned}\nabla^{\widetilde\gamma_0}f_{p_0}(x)\tau_0(x)\;=\;&\Big(1+b\,\nabla\phi_\mu(x-p_0)\cdot\tau_0(x)\Big)\tau_0(x)\\&+b\,\phi_\mu(x-p_0)\,\Big(\nabla^{\widetilde\gamma_0}\tau_0(x)\tau_0(x)\Big)\\&+\sum_{i=1}^{n-1}(a_i-\zeta^{(i)}(x))\,\Big(\nabla^{\widetilde\gamma_0}\nu_0^{(i)}(x)\tau_0(x)\Big)-\Big(\nabla^{\widetilde\gamma_0}\zeta^{(i)}(x)\tau_0(x)\Big)\,\nu_0^{(i)}(x)\,.\end{aligned} \tag{2.62}$$

We may thus prove (2.33): indeed, by (2.25), (2.36), (2.38), (2.58), (2.59), (2.62) and $\rho\le\mu^2$ we find that, if $x\in U_{p_0,\eta_0}$, then

$$\begin{aligned}J^{\widetilde\gamma_0}f_{p_0}(x)\;&=\;\Big|\nabla^{\widetilde\gamma_0}f_{p_0}(x)\tau_0(x)\Big|\ge\Big|\Big(\nabla^{\widetilde\gamma_0}f_{p_0}(x)\tau_0(x)\Big)\cdot\tau_0(x)\Big|\\&\ge\;1-C\,\Big(\frac{|b|}{\mu}+\sum_{i=1}^{n-1}|a_i|+|\zeta(x)|\Big)\ge1-C\,(\mu_0+\eta_0)\ge\frac12\,,\end{aligned} \tag{2.63}$$

provided η_0 (and thus μ_0) is small enough. Similarly, by also taking into account (2.48)

$$\Big(\nabla^{\widetilde{\gamma}_0} f_{p_0}(p_0)\tau_0(p_0)\Big)\cdot\tau(p)\geq\tau_0(p_0)\cdot\tau(p)-C(\mu_0+\eta_0)\geq\frac{1}{2}\,, \tag{2.64}$$

again, for η_0 small enough. By (2.55), there exists a continuous function $\lambda : U_{p_0,\eta_0}\to\mathbb{R}$ such that

$$\nabla^{\widetilde{\gamma}_0} f_{p_0}(x)\tau_0(x)=\lambda(x)\,\tau(f_{p_0}(x))\,,\qquad\forall x\in U_{p_0,\eta_0}\,. \tag{2.65}$$

Thus, by combining (2.65), (2.63) (which gives $|\lambda|\geq 1/2$ on U_{p_0,η_0}) and (2.64) (which gives $\lambda(p_0)\geq 1/2$), we conclude that

$$\Big(\nabla^{\widetilde{\gamma}_0} f_{p_0}(x)\tau_0(x)\Big)\cdot\tau(f_{p_0}(x))\geq\frac{1}{2}\,,\qquad\forall x\in U_{p_0,\eta_0}\,. \tag{2.66}$$

By (2.66) and (2.55) we deduce (2.34). We are thus left to prove (2.29) and (2.30). By (2.34) and by assumption (i),

$$\rho>\mathrm{hd}(\gamma,\gamma_0)\geq\mathrm{dist}(f_{p_0}(x),\gamma_0)\,,\qquad\forall x\in\gamma_0\cap U_{p_0,\eta_0}\,. \tag{2.67}$$

Let $\varepsilon_0>0$ be smaller than the maximum of the curvature of γ_0, so that

$$\mathrm{dist}\Big(x+\sum_{i=1}^{n-1}t_i\,\nu_0^{(i)}(x),\gamma_0\Big)=|\mathbf{t}|\,,\qquad\forall x\in\gamma_0\,,|\mathbf{t}|<\varepsilon_0\,. \tag{2.68}$$

By (2.60) and (2.32),

$$f_{p_0}(x)=x+\sum_{i=1}^{n-1}t_i\,\nu_0^{(i)}(x)\,,\qquad\forall x\in U_{p_0,\eta_0}\setminus B_{p_0,\mu}\,,$$

where $t_i=a_i-\zeta^{(i)}$ is such that $|\mathbf{t}|\leq C(\eta_0+\mu_0)$ by (2.37) and (2.58); by combining this fact with (2.68), (2.67) and (2.37), we thus find

$$|\zeta(x)|\leq C\,\rho\,,\qquad\forall x\in U_{p_0,\eta_0}\setminus B_{p_0,\mu}\,. \tag{2.69}$$

Given $x\in U_{p_0,\eta_0}$, let now $g(x)\in\gamma_0$ be such that $|f_{p_0}(x)-g(x)|=\mathrm{dist}(f_{p_0}(x),\gamma_0)$. We claim that

$$g(x)\in U_{p_0,\eta_0}\,,\qquad\forall x\in\gamma_0\cap B_{p_0,\mu}\,. \tag{2.70}$$

Indeed, should this not be the case, then by definition of $g(x)$, $f_{p_0}(p_0)=p$ and $|x-p_0|<\mu$, we would find

$$\begin{aligned}\eta_0\quad<\quad &\mathrm{d}_0(p_0,g(x))\overset{(2.24)}{\leq}C\,|g(x)-p_0|\leq C\Big(|g(x)-f_{p_0}(x)|+|f_{p_0}(x)-p|+|p-p_0|\Big)\\ &\overset{(2.12),(2.67)}{\leq}C\Big(\rho+|f_{p_0}(x)-f_{p_0}(p_0)|+\rho\Big)\overset{(2.28)}{\leq}C\Big(|x-p_0|+\rho\Big)\leq C\,\mu_0\,,\end{aligned}$$

which is a contradiction provided μ_0 is small enough with respect to η_0. By (2.70), we can apply (2.51) to find that

$$\sum_{i=1}^{n-1}|(g(x)-x)\cdot\nu_0^{(i)}(x)|\leq C|(g(x)-x)\cdot\tau_0(x)|^2\,,\qquad\forall x\in\gamma_0\cap B_{p_0,\mu}\,. \tag{2.71}$$

Now, by (2.67) and (2.60) we find that, if $x\in\gamma_0\cap B_{p_0,\mu}$, then

$$\begin{aligned}\rho\quad\geq\quad &\mathrm{dist}(f_{p_0}(x),\gamma_0)=|f_{p_0}(x)-g(x)|\geq|(f_{p_0}(x)-g(x))\cdot\tau_0(x)|\\ =\quad &|(x-g(x))\cdot\tau_0(x)|-|b|\,\phi_\mu(x-p_0)\end{aligned}$$

so that (2.71) and (2.38) give

$$\sum_{i=1}^{n-1}|(g(x)-x)\cdot\nu_0^{(i)}(x)|\leq C\rho^2\,,\qquad\forall x\in\gamma_0\cap B_{p_0,\mu}\,;$$

by exploiting this inequality we now deduce that if $x\in\gamma_0\cap B_{p_0,\mu}$, then

$$\begin{aligned}\rho &\ge \mathrm{dist}(f_{p_0}(x),\gamma_0)=|f_{p_0}(x)-g(x)|\ge|(f_{p_0}(x)-g(x))\cdot\nu_0^{(i)}(x)|\\ &\ge |(x-g(x))\cdot\nu_0^{(i)}(x)+(a_i-\zeta^{(i)}(x))|\ge|\zeta^{(i)}(x)|-C(\rho+\rho^2)\,,\end{aligned}$$

so that (2.69) improves into

$$\|\zeta\|_{C^0(U_{p_0,\eta_0})}\le C\,\rho\,. \tag{2.72}$$

By combining (2.72) with (2.60) and (2.38) we prove (2.29). We now claim that there exists a constant M depending on α, L and γ_0 only such that

$$f_{p_0}(x)\in[\gamma]_{3\rho}\,,\qquad \forall x\in\gamma_0\cap(U_{p_0,\eta_0}\setminus U_{p_0,M\rho})\,. \tag{2.73}$$

To this end we notice that if $x\in\gamma_0\cap(U_{p_0,\eta_0}\setminus U_{p_0,M\,\rho})$, then by (2.12), (2.24) and (2.29) we have

$$|f_{p_0}(x)-p|\ge|p_0-x|-|f_{p_0}(x)-x|-|p-p_0|\ge\frac{\mathrm{d}_0(p_0,x)}{C}-C\,\rho\ge\Big(\frac{M}{C}-C\Big)\,\rho\,,$$

while by (2.12) and $|x-p_0|\le\mathrm{d}_0(x,p_0)\le\eta_0$

$$\begin{aligned}|f_{p_0}(x)-q| &\ge |p-q|-|f_{p_0}(x)-p|=|p-q|-|f_{p_0}(x)-f_{p_0}(p_0)|\\ &\ge |p_0-q_0|-2\rho-C|x-p_0|\ge|p_0-q_0|-C(\mu_0+\eta_0)\,,\end{aligned}$$

so that we can entail $\min\{|f_{p_0}(x)-p|,|f_{p_0}(x)-q|\}\ge3\rho$ up to take M large enough and up to further decrease η_0 and μ_0. This proves (2.73). By combining assumption (iii) with (2.73) we see that

$$f_{p_0}(x)=g(x)+\psi(g(x))\,,\qquad g(x)\in[\gamma_0]_\rho\,,\qquad \forall x\in\gamma_0\cap(U_{p_0,\eta_0}\setminus U_{p_0,M\,\rho})\,. \tag{2.74}$$

(This implies, in particular, that

$$f_{p_0}(x)=x+\psi(x)\,,\qquad \forall x\in\gamma_0\cap(U_{p_0,\eta_0}\setminus U_{p_0,\mu})\,, \tag{2.75}$$

thanks to (2.32).) By (2.29) and by $|g(x)-f_{p_0}(x)|\le|x-f_{p_0}(x)|$ we find that

$$|g(x)-x|\le C\,\rho\,,\qquad \forall x\in\gamma_0\cap(U_{p_0,\eta_0}\setminus U_{p_0,M\,\rho})\,. \tag{2.76}$$

We finally exploit (2.74) and (2.76) to show that

$$\tau(f_{p_0}(x))\cdot\tau_0(x)\ge1-C\,\rho\,,\qquad \forall x\in\gamma_0\cap(U_{p_0,\eta_0}\setminus U_{p_0,M\,\rho})\,. \tag{2.77}$$

Indeed by (2.14) we have

$$\tau(x+\psi(x))=\frac{\tau_0(x)+\nabla^{\gamma_0}\psi(x)\tau_0(x)}{|\tau_0(x)+\nabla^{\gamma_0}\psi(x)\tau_0(x)|}\,,\qquad \forall x\in[\gamma_0]_\rho\,,$$

so that, by (2.15),

$$\tau(x+\psi(x))\cdot\tau_0(x)\ge1-C\,\rho\,,\qquad \forall x\in[\gamma_0]_\rho\,.$$

By combining this inequality with (2.74), we find that

$$\tau(f_{p_0}(x))\cdot\tau_0(g(x))\ge1-C\,\rho\,,\qquad \forall x\in\gamma_0\cap(U_{p_0,\eta_0}\setminus U_{p_0,M\,\rho})\,,$$

which, combined with (2.76) and (2.25), gives (2.77). We now prove (2.30). Indeed, by (2.54), $d_\gamma(f_{p_0}(x))=0$ for every $x\in U_{p_0,\eta_0}$. We differentiate this identity along U_{p_0,η_0} to find

$$0=\nabla d_\gamma(f_{p_0}(x))\Big[\nabla^{\widetilde{\gamma}_0}f_{p_0}(x)\tau_0(x)\Big]\,,\qquad \forall x\in U_{p_0,\eta_0}\,. \tag{2.78}$$

By taking (2.62) into account, and by (2.36), (2.37), (2.38) and (2.72), we find

$$\Big|\nabla d_\gamma(f_{p_0}(x))\Big[\tau_0(x)-\sum_{i=1}^{n-1}\Big(\nabla^{\widetilde{\gamma}_0}\zeta^{(i)}(x)\tau_0(x)\Big)\,\nu_0^{(i)}(x)\Big]\Big|\le C\,\frac{\rho}{\mu}\,,\qquad \forall x\in U_{p_0,\eta_0}\,. \tag{2.79}$$

By (2.21) and (2.22) we have that

$$\begin{cases} |\nabla d_\gamma(f_{p_0}(x))\,[\tau_0(x)]| \le C\,\rho\,, \\ |\nabla d_\gamma(f_{p_0}(x))\,[v]| \ge (1-C\,\rho)|v|\,, \end{cases} \qquad \text{whenever} \qquad \begin{array}{l} x\in U_{p_0,\eta_0},\ v\in\mathbb{R}^n, \\ \text{with } |\tau_0(x)\cdot\tau(f_{p_0}(x))|\ge 1-C\rho \\ \text{and } v\cdot\tau_0(x)=0\,. \end{array} \tag{2.80}$$

By (2.13), we can combine (2.79) and (2.80) at $x=p_0$ to find that

$$\sum_{i=1}^{n-1}\left|\nabla^{\widetilde{\gamma}_0}\zeta^{(i)}(p_0)\tau_0(p_0)\right| \le C\,\frac{\rho}{\mu}\,. \tag{2.81}$$

By (2.81), (2.59) and (2.72) we thus have

$$\|\zeta\|_{C^1(U_{p_0,2M\rho})} \le C\,\frac{\rho}{\mu}\,. \tag{2.82}$$

By (2.77) we can can combine (2.79) and (2.80) at every $x\in\gamma_0\cap(U_{p_0,\eta_0}\setminus U_{p_0,M\,\rho})$ to find

$$\sum_{i=1}^{n-1}\left|\nabla^{\widetilde{\gamma}_0}\zeta^{(i)}(x)\tau_0(x)\right| \le C\,\frac{\rho}{\mu}\,, \qquad \forall x\in\gamma_0\cap(U_{p_0,\eta_0}\setminus U_{p_0,M\,\rho})\,. \tag{2.83}$$

By combining (2.72), (2.82) and (2.83) we conclude that

$$\|\zeta\|_{C^1(U_{p_0,\eta_0})} \le C\,\frac{\rho}{\mu}\,. \tag{2.84}$$

By combining (2.62) with (2.25), (2.36), (2.37), (2.38), (2.84), and by taking (2.29) into account we finally conclude the proof (2.30), thus of the claim.

Conclusion of the proof: By repeating the above argument with q_0 and q in place of p_0 and p, we construct $f_{q_0}\in C^{1,\alpha}(\widetilde{\gamma}_0\cap U_{q_0,\eta_0};\widetilde{\gamma})$ such that $f_{q_0}(q_0)=q$,

$$\begin{aligned}
\|f_{q_0}\|_{C^{1,\alpha}(U(q_0,\eta_0))} &\le C\,, && (2.85)\\
\|f_{q_0}-\mathrm{Id}\|_{C^0(U(q_0,\eta_0))} &\le C\,\rho\,, && (2.86)\\
\|f_{q_0}-\mathrm{Id}\|_{C^1(U(q_0,\eta_0))} &\le \frac{C}{\mu}\,\rho\,, && (2.87)\\
\|(f_{q_0}-\mathrm{Id})\cdot\tau_0\|_{C^1(U(q_0,\eta_0))} &\le \frac{C}{\mu}\,|(q_0-q)\cdot\tau_0(q_0)|\,, && (2.88)\\
(f_{q_0}-\mathrm{Id})\cdot\tau_0 &= 0\,, \qquad \text{on } \gamma_0\cap(U_{q_0,\eta_0}\setminus B_{q_0,\mu})\,, && (2.89)\\
J^{\widetilde{\gamma}_0}f_{q_0} &\ge \frac{1}{2}\,, \qquad \text{on } U_{q_0,\eta_0}\,, && (2.90)\\
f_{q_0}(\gamma_0\cap U_{q_0,\eta_0}) &\subset \gamma\,. && (2.91)
\end{aligned}$$

Moreover, we find as in (2.75) that

$$f_{q_0}=\mathrm{Id}+\psi\,, \qquad \text{on } \gamma_0\cap(U_{q_0,\eta_0}\setminus U_{q_0,\mu})\,. \tag{2.92}$$

Let us finally define $f:\gamma_0\to\mathbb{R}^n$ by setting $f=f_{p_0}$ on $\gamma_0\cap U_{p_0,2\mu}$, $f=f_{q_0}$ on $\gamma_0\cap U_{q_0,2\mu}$ and $f=\mathrm{Id}+\psi$ on $\gamma_0\setminus(U_{p_0,2\,\mu}\cup U_{q_0,2\,\mu})$. In this way, by (2.28)–(2.34), (2.85)–(2.91), (2.14), (2.15), (2.75) and (2.92), it turns out that $f\in C^{1,\alpha}(\gamma_0;\mathbb{R}^n)$ with $f(p_0)=p$, $f(q_0)=q$, and $f(\gamma_0)\subset\gamma$, with (2.16)–(2.20) in force, and with $J^{\gamma_0}f\ge 1/2$ on γ_0 provided μ_0 is small enough. In particular, up to further decrease the value of μ_0, we may use Theorem 2.1 to deduce that f is a $C^{1,\alpha}$-diffeomorphism between γ_0 and $f(\gamma_0)$. Since $f(p_0)=p$, $f(q_0)=q$, $f(\gamma_0)\subset\gamma$, and γ is diffeomorphic to γ_0, we conclude that it must be $f(\gamma_0)=\gamma$. $\square$

3. Perimeter almost-minimizing clusters in $\mathbb{R}^n$

3.1. **Sets of finite perimeter.** A Lebesgue-measurable set $E \subset \mathbb{R}^n$ is a *set of locally finite perimeter* in an open set $A \subset \mathbb{R}^n$ if

$$\sup\Big\{\int_E \operatorname{div} T : T \in C^1_c(A;B)\Big\} < \infty\,.$$

or, equivalently, if there exists a $\mathbb{R}^n$-valued Radon measure μ on A with

$$\int_E \nabla\varphi(x)\,dx = \int_{\mathbb{R}^n} \varphi(x)\,d\mu(x)\,, \qquad \forall \varphi \in C^1_c(A)\,. \tag{3.1}$$

The Gauss–Green measure μ_E of E is defined as the Radon measure appearing in (3.1) for the largest open set A such that E is of locally finite perimeter in A. The *reduced boundary* $\partial^* E$ of E is defined as the set of those $x \in \operatorname{spt}\mu_E \subset A$ such that

$$\nu_E(x) = \lim_{r\to 0^+} \frac{\mu_E(B_{x,r})}{|\mu_E|(B_{x,r})} \qquad \text{exists and belongs to } \mathbb{S}^{n-1}\,. \tag{3.2}$$

It turns out that $\partial^* E$ is a locally $\mathcal{H}^{n-1}$-rectifiable set in A. (Here, $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure on $\mathbb{R}^n$, and $S \subset \mathbb{R}^n$ is locally k-rectifiable in A if $\mathcal{H}^k \llcorner S$ is a Radon measure on A and S is contained, modulo an $\mathcal{H}^k$-null set, into a countable union of k-dimensional C^1-surfaces.) Moreover, the Borel vector field $\nu_E : \partial^* E \to \mathbb{S}^{n-1}$ (called the *measure-theoretic outer unit normal to E*) is such that

$$\mu_E = \nu_E\, \mathcal{H}^{n-1} \llcorner \partial^* E \qquad \text{on bounded Borel sets in } A.$$

In particular, (3.1) takes the more explicit form

$$\int_E \nabla\varphi(x)\,dx = \int_{\partial^* E} \varphi(x)\,\nu_E(x)\,d\mathcal{H}^{n-1}(x)\,, \qquad \forall \varphi \in C^1_c(A)\,. \tag{3.3}$$

If $F \subset A$ is a Borel set, then the *perimeter of E relative to the Borel set F* is defined as

$$P(E;F) = |\mu_E|(F) = \mathcal{H}^{n-1}(F \cap \partial^* E)\,, \tag{3.4}$$

and we set $P(E) = P(E;\mathbb{R}^n)$. One always has

$$A \cap \operatorname{cl}(\partial^* E) = \operatorname{spt}\mu_E = \Big\{x \in A : 0 < |E \cap B_{x,r}| < \omega_n\, r^n \quad \forall r > 0\Big\} \subset A \cap \partial E\,,$$

where ω_n is the volume of the Euclidean unit ball in $\mathbb{R}^n$; moreover, μ_E is invariant by modifications of $E \cap A$ on and by a set of volume zero, and up to such modifications we can assume that

$$A \cap \operatorname{cl}(\partial^* E) = \operatorname{spt}\mu_E = A \cap \partial E\,; \tag{3.5}$$

see, for example, [Mag12, Proposition 12.19]. Throughout this paper, **all sets of finite perimeter shall be normalized so to have identity** (3.5) **in force** (where A denotes the largest open set such that E is of locally finite perimeter in A).

3.2. **A regularity criterion for (Λ, r_0)-minimizing sets.** Given $x \in \mathbb{R}^n$, $r > 0$ and $\nu \in \mathbb{S}^{n-1}$, let us set

$$\mathbf{C}^\nu_{x,r} = \Big\{y \in \mathbb{R}^n : |(y-x)\cdot\nu| < r\,, |(y-x) - ((y-x)\cdot\nu)\nu| < r\Big\}\,,$$
$$\mathbf{D}^\nu_{x,r} = \Big\{y \in \mathbb{R}^n : |(y-x)\cdot\nu| = 0\,, |(y-x) - ((y-x)\cdot\nu)\nu| < r\Big\}\,,$$

and define the *cylindrical excess* of $E \subset \mathbb{R}^n$ at x, in direction ν, and at scale r, as

$$\mathbf{exc}^\nu_{x,r}(E) = \frac{1}{r^{n-1}} \int_{\mathbf{C}^\nu_{x,r} \cap \partial^* E} |\nu_E - \nu|^2\, d\mathcal{H}^{n-1}\,,$$

provided E is of finite perimeter on $\mathbf{C}^\nu_{x,r}$. When $\nu = e_n$ and $x = 0$ we simply set

$$\mathbf{C}_r = \mathbf{C}^{e_n}_{0,r}\,,\qquad \mathbf{D}_r = \mathbf{D}^{e_n}_{0,r}\,,\qquad \mathbf{exc}_r(E) = \mathbf{exc}^{e_n}_{0,r}(E)\,.$$

The next result is a classical local regularity criterion for perimeter (Λ, r_0)-minimizing sets (from now on simply called (Λ, r_0)-*minimizing sets*).

Theorem 3.1 (Small excess regularity criterion)**.** *For every $\alpha \in (0,1)$ there exist positive constants ε_* and C_*, depending on n and α only, with the following property. If E is a (Λ, r_0)-minimizing set in $\mathbf{C}^\nu_{x_0,r}$, with $\Lambda\, r_0 \le 1$ and $r < r_0$, and if $x_0 \in \partial E$ is such that*

$$\mathbf{exc}^\nu_{x_0,r}(E) + \Lambda\, r \le \varepsilon_*\,,$$

then there exists a Lipschitz function $v : \mathbf{D}^\nu_{x_0,r/2} \to \mathbb{R}$ with $v(x_0) = 0$, $\mathrm{Lip}\,(v) \le 1$,

$$\|v\|_{C^0(\mathbf{D}^\nu_{x_0,r/2})} \le C_*\, r\, \mathbf{exc}^\nu_{x_0,r}(E)^{1/2(n-1)}\,, \tag{3.6}$$

$$\|\nabla v\|_{C^0(\mathbf{D}^\nu_{x_0,r/2})} \le C_* \Big(\mathbf{exc}^\nu_{x_0,r}(E) + \Lambda\, r\Big)^{1/2(n-1)}\,, \tag{3.7}$$

$$[\nabla v]_{C^{0,\alpha}(\mathbf{D}^\nu_{x_0,r/2})} \le \frac{C_*}{r^\alpha} \Big(\mathbf{exc}^\nu_{x_0,r}(E) + \Lambda\, r\Big)^{1/2}\,, \tag{3.8}$$

and such that

$$\mathbf{C}^\nu_{x_0,r/2} \cap \partial E = (\mathrm{Id} + v\,\nu)(\mathbf{D}^\nu_{x_0,r/2})\,. \tag{3.9}$$

Proof. This is, with the minor addition of (3.7), [Mag12, Theorem 26.3]. □

Remark 3.2. Recall that $\lim_{r\to 0^+} \inf_{\nu\in\mathbb{S}^{n-1}} \mathbf{exc}^\nu_{x,r}(E) = 0$ for every $x \in \partial^* E$; see, for example, [Mag12, Proposition 22.3]. In particular, if E is a (Λ, r_0)-minimizing set in A, then $A \cap \partial^* E$ is a $C^{1,\alpha}$-hypersurface for every $\alpha \in (0,1)$.

Theorem 3.1 can be used to locally represent the boundaries of (Λ, r_0)-minimizing sets E_k converging to a set E as graphs with respect to ∂E, at least provided ∂E is smooth enough. This basic idea is made precise in Lemma 3.4 below. Before stating this lemma, let us premise the following technical statement, where functions $u : \mathbf{D}_4 \to \mathbb{R}$ with $|u| < 4$ are considered, together with their graphs

$$\Gamma(u) = (\mathrm{Id} + u\, e_n)(\mathbf{D}_4) \subset \mathbf{C}_4\,.$$

We also set $\alpha \wedge \beta = \min\{\alpha, \beta\}$.

Lemma 3.3. *Given $n \ge 2$, $L > 0$ and $\alpha, \beta \in [0,1]$ there exist positive constants $\sigma_0 < 1$ and C_0 with the following property. If $u_1 \in C^{2,\alpha}(\mathbf{D}_4)$, $u_2 \in C^{1,\beta}(\mathbf{D}_4)$, and*

$$\max_{i=1,2} \|u_i\|_{C^1(\mathbf{D}_4)} \le \sigma_0\,,\qquad \max\Big\{\|u_1\|_{C^{2,\alpha}(\mathbf{D}_4)}, \|u_2\|_{C^{1,\beta}(\mathbf{D}_4)}\Big\} \le L\,, \tag{3.10}$$

then there exists $\psi \in C^{1,\alpha\wedge\beta}(\mathbf{C}_2 \cap \Gamma(u_1))$ such that

$$\mathbf{C}_1 \cap \Gamma(u_2) \subset (\mathrm{Id} + \psi\nu)(\mathbf{C}_2 \cap \Gamma(u_1)) \subset \Gamma(u_2)\,, \tag{3.11}$$

$$\|\psi\|_{C^{1,\alpha\wedge\beta}(\mathbf{C}_2\cap\Gamma(u_1))} \le C_0\,,\qquad \|\psi\|_{C^1(\mathbf{C}_2\cap\Gamma(u_1))} \le C_0\, \|u_1 - u_2\|_{C^1(\mathbf{D}_4)}\,. \tag{3.12}$$

Here, $\nu \in C^{1,\alpha}(\Gamma(u_1); \mathbb{S}^{n-1})$ is the normal unit vector field to $\Gamma(u_1)$ defined by

$$\nu(z, u_1(z)) = \frac{(-\nabla u_1(z), 1)}{\sqrt{1 + |\nabla u_1(z)|^2}}\,,\qquad \forall z \in \mathbf{D}_4\,. \tag{3.13}$$

Proof. We define $F : \mathbf{D}_4 \times \mathbb{R} \to \mathbb{R}^n$ and $\phi : \mathbf{D}_4 \times \mathbb{R} \to \mathbb{R}$ by setting

$$F(z,t) = \Big(z - t\, \frac{\nabla u_1(z)}{\sqrt{1 + |\nabla u_1(z)|^2}}, u_1(z) + \frac{t}{\sqrt{1 + |\nabla u_1(z)|^2}}\Big)\,, \tag{3.14}$$

$$\phi(z,t) = u_2(z) - t\,, \tag{3.15}$$

for $(z,t)\in\mathbf{D}_4\times\mathbb{R}$. Notice that $F\in C^{1,\alpha}(\mathbf{C}_4)$ and $\phi\in C^{1,\beta}(\mathbf{C}_4)$ with

$$\|F\|_{C^{1,\alpha}(\mathbf{C}_4)}\le C\,,\qquad \|\phi\|_{C^{1,\beta}(\mathbf{C}_4)}\le C\,,\tag{3.16}$$

where C is a constant depending on n, α, β and L only. Provided σ_0 is small enough we also find $F(\mathbf{C}_2)\subset\mathbf{C}_4$, so that we can define $\Phi:\mathbf{C}_2\to\mathbb{R}$ by setting

$$\Phi(z,t)=\phi(F(z,t))=u_2\Big(z-t\,\frac{\nabla u_1(z)}{\sqrt{1+|\nabla u_1(z)|^2}}\Big)-u_1(z)-\frac{t}{\sqrt{1+|\nabla u_1(z)|^2}}\,.$$

By exploiting (3.10) and (3.16) we find that, provided σ_0 is small enough,

$$\|\Phi\|_{C^{1,\alpha\wedge\beta}(\mathbf{C}_2)}\le C\,,\quad \Phi(z,2)\le -1\,,\quad \Phi(z,-2)\ge 1\,,\quad \frac{\partial\Phi}{\partial t}(z,t)\le-\frac12\,,$$

for every $(z,t)\in\mathbf{C}_2$; hence there exists $\zeta\in C^{1,\alpha\wedge\beta}(\mathbf{D}_2;(-1,1))$ with

$$\|\zeta\|_{C^{1,\alpha\wedge\beta}(\mathbf{D}_2)}\le C\,,\qquad \Phi(z,\zeta(z))=0\,,\qquad \forall z\in\mathbf{D}_2\,.\tag{3.17}$$

By (3.13) and (3.17) we find

$$\Big\{(z,u_1(z))+\zeta(z)\,\nu(z,u_1(z)):z\in\mathbf{D}_2\Big\}\subset\Gamma(u_2)\,.\tag{3.18}$$

Again by $\Phi(z,\zeta(z))=0$ we deduce that

$$\zeta(z)=\sqrt{1+|\nabla u_1(z)|^2}\bigg(u_2\Big(z-\zeta(z)\,\frac{\nabla u_1(z)}{\sqrt{1+|\nabla u_1(z)|^2}}\Big)-u_1(z)\bigg)\,,\tag{3.19}$$

so that, by (3.10),

$$\|\zeta\|_{C^0(\mathbf{D}_2)}\le\sqrt{1+\sigma_0^2}\,\Big(\|u_2-u_1\|_{C^0(\mathbf{D}_2)}+\sigma_0^2\,\|\zeta\|_{C^0(\mathbf{D}_2)}\Big)$$

and thus $\|\zeta\|_{C^0(\mathbf{D}_2)}\le C\,\|u_1-u_2\|_{C^0(\mathbf{D}_2)}$. Similarly, by differentiating (3.19), by exploiting the fact that $u_1\in C^{2,\alpha}(\mathbf{D}_2)$ and thanks to (3.10), one finds that

$$\|\zeta\|_{C^1(\mathbf{D}_2)}\le C\,\|u_1-u_2\|_{C^1(\mathbf{D}_2)}\,.\tag{3.20}$$

We finally define $\psi\in C^{1,\alpha\wedge\beta}(\mathbf{C}_2\cap\Gamma(u_1))$ by the identity $\psi(z,u_1(z))=\zeta(z)$, $z\in\mathbf{D}_2$. In this way (3.12) follows immediately from (3.10), (3.17) and (3.20), whereas (3.18) gives the second inclusion in (3.11). The first inclusion in (3.11) is obtained by noticing that: (i) up to further decrease the value of σ_0 we have

$$\begin{cases} x\in\mathbf{C}_2\cap\Gamma(u_1)\,,\\ x+t\,\nu(x)\,,x+s\,\nu(x)\in\Gamma(u_2)\end{cases}\qquad\Rightarrow\qquad t=s\,;\tag{3.21}$$

(ii) there exists $\eta>0$ (depending on L only) such that every $y\in N_\eta(\mathbf{C}_2\cap\Gamma(u_1))$ has a unique projection over $\mathbf{C}_2\cap\Gamma(u_1)$. Since (by (3.10) and provided σ_0 is small enough) we can entail

$$\mathbf{C}_1\cap\Gamma(u_2)\subset N_\eta(\mathbf{C}_2\cap\Gamma(u_1))\,,$$

by (ii) we find that for every $y\in\mathbf{C}_1\cap\Gamma(u_2)$ there exists a unique $\hat y\in\mathbf{C}_2\cap\Gamma(u_1)$ such that

$$y=\hat y+\mathrm{dist}(y,\mathbf{C}_2\cap\Gamma(u_1))\,\nu(\hat y)\,.$$

Furthermore, by the second inclusion in (3.11) we find, $\hat y+\psi(\hat y)\,\nu(\hat y)\in\Gamma(u_2)$, and thus, by (i), $y=\hat y+\psi(\hat y)\,\nu(\hat y)$. The first inclusion in (3.11) is thus proved. □

Lemma 3.4. *If $\alpha,\beta\in[0,1]$, $\Lambda\ge0$, and E is a set of finite perimeter in $\mathbf{C}_1$ with $0\in\partial E$ and*

$$\mathbf{C}_1\cap E=\Big\{z+s\,e_n:z\in\mathbf{D}_1\,,v(z)<s<1\Big\}\,,\tag{3.22}$$

where $v\in C^{2,\alpha}(\mathbf{D}_1)$ with $v(0)=0$ and $\nabla v(0)=0$, then there exists $r\in(0,1/64)$ (depending on α, β, Λ and E) with the following property. If $\{E_k\}_{k\in\mathbb{N}}$ is a sequence of (Λ,r_0)-minimizing *sets in* $B_{32\,r}$ *with* $|B_{32\,r}\cap(E_k\Delta E)|\to 0$ *as* $k\to\infty$*, then there exist* $k_0\in\mathbb{N}$ *and* $\{\psi_k\}_{k\ge k_0}\subset C^{1,\alpha\wedge\beta}(\mathbf{C}_{2\,r}\cap\partial E)$ *such that*

$$\mathbf{C}_r\cap\partial E_k\subset(\mathrm{Id}+\psi_k\nu_E)(\mathbf{C}_{2r}\cap\partial E)\subset\mathbf{C}_{4\,r}\cap\partial E_k\,,\qquad\forall k\ge k_0\,,\tag{3.23}$$

$$\|\psi_k\|_{C^{1,\alpha\wedge\beta}(\mathbf{C}_{2r}\cap\partial E)}\le C_0\,,\qquad\lim_{k\to\infty}\|\psi_k\|_{C^1(\mathbf{C}_{2r}\cap\partial E)}=0\,,\tag{3.24}$$

where C_0 *is a constant depending on* α*,* β*,* Λ *and* E*.*

Proof. We first notice for future reference that by (3.22),

$$\mathbf{C}_1\cap\partial E=\Big\{z+v(z)\,e_n:z\in\mathbf{D}_1\Big\}\,.\tag{3.25}$$

Let now ε_* and C_* be determined in dependence of n and β as in Theorem 3.1, and set

$$L=\|v\|_{C^{2,\alpha}(\mathbf{D}_1)}\,,\tag{3.26}$$

so that L depends on E. For a parameter σ to be chosen later on in dependence of α, β, Λ and E, and using the fact that $v(0)=0$ and $\nabla v(0)=0$, we can find $r\in(0,1/64)$ (depending on α, Λ, and E) such that

$$\mathbf{exc}_{64\,r}(E)+\Lambda\,(64r)\le\frac{\sigma}{4^n}\,,\tag{3.27}$$

$$\|v\|_{C^1(\mathbf{D}_{4r})}\le\sigma\,.\tag{3.28}$$

Since $0\in\partial E$, E_k is a (Λ,r_0)-minimizing set in $B_{32\,r}$, and $|(E_k\Delta E)\cap B_{32r}|\to 0$ as $k\to\infty$, by [Mag12, Theorem 21.14-(ii)] there exists $\{x_k\}_{k\in\mathbb{N}}$ with $x_k\in\partial E_k$ and $x_k\to 0$ as $k\to\infty$. By [Mag12, Proposition 22.6], for a.e. $t<32\,r$,

$$\mathbf{exc}_t(E)=\lim_{k\to\infty}\mathbf{exc}_t(E_k-x_k)=\lim_{k\to\infty}\mathbf{exc}_{x_k,t}(E_k)\,.$$

We may thus pick $t\in(16\,r,32\,r)$ such that

$$\lim_{k\to\infty}\mathbf{exc}_{x_k,t}(E_k)\le\Big(\frac{32\,r}{t}\Big)^{n-1}\mathbf{exc}_{32\,r}(E)\le 2^{n-1}\,\mathbf{exc}_{32\,r}(E)\,.$$

By (3.27) there exists $k_0\in\mathbb{N}$ such that

$$\mathbf{exc}_{x_k,t}(E_k)+\Lambda\,t<\sigma\,,\qquad\forall k\ge k_0\,.\tag{3.29}$$

By requiring $\sigma<\varepsilon_*$, by (3.29) and by Theorem 3.1 for every $k\ge k_0$ there exists $w_k\in C^{1,\beta}(\mathbf{D}_{x_k,t/2})$ such that

$$\begin{aligned}\mathbf{C}_{x_k,t/2}\cap E_k&=\Big\{z+s\,e_n:z\in\mathbf{D}_{x_k,t/2}\,,w_k(z)\le s\le\frac{t}{2}\Big\}\,,\\ \mathbf{C}_{x_k,t/2}\cap\partial E_k&=\Big\{z+w_k(z)\,e_n:z\in\mathbf{D}_{x_k,t/2}\Big\}\,,\end{aligned}\tag{3.30}$$

and

$$\|w_k\|_{C^{1,\beta}(\mathbf{D}_{x_k,t/2})}\le C_*\,\max\Big\{\frac{t}{2},\frac{1}{(t/2)^\beta}\Big\}\,\sigma^{1/2(n-1)}\le C\,\sigma^{1/2(n-1)}\,.\tag{3.31}$$

where C depends on β, Λ and E. By composing the functions w_k with vanishing horizontal and vertical translations, and since $t/2>8\,r$, we actually find that, up to further increase the value of k_0, then for every $k\ge k_0$ there exists $v_k\in C^{1,\beta}(\mathbf{D}_{8\,r})$ such that

$$\mathbf{C}_{8\,r}\cap E_k=\Big\{z+s\,e_n:z\in\mathbf{D}_{8\,r}\,,v_k(z)\le s\le 8\,r\Big\}\,,\tag{3.32}$$

$$\mathbf{C}_{8\,r}\cap\partial E_k=\Big\{z+v_k(z)\,e_n:z\in\mathbf{D}_{8\,r}\Big\}\,,\tag{3.33}$$

and, thanks to (3.31),

$$\|v_k\|_{C^{1,\beta}(\mathbf{D}_{8\,r})}\le C\,\sigma^{1/2(n-1)}\le L\,,\tag{3.34}$$

provided σ is small enough (depending on β, Λ and E). By (3.22) and (3.32) we have

$$\|v_k - v\|_{L^1(\mathbf{D}_{8\,r})} \le |\mathbf{C}_{8\,r} \cap (E_k \Delta E)|\,, \qquad \forall k \ge k_0\,.$$

By (3.34) and by interpolation there exists $\theta \in (0,1)$ (depending on n, α, and β only) such that

$$\|v_k - v\|_{C^1(\mathbf{D}_{4r})} \le C\,|\mathbf{C}_{8\,r} \cap (E_k \Delta E)|^\theta\,, \qquad \forall k \ge k_0\,. \tag{3.35}$$

By (3.28) and (3.35), provided we further decrease the value of σ and possibly up to increase the value of k_0 we entail that

$$\max\Big\{\|v\|_{C^1(\mathbf{D}_{4\,r})}, \|v_k\|_{C^1(\mathbf{D}_{4\,r})}\Big\} \le \sigma_0\,, \qquad \forall k \ge k_0\,,$$

where σ_0 is determined as in Lemma 3.3 in dependence of n, L, β and α. Since, by (3.34),

$$\max\Big\{\|v\|_{C^{2,\alpha}(\mathbf{D}_{4\,r})}, \|v\|_{C^{1,\beta}(\mathbf{D}_{4\,r})}\Big\} \le L\,, \qquad \forall k \ge k_0\,,$$

we can indeed apply Lemma 3.3 to find $\psi_k \in C^{1,\alpha\wedge\beta}(\mathbf{C}_{2\,r} \cap \partial E)$ with the required properties. $\square$

3.3. **Regularity of (Λ, r_0)-minimizing clusters.** We gather here some basic regularity properties of (Λ, r_0)-minimizing clusters. In doing so it is convenient to first localize to an open set $A \subset \mathbb{R}^n$ the terminology introduced in section 1.4.

Let $\mathcal{E} = \{\mathcal{E}(h)\}_{h=1}^N$ be a family of Lebesgue-measurable sets in $\mathbb{R}^n$ with

$$|\mathcal{E}(h)| < \infty \qquad \forall h = 1, ..., N\,, \qquad |\mathcal{E}(h) \cap \mathcal{E}(k)| = 0 \qquad \forall 1 \le h < k \le N\,,$$

and set $\mathcal{E}(0) = \mathbb{R}^n \setminus \bigcup_{h=1}^N \mathcal{E}(h)$. One says that $\mathcal{E}$ is an *N-cluster in A* if each $\mathcal{E}(h)$ is a set of locally finite perimeter in A and

$$|\mathcal{E}(h) \cap A| > 0 \qquad \forall h = 1, ..., N\,.$$

If A is the largest open set such that $\mathcal{E}$ is a cluster in A, then, according to (3.2), $\partial^*\mathcal{E}(h)$ is well-defined as a subset of A for every $h = 0, ..., N$, and so are the interfaces $\mathcal{E}(h,k) = \partial^*\mathcal{E}(h) \cap \partial^*\mathcal{E}(k)$ whenever $0 \le h < k \le N$; we may thus set

$$\partial^*\mathcal{E} = \bigcup_{0 \le h < k \le N} \mathcal{E}(h,k)\,,$$

so that $\partial^*\mathcal{E}$ is automatically a subset of A. By (3.5), we are always assuming that

$$\mathrm{cl}\,(\partial^*\mathcal{E}) = A \cap \bigcup_{h=1}^N \mathrm{spt}\mu_{\mathcal{E}(h)} = \bigcup_{h=1}^N \Big\{x \in A : 0 < |\mathcal{E}(h) \cap B_{x,r}| < \omega_n\, r^n\ \forall r > 0\Big\} = A \cap \partial\mathcal{E}\,,$$

where $\partial\mathcal{E} = \bigcup_{h=1}^N \partial\mathcal{E}(h)$. We also set

$$\Sigma_F(\mathcal{E}) = (F \cap \partial\mathcal{E}) \setminus \partial^*\mathcal{E} \qquad \forall F \subset A\,, \qquad \Sigma(\mathcal{E}) = \Sigma_{\mathbb{R}^n}(\mathcal{E})\,.$$

Finally, one says that $\mathcal{E}$ is a *(Λ, r_0)-minimizing cluster in A* if (1.17) holds whenever $x \in \mathbb{R}^n$, $r < r_0$ and $\mathcal{E}(h)\Delta\mathcal{F}(h) \subset\subset B_{x,r} \subset\subset A$ for every $h = 1, ..., N$. We now prove the following lemma, which is a special case of [LT02, Lemma 4.6] (see also [Leo01, Theorem 3.1] for a similar result in the context of immiscible fluids).

Lemma 3.5 (Infiltration lemma)**.** *There exists a positive constant $\eta_0 = \eta_0(n) < \omega_n$ with the following property: if $\mathcal{E}$ is a (Λ, r_0)-minimizing cluster in A, then there exists a positive constant $r_1 \le r_0$ (depending on Λ and r_0 only) such that, if*

$$\sum_{h \in H} |\,\mathcal{E}(h) \cap B_{x,r}| \le \eta_0\, r^n\,, \tag{3.36}$$

for some $r \le r_1$, $H \subset \{0, \dots, N\}$, and $x \in \mathbb{R}^n$ with $B_{x,r} \subset\subset A$, then

$$\sum_{h \in H} |\,\mathcal{E}(h) \cap B_{x,r/2}| = 0\,. \tag{3.37}$$

Proof. By arguing as in [Mag12, Lemma 30.2] one sees that if $\mathcal{E}$ is a N-cluster in A such that

$$P(\mathcal{E};B_{x,r})\le P(\mathcal{F};B_{x,r})+C_0\,|\,\mathrm{vol}\,(\mathcal{E})-\mathrm{vol}\,(\mathcal{F})|\,,\tag{3.38}$$

whenever $\mathcal{E}(h)\Delta\mathcal{F}(h)\subset\subset B_{x,r}\subset\subset A$ for some $x\in\mathbb{R}^n$, $r<r_0$ and every $h=1,...,N$, then (3.36) implies (3.37) with $r_1=\min\{r_0,1/8C_0\}$. This is achieved by exploiting the perturbed minimality inequality (3.38) on comparison clusters $\mathcal{F}$ having the property that, if $0\le h\le N$, then either $\mathcal{F}(h)\subset\mathcal{E}(h)$ or $\mathcal{E}(h)\subset\mathcal{F}(h)$. We now notice that, on such clusters $\mathcal{F}$ one has

$$\mathrm{d}(\mathcal{E},\mathcal{F})=\sum_{h=1}^N\big|\,|\mathcal{E}(h)|-|\mathcal{F}(h)|\,\big|\le\sqrt{N}\,|\,\mathrm{vol}\,(\mathcal{E})-\mathrm{vol}\,(\mathcal{F})\,|\,.$$

Therefore, if $\mathcal{E}$ is a (Λ,r_0)-minimizing cluster in A, then (3.38) holds on every comparison cluster $\mathcal{F}$ as above with $C_0=\sqrt{N}\Lambda$, and we can argue as in [Mag12, Lemma 30.2] to prove the lemma (with $r_1=\min\{r_0,1/8\sqrt{N}\Lambda\}$). □

We now deduce some corollaries of Lemma 3.5 and Theorem 3.1.

Corollary 3.6 (Density estimates). *If $\mathcal{E}$ is a (Λ,r_0)-minimizing cluster in A, then there exist positive constants c_0, $c_1<1$, and c (depending on n only), C (depending on n and Λ only) and $r_1\le r_0$ (depending on $\mathcal{E}$), such that, if $0\le h\le N$, $x\in\partial\mathcal{E}(h)$, and $r<r_1$ is such that $B_{x,r}\subset\subset A$, then*

$$c_0\le\frac{|\mathcal{E}(h)\cap B_{x,r}|}{\omega_n\,r^n}\le c_1\,,\tag{3.39}$$

$$c\,\le\frac{P(\mathcal{E}(h);B_{x,r})}{r^{n-1}}\le C(1+r)\,.\tag{3.40}$$

Proof. Lemma 3.5 implies (3.39) with $c_1=1-c_0$ and $c_0=\eta_0/\omega_n$; see [Mag12, Section 30.2]. The lower bound in (3.40) follows from (3.39) and the relative isoperimetric inequality on balls, see [Mag12, Proposition 12.37]. Finally, by testing (1.17) on $\mathcal{F}(h)=\mathcal{E}(h)\setminus B_{x,r}$, $1\le h\le N$, we find that $P(\mathcal{E};B_{x,r})\le n\omega_n\,r^{n-1}+\Lambda\,\omega_n\,r^n$, whence the upper bound in (3.40). □

In general, if $\mathcal{E}$ is a (Λ,r_0)-minimizing cluster in A, then its chambers $\mathcal{E}(h)$ are not necessarily (Λ,r_0)-minimizing sets in A; however, they are (Λ,r_0)-minimizing sets in suitably small neighborhoods of any interface point.

Corollary 3.7 (Almost everywhere regularity). *If $\mathcal{E}$ is a (Λ,r_0)-minimizing cluster in A and $0\le h<k\le N$, then for every $x\in\mathcal{E}(h,k)$ there exists a positive $r_x\le r_0$ such that $|\mathcal{E}(j)\cap B_{x,r_x}|=0$ if $j\ne h,k$ and $B_{x,r_x}\subset\subset A$: in particular, $\mathcal{E}(h)$ and $\mathcal{E}(k)$ are both (Λ,r_0)-minimizing sets in B_{x,r_x}. As a consequence, $\partial^*\mathcal{E}$ is a $C^{1,\alpha}$-hypersurface for every $\alpha\in(0,1)$, it is relatively open inside $A\cap\partial\mathcal{E}$, and $\mathcal{H}^{n-1}(\Sigma_A(\mathcal{E}))=0$. Finally, if $n=2$, we can replace $C^{1,\alpha}$ with $C^{1,1}$.*

Proof. Since $x\in\mathcal{E}(h,k)=\partial^*\mathcal{E}(h)\cap\partial^*\mathcal{E}(k)$, by standard density estimates (see [Mag12, Exercise 29.6]), we have

$$\lim_{r\to0^+}\frac{|\,\mathcal{E}(h)\cap B_{x,r}|}{\omega_n\,r^n}+\frac{|\,\mathcal{E}(k)\cap B_{x,r}|}{\omega_n\,r^n}=1\,.$$

Therefore, by Lemma 3.5, $|\mathcal{E}(j)\cap B_{x,r_x}|=0$ for some $r_x>0$ and for every $j\ne h,k$. Exploiting (1.17) we easily infer that $\mathcal{E}(h)$ and $\mathcal{E}(k)$ are (Λ,r_0)-minimizing sets on B_{x,r_x}. By [Mag12, Theorem 21.8], $\partial^*\mathcal{E}$ is a $C^{1,\alpha}$-hypersurface for every $\alpha\in(0,1)$ (with $C^{1,1}$ in place of $C^{1,\alpha}$ if $n=2$), relatively open inside $A\cap\partial\mathcal{E}$. Finally, the lower $(n-1)$-dimensional estimate in (3.40) implies $\mathcal{H}^{n-1}(\Sigma_A(\mathcal{E}))=0$ by Federer's theorem (see [Mag12, Theorem 16.4]). □

Corollary 3.8 (Local finiteness away from the singular set). *If $\mathcal{E}$ is a (Λ,r_0)-minimizing N-cluster in A, $\rho>0$, and $A'\subset\subset A$ is open, then $(A'\cap\partial\mathcal{E})\setminus\mathrm{cl}\,(I_\rho(\Sigma_A(\mathcal{E})))$ is the union of finitely many disjoint connected hypersurfaces.*

Proof. By Corollary 3.7, we can directly assume that $\partial^*\mathcal{E}=\bigcup_{i\in\mathbb{N}} S_i$, where each S_i is a nonempty connected C^1-hypersurface with $S_i\cap S_j=\emptyset$ for $i\neq j$. If we set $S_i^\rho=(A'\cap S_i)\setminus \mathrm{cl}\,(I_\rho(\Sigma_A(\mathcal{E})))$ then $\{S_i^\rho\}_{i\in\mathbb{N}}$ is a disjoint family of connected C^1-hypersurfaces whose union is equal to $(A'\cap\partial\mathcal{E})\setminus \mathrm{cl}\,(I_\rho(\Sigma_A(\mathcal{E})))$. We claim that only finitely many elements of $\{S_i^\rho\}_{i\in\mathbb{N}}$ are nonempty. If this were not the case, then, up to extracting subsequences, we could find $\{x_i\}_{i\in\mathbb{N}}\subset(A'\cap\partial\mathcal{E})\setminus \mathrm{cl}\,(I_\rho(\Sigma_A(\mathcal{E})))$ with $x_i\in S_i$ for every $i\in\mathbb{N}$ and $x_i\to x$ for some $x\in(\mathrm{cl}\,(A')\cap\partial\mathcal{E})\setminus I_\rho(\Sigma_A(\mathcal{E}))$. Since $x\in\partial^*\mathcal{E}$, by Theorem 3.1 and Corollary 3.7, there exists $r_x>0$ and $\nu\in\mathbb{S}^{n-1}$ such that $\partial\mathcal{E}\cap\mathbf{C}^\nu_{x,r_x}=\partial^*\mathcal{E}\cap\mathbf{C}^\nu_{x,r_x}=(\mathrm{Id}+v\,\nu)(\mathbf{D}^\nu_{x,r_x})$ for some $v\in C^1(\mathbf{D}^\nu_{x,r_x})$. By connectedness, we infer that $S_i\cap\mathbf{C}^\nu_{x,r_x}=S_j\cap\mathbf{C}^\nu_{x,r_x}$, which contradicts the assumption on S_i and S_j. □

We finally prove that the boundary of a (Λ,r_0)-minimizing cluster has bounded mean curvature (in distributional sense). Let us recall here that if S is locally $\mathcal{H}^k$-rectifiable then for $\mathcal{H}^k$-a.e. $x\in S$ there exists a k-plane T_xS in $\mathbb{R}^n$, the *approximate tangent space to S at x*, with

$$\mathcal{H}^k\llcorner\Big(\frac{S-x}{r}\Big)\overset{*}{\rightharpoonup}\mathcal{H}^k\llcorner T_xS\,,\qquad\text{as } r\to0^+\,,$$

in the weak-star convergence of Radon measures; see [Mag12, Theorem 10.2]. Given such $x\in S$, $T\in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$, and $\{\tau_i(x)\}_{i=1}^k$ an orthonormal basis of T_xS, the *tangential divergence* $\mathrm{div}_S T$ of T over S at x is defined by $\mathrm{div}_S\,T(x)=\sum_{i=1}^k\tau_i(x)\cdot(\nabla T(x)\tau_i(x))$. One says that S has *generalized mean curvature* $\mathbf{H}_S\in L^1_{\mathrm{loc}}(\mathcal{H}^k\llcorner(A\cap S);\mathbb{R}^n)$ in $A\subset\mathbb{R}^n$ open, if

$$\int_S \mathrm{div}_S\,T\,d\mathcal{H}^k=\int_S T\cdot\mathbf{H}_S\,d\mathcal{H}^k\,,\qquad\forall T\in C^1_c(A;\mathbb{R}^n)\,. \tag{3.41}$$

If $\mathbf{H}_S\in L^\infty(\mathcal{H}^k\llcorner(A\cap S);\mathbb{R}^n)$ one says that S has bounded generalized mean curvature. With this terminology at hand, we prove the following property of (Λ,r_0)-minimizing clusters.

Corollary 3.9 (Bounded mean curvature)**.** *If $\mathcal{E}$ is a (Λ,r_0)-minimizing cluster in A, then $A\cap\partial\mathcal{E}$ is a locally $\mathcal{H}^{n-1}$-rectifiable set with bounded mean curvature in A, and*

$$\|\mathbf{H}_{\partial\mathcal{E}}\|_{L^\infty(\mathcal{H}^{n-1}\llcorner(A\cap\partial\mathcal{E}))}\le\Lambda\,. \tag{3.42}$$

Proof. Clearly $A\cap\partial\mathcal{E}$ is a locally $\mathcal{H}^{n-1}$-rectifiable set in A as $\partial^*\mathcal{E}$ is a locally $\mathcal{H}^{n-1}$-rectifiable in A and $\mathcal{H}^{n-1}(\Sigma_A(\mathcal{E}))=0$. Let now $x\in A$, $r<\min\{r_0,\mathrm{dist}(x,\partial A)\}$, and $T\in C^1_c(B_{x,r};\mathbb{R}^n)$ with $|T|\le1$ be given, and let $\{f_t\}_{|t|<\varepsilon}$ be the flow with initial velocity T, so that (see, e.g., [Mag12, Theorem 17.5])

$$P(f_t(E);B_{x,r})=P(E;B_{x,r})+t\int_{\partial^*E}\mathrm{div}_{\partial E}T\,d\mathcal{H}^{n-1}+O(t^2)\,,$$

for every set E of finite perimeter in $B_{x,r}$. By Lemma C.2 (see Appendix C) one sees that for every $\eta>0$ it is possible to decrease $\varepsilon>0$ in such a way that

$$|f_t(E)\Delta E|\le(1+\eta)\,P(E;B_{x,r})\,|t|\,,\qquad\forall|t|<\varepsilon\,,$$

for every Borel set $E\subset\mathbb{R}^n$. Up to further decrease the value of ε we have $f_t(\mathcal{E}(h))\Delta\mathcal{E}(h)\subset\subset B_{x,r}$ for every $h=1,...,N$, so that by (1.17) one finds

$$\begin{aligned}P(\mathcal{E};B_{x,r})&\le P(f_t(\mathcal{E});B_{x,r})+\frac{\Lambda}{2}\sum_{h=0}^N|\mathcal{E}(h)\Delta f_t(\mathcal{E}(h))|\\&=P(\mathcal{E};B_{x,r})+t\int_{\partial\mathcal{E}}\mathrm{div}_{\partial\mathcal{E}}T\,d\mathcal{H}^{n-1}+O(t^2)+(1+\eta)\,\Lambda\,|t|\,P(\mathcal{E};B_{x,r})\,.\end{aligned}$$

We conclude that, if $B_{x,r}\subset\subset A$ with $r<r_0$ and $T\in C^1_c(B_{x,r};\mathbb{R}^n)$ with $|T|\le1$, then

$$\Big|\int_{\partial\mathcal{E}}\mathrm{div}_{\partial\mathcal{E}}T\,d\mathcal{H}^{n-1}\Big|\le(1+\eta)\,\Lambda\,P(\mathcal{E};B_{x,r})\,,$$

so that (3.42) follows by Riesz theorem and Lebesgue–Besicovitch differentiation theorem. □

3.4. **Convergence properties of boundaries.** We now exploit the infiltration lemma and the small excess regularity criterion to prove Hausdorff convergence of boundaries. We localize the cluster distance d defined in (1.18) to $A \subset \mathbb{R}^n$ by setting

$$\mathrm{d}_A(\mathcal{E},\mathcal{F}) = \frac{1}{2}\sum_{h=0}^{N} |A \cap (\mathcal{E}(h)\Delta\mathcal{F}(h))|\,,$$

and, similarly, we localize the Hausdorff distance hd to A (see section 2.1) by setting

$$\mathrm{hd}_A(S,T) = \max\Big\{\max_{x\in A\cap S}\mathrm{dist}(x,T),\,\max_{x\in A\cap T}\mathrm{dist}(x,S)\Big\}\,,$$

for every pair S, T of compact sets in A. (It is useful to keep in mind that $\mathrm{hd}_A(S,T) < \delta$ if and only if $A\cap S \subset I_\delta(T)$ and $A \cap T \subset I_\delta(S)$.)

Theorem 3.10 (Hausdorff convergence of boundaries)**.** *If $\mathcal{E}$ is a N-cluster in A and $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ is a sequence of (Λ, r_0)-minimizing N-clusters in A with $\mathrm{d}_A(\mathcal{E}_k,\mathcal{E}) \to 0$ as $k\to\infty$, then $\mathcal{E}$ is a (Λ, r_0)-minimizing cluster in A. Moreover, for every $0 \le i < j \le N$ and $A' \subset\subset A$ one has*

$$\lim_{k\to\infty}\mathrm{hd}_{A'}\Big(\partial\mathcal{E}_k(i)\cap\partial\mathcal{E}_k(j),\partial\mathcal{E}(i)\cap\partial\mathcal{E}(j)\Big) = 0\,, \tag{3.43}$$

so that, in particular, $\mathrm{hd}_{A'}(\partial\mathcal{E}_k,\partial\mathcal{E}) \to 0$ as $k\to\infty$. Finally, for every $\varepsilon > 0$ there exist $k(\varepsilon)\in\mathbb{N}$ such that

$$\Sigma_{A'}(\mathcal{E}_k) \subset I_\varepsilon(\Sigma_A(\mathcal{E}))\,,\qquad \forall k \ge k(\varepsilon)\,. \tag{3.44}$$

Remark 3.11. We are not able, in general, to prove the inclusion $\Sigma_{A'}(\mathcal{E}) \subset I_\varepsilon(\Sigma_A(\mathcal{E}_k))$ for k large, and thus infer the full Hausdorff convergence $\Sigma_A(\mathcal{E}_k)$ to $\Sigma_A(\mathcal{E})$ in every $A' \subset\subset A$. We can achieve this if $n = 2$, see Theorem 3.19 below, and if $n = 3$, see [CLMa].

Proof of Theorem 3.10. The fact that $\mathcal{E}$ is a (Λ, r_0)-minimizing cluster in A is obtained by arguing exactly as in the proof of [Mag12, Theorem 21.14], so we shall omit the details. The remaining part of the theorem also follows by a rather standard argument.

Step one: We prove (3.43). To this end, let us fix $0 \le i < j \le N$, set

$$S_{i,j}^k = \partial\mathcal{E}_k(i)\cap\partial\mathcal{E}_k(j)\,,\qquad S_{i,j} = \partial\mathcal{E}(i)\cap\partial\mathcal{E}(j)\,,$$

and show that for every $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that

$$A'\cap S_{i,j}^k \subset I_\varepsilon(S_{i,j})\,,\qquad A'\cap S_{i,j} \subset I_\varepsilon(S_{i,j}^k)\,,\qquad \forall k\ge k_0\,. \tag{3.45}$$

To prove the first inclusion in (3.45) we argue by contradiction, and consider $x_k \in A'\cap S_{i,j}^k$ with $\mathrm{dist}(x_k, S_{i,j}) > \varepsilon$ for every $k\in\mathbb{N}$. Up to extracting subsequences, we may assume that $x_k \to x$ as $k\to\infty$ for some $x \in \mathrm{cl}\,(A') \subset A$. Since $\mathrm{dist}(x, S_{i,j}) \ge \varepsilon$, by (3.5) there exists $r_x < \mathrm{dist}(x, \partial A)$ such that

$$\begin{aligned} &\text{either} \qquad |B_{x,r_x}\cap\mathcal{E}(i)| = 0\,, \qquad \text{or} \qquad |B_{x,r_x}\cap\mathcal{E}(i)| = \omega_n\, r_x^n\,,\\ &\text{or} \qquad |B_{x,r_x}\cap\mathcal{E}(j)| = 0\,, \qquad \text{or} \qquad |B_{x,r_x}\cap\mathcal{E}(j)| = \omega_n\, r_x^n\,. \end{aligned}$$

Let $s_x = \min\{r_x, r_1\}/2$, then for $k\ge k_0$ one has

$$\begin{aligned} &\text{either} \qquad |B_{x_k,2s_x}\cap\mathcal{E}_k(i)| < \eta_0\,(2s_x)^n\,, \qquad \text{or} \qquad |B_{x_k,2s_x}\cap\mathcal{E}_k(i)| > (\omega_n-\eta_0)\,(2s_x)^n\,,\\ &\text{or} \qquad |B_{x_k,2s_x}\cap\mathcal{E}_k(j)| < \eta_0\,(2s_x)^n\,, \qquad \text{or} \qquad |B_{x_k,2s_x}\cap\mathcal{E}_k(j)| > (\omega_n-\eta_0)\,(2s_x)^n\,, \end{aligned}$$

and thus, by Lemma 3.5,

$$\begin{aligned} &\text{either} \qquad |B_{x_k,s_x}\cap\mathcal{E}_k(i)| = 0\,, \qquad \text{or} \qquad |B_{x_k,s_x}\cap\mathcal{E}_k(i)| = \omega_n\, s_x^n\,,\\ &\text{or} \qquad |B_{x_k,s_x}\cap\mathcal{E}_k(j)| = 0\,, \qquad \text{or} \qquad |B_{x_k,s_x}\cap\mathcal{E}_k(j)| = \omega_n\, s_x^n\,. \end{aligned}$$

By (3.5), $x_k\in A'\setminus S^k_{i,j}$ for k large, a contradiction. We now prove the second inclusion in (3.45): by contradiction, there exist $x\in A'\cap S_{i,j}$ and $\varepsilon>0$ such that $B_{x,\varepsilon}\cap S^k_{i,j}=\emptyset$, i.e., by (3.5),

$$\begin{aligned} &\text{either} && |B_{x,\varepsilon}\cap\mathcal{E}_k(i)|=0\,, && \text{or} && |B_{x,\varepsilon}\cap\mathcal{E}_k(i)|=\omega_n\,\varepsilon^n\,,\\ &\text{or} && |B_{x,\varepsilon}\cap\mathcal{E}_k(j)|=0\,, && \text{or} && |B_{x,\varepsilon}\cap\mathcal{E}_k(j)|=\omega_n\,\varepsilon^n\,, \end{aligned}$$

for infinitely many values of k; by letting $k\to\infty$ along such values we thus find that $x\notin S_{i,j}$.

Step two: We prove (3.44). Should (3.44) fail, we could find $\varepsilon>0$ and $x_k\in A'\cap\Sigma(\mathcal{E}_k)$ with $\mathrm{dist}(x_k,\Sigma(\mathcal{E}))>\varepsilon$ for infinitely many $k\in\mathbb{N}$. By step one, up to extracting subsequences, $x_k\to x$ as $k\to\infty$ for some $x\in A\cap\partial\mathcal{E}$. Since $\mathrm{dist}(x,\Sigma(\mathcal{E}))\ge\varepsilon$, we have $x\in\partial^*\mathcal{E}$. By Corollary 3.7, there exist $0\le h<h'\le N$ and $2\,r_*<\min\{r_1,\mathrm{dist}(x,\partial A)\}$ such that $x\in\mathcal{E}(h,h')$ and $B_{x,2\,r_*}\subset\mathcal{E}(h)\cup\mathcal{E}(h')$. Hence, for some $k_0\in\mathbb{N}$ we have

$$|\mathcal{E}_k(h)\cap B_{x_k,2\,r_*}|+|\mathcal{E}_k(h')\cap B_{x_k,2\,r_*}|\ge(\omega_n-\eta_0)\,r_*^n\,,\qquad\forall k\ge k_0\,.$$

By Lemma 3.5, $\mathcal{E}_k(j)\cap B_{x_k,r_*}=\emptyset$ for every $k\ge k_0$ and $j\ne h,h'$, so that $\mathcal{E}_k(h)$ is a (Λ,r_0)-minimizing set in B_{x_k,r_*}. By arguing as in Lemma 3.4 we find that

$$\mathbf{exc}^\nu_{x,r}(\mathcal{E}(h))=\lim_{k\to\infty}\mathbf{exc}^\nu_{x_k,r}(\mathcal{E}_k(h))\,,\qquad\text{for a.e. } r<r_*\,. \tag{3.46}$$

Since $x\in\mathcal{E}(h,h')$, by Remark 3.2 there exist $r_{**}<\min\{r_*,r_0\}$ and $\nu\in\mathbb{S}^{n-1}$ such that

$$\mathbf{exc}^\nu_{x,r_{**}}(\mathcal{E}(h))+\Lambda\,r_{**}\le\frac{\varepsilon_*}{2^n}\,, \tag{3.47}$$

where ε_* is defined (depending on n and $\alpha\in(0,1)$) as in Theorem 3.1. Since, trivially, $\mathbf{exc}^\nu_{x,r}(\mathcal{E}(h))\le(r_{**}/r)^{n-1}\,\mathbf{exc}^\nu_{x,r_{**}}(\mathcal{E}(h))$ for every $r<r_{**}$, by (3.46) and (3.47) we conclude that, for some $r\in(r_{**}/2,r_{**})$ and up to increase k_0, $\mathbf{exc}^\nu_{x_k,r}(\mathcal{E}_k(h))+\Lambda\,r<\varepsilon_*$ for every $k\ge k_0$. By Theorem 3.1, $B_{x_k,r/2}\cap\partial^*\mathcal{E}_k(h)$ is a $C^{1,\alpha}$-hypersurface, against $x_k\in\Sigma_{A'}(\mathcal{E}_k)$. $\square$

3.5. **Normal representation theorem away from singularities.** Given a cluster $\mathcal{E}$ in A, let us now set for the sake of brevity

$$[\partial\mathcal{E}]_\rho=(A\cap\partial\mathcal{E})\setminus I_\rho(\Sigma_A(\mathcal{E}))\,,$$

and combine Theorem 3.1 and Theorem 3.10 to show that if $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ is a sequence of (Λ,r_0)-minimizing clusters in A with $\mathrm{d}_A(\mathcal{E}_k,\mathcal{E})\to0$ for some $\mathcal{E}$ with $\partial^*\mathcal{E}$ of class $C^{2,1}$, then, for every $\rho<\rho_0$ and $A'\subset\subset A$ we can cover $(A'\cap\partial\mathcal{E}_k)\setminus I_{2\rho}(\Sigma_A(\mathcal{E}))$ with $(\mathrm{Id}+\psi_k\nu_{\mathcal{E}})(A'\cap[\partial\mathcal{E}]_\rho)\subset\partial^*\mathcal{E}_k$, where $\psi_k\to0$ in $C^1(A'\cap[\partial\mathcal{E}]_\rho)$ as $k\to\infty$ and $\nu_{\mathcal{E}}$ is a $C^{1,1}$-unit normal vector field to $\partial^*\mathcal{E}$.

Theorem 3.12 (Normal representation theorem)**.** *If $\Lambda\ge0$, $\alpha\in(0,1)$ and $\mathcal{E}$ is a N-cluster in A such that $\partial^*\mathcal{E}$ is a $C^{2,1}$-hypersurface, then there exist positive constants ρ_0 (depending on $\mathcal{E}$) and C (depending on α, Λ and $\mathcal{E}$) with the following property.*

If $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ is a sequence of (Λ,r_0)-minimizing clusters in A with $\mathrm{d}_A(\mathcal{E}_k,\mathcal{E})\to0$ as $k\to\infty$, then for every $A'\subset\subset A$ and $\rho<\rho_0$ there exist $k_0\in\mathbb{N}$, $\varepsilon>0$, and $\{\psi_k\}_{k\ge k_0}\subset C^{1,\alpha}(A'\cap[\partial\mathcal{E}]_\rho)$ such that

$$(A'\cap\partial\mathcal{E}_k)\setminus I_{2\rho}(\Sigma_A(\mathcal{E}))\subset(\mathrm{Id}+\psi_k\nu_{\mathcal{E}})(A'\cap[\partial\mathcal{E}]_\rho)\subset\partial^*\mathcal{E}_k\,, \tag{3.48}$$

$$N_\varepsilon(A'\cap[\partial\mathcal{E}]_\rho)\cap\partial\mathcal{E}_k=(\mathrm{Id}+\psi_k\,\nu_{\mathcal{E}})(A'\cap[\partial\mathcal{E}]_\rho)\,, \tag{3.49}$$

with

$$\lim_{k\to\infty}\|\psi_k\|_{C^1(A'\cap[\partial\mathcal{E}]_\rho)}=0\,,\qquad\sup_{k\ge k_0}\|\psi_k\|_{C^{1,\alpha}(A'\cap[\partial\mathcal{E}]_\rho)}\le C\,. \tag{3.50}$$

Moreover, when $n=2$ one can set $\alpha=1$.

Proof. Since $\partial^*\mathcal{E}$ is a $C^{2,1}$-hypersurface, for every $x \in \partial^*\mathcal{E}$ there exist $r_x > 0$, $\nu_x \in \mathbb{S}^{n-1}$ and $v_x \in C^{2,1}(\mathbf{D}^{\nu_x}_{x,64\,r_x})$ with $v_x(x) = 0$, $\nabla v_x(x) = 0$, and

$$\partial\mathcal{E} \cap \mathbf{C}^{\nu_x}_{x,64\,r_x} = (\mathrm{Id} + v_x\,\nu_x)(\mathbf{D}^{\nu_x}_{x,64\,r_x})\,, \qquad \mathbf{C}^{\nu_x}_{x,64\,r_x} \subset\subset A\,. \tag{3.51}$$

By Theorem 3.10, $\mathcal{E}$ is a (Λ, r_0)-minimizing cluster in A, so that by Corollary 3.7 there also exist $0 \le h_x < h'_x \le N$ such that, up to further decrease r_x, one has

$$|\mathcal{E}(j) \cap \mathbf{C}^{\nu_x}_{x,64\,r_x}| = 0\,, \qquad \forall j \ne h_x, h'_x\,, \tag{3.52}$$

and thus, taking (3.51) into account and without loss of generality,

$$\mathbf{C}^{\nu_x}_{x,64\,r_x} \cap \mathcal{E}(h_x) = \Big\{ z + s\,\nu_x : z \in \mathbf{D}^{\nu_x}_{x,64\,r_x}\,, v_x(z) < s < 64\,r_x \Big\}\,. \tag{3.53}$$

By Lemma 3.5 and by (3.52) there exists $k_x \in \mathbb{N}$ such that

$$|\mathcal{E}_k(j) \cap B_{x,32\,r_x}| = 0\,, \qquad \forall j \ne h_x, h'_x\,, \qquad \forall k \ge k_x\,, \tag{3.54}$$

so that $\mathcal{E}_k(h_x)$ is a (Λ, r_0)-minimizing set in $B_{x,32\,r_x}$ for $k \ge k_x$. By Lemma 3.4 there exist $s_x \in (0, r_x)$ and, up to increase k_x, functions $\psi_{x,k} \in C^{1,\alpha}(\mathbf{C}^{\nu_x}_{x,2\,s_x} \cap \partial\mathcal{E}_k(h_x))$ with

$$\mathbf{C}^{\nu_x}_{x,s_x} \cap \partial\mathcal{E}_k(h_x) \subset (\mathrm{Id} + \psi_{x,k}\,\nu_{\mathcal{E}_k(h_x)})(\mathbf{C}^{\nu_x}_{x,2\,s_x} \cap \partial\mathcal{E}(h_x)) \subset \mathbf{C}^{\nu_x}_{x,4\,s_x} \cap \partial\mathcal{E}_k(h_x) \tag{3.55}$$

$$\|\psi_{x,k}\|_{C^{1,\alpha}(\mathbf{C}^{\nu_x}_{x,2\,s_x} \cap \partial\mathcal{E}_k(h_x))} \le C\,, \qquad \lim_{k\to\infty} \|\psi_{x,k}\|_{C^1(\mathbf{C}^{\nu_x}_{x,2\,s_x} \cap \partial\mathcal{E}_k(h_x))} = 0\,, \tag{3.56}$$

where C depends on α, Λ and $\mathcal{E}$.

Let $\rho_0 > 0$ be such that $[\partial\mathcal{E}]_{\rho_0} \ne \emptyset$. By compactness, for every $\rho < \rho_0$ we can find $\{x_i\}_{i=1}^M \subset A' \cap [\partial\mathcal{E}]_\rho \subset \partial^*\mathcal{E}$ such that (for $h_i = h_{x_i}$, $r_i = r_{x_i}$, $s_i = s_{x_i}$, and $\nu_i = \nu_{x_i}$) one has

$$A' \cap [\partial\mathcal{E}]_\rho \subset \bigcup_{i=1}^M \mathbf{C}^{\nu_i}_{x_i,s_i}\,, \qquad \mathbf{C}^{\nu_i}_{x_i,64\,s_i} \subset\subset A\,. \tag{3.57}$$

Since $\partial^*\mathcal{E}$ is a C^2-hypersurface we can find $\varepsilon(\rho)$ such that every point in $N_{\varepsilon(\rho)}(A' \cap [\partial\mathcal{E}]_\rho)$ has a unique projection onto $A' \cap [\partial\mathcal{E}]_\rho$ and

$$N_{\varepsilon(\rho)}(A' \cap [\partial\mathcal{E}]_\rho) \subset I_{\varepsilon(\rho)}(A' \cap [\partial\mathcal{E}]_\rho) \subset \bigcup_{i=1}^M \mathbf{C}^{\nu_i}_{x_i,s_i}\,. \tag{3.58}$$

By arguing as in the proof of Lemma 3.3, we see that $\psi_{i,k} = \psi_{j,k}$ on $\mathbf{C}^{\nu_i}_{x_i,2\,s_i} \cap \mathbf{C}^{\nu_j}_{x_j,2\,s_j} \cap \partial\mathcal{E}$ for every i, j. In particular, if we set

$$\Omega = \bigcup_{i=1}^M \mathbf{C}^{\nu_i}_{x_i,2\,s_i}\,,$$

then it makes sense to define $\psi_k \in C^{1,\alpha}(\Omega \cap \partial\mathcal{E})$ for every $k \ge k_0 = \max\{k_i : 1 \le i \le M\}$ by letting $\psi_k = \psi_{x_i,k}$ on $\mathbf{C}^{\nu_i}_{x_i,2\,s_i} \cap \partial\mathcal{E}$. In this way,

$$\partial\mathcal{E}_k \cap \bigcup_{i=1}^M \mathbf{C}^{\nu_i}_{x_i,s_i} \subset (\mathrm{Id} + \psi_k\nu_{\mathcal{E}})(\Omega \cap \partial\mathcal{E}) \subset \partial^*\mathcal{E}_k\,, \tag{3.59}$$

$$\|\psi_k\|_{C^{1,\alpha}(\Omega\cap\partial\mathcal{E})} \le C\,, \qquad \lim_{k\to\infty} \|\psi_k\|_{C^1(\Omega\cap\partial\mathcal{E})} = 0\,. \tag{3.60}$$

By (3.58), (3.59), $A' \cap [\partial\mathcal{E}]_\rho \subset \Omega$, and since $\mathrm{Id} + \psi_k\nu_{\mathcal{E}}$ is a normal deformation of $\Omega \cap \partial\mathcal{E}$,

$$\begin{aligned} N_{\varepsilon(\rho)}(A' \cap [\partial\mathcal{E}]_\rho) \cap \partial\mathcal{E}_k \quad &\subset \quad (\mathrm{Id} + \psi_k\nu_{\mathcal{E}})(\Omega \cap \partial\mathcal{E}) \cap N_{\varepsilon(\rho)}(A' \cap [\partial\mathcal{E}]_\rho) \\ &= \quad (\mathrm{Id} + \psi_k\nu_{\mathcal{E}})(A' \cap [\partial\mathcal{E}]_\rho) \subset N_{\varepsilon(\rho)}(A' \cap [\partial\mathcal{E}]_\rho) \cap \partial\mathcal{E}_k\,, \end{aligned}$$

where the last inclusion follows by the second inclusion in (3.59) provided $\|\psi_k\|_{C^0(\Omega)} < \varepsilon(\rho)$; this proves (3.49). Finally, by Theorem 3.10, up to increase k_0, $A' \cap \partial\mathcal{E}_k \subset I_{\varepsilon(\rho)}(\partial\mathcal{E})$ for every $k \geq k_0$, so that, up to require that $\varepsilon(\rho) < \rho$, we find

$$(A' \cap \partial\mathcal{E}_k) \setminus I_{2\rho}(\Sigma_A(\mathcal{E})) \subset A' \cap \Big(I_{\varepsilon(\rho)}(\partial\mathcal{E}) \setminus I_{2\rho}(\Sigma_A(\mathcal{E}))\Big) \subset A' \cap I_{\varepsilon(\rho)}([\partial\mathcal{E}]_\rho) \subset I_{\varepsilon(\rho)}(A' \cap [\partial\mathcal{E}]_\rho)\,.$$

By combining this last inclusion with (3.58) we find that

$$(A' \cap \partial\mathcal{E}_k) \setminus I_{2\rho}(\Sigma_A(\mathcal{E})) \subset \partial\mathcal{E}_k \cap \bigcup_{i=1}^M \mathbf{C}_{x_i,s_i}^{\nu_i}\,,$$

and thus deduce (3.48) from (3.59). □

3.6. **Blow-ups of** (Λ, r_0)**-minimizing clusters.** We finally comment on the monotonicity properties of density ratios and the existence of blow-up limits. We set

$$\theta(\partial\mathcal{E}, x, r) = \frac{P(\mathcal{E}; B_{x,r})}{r^{n-1}}\,, \qquad \forall x \in \mathbb{R}^n\,, r > 0\,.$$

Theorem 3.13 (Monotonicity of density ratios)**.** *If $\mathcal{E}$ is a (Λ, r_0)-minimizing N-cluster in A, $x \in A \cap \partial\mathcal{E}$, and $r_* \in (0, r_0)$ is such that $\omega_n\, r_*^n < \min\{|\mathcal{E}(h) \cap A| : 1 \leq h \leq N\}$, then*

$$\theta(\partial\mathcal{E}, x, r)\, e^{(n-1)\,\omega_n\,\Lambda\, r} \qquad \textit{is increasing on } (0, r_*), \tag{3.61}$$

In particular, the density $\theta(\partial\mathcal{E}, x) = \theta(\partial E, x, 0^+)$ is well defined for every $x \in A \cap \partial\mathcal{E}$. Moreover, if $\Lambda = 0$ and $\theta(\partial\mathcal{E}, x, \cdot)$ is constant on $(0, r_*)$, then $B_{x,r_*} \cap \partial\mathcal{E}$ is a cone with vertex at x (that is, $x + t(y - x) \in \partial\mathcal{E}$ for every $y \in B_{x,r_*} \cap \partial\mathcal{E}$ and $t \in [0, 1]$).

Proof. This comes from a classical argument (see for example [Mag12, Theorem 28.4] in the case $\Lambda = 0$), that is sketched here, for the reader's convenience, under the assumption that $P(\mathcal{E}; \partial B_{x,r}) = 0$ for every $r < r_*$. Given $r < r_*$, define N-clusters $\mathcal{F}$ with $\mathcal{E}(h)\Delta\mathcal{F}(h) \subset\subset B_{x,r}$ for every $h = 1, ..., N$, by setting

$$\mathcal{F}(h) = \Big(\mathcal{E}(h) \setminus B_{x,r}\Big) \cup \Big\{x + t\,(y - x) : 0 < t < 1\,, y \in \mathcal{E}(h) \cap \partial B_{x,r}\Big\}\,, \qquad h = 1, ..., N\,.$$

(Notice that $|\mathcal{F}(h)| > 0$ for every $h = 1, ..., N$ thanks to the definition of r_*.) If $\nu : \partial^*\mathcal{E} \to \mathbb{S}^{n-1}$ is a Borel normal vector field to $\partial^*\mathcal{E}$, then, by applying the coarea formula to $\partial^*\mathcal{E}$ and since $P(\mathcal{E}; \partial B_{x,s}) = 0$ for every $s < r_*$, we find that

$$P(\mathcal{E}; B_{x,r}) = \int_0^r ds \int_{\partial^*\mathcal{E} \cap \partial B_{x,s}} \frac{d\mathcal{H}^{n-2}(y)}{\sqrt{1 - (\nu(y) \cdot ((x - y)/|x - y|))^2}}\,, \qquad \forall r < r_*\,.$$

Thus, $p(r) = P(\mathcal{E}; B_{x,r})$ is absolutely continuous on $(0, r_*)$, with $p'(r) \geq \mathcal{H}^{n-2}(\partial^*\mathcal{E} \cap \partial B_{x,r})$ for a.e. $r < r_*$. At the same time, by applying the coarea formula to $\partial^*\mathcal{F}$ we find that

$$P(\mathcal{F}; B_{x,r}) = \int_0^r \mathcal{H}^{n-2}(\partial^*\mathcal{F} \cap \partial B_{x,s})\, ds = \mathcal{H}^{n-2}(\partial\mathcal{E} \cap \partial B_{x,r}) \int_0^r \Big(\frac{s}{r}\Big)^{n-2} ds\,,$$

so that $P(\mathcal{F}; B_{x,r}) \leq r\, p'(r)/(n-1)$. By (1.17) we find that $(n-1)\, p(r) \leq r\, p'(r) + (n-1)\,\Lambda\,\omega_n\, r^n$ for every $r < r_*$. This proves (3.61), and the rigidity assertion is easily inferred by a careful inspection of the above argument. □

If $\mathcal{E}$ is a N-cluster in A and $x \in A$, then the blow-up of $\mathcal{E}$ at x at scale r is the N-cluster $\mathcal{E}_{x,r}$ in $(A - x)/r$ defined by the equations

$$\mathcal{E}_{x,r}(h) = \frac{\mathcal{E}(h) - x}{r}\,, \qquad 1 \leq h \leq N\,.$$

(Notice that $(A - x)/r$ eventually contains any given compact set in $\mathbb{R}^n$ as $r \to 0^+$.) In order to describe blow-up limits (as $r \to 0^+$) of (Λ, r_0)-minimizing clusters, we need to introduce the following terminology. One says that $\mathcal{K} = \{\mathcal{K}(h)\}_{h=1}^M$ is an *improper M-cluster* in $\mathbb{R}^n$ if $\mathcal{K}(h)$ is of locally finite perimeter in $\mathbb{R}^n$ for every $h = 1, \dots, M$, with $|\mathcal{K}(h) \cap \mathcal{K}(k)| = 0$ whenever $1 \le h < k \le M$ and $\left|\mathbb{R}^n \setminus \bigcup_{h=1}^M \mathcal{K}(h)\right| = 0$. If F is a bounded set in $\mathbb{R}^n$, then the relative perimeter of $\mathcal{K}$ in F is defined as

$$P(\mathcal{K}; F) = \frac{1}{2}\sum_{h=1}^M P(\mathcal{K}(h); F) = \sum_{1\le h<k\le M} \mathcal{H}^{n-1}\Big(F \cap \partial^*\mathcal{K}(h) \cap \partial^*\mathcal{K}(k)\Big) .$$

Correspondingly, one says that $\mathcal{K}$ is a *cone-like minimizing M-cluster* (with vertex at 0) if $\mathcal{K}(h)$ is an open cone with vertex at 0 for every $h = 1, ..., M$, and if

$$P(\mathcal{K}; B_R) \le P(\mathcal{F}; B_R) , \tag{3.62}$$

whenever $\mathcal{F}$ is an improper M-cluster in $\mathbb{R}^n$ with $\mathcal{F}(h)\Delta\mathcal{K}(h) \subset\subset B_R$ for some $R > 0$ and every $h = 1, ..., M$.

Corollary 3.14 (Tangent cone-like minimizing clusters)**.** *If $\mathcal{E}$ is a (Λ, r_0)-minimizing N-cluster in A, $x \in A \cap \partial\mathcal{E}$, and $r_k \to 0$ as $k \to \infty$, then there exist $\{k(j)\}_{j\in\mathbb{N}}$ with $k(j) \to \infty$ as $j \to \infty$, and a cone-like minimizing M-cluster $\mathcal{K}$ (with $2 \le M \le N$) such that*

$$\theta(\partial\mathcal{E}, x) = \theta(\partial\mathcal{K}, 0) , \qquad \mathcal{E}_{x,r_{k(j)}} \overset{\rm loc}{\to}_\sigma \mathcal{K} \qquad \text{as } j \to \infty ,$$

that is, there exists an injective map $\sigma : \{1, \dots, M\} \to \{0, \dots, N\}$ such that

$$\lim_{j\to\infty} \sum_{i=1}^M \left|\Big(\mathcal{K}(i)\Delta\mathcal{E}_{x,r_{k(j)}}(\sigma(i))\Big) \cap B_R\right| = 0 , \qquad \forall R > 0 . \tag{3.63}$$

Proof. The upper density estimates in (3.40) implies the compactness of $\{\mathcal{E}_{x,r_k}\}_{k\in\mathbb{N}}$ in the $\overset{\rm loc}{\to}_\sigma$ convergence to a limit which is shown to be a cone-like cluster by adapting the argument used in proving Theorem 3.13. Since this proof is classical we omit the details, and refer for example to [Mag12, Theorem 28.6] for the case of (Λ, r_0)-minimizing sets. □

3.7. **(Λ, r_0)-minimizer clusters in $\mathbb{R}^2$.** In view of Corollary 3.14, the starting point in the analysis of almost-minimizing clusters near their singular sets is the classification of cone-like minimizing clusters. Such a classification is currently known only in $\mathbb{R}^2$ and $\mathbb{R}^3$. Referring to [CLMa] for the latter case, we work from now on in $\mathbb{R}^2$. Let us denote by $\mathcal{Y}_2$ the cone-like minimizing 3-cluster in $\mathbb{R}^2$ defined by

$$\mathcal{Y}_2(i) = \Big\{(t\cos\theta, t\sin\theta) : \ t > 0 , (i-1)\,\frac{2\pi}{3} < \theta < i\,\frac{2\pi}{3}\Big\} , \qquad i = 1, 2, 3 .$$

Up to rotations around the origin, $\mathcal{Y}_2$ is the only cone-like minimizing cluster in $\mathbb{R}^2$ (other than the one defined by a pair of complementary half-planes, of course); see, for example, [Mag12, Proposition 30.9]. As a consequence, by Corollary 3.14 one has that if $\mathcal{E}$ is a (Λ, r_0)-minimizing cluster in $A \subset \mathbb{R}^2$, then $\partial^*\mathcal{E} = \{x \in A \cap \partial\mathcal{E} : \theta(\partial\mathcal{E}, x) = 2\}$ and

$$\Sigma_A(\mathcal{E}) = \Big\{x \in A \cap \partial\mathcal{E} : \theta(\partial\mathcal{E}, x) = \theta(\mathcal{Y}_2, 0) = 3\Big\} . \tag{3.64}$$

We now localize Definition 1.2, and then, in Theorem 3.16, describe the structure of planar almost-minimizing clusters.

Definition 3.15. Let $\mathcal{E}$ be a cluster in $A \subset \mathbb{R}^2$ open. One says that $\mathcal{E}$ is a $C^{k,\alpha}$*-cluster in A* if there exist at most countable families $\{\gamma_i\}_{i\in I}$ of connected $C^{k,\alpha}$-curves with boundary relatively closed in A, and $\{p_j\}_{j\in J}$ of points of A, which are both locally finite in A (that is, given $A'\subset\subset A$ we have $\gamma_i\cap A'\neq\emptyset$ and $p_j\in A'$ only for finitely many $i\in I$ and $j\in J$), and such that

$$\begin{aligned} A\cap\partial\mathcal{E}&=\bigcup_{i\in I}\gamma_i\,,\qquad \partial^*\mathcal{E}=\bigcup_{i\in I}\mathrm{int}\,(\gamma_i)\,,\\ \Sigma_A(\mathcal{E})&=A\cap\bigcup_{i\in I}\mathrm{bd}\,(\gamma_i)=A\cap\bigcup_{j\in J}\{p_j\}\,. \end{aligned}\tag{3.65}$$

Theorem 3.16. *If $\mathcal{E}$ is a (Λ,r_0)-minimizing cluster in $A\subset\mathbb{R}^2$, then $\mathcal{E}$ is a $C^{1,1}$-cluster in A. Moreover, each γ_i has distributional curvature bounded by Λ and each p_j is a common boundary point of exactly three different curves from $\{\gamma_i\}_{i\in I}$ which form three 120 degrees angles at p_j. Finally,* $\mathrm{diam}(\gamma_i)\ge 1/2\Lambda$ *for every $i\in I$ such that $\gamma_i\subset\subset A$ and* $\mathrm{bd}\,(\gamma_i)=\emptyset$*. (In particular, if $\Lambda=0$, then* $\mathrm{bd}\,(\gamma_i)\neq\emptyset$ *for every $i\in I$.)*

Proof. By exploiting the argument of [Mag12, Theorem 30.7] (which addresses the case of planar isoperimetric clusters, but actually uses only a minimality condition of the form (1.17), and that can be easily localized to a given open set) we just need to prove that the curves γ_i have distributional curvature bounded by Λ and the diameter lower bound when $\gamma_i\subset\subset A$ with $\mathrm{bd}\,(\gamma_i)=\emptyset$. By Corollary 3.9 we have that

$$\int_{\partial\mathcal{E}}\mathrm{div}_{\partial\mathcal{E}}T\,d\mathcal{H}^1=\int_{\partial\mathcal{E}}T\cdot\mathbf{H}_{\partial\mathcal{E}}\,d\mathcal{H}^1\,,\qquad\forall T\in C^1_c(A;\mathbb{R}^2)\,,\tag{3.66}$$

where $|\mathbf{H}_{\partial\mathcal{E}}|\le\Lambda$. In particular,

$$\int_{\gamma_i}\mathrm{div}_{\gamma_i}T\,d\mathcal{H}^1=\int_{\gamma_i}T\cdot\mathbf{H}_{\partial\mathcal{E}}\,d\mathcal{H}^1\,,\tag{3.67}$$

for every $T\in C^1_c(A';\mathbb{R}^2)$ such that $\mathrm{spt}\,T\cap\partial\mathcal{E}=\mathrm{spt}\,T\cap\mathrm{int}\,(\gamma_i)$. Since $|\mathbf{H}_{\partial\mathcal{E}}|\le\Lambda$ this proves that each $A'\cap\gamma_i$ has distributional mean curvature bounded by Λ. If, in addition, $\gamma_i\subset\subset A'\subset\subset A$ and $\mathrm{bd}\,(\gamma_i)=\emptyset$, then we can test (3.67) with $T(x)=\zeta(x)(x-x_0)$ where $x_0\in\mathbb{R}^2$ is such that $\gamma_i\subset B_{x_0,2\,\mathrm{diam}(\gamma_i)}$ and $\zeta\in C^1_c(A')$ with $\zeta=1$ on γ_i and $\mathrm{spt}\zeta\cap\partial\mathcal{E}=\mathrm{spt}\zeta\cap\gamma_i$, to find that $\mathcal{H}^1(\gamma_i)\le 2\Lambda\,\mathrm{diam}(\gamma_i)\,\mathcal{H}^1(\gamma_i)$, as required. □

Remark 3.17 (Topology of boundaries of planar (Λ,r_0)-minimizing clusters)**.** If $\mathcal{E}$ is a bounded (Λ,r_0)-minimizing cluster in $\mathbb{R}^2$, then Theorem 3.16 implies the existence of *finite* families of closed connected $C^{1,1}$-curves with boundary $\{\gamma_i\}_{i\in I}$ (whose distributional curvature is bounded by Λ) and of finitely many points $\{p_j\}_{j\in J}$ (such that each p_j is the common end-point of three different curves from $\{\gamma_i\}_{i\in I}$, which form three 120 degrees angles at p_j). Moreover, (3.65) takes the form

$$\partial\mathcal{E}=\bigcup_{i\in I}\gamma_i\,,\qquad\partial^*\mathcal{E}=\bigcup_{i\in I}\mathrm{int}\,(\gamma_i)\,,\qquad\Sigma(\mathcal{E})=\bigcup_{i\in I}\mathrm{bd}\,(\gamma_i)=\bigcup_{j\in J}\{p_j\}\,.\tag{3.68}$$

We also notice that if, in addition,

$$\gamma_i\text{ is diffeomorphic to }[0,1]\text{ for every }i\in I\,,\tag{3.69}$$

then

$$\#(I)=3(N'-2)\,,\qquad\mathcal{H}^0(\Sigma(\mathcal{E}))=2(N'-2)\,,\tag{3.70}$$

where N' is the sum of the numbers of connected components of the chambers $\mathcal{E}(h)$ of $\mathcal{E}$ over $h=0,\dots,N$. (In particular, $N'=N+1$ if every chamber, including the exterior chamber, is connected.) Indeed, by (3.69) and by (3.68) each γ_i has exactly two end-points, both belonging to $\Sigma(\mathcal{E})$, and for every $x\in\Sigma(\mathcal{E})$ there exist three curves from $\{\gamma_i\}_{i\in I}$ sharing x as a common end-point: therefore we find $\#(I)=(3/2)\,\mathcal{H}^0(\Sigma(\mathcal{E}))$. If we now apply Euler's formula to the planar graph having the singular points from $\Sigma(\mathcal{E})$ as its vertexes, the curves $\{\gamma_i\}_{i\in I}$ as its edges, and the N' connected components of the chambers of $\mathcal{E}$ as its faces, then we find $2=\mathcal{H}^0(\Sigma(\mathcal{E}))-\#(I)+N'$. Since $\#(I)=(3/2)\,\mathcal{H}^0(\Sigma(\mathcal{E}))$, we have proved (3.70).

Remark 3.18. We notice that (3.69) holds true whenever $\mathcal{E}$ is a planar isoperimetric cluster (that is, $\mathcal{E}$ is a minimizer in (1.15) with $N \ge 2$ and $n = 2$; notice that $\mathcal{E}$ is necessarily bounded). By contradiction, let us assume there exists $i \in I$ such that γ_i is C^1-diffeomorphic to a circle. Since $\gamma_i \cap \Sigma(\mathcal{E}) = \emptyset$, the constant curvature condition on interfaces of $\mathcal{E}$ implies that γ_i is, in fact, a circle. Moreover, since $N \ge 2$, we must have $\#(I) \ge 2$. Since $\#(I) \ge 2$, we can translate γ_i along a suitable direction until it intersects for the first time $\partial\mathcal{E} \setminus \gamma_i$ at some point x. Denoting by $\mathcal{E}'$ the resulting cluster, we have that $P(\mathcal{E}') = P(\mathcal{E})$ and $\mathrm{vol}\,(\mathcal{E}') = \mathrm{vol}\,(\mathcal{E})$, so that $\mathcal{E}'$ is a minimizing cluster in $\mathbb{R}^2$. Therefore, the fact that, in a neighborhood of x, $\partial\mathcal{E}'$ is the union of two tangent circular arcs, leads to a contradiction with Theorem 3.16 applied to $\mathcal{E}'$.

We now close this section by upgrading (3.44) to the full Hausdorff convergence of singular sets, at least in the special case of planar clusters.

Theorem 3.19 (Hausdorff convergence of singular sets)**.** *If $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ is a sequence of (Λ, r_0)-minimizing clusters in $A \subset \mathbb{R}^2$ with $\mathrm{d}_A(\mathcal{E}_k, \mathcal{E}) \to 0$ as $k \to \infty$, then*

$$\lim_{k\to\infty} \mathrm{hd}_{A'}(\Sigma_A(\mathcal{E}_k), \Sigma_A(\mathcal{E})) = 0 \qquad \forall A' \subset\subset A\,.$$

Proof. By (3.44) in Theorem 3.10, and arguing by contradiction, we may directly assume the existence of $x_0 \in \Sigma_{A'}(\mathcal{E})$ and $\varepsilon > 0$ such that $B_{x_0,\varepsilon} \subset\subset A$ and, up to subsequences,

$$\Sigma_{B_{x_0,\varepsilon}}(\mathcal{E}_k) = B_{x_0,\varepsilon} \cap \Sigma_A(\mathcal{E}_k) = \emptyset\,, \qquad \forall k \in \mathbb{N}\,. \tag{3.71}$$

By Theorem 3.10 we have $x_k \to x_0$ as $k \to \infty$ for some $x_k \in A \cap \partial\mathcal{E}_k$, so that, for k large enough, we must have $x_k \in A' \cap \partial^*\mathcal{E}_k$. Up to translations we have thus reduced to consider the following situation: $\mathcal{E}_k$ are (Λ, r_0)-minimizing clusters in $B_{x_0,\varepsilon} \subset\subset A$ with $\mathrm{d}_{B_{x_0,\varepsilon}}(\mathcal{E}_k, \mathcal{E}) \to 0$, $x_0 \in \Sigma_A(\mathcal{E})$, and $x_0 \in \partial^*\mathcal{E}_k$ for every k. We now fix a sequence $s_j \to 0^+$ as $j \to \infty$, and correspondingly define $k(j) \to \infty$ as $j \to \infty$ in such a way that

$$\mathrm{d}_{B_{x_0,\varepsilon}}(\mathcal{E}_{k(j)}, \mathcal{E}) = o(s_j^n) \qquad \text{as } j \to \infty\,. \tag{3.72}$$

By $x_0 \in \Sigma_A(\mathcal{E})$ and by Corollary 3.14, up to extracting a subsequence in j and up to apply the same rotation to $\mathcal{E}_0$ and to each $\mathcal{E}_k$, we can also entail

$$\mathcal{E}_{x_0,s_j} \overset{\mathrm{loc}}{\to}_\sigma \mathcal{Y}_2\,, \qquad \text{as } j \to \infty\,, \tag{3.73}$$

for an injective map $\sigma : \{1,2,3\} \to \{0,...,N\}$. Let us now define $\mathcal{G}_j = \{\mathcal{G}_j(i)\}_{i=1}^3$, $\mathcal{G}_j^* = \{\mathcal{G}_j^*(i)\}_{i=1}^3$, and $\mathcal{G} = \{\mathcal{G}(i)\}_{i=1}^3$ by setting

$$\mathcal{G}_j(i) = B_2 \cap \Big(\frac{\mathcal{E}_{k(j)}(\sigma(i)) - x_0}{s_j}\Big)\,, \qquad \mathcal{G}_j^*(i) = B_2 \cap \Big(\frac{\mathcal{E}(\sigma(i)) - x_0}{s_j}\Big)$$

and $\mathcal{G}(i) = B_2 \cap \mathcal{Y}_2(i)$ for $i = 1,2,3$. By (3.73) we find $\mathrm{d}_{B_2}(\mathcal{G}_j^*, \mathcal{G}) \to 0$ as $j \to \infty$, so that by (3.72) and by triangular inequality $\mathrm{d}_{B_2}(\mathcal{G}_j, \mathcal{G}) \to 0$ as $j \to \infty$. In particular, $\mathcal{G}_j$ defines a 3-cluster in B_2 for j large enough and, actually, $\mathcal{G}_j$ is a $(\Lambda\, s_j, r_0/s_j)$-minimizing 3-cluster in B_2. Again by $\mathrm{d}_{B_2}(\mathcal{G}_j, \mathcal{G}) \to 0$ as $j \to \infty$, Theorem 3.10 gives

$$\lim_{j\to\infty} \max_{1\le i<\ell\le 3} \mathrm{hd}_B\Big(\partial\mathcal{G}_j(i) \cap \partial\mathcal{G}_j(\ell), \partial\mathcal{Y}_2(i) \cap \partial\mathcal{Y}_2(\ell)\Big) = 0\,, \tag{3.74}$$

while, by Theorem 3.12, for every δ small enough one can find $j(\delta) \in \mathbb{N}$ and $\{\psi_j\}_{j\ge j(\delta)} \subset C^1(B \cap [\partial\mathcal{Y}_2]_\delta)$ such that (on taking into account that $I_{2\delta}(\Sigma(\mathcal{Y}_2)) = B_{2\delta}$)

$$\partial\mathcal{G}_j \cap (B \setminus B_{2\delta}) \subset (\mathrm{Id} + \psi_j \nu)(B \cap [\partial\mathcal{Y}_2]_\delta)\,, \qquad \forall j \ge j(\delta)\,, \tag{3.75}$$

where ν denotes a continuous normal vector field to $\partial^*\mathcal{Y}_2$. Finally, we notice that by (3.71), as soon as j is large enough to give $2\,s_j < \varepsilon$, one has

$$\Sigma_{B_2}(\mathcal{G}_j) = \emptyset\,. \tag{3.76}$$

By Theorem 3.16 there exists a finite family of connected $C^{1,1}$-curves with boundary $\{\gamma_i\}_{i\in I}$, relatively closed in B, such that $B\cap\partial\mathcal{G}_j = B\cap\bigcup_{i\in I}\gamma_i$ and $\Sigma_B(\mathcal{G}_j)=\bigcup_{i\in I}B\cap\mathrm{bd}\,(\gamma_i)$, so that, by (3.76), $B\cap\mathrm{bd}\,(\gamma_i)=\emptyset$ for every $i\in I$. Let $\gamma_{i\ell}$ denote the connected curve in $\partial\mathcal{G}_j$ that contains $(\mathrm{Id}+\psi_j\nu)(B\cap[\partial\mathcal{Y}_2(i)\cap\partial\mathcal{Y}_2(\ell)]_\delta)$, for $1\le i<\ell\le 3$. By (3.75) we notice that

$$\partial\mathcal{G}_j\cap(B\setminus B_{2\delta})=\bigcup_{1\le i<\ell\le 3}\gamma_{i\ell}\cap(B\setminus B_{2\delta}) \tag{3.77}$$

while by (3.74) we get

$$\gamma_{i\ell}\cap B\subset I_\delta(\partial\mathcal{Y}_2(i)\cap\partial\mathcal{Y}_2(\ell))\qquad\text{for all } 1\le i<\ell\le 3\,. \tag{3.78}$$

By (3.77) and (3.78) we deduce that $\mathrm{bd}\,(\gamma_{i\ell})\cap B_{2\delta}\neq\emptyset$, against the fact that $B\cap\mathrm{bd}\,(\gamma_i)=\emptyset$ for every $i\in I$. $\square$

4. Proof of the improved convergence theorem for planar clusters

We shall use the following theorem in order to deduce Theorem 1.5 from Theorem 2.6.

Theorem 4.1. *Under the assumptions of Theorem 1.5, there exist $k_0\in\mathbb{N}$ and $C_0,\rho_0>0$ such that the following properties hold:*

(i) *if $\mathcal{E}$ and $\mathcal{E}_k$ satisfy (3.68) with $\{\gamma_i\}_{i\in I}$ and $\{p_j\}_{j\in J}$, and with $\{\gamma_i^k\}_{i\in I_k}$ and $\{p_j^k\}_{j\in J_k}$ respectively, then for $k\ge k_0$ and up to a relabeling, one has $I=I_k$, $J=J_k$, $\mathrm{bd}\,(\gamma_i)\neq\emptyset$ if and only if $\mathrm{bd}\,(\gamma_i^k)\neq\emptyset$ for every $i\in I$, and*

$$\lim_{k\to\infty}|p_j^k-p_j|+\mathrm{hd}(\gamma_i^k,\gamma_i)=0\,,\qquad\forall i\in I, j\in J\,; \tag{4.1}$$

moreover, for every $i\in I$ there exists an extension by foliation (ε_i^k,d_i^k) of γ_i^k with

$$\max\Big\{\frac{1}{\varepsilon_i^k},\|d_i^k\|_{C^{1,1}(\mathbb{R}^2)}\Big\}\le C_0\,; \tag{4.2}$$

(ii) *for every $i\in I$ and $k\ge k_0$, if $\mathrm{bd}\,(\gamma_i)=\{p_j,p_{j'}\}$, $\mathrm{bd}\,(\gamma_i^k)=\{p_j^k,p_{j'}^k\}$, and if $\tau_i^k\in C^{0,\alpha}(\gamma_i^k;\mathbb{S}^1)$ and $\tau_i\in C^{1,1}(\gamma_i;\mathbb{S}^1)$ denote tangent unit vector fields to γ_i^k and γ_i respectively, then, up to a change of orientation,*

$$\lim_{k\to\infty}|\tau_i(p_j)-\tau_i^k(p_j^k)|+|\tau_i(p_{j'})-\tau_i^k(p_{j'}^k)|=0\,; \tag{4.3}$$

(iii) *for every $\rho\le\rho_0$ there exist $k(\rho)\ge k_0$ and $\{\psi_k\}_{k\ge k(\rho)}\subset C^{1,1}([\partial\mathcal{E}]_\rho)$ such that*

$$[\partial\mathcal{E}_k]_{3\rho}\subset(\mathrm{Id}+\psi_k\nu)([\partial\mathcal{E}]_\rho)\subset\partial^*\mathcal{E}_k\,, \tag{4.4}$$

where ν is a $C^{1,1}$ normal unit vector field to $\partial^*\mathcal{E}$ and

$$\lim_{k\to\infty}\|\psi_k\|_{C^1([\partial\mathcal{E}]_\rho)}=0\,,\qquad\sup_{k\ge k(\rho)}\|\psi_k\|_{C^{1,1}([\partial\mathcal{E}]_\rho)}\le C_0\,. \tag{4.5}$$

Proof. Step one: We prove statement (iii). By Theorem 3.12 (applied with $A=\mathbb{R}^2$ and A' an open ball such that $\mathcal{E}(h)\subset\subset A'$ for every $h=1,...,N$) there exist $\rho_0,C_0>0$ such that for every $\rho<\rho_0$ one can find $k(\rho)\in\mathbb{N}$, $\varepsilon(\rho)>0$ and $\{\psi_k\}_{k\ge k(\rho)}\subset C^{1,1}([\partial\mathcal{E}]_\rho)$ such that (4.5) holds, with

$$\partial\mathcal{E}_k\setminus I_{2\rho}(\Sigma(\mathcal{E}))\subset(\mathrm{Id}+\psi_k\nu)([\partial\mathcal{E}]_\rho)\subset\partial^*\mathcal{E}_k\,, \tag{4.6}$$

$$N_{\varepsilon(\rho)}([\partial\mathcal{E}]_\rho)\cap\partial\mathcal{E}_k=(\mathrm{Id}+\psi_k\nu)([\partial\mathcal{E}]_\rho)\,. \tag{4.7}$$

In turn, by Theorem 3.19 (applied with $A=\mathbb{R}^2$ and A' as above), we have $\mathrm{hd}(\Sigma(\mathcal{E}_k),\Sigma(\mathcal{E}))\to 0$ as $k\to\infty$. Hence, up to increase the value of $k(\rho)$ we find $\Sigma(\mathcal{E})\subset I_\rho(\Sigma(\mathcal{E}_k))$ for $k\ge k(\rho)$, and thus $[\partial\mathcal{E}_k]_{3\rho}=\partial\mathcal{E}_k\setminus I_{3\rho}(\Sigma(\mathcal{E}_k))\subset\partial\mathcal{E}_k\setminus I_{2\rho}(\Sigma(\mathcal{E}))$. Thus (4.4) follows from (4.6).

Step two: We prove statement (i) up to (4.1). Since $\mathrm{hd}(\Sigma(\mathcal{E}_k),\Sigma(\mathcal{E}))\to 0$, we can assume without loss of generality that $J=J_k$ with

$$\lim_{k\to\infty}|p_j^k-p_j|=0\,,\qquad\forall j\in J\,.\tag{4.8}$$

Let now I' and I'' be the sets of those $i\in I$ such that γ_i is homeomorphic, respectively, either to $\mathbb{S}^1$ or to $[0,1]$, and similarly define I'_k and I''_k starting from I_k. By intersecting with $N_{\varepsilon(\rho)}([\gamma_i]_\rho)$ in (4.7) and by directly assuming that $\|\psi_k\|_{C^0([\partial\mathcal{E}]_\rho)}<\rho$ for every $k\ge k(\rho)$ we find

$$N_{\varepsilon(\rho)}([\gamma_i]_\rho)\cap\partial\mathcal{E}_k=(\mathrm{Id}+\psi_k\nu)([\gamma_i]_\rho)\,,\qquad\forall i\in I\,,k\ge k(\rho)\,.$$

In particular, by exploiting the connectedness of the curves $\{\gamma_k^i\}_{i\in I_k}$, one defines for every $k\ge k(\rho)$ a map $\sigma_k:I\to I_k$ in such a way that

$$\begin{aligned}&(\mathrm{Id}+\psi_k\nu)([\gamma_i]_\rho)\subset\gamma^k_{\sigma_k(i)}\,,\\&(\mathrm{Id}+\psi_k\nu)([\gamma_i]_\rho)\cap\gamma^k_{i'}=\emptyset\,,\qquad\forall i\in I\,,\forall i'\in I_k\setminus\{\sigma_k(i)\}\,;\end{aligned}$$

hence,

$$(\mathrm{Id}+\psi_k\nu)([\gamma_i]_\rho)=N_{\varepsilon(\rho)}([\gamma_i]_\rho)\cap\partial\mathcal{E}_k=N_{\varepsilon(\rho)}([\gamma_i]_\rho)\cap\gamma^k_{\sigma_k(i)}\,,\qquad\forall k\ge k(\rho)\,,i\in I\,.\tag{4.9}$$

To complete the proof of (4.1) it will suffice to show that

$$\sigma_k\text{ is a bijection with }\sigma_k(I')=I'_k\text{ and }\sigma_k(I'')=I''_k\,,\tag{4.10}$$

$$\lim_{k\to\infty}\mathrm{hd}(\gamma_i,\gamma^k_{\sigma_k(i)})=0\,,\qquad\forall i\in I\,.\tag{4.11}$$

We start by choosing $\eta>0$ such that

$$I_\eta(\gamma_i)\cap I_\eta(\gamma_{i'})=\emptyset\,,\qquad\forall i,i'\in I'\,.\tag{4.12}$$

If $i\in I'$, then $[\gamma_i]_\rho=\gamma_i$ and $N_{\varepsilon(\rho)}(\gamma_i)=I_{\varepsilon(\rho)}(\gamma_i)$ for every $\rho>0$, so that (4.9) gives

$$(\mathrm{Id}+\psi_k\nu)(\gamma_i)=I_{\varepsilon(\rho)}(\gamma_i)\cap\partial\mathcal{E}_k=I_{\varepsilon(\rho)}(\gamma_i)\cap\gamma^k_{\sigma_k(i)}\,,\qquad\forall k\ge k(\rho)\,,i\in I'\,.\tag{4.13}$$

Since $(\mathrm{Id}+\psi_k\nu)(\gamma_i)$ is homeomorphic to $\mathbb{S}^1$ and is contained in $\gamma^k_{\sigma_k(i)}$, by connectedness of $\gamma^k_{\sigma_k(i)}$ we conclude that $\sigma_k(i)\in I'_k$ with

$$(\mathrm{Id}+\psi_k\nu)(\gamma_i)=I_{\varepsilon(\rho)}(\gamma_i)\cap\partial\mathcal{E}_k=\gamma^k_{\sigma_k(i)}\,,\tag{4.14}$$

$$\mathrm{hd}(\gamma_i,\gamma^k_{\sigma_k(i)})\le\|\psi_k\|_{C^1([\partial\mathcal{E}]_\rho)}<\rho\,,\qquad\forall k\ge k(\rho)\,,i\in I'\,.\tag{4.15}$$

By combining (4.12), (4.14) and (4.5) we conclude that

$$\text{(4.11) holds for every } i\in I',\ \sigma_k(I')\subset I'_k,\ \sigma_k\text{ is injective on }I'\,.\tag{4.16}$$

Our next goal is proving that

$$\sigma_k\text{ is a bijection between }I''\text{ and }I''_k\,.\tag{4.17}$$

To this end, we shall first need to prove (4.18) and (4.22) below. In order to formulate (4.18) we introduce the following notation: given $j\in J$, let us denote by $a_j(1)$, $a_j(2)$, and $a_j(3)$ the three distinct elements in I'' such that the curves $\{\gamma_{a_j(\ell)}\}_{\ell=1}^3$ share p_j as a common boundary point (as described in Theorem 3.16), and let $\{a_j^k(\ell)\}_{\ell=1}^3\subset I''_k$ be defined analogously starting from p_j^k. We claim that, up to permutations in the index $\ell\in\{1,2,3\}$, one has

$$a_j^k(\ell)=\sigma_k(a_j(\ell))\,,\qquad\forall j\in J\,,k\ge k(\rho)\,,\ell\in\{1,2,3\}\,.\tag{4.18}$$

Indeed, by Theorem 3.16, up to decrease the value of $\eta>0$, we find that, for every $j\in J$,

$$\partial\mathcal{E}\cap B_{p_j,\eta}=\bigcup_{\ell=1}^3\gamma_{a_j(\ell)}\cap B_{p_j,\eta}\,,\qquad\{p_j\}=\Sigma(\mathcal{E})\cap B_{p_j,\eta}=\bigcup_{\ell=1}^3\mathrm{bd}\,(\gamma_{a_j(\ell)})\cap B_{p_j,\eta}\,.\tag{4.19}$$

Assuming without loss of generality that $\varepsilon(\rho)<\rho$ and by taking ρ_0 small enough with respect to η, we can entail by Theorem 3.10 and (4.8) that

$$\partial\mathcal{E}_k\subset I_{\varepsilon(\rho)}(\partial\mathcal{E})\,,\qquad \Sigma(\mathcal{E}_k)\cap B_{p_j,\eta}=\{p_j^k\}\subset B_{p_j,\varepsilon(\rho)}\,,\qquad \forall j\in J\,,k\ge k(\rho)\,. \tag{4.20}$$

By (4.19) and provided ρ_0 is small enough,

$$\begin{aligned} I_{\varepsilon(\rho)}(\partial\mathcal{E})\cap B_{p_j,\eta} &= \bigcup_{\ell=1}^3 I_{\varepsilon(\rho)}(\gamma_{a_j(\ell)})\cap B_{p_j,\eta} \\ &\subset B_{p_j,2\rho}\cup\bigcup_{\ell=1}^3\Big(N_{\varepsilon(\rho)}([\gamma_{a_j(\ell)}]_\rho)\cap B_{p_j,\eta}\Big)\,,\qquad \forall j\in J\,. \end{aligned}$$

By $\partial\mathcal{E}_k\subset I_{\varepsilon(\rho)}(\partial\mathcal{E})$ and by (4.9) one thus finds

$$\partial\mathcal{E}_k\cap B_{p_j,\eta}\subset\Big(\partial\mathcal{E}_k\cap B_{p_j,2\rho}\Big)\cup\bigcup_{\ell=1}^3\Big(\gamma^k_{\sigma_k(a_j(\ell))}\cap B_{p_j,\eta}\Big)\,. \tag{4.21}$$

Let now ω be the connected component of $\gamma^k_{a_j^k(1)}\cap\mathrm{cl}\,(B_{p_j,\eta})$ which contains p_j^k. In this way, ω is a connected $C^{1,1}$-curve with boundary, homeomorphic to $[0,1]$, with $p_j^k\in\mathrm{bd}\,(\omega)\cap B_{p_j,\eta}$. It cannot be $\omega\subset\subset B_{p_j,\eta}$, because otherwise it would be $\omega=\gamma^k_{a_j^k(1)}\subset\subset B_{p_j,\eta}$, and thus $\Sigma(\mathcal{E}_k)\cap B_{p_j,\eta}\setminus\{p_j^k\}\ne\emptyset$, against (4.20). Hence $\omega\cap\partial B_{p_j,\eta}\ne\emptyset$. At the same time, by (4.21),

$$\omega\cap B_{p_j,\eta}\subset\Big(\omega\cap B_{p_j,2\rho}\Big)\cup\bigcup_{\ell=1}^3\Big(\omega\cap\gamma^k_{\sigma_k(a_j(\ell))}\cap B_{p_j,\eta}\Big)\,,$$

and since ω is connected with $\omega\cap\partial B_{p_j,\eta}\ne\emptyset$, it must be $\omega\cap\gamma^k_{\sigma_k(a_j(\ell))}\ne\emptyset$ for some $\ell\in\{1,2,3\}$, thus $\gamma^k_{a_j^k(1)}\cap\gamma^k_{\sigma_k(a_j(\ell))}\ne\emptyset$. Up to relabeling $\ell\in\{1,2,3\}$, we have thus proved that

$$\gamma^k_{a_j^k(\ell)}\cap\gamma^k_{\sigma_k(a_j(\ell))}\ne\emptyset\,,\qquad\forall j\in J\,,k\ge k(\rho)\,,\ell\in\{1,2,3\}\,,$$

from which (4.18) follows by connectedness of the curves $\{\gamma_i^k\}_{i\in I}$. Having proved (4.18), we now introduce the notation needed to formulate (4.22): given $i\in I''$, let $b_i(1)$ and $b_i(2)$ denote the two distinct elements of J such that $\mathrm{bd}\,(\gamma_i)=\{b_i(1),b_i(2)\}$, and define similarly $b_i^k(m)$ $(m=1,2)$ for each $i\in I''_k$. Then, up to permutations in the index $m\in\{1,2\}$,

$$b^k_{\sigma_k(i)}(m)=b_i(m)\,,\qquad\forall i\in I''\,,k\ge k(\rho)\,,m=1,2\,. \tag{4.22}$$

Indeed, if $i\in I''$ then $i=a_{b_i(1)}(\ell)$ for some $\ell\in\{1,2,3\}$, therefore, by (4.18),

$$\sigma_k(i)=\sigma_k(a_{b_i(1)}(\ell))=a^k_{b_i(1)}(\ell)\,,$$

that is,

$$p^k_{b_i(1)}\in\mathrm{bd}\,(\gamma_{\sigma_k(i)})=\{p_{b^k_{\sigma_k(i)}(1)},p_{b^k_{\sigma_k(i)}(2)}\}\,,\qquad\text{thus}\qquad b_i(1)\in\{b^k_{\sigma_k(i)}(1),b^k_{\sigma_k(i)}(2)\}\,,$$

as required. With (4.18) and (4.22) in force, we now prove (4.17). The fact that $\sigma_k(I'')\subset I''_k$ is immediate from $I''=\{a_j(\ell):j\in J\,,\ell\in\{1,2,3\}\}$ and (4.18). If now $i,i'\in I''$ are such that $\sigma_k(i)=\sigma_k(i')$ then by (4.22)

$$\begin{aligned}\{j\in J:p_j\in\mathrm{bd}\,(\gamma_i)\} &= \{b_i(m)\}_{m=1}^2=\{b^k_{\sigma_k(i)}(m)\}_{m=1}^2=\{b^k_{\sigma_k(i')}(m)\}_{m=1}^2\\ &= \{b_{i'}(m)\}_{m=1}^2=\{j\in J:p_j\in\mathrm{bd}\,(\gamma_{i'})\}\,,\end{aligned}$$

so that $\mathrm{bd}\,(\gamma_i)=\mathrm{bd}\,(\gamma_{i'})$, and thus $i=i'$; this proves that σ_k is injective on I''. Finally, by Remark 3.17, it must be $\#\,I''=(3/2)\,\#\,J=(3/2)\,\#\,J_k=\#\,I''_k$, so that σ_k is actually a bijection between I'' and I''_k, and (4.17) is proved.

Let us now show that

$$\lim_{k\to\infty} \mathrm{hd}(\gamma_i,\gamma^k_{\sigma_k(i)}) = 0\,, \qquad \forall i\in I''\,. \tag{4.23}$$

We first notice that, by (4.22),

$$\{j\in J: p^k_j\in \mathrm{bd}\,(\gamma^k_{\sigma_k(i)})\} = \{b^k_{\sigma_k(i)}(m)\}_{m=1}^2 = \{b_i(m)\}_{m=1}^2 = \{j\in J: p_j\in\mathrm{bd}\,(\gamma_i)\}\,,$$

so that (4.8) gives

$$\lim_{k\to\infty} \mathrm{hd}(\mathrm{bd}\,(\gamma_i),\mathrm{bd}\,(\gamma^k_{\sigma_k(i)})) = 0\,, \qquad \forall i\in I''\,. \tag{4.24}$$

Next, if $i\in I''$, then by (4.13) one has $\gamma^k_{\sigma_k(i)}\cap I_{\varepsilon(\rho)}(\gamma_{i'})=\emptyset$ for every $i'\in I'$, while (4.9) gives $\gamma^k_{\sigma_k(i)}\cap N_{\varepsilon(\rho)}([\gamma_{i'}]_\rho)=\emptyset$ for every $i'\in I''\setminus\{i\}$; since $\partial\mathcal{E}_k\subset I_{\varepsilon(\rho)}(\partial\mathcal{E})$ for $k\ge k(\rho)$, we thus find

$$\gamma^k_{\sigma_k(i)}\subset I_{2\,\rho}(\gamma_i)\cup\bigcup_{i'\in I''} I_{2\rho}(\mathrm{bd}\,(\gamma_{i'}))\,, \qquad \forall i\in I''\,, k\ge k(\rho)\,.$$

Since $I_{2\,\rho}(\gamma_i)$ is disjoint from $\bigcup_{i'\in I''}I_{2\rho}(\mathrm{bd}\,(\gamma_{i'}))$ thanks to (4.19), we conclude that $\gamma^k_{\sigma_k(i)}\subset I_{2\,\rho}(\gamma_i)$ for every $i\in I''$ and $k\ge k(\rho)$. At the same time, by (4.9), (4.5), and (4.24)

$$[\gamma_i]_\rho\subset I_\rho(\gamma^k_{\sigma_k(i)})\,, \qquad I_\rho(\mathrm{bd}\,(\gamma_i))\subset I_{2\rho}(\gamma^k_{\sigma_k(i)})\,, \qquad \forall i\in I''\,, k\ge k(\rho)\,,$$

that is, $\gamma_i\subset I_{2\rho}(\gamma^k_{\sigma_k(i)})$ for every $i\in I''$ and $k\ge k(\rho)$. We have thus proved (4.23).

In order to complete the proof of (4.10) and (4.11) we are thus left to show that $\sigma_k(I')=I'_k$. We argue by contradiction, and assume the existence of $i_*\in I'_k\setminus\sigma_k(I')$. Since $I_{\varepsilon(\rho)}(\gamma_i)\cap\partial\mathcal{E}_k=\gamma^k_{\sigma_k(i)}$ for every $i\in I'$ (recall (4.14)), by connectedness we deduce that

$$\gamma^k_{i_*}\cap\bigcup_{i\in I'} I_{\varepsilon(\rho)}(\gamma_i)=\emptyset\,. \tag{4.25}$$

Since $N_{\varepsilon(\rho)}([\gamma_i]_\rho)\cap\partial\mathcal{E}_k = N_{\varepsilon(\rho)}([\gamma_i]_\rho)\cap\gamma^k_{\sigma_k(i)}$ for every $i\in I$ (recall (4.9)), if $\gamma^k_{i_*}\cap N_{\varepsilon(\rho)}([\gamma_i]_\rho)\ne\emptyset$, then, by connectedness of $\gamma^k_{\sigma_k(i)}$, one finds $i_*=\sigma_k(i)\in\sigma_k(I)$, a contradiction: hence,

$$\gamma^k_{i_*}\cap\bigcup_{i\in I''} N_{\varepsilon(\rho)}(\gamma_i)=\emptyset\,. \tag{4.26}$$

Since $\partial\mathcal{E}_k\subset I_{\varepsilon(\rho)}(\partial\mathcal{E})=\bigcup_{i\in I}I_{\varepsilon(\rho)}(\gamma_i)$, by (4.25) and (4.26) we find

$$\gamma^k_{i_*}\subset\bigcup_{i\in I''}I_{\varepsilon(\rho)}(\gamma_i)\setminus\bigcup_{i\in I''}N_{\varepsilon(\rho)}(\gamma_i)\subset\bigcup_{j\in J}B_{p_j,\eta}\,,$$

and since the balls $\{B_{p_j,\eta}\}_{j\in J}$ are disjoint by (4.19), we conclude that for every $i_*\in I'_k\setminus\sigma_k(I')$ there exists a unique $j\in J$ such that $\gamma^k_{i_*}\subset B_{p_j,2\rho}$; however, by Theorem 3.16,

$$\frac{1}{\Lambda}\le\mathrm{diam}(\gamma^k_{i_*})<2\rho\,,$$

which leads to a contradiction if ρ_0 is sufficiently small.

Step three: We prove (4.2) by exploiting Proposition B.2 in Appendix B. We directly consider the case when $\mathrm{bd}\,(\gamma^k_i)\ne\emptyset$, and omit the (analogous) details for the case $\mathrm{bd}\,(\gamma^k_i)=\emptyset$. Let us set $\ell^k_i=\mathcal{H}^1(\gamma^k_i)$, consider $\alpha^k_i\in C^{1,1}([0,\ell^k_i];\mathbb{R}^2)$ to be an arc-length parametrization of γ^k_i, and define unit normal vector fields $\nu^k_i\in C^{0,1}(\gamma_k;\mathbb{S}^1)$ by setting $\nu^k_i(\alpha^k_i(t))=(\alpha^k_i)'(t)^\perp$, with the convention that $v^\perp=(v_2,-v_1)$ for every $v=(v_1,v_2)\in\mathbb{R}^2$. We claim that

$$|\nu^k_i(x)\cdot(y-x)|\le C\,|x-y|^2\,, \qquad |\nu^k_i(x)-\nu^k_i(y)|\le C\,|x-y|\,, \qquad \forall x,y\in\gamma^k_i\,. \tag{4.27}$$

Indeed, if $x,y\in\gamma^k_i$ with $s,t\in[0,\ell^k_i]$ such that $x=\alpha^k_i(s)$ and $y=\alpha^k_i(t)$, then, by $\mathrm{Lip}\,((\alpha^k_i)')\le\Lambda$,

$$|\nu^k_i(x)\cdot(y-x)|\le C\,|s-t|^2\,, \qquad |\nu^k_i(x)-\nu^k_i(y)|\le C\,|s-t|\,;$$

we are thus left to show that

$$|s-t|\le C\,|\alpha_i^k(s)-\alpha_i^k(t)|\,,\qquad \forall s,t\in[0,\ell_i^k]\,. \tag{4.28}$$

If $|s-t|\le 1/\Lambda$, then (4.28) follows with $C\ge 2$ by noticing that

$$|\alpha_i^k(s)-\alpha_i^k(t)|=\Big|\int_t^s(\alpha_i^k)'(r)\,dr\Big|\ge|t-s|-\Lambda\,\frac{|t-s|^2}{2}\,,$$

once again thanks to $\mathrm{Lip}\,((\alpha_i^k)')\le\Lambda$. If $\gamma_i[x,y]$ denote the arc of γ_i with end-points $x,y\in\gamma_i$, then by compactness

$$\min_{i\in I}\,\inf\Big\{|x-y|:x,y\in\gamma_i\,,\mathcal{H}^1(\gamma_i[x,y])\ge\frac{1}{2\Lambda}\Big\}\ge c\,,$$

where $c>0$ depends on $\mathcal{E}$ and Λ only. Since for every $i\in I$ we have $\mathrm{hd}(\gamma_i^k,\gamma_i)\to 0$ as $k\to\infty$, we can thus entail

$$\min_{i\in I}\,\inf\Big\{|x-y|:x=\alpha_i^k(s)\,,y=\alpha_i^k(t)\,,|s-t|\ge\frac{1}{\Lambda}\Big\}\ge\frac{c}{2}\,,$$

so that (4.28) holds on $|s-t|>1/\Lambda$ provided $C\ge 2\Lambda/c$. This completes the proof of (4.28), thus of (4.27). By (4.27) we can apply Proposition B.2 to deduce (4.2).

Step four: We prove statement (ii). Let us fix $j\in J$, and consider $p_j^k\in\Sigma(\mathcal{E}_k)$ and $i_1,i_2,i_3\in I$ such that $\{p_j^k\}=\mathrm{bd}\,(\gamma_{i_1}^k)\cap\mathrm{bd}\,(\gamma_{i_2}^k)\cap\mathrm{bd}\,(\gamma_{i_3}^k)$. Since each γ_i^k is a compact connected $C^{1,1}$-curve with distributional curvature bounded by Λ one finds that, for every $i=i_1,i_2,i_3$,

$$\lim_{r\to 0^+}\sup_{k\in\mathbb{N}}\mathrm{hd}_B\Big(\frac{\gamma_i^k-p_j^k}{r},\mathbb{R}_+\,[\tau_i^k(p_j^k)]\Big)=0\,, \tag{4.29}$$

where we have set $\mathbb{R}_+[\tau]=\{t\,\tau:t\ge 0\}$ for every $\tau\in\mathbb{S}^1$. We thus find

$$\begin{aligned}\mathrm{hd}_B\Big(\mathbb{R}_+\,[\tau_i(p_j)],\mathbb{R}_+\,[\tau_i^k(p_j^k)]\Big)\ &\le\ \sup_{k\in\mathbb{N}}\mathrm{hd}_B\Big(\frac{\gamma_i^k-p_j^k}{r},\mathbb{R}_+\,[\tau_i^k(p_j^k)]\Big)\\ &\quad+\mathrm{hd}_B\Big(\frac{\gamma_i-p_j}{r},\mathbb{R}_+\,[\tau_i(p_j)]\Big)+2\,\frac{\mathrm{hd}(\gamma_i^k,\gamma_i+(p_j^k-p_j))}{r}\,,\end{aligned} \tag{4.30}$$

where we have also used the fact that, for k large enough,

$$\mathrm{hd}_B\Big(\frac{\gamma_i^k-p_j^k}{r},\frac{\gamma_i-p_j}{r}\Big)\le 2\,\frac{\mathrm{hd}(\gamma_i^k,\gamma_i+(p_j^k-p_j))}{r}\,.$$

Let first $k\to\infty$ and then $r\to 0^+$ in (4.30). By exploiting (4.1) and (4.29), this gives $\mathrm{hd}_B(\mathbb{R}_+\,[\tau_i(p_j)],\mathbb{R}_+\,[\tau_i^k(p_j^k)])\to 0$ as $k\to\infty$, that is (4.3). □

Proof of Theorem 1.5. Let $\mathcal{E}$ be a $C^{2,1}$-cluster in $\mathbb{R}^2$, $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ be a sequence of (Λ,r_0)-minimizing clusters such that $\mathrm{d}(\mathcal{E}_k,\mathcal{E})\to 0$ as $k\to\infty$, and let k_0 and ρ_0 be the constants given by Theorem 4.1. Denote by μ_0 and C_0 two positive constants depending on Λ and $\mathcal{E}$ only, and let $\mu<\mu_0$ be fixed. We want to find $k(\mu)\in\mathbb{N}$ such that for every $k\ge k(\mu)$ there exist a $C^{1,1}$-diffeomorphism f_k between $\partial\mathcal{E}$ and $\partial\mathcal{E}_k$ with

$$\|f_k\|_{C^{1,1}(\partial\mathcal{E})}\ \le\ C_0\,, \tag{4.31}$$

$$\lim_{k\to\infty}\|f_k-\mathrm{Id}\|_{C^1(\partial\mathcal{E})}\ =\ 0\,, \tag{4.32}$$

$$\|\boldsymbol{\tau}_{\mathcal{E}}(f_k-\mathrm{Id})\|_{C^1(\partial^*\mathcal{E})}\ \le\ \frac{C_0}{\mu}\,\|f_k-\mathrm{Id}\|_{C^0(\Sigma(\mathcal{E}))}\,, \tag{4.33}$$

$$\boldsymbol{\tau}_{\mathcal{E}}(f_k-\mathrm{Id})\ =\ 0\,,\quad\text{on }[\partial\mathcal{E}]_\mu\,. \tag{4.34}$$

Let us fix $i\in I$ such that $\mathrm{bd}\,(\gamma_i)\ne\emptyset$. If $\mu_0^2<\rho_0$, then we can apply Theorem 4.1 to $\mathcal{E}$ and $\mathcal{E}_k$ and any $\rho\in(0,\mu^2)$. As a consequence we can apply Theorem 2.6 and prove the existence of

$k(\mu) \geq k_0$, $\{k_*(\rho)\}_{\rho<\mu^2} \subset \mathbb{N}$, and of maps $\{f_i^k\}_{k\geq k(\mu)}$ such that for every $k \geq k(\mu)$ one has that f_i^k is a $C^{1,1}$-diffeomorphism between γ_i and γ_i^k with $f_i^k(p_j) = p_j^k$, $f_i^k(p_{j'}) = p_{j'}^k$ (j and j' as in statement (ii) of Theorem 4.1) and

$$\|f_i^k\|_{C^{1,1}(\gamma_i)} \leq C_0\,, \tag{4.35}$$

$$\|(f_i^k - \mathrm{Id})\cdot\tau_i\|_{C^1(\gamma_i)} \leq C_0\,\frac{\|f_i^k - \mathrm{Id}\|_{C^0(\mathrm{bd}\,(\gamma_i))}}{\mu}\,, \tag{4.36}$$

$$(f_i^k - \mathrm{Id})\cdot\tau_i = 0 \qquad \text{on } [\gamma_i]_\mu\,; \tag{4.37}$$

moreover, if $k \geq k_*(\rho)$, then

$$\sup_{k\geq k_*(\rho)} \|f_i^k - \mathrm{Id}\|_{C^1(\gamma_i)} \leq C_0\,\frac{\rho}{\mu}\,, \tag{4.38}$$

which of course implies

$$\lim_{k\to\infty} \|f_i^k - \mathrm{Id}\|_{C^1(\gamma_i)} = 0\,. \tag{4.39}$$

Let us now fix $i \in I$ such that $\mathrm{bd}\,(\gamma_i) = \emptyset$. Up to further decrease μ_0, γ_i is a connected component of $[\partial\mathcal{E}]_\mu$, and thus by statement (iii) in Theorem 4.1, $\{\psi_k\}_{k\geq k(\rho)} \subset C^{1,1}([\partial\mathcal{E}_0]_\rho)$ are such that

$$\gamma_i^k = (\mathrm{Id} + \psi_k\nu)(\gamma_i)\,, \qquad \lim_{k\to\infty} \|\psi_k\|_{C^1(\gamma_i)} = 0\,, \qquad \sup_{k\in\mathbb{N}} \|\psi_k\|_{C^{1,1}(\gamma_i)} \leq C_0\,. \tag{4.40}$$

We set $f_i^k = \mathrm{Id} + \psi_k\,\nu$ for every $i \in I$ such that $\mathrm{bd}\,(\gamma_i) = \emptyset$, and finally define $f_k(x) = f_i^k(x)$ for $x \in \gamma_i$. The resulting map f_k defines a $C^{1,1}$-diffeomorphism between $\partial\mathcal{E}$ and $\partial\mathcal{E}_k$ (see Definition 1.3) with (4.31)–(4.34) in force. $\square$

5. Some applications of the improved convergence theorem

We now prove Theorem 1.8 and Theorem 1.9. To this end, let us notice that if $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ is a sequence of planar isoperimetric clusters with $\sup_{k\in\mathbb{N}} P(\mathcal{E}_k) < \infty$, then there exist $x_k \in \mathbb{R}^2$ and a planar N-cluster $\mathcal{E}_0$ such that, up to extracting subsequences, $x_k + \mathcal{E}_k \to \mathcal{E}_0$. This is a simple consequence of (i) the inequality $2\,\mathrm{diam}(E) \leq P(E)$, which holds for every indecomposable set of finite perimeter E in $\mathbb{R}^2$ (this, of course, after the normalization (3.5)); (ii) the fact that $\mathbb{R}^2 \setminus \mathcal{E}(0)$ is indecomposable whenever $\mathcal{E}$ is an isoperimetric cluster (as it can be easily inferred by arguing as in Remark 3.18).

Proof of Theorem 1.8. We argue by contradiction, and assume that there exists a sequence $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ of isoperimetric N-clusters with $\mathrm{vol}\,(\mathcal{E}_k) \to m_0$ such that $[\mathcal{E}_k]_\approx \neq [\mathcal{E}_j]_\approx$ whenever $k \neq j$. Let $\phi : \mathbb{R}^N_+ \to (0,\infty)$ denote the infimum in (1.15), then it is easily seen that ϕ is locally bounded. In particular, $\sup_{k\in\mathbb{N}} P(\mathcal{E}_k) < \infty$, and thus there exists a N-cluster $\mathcal{E}_0$ and $x_k \in \mathbb{R}^2$ such that, up to extracting subsequences, $x_k + \mathcal{E}_k \to \mathcal{E}_0$ as $k \to \infty$. We claim that, for k large enough, $x_k + \mathcal{E}_k$ is a (Λ, r_0)-minimizing cluster in $\mathbb{R}^2$, where Λ and r_0 are independent from k. To this end, let ε_0, r_0, and C_0 be the constants associated with $\mathcal{E}_0$ by Theorem C.1 and let k_0 be such that $\mathrm{d}(x_k + \mathcal{E}_k, \mathcal{E}_0) < \varepsilon_0$ for $k \geq k_0$. Given $\mathcal{F}$ with $\mathcal{F}(h)\Delta(x_k + \mathcal{E}_k(h)) \subset\subset B_{x,r_0}$ for $h = 1, ..., N$, by applying Theorem C.1 with $\mathcal{E} = x_k + \mathcal{E}_k$ we find $\mathcal{F}_k'$ such that

$$\mathrm{vol}\,(\mathcal{F}_k') = \mathrm{vol}\,(x_k + \mathcal{E}_k) = \mathrm{vol}\,(\mathcal{E}_k)\,, \qquad P(\mathcal{F}_k') \leq P(\mathcal{F}) + C_0\,\mathrm{d}(x_k + \mathcal{E}_k, \mathcal{F})\,.$$

so that, by the isoperimetric property of $\mathcal{E}_k$, $P(x_k + \mathcal{E}_k) \leq P(\mathcal{F}_k') \leq P(\mathcal{F}) + C_0\,\mathrm{d}(x_k + \mathcal{E}_k, \mathcal{F})$. Thus $x_k + \mathcal{E}_k$ is a (Λ, r_0)-minimizing cluster in $\mathbb{R}^2$ for k large enough. By Theorem 3.10 we infer that $\mathcal{E}$ is also a (Λ, r_0)-minimizing cluster in $\mathbb{R}^2$, and thus conclude by Theorem 1.5 that $x_k + \mathcal{E}_k \approx \mathcal{E}$ for k large enough. Since $x_k + \mathcal{E}_k \approx \mathcal{E}_k$, we have found a contradiction to $[\mathcal{E}_k]_\approx \neq [\mathcal{E}_j]_\approx$ for $k \neq j$. $\square$

Proof of Theorem 1.9. Step one: We first prove that, if $\mathcal{E}$ is a minimizer in (1.22) with $\delta \in (0,\delta_0)$ and $|m-m_0|<\delta_0$, then $\mathcal{E}\approx\mathcal{E}_0$. We argue by contradiction, and consider a sequence $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ of minimizers in

$$\lambda_k = \inf\Big\{P(\mathcal{E})+\delta_k\sum_{h=1}^N\int_{\mathcal{E}(h)} g(x)\,dx : \mathrm{vol}\,(\mathcal{E})=m_k\Big\}\,,\qquad k\in\mathbb{N}\,, \tag{5.1}$$

where $\delta_k\to 0$ and $m_k\to m_0$ as $k\to\infty$, and $[\mathcal{E}_k]_\approx\neq[\mathcal{E}_0]_\approx$ for every $k\in\mathbb{N}$. Let $\{\mathcal{F}_k\}_{k\in\mathbb{N}}$ be a sequence of isoperimetric clusters with $\mathrm{vol}\,(\mathcal{F}_k)=m_k$. Since $m_k\to m_0$ implies $\sup_{k\in\mathbb{N}}P(\mathcal{F}_k)<\infty$, by the argument presented at the beginning of this section there exists $R>0$ such that, up to translations, $\mathcal{F}_k(h)\subset\subset B_R$ for every $h=1,...,N$ and $k\in\mathbb{N}$. By comparing $\mathcal{E}_k$ and $\mathcal{F}_k$ in (5.1) we find

$$P(\mathcal{E}_k)+\delta_k\sum_{h=1}^N\int_{\mathcal{E}_k(h)} g\le P(\mathcal{F}_k)+\delta_k\sum_{h=1}^N\int_{\mathcal{F}_k(h)} g\le P(\mathcal{F}_k)+\delta_k\,|m_k|\,\sup_{B_R} g \tag{5.2}$$

and since $P(\mathcal{F}_k)\le P(\mathcal{E}_k)$ we thus find that for every $r>0$

$$\inf_{\mathbb{R}^2\setminus B_r} g\sum_{h=1}^N|\mathcal{E}_k(h)\setminus B_r|\le|m_k|\,\sup_{B_R} g\,.$$

By $g(x)\to\infty$ as $|x|\to\infty$, we conclude that

$$\lim_{r\to\infty}\sup_{k\in\mathbb{N}}\sum_{h=1}^N|\mathcal{E}_k(h)\setminus B_r|=0\,. \tag{5.3}$$

Since (5.2) also implies $\sup_{k\in\mathbb{N}}P(\mathcal{E}_k)<\infty$, by (5.3) we conclude that up to extracting subsequences, $\mathrm{d}(\mathcal{E}_k,\mathcal{E})\to 0$ as $k\to\infty$, where $\mathcal{E}$ is a planar cluster with $\mathrm{vol}\,(\mathcal{E})=m_0$. In particular, recalling that $\mathcal{E}_0$ denotes the unique isoperimetric cluster with $\mathrm{vol}\,(\mathcal{E}_0)=m_0$, we have

$$P(\mathcal{E}_0)\le P(\mathcal{E})\le\liminf_{k\to\infty}P(\mathcal{E}_k)\,. \tag{5.4}$$

Now, by [Mag12, Theorem 29.14] there exist positive constants ε, η and C, a smooth map $\Phi\in C^1((-\eta,\eta)^N\times\mathbb{R}^2;\mathbb{R}^2)$, and a disjoint family of balls $\{B_{z_i,\varepsilon}\}_{i=1}^M$ such that, for every $v\in(-\eta,\eta)^N$, the N-cluster defined by $\mathcal{E}_{0,v}(h):=\Phi(v,\mathcal{E}_0(h))$, $h=1,...,N$, satisfies

$$\mathcal{E}_{0,v}(h)\Delta\mathcal{E}_0(h)\subset\subset A=\bigcup_{i=1}^M B_{z_i,\varepsilon}\,,\quad P(\mathcal{E}_{0,v})\le P(\mathcal{E}_0)+C\,|v|\,,\quad \mathrm{vol}\,(\mathcal{E}_{0,v})=\mathrm{vol}\,(\mathcal{E}_0)+v\,.$$

For k large, $v_k=\mathrm{vol}\,(\mathcal{E}_k)-\mathrm{vol}\,(\mathcal{E}_0)\in(-\eta,\eta)^N$, so that $\mathrm{vol}\,(\mathcal{E}_{0,v_k})=m_k$ and, by $g\ge 0$

$$P(\mathcal{E}_k)+\delta_k\sum_{h=1}^N\int_{\mathcal{E}_k(h)} g\le P(\mathcal{E}_{0,v_k})+\delta_k\,\sum_{h=1}^N\int_{\mathcal{E}_{0,v_k}(h)} g\le P(\mathcal{E}_0)+C\,|v_k|+\delta_k\,\sup_{B_{2S}} g$$

where S is such that $\bigcup_{h=1}^N\mathcal{E}_0(h)\cup A\subset\subset B_S$. Letting $k\to\infty$ we find that

$$\limsup_{k\to\infty}P(\mathcal{E}_k)\le P(\mathcal{E}_0)\,,$$

so that, by (5.4), $P(\mathcal{E})=P(\mathcal{E}_0)$. Since $\mathrm{vol}\,(\mathcal{E})=m_0$, we find $\mathcal{E}\approx\mathcal{E}_0$ (through an isometry), and we may thus assume, without loss of generality, that $\mathcal{E}=\mathcal{E}_0$. By arguing as in the previous proof (with some minor modification because of the presence of the potential), we see that, for k large enough, $\mathcal{E}_k$ is a (Λ,r_0)-minimizer with Λ and r_0 uniform in k. Since $\mathrm{d}(\mathcal{E}_k,\mathcal{E}_0)\to 0$ as $k\to\infty$, by Theorem 1.5 we find that $\mathcal{E}_k\approx\mathcal{E}_0$ for k large enough, a contradiction.

Step two: The argument of step one can be easily adapted to show the existence of minimizers in (1.22), together with the existence of R_0 (depending on $\mathcal{E}_0$, δ_0 and g only) such that $\mathcal{E}(h)\subset B_{R_0}$

for every $h = 1, ..., N$ and every minimizer $\mathcal{E}$. In particular, there exists C_0 depending on g and R_0 only such that

$$P(\mathcal{E}) \le P(\mathcal{F}) + C_0\,\delta\,\mathrm{d}(\mathcal{E},\mathcal{F})\,, \tag{5.5}$$

whenever $\mathrm{vol}\,(\mathcal{E}) = \mathrm{vol}\,(\mathcal{F})$ and $\mathcal{F}(h) \subset B_{2R_0}$. Let us fix $x_1, x_2 \in \mathcal{E}(h,k)$, $T_i \in C^1_c(B_{x_i,r};\mathbb{R}^n)$ $(i = 1,2)$ with $|\mathcal{E}(j) \cap B_{x_i,r}| = 0$ if $i \ne h,k$ and $r < |x_1 - x_2|$, and with

$$\int_{\partial^*\mathcal{E}(h)} T_i \cdot \nu_{\mathcal{E}(h)}\,d\mathcal{H}^{n-1} = \eta_i > 0\,, \qquad \sup_{\mathbb{R}^n} |T_i| \le 1\,.$$

By a standard argument we can construct a one-parameter family of diffeomorphisms f_t with $f_t(x) = x + t\,(T_1(x) - (\eta_1/\eta_2)T_2(x)) + O(t^2)$ such that $\mathrm{vol}\,(f_t(\mathcal{E})) = \mathrm{vol}\,(\mathcal{E})$. For t small enough $\mathcal{F} = f_t(\mathcal{E})$ is admissible in (5.5), with

$$\mathrm{d}(\mathcal{E}, f_t(\mathcal{E})) \le 2|f_t(\mathcal{E}(h))\Delta\mathcal{E}(h)| \le 2\,P(\mathcal{E}(h); B_{x_1,r} \cup B_{x_2,r})\,|t|\,,$$

by Lemma C.2. Since

$$P(f_t(\mathcal{E})) = P(\mathcal{E}) + t\int_{\partial^*\mathcal{E}(h)} (T_1 - (\eta_1/\eta_2)T_2)\cdot\nu_{\mathcal{E}(h)}\,H_{\mathcal{E}(h,k)} + O(t^2)\,,$$

and $P(\mathcal{E}(h); B_{x_1,s} \cup B_{x_2,s}) = \omega_{n-1}\,s^{n-1}(1 + O(1))$ as $s \to 0^+$, by (5.5) we conclude that

$$\int_{\partial^*\mathcal{E}(h)} (T_1 - (\eta_1/\eta_2)T_2)\cdot\nu_{\mathcal{E}(h)}\,H_{\mathcal{E}(h,k)} \le 2\,C_0\,\delta\,\omega_{n-1}r^{n-1}(1 + O(1))\,.$$

Let now $T_i = T_i^j \to 1_{B_{x_i,r}}\nu_{\mathcal{E}(h)}$ in $L^1(\mathcal{H}^1 \llcorner \partial\mathcal{E}(h))$ as $j \to \infty$, so that

$$\int_{B_{x_1,r}\cap\partial^*\mathcal{E}(h)} H_{\mathcal{E}(h,k)} - \frac{\eta_1}{\eta_2}\int_{B_{x_1,r}\cap\partial^*\mathcal{E}(h)} H_{\mathcal{E}(h,k)} \le 2\,C_0\,\delta\,\omega_{n-1}r^{n-1}(1 + O(1))\,.$$

By the mean value theorem, as $r \to 0^+$, we find that $H_{\mathcal{E}(h,k)}(x_1) - H_{\mathcal{E}(h,k)}(x_2) \le 2\,C_0\,\delta$, that is,

$$\max_{0\le h<k\le N} \|H_{\mathcal{E}(h,k)} - H^\delta_{h,k}\|_{C^0(H_{\mathcal{E}(h,k)})} \le C\,\delta\,,$$

for some $H^\delta_{h,k} \in \mathbb{R}$. At the same time, by arguing for example as in [CL12, Lemma 3.7(ii)], one see that $H_{\mathcal{E}(h,k)}$ has to converge in the sense of distributions to $H_{\mathcal{E}_0(h,k)}$ as $\delta \to 0^+$, and thus prove (1.23). □

Appendix A. Proof of Theorem 2.1 and Theorem 2.3

Proof of Theorem 2.1. In the following, we denote by C a generic constant depending on α, L, M and k only. Let us set $\lambda_{\min}, \lambda_{\max} : S_0 \to \mathbb{R}$ as $\lambda_{\min}(x) = \inf\{|\nabla^{S_0} f(x)v| : v \in T_xS_0, |v| = 1\}$ and $\lambda_{\max}(x) = \|\nabla^{S_0} f(x)\|$, and then exploit (2.6) to find that

$$\frac{1}{L} \le J^{S_0} f(x) \le \lambda_{\min}(x)\,\lambda_{\max}(x)^{k-1} \le \lambda_{\min}(x)\,L^{k-1}\,,$$

that is $\lambda_{\min}(x) \ge L^{-k}$ for every $x \in S_0$. In particular, by also using (2.4) we find that

$$|\nabla^{S_0} f(x)(y - x)| = |\nabla^{S_0} f(x)\pi^0_x(y - x)| \ge \frac{|\pi^0_x(y - x)|}{L^k} \ge \frac{|y - x|}{2L^k}\,, \qquad \forall y \in B_{x,1/M} \cap S_0\,. \tag{A.1}$$

We now assume $\varepsilon_0 < 1/M$ and fix $y \in B_{x,\varepsilon_0} \cap S_0 \setminus \{x\}$. Since $\mathrm{dist}_{S_0}(x,y) > 0$ we can find $\gamma \in C^1([0,1]; S_0)$ such that $\gamma(0) = x$, $\gamma(1) = y$ and

$$\mathrm{dist}_{S_0}(x,y) \le \int_0^1 |\dot\gamma(t)|\,dt \le 2\,\mathrm{dist}_{S_0}(x,y)\,. \tag{A.2}$$

By (A.1),

$$
\begin{aligned}
|f(y)-f(x)| &= \left|\int_0^1 \nabla^{S_0} f(\gamma(t))\dot\gamma(t)\,dt\right| \\
&= \left|\nabla^{S_0} f(x)(y-x) - \int_0^1 (\nabla^{S_0} f(\gamma(t)) - \nabla^{S_0} f(x))\dot\gamma(t)\,dt\right| \\
&\geq \frac{|y-x|}{2L^k} - \int_0^1 \|\nabla^{S_0} f(\gamma(t)) - \nabla^{S_0} f(x)\||\dot\gamma(t)|\,dt\,.
\end{aligned}
$$

By (2.6), (A.2), and (2.3)

$$
\begin{aligned}
\int_0^1 \|\nabla^{S_0} f(\gamma(t)) - \nabla^{S_0} f(x)\||\dot\gamma(t)|\,dt &\leq L\int_0^1 |x-\gamma(t)|^\alpha\,|\dot\gamma(t)|\,dt \\
&\leq L\int_0^1 \Big(\int_0^t |\dot\gamma(s)|\,ds\Big)^\alpha |\dot\gamma(t)|\,dt \\
&\leq L\,2^{1+\alpha}\,\mathrm{dist}_{S_0}(x,y)^{1+\alpha} \leq L\,(2M)^{1+\alpha}\,|x-y|^{1+\alpha}\,.
\end{aligned}
$$

We thus conclude (up to further decrease the value of ε_0) that if $x\in S_0$ and $y\in B_{x,\varepsilon_0}\cap S_0$, then

$$
|f(x)-f(y)| \geq |y-x|\left(\frac{1}{2L^k} - L\,(2M)^{1+\alpha}\varepsilon_0^\alpha\right) \geq \frac{|y-x|}{4L^k}\,. \tag{A.3}
$$

This shows that f is injective on $B_{x,\varepsilon_0}\cap S_0$ for every $x\in S_0$. If now (2.7) is in force, then we notice that (provided $\rho\leq\varepsilon_0/4$ and by $\mathrm{diam}(S_0)\leq M$) for every $x,y\in S_0$ with $|x-y|\geq\varepsilon_0$ one trivially has

$$
|f(x)-f(y)| \geq |x-y| - |f(x)-x| - |f(y)-y| \geq \varepsilon_0 - 2\,\rho_0 \geq \frac{\varepsilon_0}{2} \geq \frac{\varepsilon_0}{2\,M}\,|x-y|\,.
$$

We have proved that if (2.7) holds true, then f is injective on S_0 with

$$
|f^{-1}(p_1)-f^{-1}(p_2)| \leq C\,|p_1-p_2|\,, \qquad \forall p_1,p_2\in S=f(S_0)\,. \tag{A.4}
$$

We are thus left to prove that

$$
\|\nabla^S f^{-1}(p_1) - \nabla^S f^{-1}(p_2)\| \leq C\,|p_1-p_2|^\alpha\,, \qquad \forall p_1,p_2\in S\,. \tag{A.5}
$$

Indeed, if π_p denotes the projection of $\mathbb{R}^n$ onto T_pS, then by (2.5) we can entail

$$
\|\pi_p-\pi_q\| \leq C\,|p-q|^\alpha\,, \qquad \forall p,q\in S\,. \tag{A.6}
$$

Let us now fix $p_1,p_2\in S$ and set

$$
M_i=\nabla^S f^{-1}(p_i)\,,\quad \pi_i=\pi_{p_i}\,,\quad x_i=f^{-1}(p_i)\,,\quad N_i=\nabla^{S_0}f(x_i)\,,\quad \pi_i^0=\pi_{x_i}^0\,.
$$

By exploiting the relations

$$
\pi_i^0 M_i = M_i = M_i\pi_i\,, \qquad \pi_i N_i = N_i = N_i\pi_i^0\,, \tag{A.7}
$$

$$
N_1M_1\pi_1=\pi_1\,, \qquad N_2M_2\pi_2=\pi_2\,, \qquad M_1N_1\pi_1^0=\pi_1^0\,, \qquad M_2N_2\pi_2^0=\pi_2^0\,, \tag{A.8}
$$

one finds that

$$
\begin{aligned}
&M_1(N_2-N_1)M_2 + M_2(N_2-N_1)M_1 \\
=\;& M_1N_2M_2 - M_1N_1M_2 + M_2N_2M_1 - M_2N_1M_1 \\
=\;& M_1N_2M_2\pi_2 - M_1N_1\pi_1^0M_2 + M_2N_2\pi_2^0M_1 - M_2N_1M_1\pi_1 \\
=\;& M_1\pi_2 - \pi_1^0M_2 + \pi_2^0M_1 - M_2\pi_1 \\
=\;& 2(M_1-M_2) + (M_1+M_2)(\pi_2-\pi_1) + (\pi_2^0-\pi_1^0)(M_1+M_2)\,.
\end{aligned}
$$

By (2.5) and (A.6), and since $\|M_i\|\le C$ by (A.4), we thus find

$$\begin{aligned}2\|M_2-M_1\| &\le 2\,\|M_1\|\|M_2\|\|N_2-N_1\|+\|M_1+M_2\|\Big(\|\pi_2-\pi_1\|+\|\pi_2^0-\pi_1^0\|\Big)\\ &\le C\,\Big(\|N_2-N_1\|+|p_2-p_1|^\alpha+|x_2-x_1|^\alpha\Big)\\ &\le C\,\Big((1+L)\,|x_2-x_1|^\alpha+|p_2-p_1|^\alpha\Big)\le C\,|p_2-p_1|^\alpha\,,\end{aligned}$$

where in the last line we have first used $[\nabla^{S_0}f]_{C^{0,\alpha}(S_0)}\le L$ and then (A.4). This completes the proof of (A.5), thus of the theorem. □

Appendix B. Whitney's extension theorem

We quickly review here some basic fact concerning Whitney's extension theorem. Let $\mathbf{k}=(k_1,...,k_n)$ denote the generic element of $\mathbb{N}^n$, and set

$$|\mathbf{k}|=\sum_{i=1}^n k_i\,,\qquad \mathbf{k}!=\prod_{i=1}^n k_i\,,\qquad z^{\mathbf{k}}=\prod_{i=1}^n z_i^{k_i}\,,$$

for every $\mathbf{k}\in\mathbb{N}^n$ and $z\in\mathbb{R}^n$. If f is $|\mathbf{k}|$-times differentiable at $x\in\mathbb{R}^n$, we let

$$D^{\mathbf{k}}\,f(x)=\frac{\partial^{|\mathbf{k}|}f}{\partial x_1^{k_1}...\partial x_n^{k_n}}(x)=\frac{\partial^{|\mathbf{k}|}f}{\partial x^{\mathbf{k}}}(x)\,,$$

denote the $\mathbf{k}$-partial derivative of f, with the convention that $D^{\mathbf{0}}f=f$ (here, $\mathbf{0}=(0,...,0)\in\mathbb{N}^n$).

A *jet of order h on X* is a family of continuous functions $\mathcal{F}=\{F^{\mathbf{k}}\}_{|\mathbf{k}|\le h}$ on X. We denote by $J^h(X)$ the vector space of jets of order h on X, and set

$$\|\mathcal{F}\|_{J^h(X)}=\max_{|\mathbf{k}|\le h}\|F^{\mathbf{k}}\|_{C^0(X)}\,.$$

A jet of infinite order on X is just a family of continuous functions $\mathcal{F}=\{F^{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}^n}$ on X, and in this case we set $\mathcal{F}\in J^\infty(X)$. One says that $\mathcal{F}\in J^h(X)$ is a *Whitney's jet of order h on X* if, for every $|\mathbf{k}|\le h$,

$$\sup_{x,y\in X\,,0<|x-y|<r}\Big|F^{\mathbf{k}}(y)-F^{\mathbf{k}}(x)-\sum_{|j|=1}^{h-|\mathbf{k}|}F^{\mathbf{k}+\mathbf{j}}(x)(y-x)^{\mathbf{k}+\mathbf{j}}\Big|=o(r^{h-|\mathbf{k}|})\,.$$

We denote by $WJ^h(X)$ the space of Whitney's jets of order h on X, and set

$$\|\mathcal{F}\|_{WJ^h(X)}=\max_{|\mathbf{k}|\le h}\|F^{\mathbf{k}}\|_{C^0(X)}+\max_{|\mathbf{k}|\le h}\sup_{x\,,y\in X\,,x\ne y}\frac{|F^{\mathbf{k}}(y)-F^{\mathbf{k}}(x)-\sum_{|j|=1}^{h-|\mathbf{k}|}F^{\mathbf{k}+\mathbf{j}}(x)(y-x)^{\mathbf{k}+\mathbf{j}}|}{|x-y|^{h-|\mathbf{k}|}}\,.$$

Finally, given $\alpha\in(0,1]$ we define $WJ^{h,\alpha}(X)$ as the space of those jets $\mathcal{F}\in J^h(X)$ such that

$$\|\mathcal{F}\|_{WJ^{h,\alpha}(X)}=\max_{|\mathbf{k}|\le h}\|F^{\mathbf{k}}\|_{C^0(X)}+\max_{|\mathbf{k}|\le h}\sup_{x\,,y\in X\,,x\ne y}\frac{|F^{\mathbf{k}}(y)-F^{\mathbf{k}}(x)-\sum_{|j|=1}^{h-|\mathbf{k}|}F^{\mathbf{k}+\mathbf{j}}(x)(y-x)^{\mathbf{k}+\mathbf{j}}|}{|x-y|^{h-|\mathbf{k}|+\alpha}}\,,$$

is finite. Notice that $WJ^{h+1}(X)\subset WJ^{h,\alpha}(X)\subset WJ^h(X)$ for every $h\in\mathbb{N}$ and $\alpha\in(0,1]$, so we also set $WJ^h(X)=WJ^{h,0}(X)$. We are now ready to state Whitney's extension theorem, and to prove Proposition B.2 (which was used in the proof of Theorem 4.1).

Theorem B.1 (Whitney's extension theorem, $C^{1,\alpha}$-case)**.** *For every $n\ge1$, $\alpha\in[0,1]$ and $R>0$ there exists a constant C_0 depending on n, α and R only with the following property. If X is a compact set in $\mathbb{R}^n$, $X\subset B_R$, and $\mathcal{F}\in WJ^{1,\alpha}(X)$, then there exists $f\in C^\infty(\mathbb{R}^n\setminus X)\cap C^{1,\alpha}(\mathbb{R}^n)$ such that*

$$D^{\mathbf{k}}f=F^{\mathbf{k}}\ \textit{on } X \textit{ for every } |\mathbf{k}|\le1\,,\tag{B.1}$$

$$\|f\|_{C^{1,\alpha}(\mathbb{R}^n)} \le C_0 \, \|\mathcal{F}\|_{WJ^{1,\alpha}(X)} \,, \tag{B.2}$$

Proof. The classical construction introduced by Whitney (see [Ste70, Theorem 4, Chapter VI] or [Bie80, Theorem 2.3]) gives a function $g \in C^\infty(\mathbb{R}^n \setminus X) \cap C^{1,\alpha}(B_{2R})$ with

$$D^{\mathbf{k}} g = F^{\mathbf{k}} \text{ on } X \text{ for every } |\mathbf{k}| \le 1 \,, \tag{B.3}$$

$$\|g\|_{C^{1,\alpha}(B_{2R})} \le C \, \|\mathcal{F}\|_{WJ^{1,\alpha}(X)} \,, \tag{B.4}$$

where the constant C depends on n, h, α and R. If we now pick $\eta \in C^\infty_c(B_{2R};[0,1])$ with $\eta = 1$ on B_R, then by setting $f = g\,\eta$ we find that (B.1) holds with

$$\|f\|_{C^{1,\alpha}(\mathbb{R}^n)} \le C \, \|g\|_{C^{1,\alpha}(B_{2R})} \,.$$

In particular, (B.2) follows from (B.4). □

Proposition B.2. *If γ is a compact $C^{1,1}$-curve with boundary in $\mathbb{R}^2$ and $K > 0$ is such that $\gamma \subset B_K$ and, for a normal vector field $\nu \in C^{0,1}(\gamma;\mathbb{S}^1)$ to γ,*

$$|\nu(x)\cdot(y-x)| \le K\,|x-y|^2 \,, \qquad |\nu(x)-\nu(y)| \le K\,|x-y| \,, \qquad \forall x,y \in \gamma \,, \tag{B.5}$$

then there exists a function $d \in C^\infty(\mathbb{R}^2 \setminus \gamma) \cap C^{1,1}(\mathbb{R}^2)$ with $d = 0$ and $\nabla d = \nu$ on γ, such that $\|d\|_{C^{1,1}(\mathbb{R}^2)} \le C$ for a constant C depending only on K. In particular, there exists $\varepsilon > 0$ (depending on K only) such that (ε, d) is an extension by foliation of γ (see Definition 2.4).

Proof. We define a jet $\mathcal{F}$ of order 1 on γ by setting $F^{\mathbf{0}}(x) = 0$, $F^{e_1}(x) = \nu(x)\cdot e_1$ and $F^{e_2}(x) = \nu(x)\cdot e_2$ for every $x \in \gamma$. By (B.5) we find that, for every $x, y \in \gamma$,

$$\Big| F^{\mathbf{0}}(y) - F^{\mathbf{0}}(x) - \sum_{i=1}^{2} F^{\varepsilon_i}(x)(y-x)\cdot e_i \Big| \le K\,|x-y|^2 \,,$$
$$\max_{i=1,2} |F^{e_i}(x) - F^{e_i}(y)| \le K|x-y| \,,$$

so that $\mathcal{F} \in WJ^{1,1}(\gamma)$ with $\|\mathcal{F}\|_{WJ^{1,1}(\gamma)} \le C(K)$. Since $\gamma \subset B_K$, Theorem B.1 immediately gives us a function d with the required properties. (We notice that the existence of $\varepsilon > 0$ such that (ε, d) is an extension by foliation of γ – see Definition 2.4 – follows by the uniform implicit function theorem Theorem 2.3.) □

Appendix C. Volume-fixing variations

Comparison sets used in variational arguments usually arise as compactly supported perturbations of the considered minimizer. In order to use these constructions in volume constrained variational problems, one needs to restore changes in volume due to such local variations. In the study of minimizing clusters, this kind of tool is provided in [Alm76, Proposition VI.12]; see also [Mag12, Section 29.6]. The following theorem is a version of Almgren's result which is suitably adapted to the problems considered in here. In particular, it adds to [Mag12, Corollary 29.17] the conclusions (C.6) and (C.7).

Theorem C.1 (Volume-fixing variations)**.** *If $\mathcal{E}_0$ is a N-cluster in $\mathbb{R}^n$, then there exist positive constants r_0, ε_0, R_0 and C_0 (depending on $\mathcal{E}_0$) with the following property: if $\mathcal{E}$ and $\mathcal{F}$ are N-clusters in $\mathbb{R}^n$ with*

$$\mathrm{d}(\mathcal{E},\mathcal{E}_0) \quad \le \quad \varepsilon_0 \,, \tag{C.1}$$

$$\mathcal{F}(h)\Delta\mathcal{E}(h) \quad \subset\subset \quad B_{x,r_0} \,, \qquad \forall h = 1,...,N \,, \tag{C.2}$$

for some $x \in \mathbb{R}^n$, then there exists a N-cluster $\mathcal{F}'$ such that

$$\begin{aligned} \mathcal{F}'(h)\Delta\mathcal{F}(h) \quad &\subset\subset \quad B_{R_0} \setminus \overline{B_{x,r_0}}\,, \qquad \forall h = 1,...,N\,, && \text{(C.3)}\\ \mathrm{vol}\,(\mathcal{F}') \quad &= \quad \mathrm{vol}\,(\mathcal{E})\,, && \text{(C.4)}\\ |P(\mathcal{F}') - P(\mathcal{F})| \quad &\le \quad C_0\, P(\mathcal{E})\, |\mathrm{vol}\,(\mathcal{F}) - \mathrm{vol}\,(\mathcal{E})|\,, && \text{(C.5)}\\ |\mathrm{d}(\mathcal{F}', \mathcal{E}) - \mathrm{d}(\mathcal{F}, \mathcal{E})| \quad &\le \quad C_0\, P(\mathcal{E})\, |\mathrm{vol}\,(\mathcal{F}) - \mathrm{vol}\,(\mathcal{E})|\,. && \text{(C.6)} \end{aligned}$$

Moreover, if $g : \mathbb{R}^n \to [0,\infty)$ is locally bounded, then

$$\sum_{h=0}^{N} \int_{\mathcal{F}'(h)\Delta\mathcal{F}(h)} g \quad \le \quad C_0\, \|g\|_{L^\infty(B_R)}\, P(\mathcal{E})\; |\mathrm{vol}\,(\mathcal{F}) - \mathrm{vol}\,(\mathcal{E})|\,. \tag{C.7}$$

We shall need the following slight refinement of [Mag12, Lemma 17.9].

Lemma C.2. *If $g : \mathbb{R}^n \to [0,\infty)$ is locally bounded, E is a set of locally finite perimeter in an open set A and $T \in C^1_c(A;\mathbb{R}^n)$, then for every $\eta > 0$ there exist $K \subset A$ compact and $\varepsilon > 0$ (depending on T) such that if $\{f_t\}_{|t|<\varepsilon}$ is a flow with initial velocity T, then*

$$\int_{f_t(E)\Delta E} g \le (1+\eta)\, \|T\|_{C^0(\mathbb{R}^n)}\, \|g\|_{L^\infty(K)}\, P(E;K)\, |t|\,, \qquad \forall |t| < \varepsilon\,. \tag{C.8}$$

Proof. Since $(d(f_t)^{-1}/dt)|_{t=0} = -T$, if we set $\Phi_{s,t}(x) = s\,x + (1-s)\,(f_t)^{-1}(x)$ for $x \in \mathbb{R}^n$ and $s \in (0,1)$, then for every $\eta > 0$ there exists $\varepsilon > 0$ such that $\{\Phi_{s,t}\}_{|t|<\varepsilon}$ is a family of diffeomorphism on $\mathbb{R}^n$ with

$$\inf_{x\in\mathbb{R}^n} J\Phi_{s,t}(x) \ge 1 - \eta\,, \qquad \|\mathrm{Id} - (f_t)^{-1}\|_{C^0(\mathbb{R}^n)} \le (1+\eta)\,|t|\,\|T\|_{C^0(\mathbb{R}^n)}\,, \qquad \forall |t| < \varepsilon\,.$$

Let $K \subset A$ compact be such that $\{f_t \neq \mathrm{Id}\} \subset K$ for every $|t| < \varepsilon$. By Fubini's theorem and by the area formula, if $u \in C^1(\mathbb{R}^n)$, then

$$\begin{aligned} \int_{\mathbb{R}^n} g\,|u - u((f_t)^{-1})| \quad &\le \quad (1+\eta)\,|t|\,\|T\|_{C^0(\mathbb{R}^n)} \int_K g(x)\,dx \int_0^1 |\nabla u(\Phi_{s,t}(x))|\,ds \\ &= \quad (1+\eta)\,|t|\,\|T\|_{C^0(\mathbb{R}^n)} \|g\|_{L^\infty(K)} \int_0^1 ds \int_K \frac{|\nabla u(y)|}{J\Phi_{s,t}(\Phi^{-1}_{s,t}(y))}\,dy \\ &\le \quad \frac{1+\eta}{1-\eta}\,|t|\,\|T\|_{C^0(\mathbb{R}^n)} \|g\|_{L^\infty(K)} \int_K |\nabla u|\,. \end{aligned}$$

By [Mag12, Theorem 13.8] there exists $\{u_h\}_{h\in\mathbb{N}} \subset C^1(\mathbb{R}^n)$ such that $u_h \to 1_E$ a.e. on A and $\limsup_{h\to\infty} \int_K |\nabla u_h| \le P(E;\overline{K})$. Since $|u_h - u_h((f_t)^{-1})| \to 1_{E\Delta f_t(E)}$ a.e. on A, we conclude the proof by Fatou's lemma. □

Proof of Theorem C.1. One repeats the proof of [Mag12, Corollary 29.17], exploiting Lemma C.2 in place of [Mag12, Lemma 17.9] in order to obtain (C.6) and (C.7). We thus omit the details. □

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