ARE MATHEMATICS STUDENTS THINKING AS KEPLER? CONICS AND MATHEMATICAL MACHINES

Francesca Ferrara*, Michela Maschietto°

* Dipartimento di Matematica "Giuseppe Peano", Torino, Italy

° Dipartimento di Educazione e Scienze Umane, Modena e Reggio Emilia, Italy

Our interest is the analysis of the thinking processes of some university students who worked on the design of a machine that uses a tightened thread to draw a hyperbola. Previously, the students worked with other machines for conics. We focus on the way past experience becomes part of a new experience, in which making of the machine is the end point of the task. This implies the presence of technological and scientific aspects, whose interplay is fundamental to shape thinking.

Keywords: instrumental genesis, mathematical machine, transfer.

INTRODUCTION

This paper centres on an activity that asked some university students to think of the way to design a machine using a tightened thread to draw hyperbolas. The activity is part of a course on Elementary Mathematics from an Advanced Standpoint (in the tradition of Klein; Bartolini Bussi *et al*., 2010) that the students attend at the second year of their Master's Degree in Mathematics. The topic of the course considers conic sections and their properties, starting from Greek Mathematics. During the course, the students worked through laboratory activities with some mathematical machines for drawing conics. A mathematical machine is defined as a tool that forces a point to follow a trajectory or to be transformed according to a given law (Maschietto, 2005; Maschietto & Bartolini Bussi, 2011). The students dealt with the use of machines in two manners: first, they explored some curve drawers for ellipse and parabola, and then they were given the task to construct a machine to trace hyperbolas.

Our interest in this paper is focused on how aspects and elements coming from the previous activities with mathematical machines are transferred in the new situation, and on how they imply and originate new ways of writing, new ways of drawing, new ways of thinking.

To this aim, we will cite Kepler's thought about the construction of a new machine. We will also refer to theoretical elements relative to transfer of learning and the utilization schemes that are concerned with the use of an artefact. In the analysis, attention will be drawn to how acquired schemes shape new schemes for the new machine. Additionally, we will address other matters that intervene in the task.

KEPLER, ANALOGY, AND TRANSFER

In his *Ad Vitellionem paralipomena*, Kepler (1604) considers the way to draw conics:

> Analogy also helped me a lot to draw conic sections. From reading Propositions 51 and 52 [*concerning the metric properties involving the foci*] from Apollonius's Third Book,

> one can easily see how to trace ellipses and hyperbolas: these tracings can be made with a thread (Figures 1A and 1B). […] I regretted that for long I wasn't able to describe the parabola in the same way. At the end, the analogy revealed to me that to trace this curve is not much more difficult (and the geometric theory does confirm it) (Figure 1C). (Kepler, 1604, Italian version, pp. 4-5; our translation)

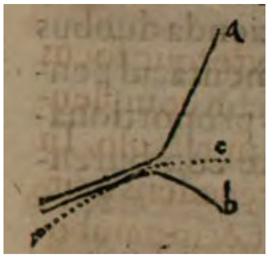
A  B  C 

Figure 1. Kepler's drawings: A) Hyperbola; B) Ellipse; C) Parabola.

Kepler's text (1604) is an example of the use of *analogy* in mathematics. We were already in the final part of our university course when we reencountered Kepler's text. We were surprised of the relevance given to analogy in geometrical reasoning:

> I love analogies a lot, considering them as my very reliable masters, experts of all the mysteries of nature; in geometry, one has to pay attention to them, especially when they enclose –even if with expressions that seem absurd– infinite cases intermediate between their extremes (and a centre), and thus put before our eyes, in full light, the true essence of an object. (Kepler, 1604, Italian version, pp. 3-4; our translation)

Mathematics Education research had studied analogy and analogical reasoning a lot (English, 1997), so Kepler's use was strikingly meaningful. But our aim is not to adopt a specific sense for analogy over the many that have been pointed out in the literature. Instead, here we want to refer to analogy in a naïve manner, following Kepler. He explains how, starting from reading Apollonius, he could trace hyperbolas and ellipses using a tightened thread, but not describe a parabola as easily. Analogy (the main form of reasoning in mathematics at that time) helped him in the case of the parabola. We can think of analogy as "continuity" and extension of thinking, say, transfer of knowledge, since knowledge acquired about the other conics is used in order to describe the parabola. Kepler's analogy as if he were thinking of a machine with tightened thread to obtain the tracing is significant to us.

The type of reasoning adopted by Kepler is interesting with respect to the kind of task given to our students (a kind of task that was chosen with the aim to study the effect of previous tasks). In a manner similar to Kepler, our students can recall aspects and elements of their past experiences with the other machines in order to face the task at hand about the new machine. In these terms, we can think of reasoning by analogy as a sort of 'transfer' of knowledge, of learning. Nemirovsky (2011) speaks of *transfer of learning* "in the context of common and experiential phenomena of learning":

> I see transfer as part of the study of how one experience becomes part of another. People can all sense that experiences do become part of other experiences. It is also clear, I

> think, that such participation can be lived in numerous ways, some of which I suggest calling "transfer". (p. 309)

Nemirovsky suggests the importance of developing case studies describing learning experiences as "instances in which an experience clearly comes to be part of another in the view of the subject and/or the authors of the case study." (p. 310).

We present a case study investigating the question: How does previous experience with a parabola drawer and an ellipsograph with tightened thread become part of a new experience when students are asked to think of a machine to draw hyperbolas?

In past tasks, the students were required to study the functioning of a machine. In the new task they are challenged to make a machine to draw hyperbolas. An artefact is being built, so the ways to use it, the subject's utilization schemes, are also decided. Following Koyré (1967), the construction of the new machine implies a "creation of scientific thought or, better yet, the conscious realization of a theory" (p. 106).

UTILIZATION SCHEMES AND MATHEMATICS LABORATORY

Knowledge of a machine involves knowledge of the *utilization schemes* that can be activated with it, for reaching the task's goals. A machine is an artefact, according to the instrumental approach (Rabardel, 1995; Rabardel & Samurçay, 2001). An artefact is a material or abstract object produced by human activity and aimed to sustain new human activity for facing tasks. It is designed and constructed with a purpose and given to a subject. In the hands of the subject, as part of an educational task, the artefact becomes an instrument, a mixed entity composed of both the artefact (object) and the utilization schemes developed by the subject to reach the specific goal of the task. So, the instrument has a subjective and cognitive character. The development of the instrument, that is, the *instrumental genesis*, is composed of complex processes, instrumentalization and instrumentation, linked to the artefact's potentialities and constraints, and to the contextual activity, background and knowledge of the subject. In addition, Guin *et al*. (2005) asserted that instrumental geneses are conceptual geneses, and stressed then their importance for learning, in particular in mathematics. Furthermore, the recognition of the importance of the teacher's action in encouraging and guiding instrumental geneses, gave rise to various directions of study within the instrumental approach (e.g. Trouche, 2004; Trouche & Drijvers, 2010).

From the methodological point of view, activities with artefacts are typical of the *mathematics laboratory* deeply rooted in the Italian teaching tradition, in research studies about innovation in mathematics education, and in the mathematical tradition concerning the use of tools (Maschietto & Trouche, 2010). The laboratory is not meant as a physical space with equipment, but as a structured set of activities aimed to the construction of meanings for mathematical objects (Anichini *et al*., 2004). In laboratory activities are essential the task(s) to be addressed, the presence of tools that one can use and manipulate, the work methodology that affects relationships and interactions (students and teacher), and the presence and role of the teacher.

In this paper, the perspectives of the instrumental genesis and mathematics laboratory are relevant at least at two levels. First, regarding utilization schemes that a student activates to perform a given task with a given mathematical machine. Second, in terms of knowledge that a student uses when a mathematical machine is involved in the task. In the first case, the specific purpose of the activity guides the choice of certain schemes so action is aimed to reach a purpose. In the second case, previous knowledge is used to create new knowledge and meanings so that utilization schemes acquired in the past may be purposefully adapted to the new situation.

The mathematical machines studied by the students make use of a tightened thread and a pencil attached to draw curves for particular conic sections on a flat surface: ellipse, hyperbola, and parabola. These machines work in accordance to determined rules that are connected to the definitions of the curves under consideration.

The laboratory use of a mathematical machine, to study the way utilization schemes previously met are transposed (transferred) in the new situation and how this originates new ways of drawing and thinking, is interesting. To this aim, we need first to look at the kind of tasks given to the university students.

MATHEMATICAL MACHINES AND TASKS

In the course on Elementary Mathematics from an Advanced Standpoint, the students worked with curve drawers for conics [1] in the context of mathematics laboratory, after studying Apollonius's theory of conics (Heath, 1931). In five laboratory sessions we proposed five machines (we just describe in details drawers with tightened thread).

D1: Cavalieri's drawer for parabola [2]. It concerns Menaechmus's definition in which the parabola is identified by a proportion among segments.

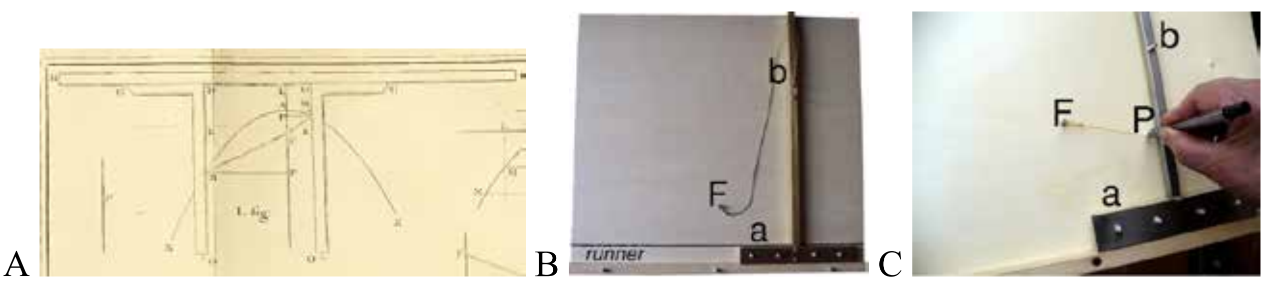
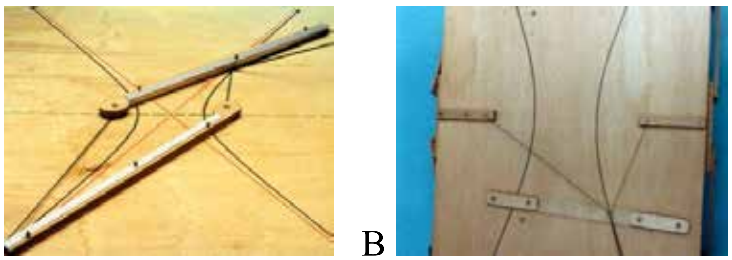


Figure 2. Parabola drawer with tightened thread

D2: Parabola drawer with tightened thread [2] (Figures 2B and 2C). It is composed of two perpendicular rods shaping a T. With respect to Figure 2B, the shortest rod (a) slides on a runner fixed to a wooden flat surface, a thread is linked to the free end of the longest rod (b) and to a pin (F) on the surface. In order to trace a parabola, a pen always has to stretch the thread near the rod (b) as the rod (a) is pushed and moves (Figure 2C). F is the focus of the parabola. When the thread's length is the same as the rod's (b), the runner is the directrix. In his *Traité analytique des sections coniques*, de l'Hôpital (1720) defined conic sections as curves drawn in the plane by particular tools with tightened thread (Figure 2A).

D3: Ellipsograph with tightened thread (gardener's method to draw elliptical flower beds) [2]. Two pins (foci of the ellipse) are fixed to a wooden flat surface. A pen stretches a thread linked to the pins and traces a curve when moving around them.

D4: Hyperbola drawer with tightened thread [2]. From an historical point of view, two kinds of machines were created. One of them (Figure 3B) mainly corresponds to Kepler's description. In both models, two pins (foci of the curve) are fixed to a wooden flat surface. The model in Figure3A shows two foci, two rods constrained each to turn around one focus, and a thread fixed to the other focus and to the free end of the rod. For every rod, a pen stretches the thread near it and traces a branch of the curve when moving around a pin.



A B

Figure 3. Hyperbola drawers

D5: Ellipse drawer with articulated crossed parallelogram [2]. It is composed of a crossed parallelogram with two couples of unequal rods.

In each session, the students worked following the methodology of the mathematics laboratory: group work, collective discussion and individual work. The teacher proposed the exploration of the drawers D1, D2, D3 and D5 through three general questions (in line with other teaching experiments that used mathematical machines). The first question "how it is made" aims to describe the physical structure of the machine and to detect parts and spatial relations. This task supports processes of instrumentalization. The second question "what it traces" focuses on the product of the machine. After putting a lead into a hole or keeping the thread well tightened by the means of a pen (see Figure 2C), the students can trace a curve (they can act on the machine) and analyze it. The third question "why" encourages the students to produce conjectures, to argue them, and to construct proofs.

The task was changed in the case of the drawer D4. The students were asked to imagine a mathematical machine with tightened thread to draw hyperbolas. Different from the previous tasks, the students were faced with a problem solving situation, in which they knew the final curve to obtain, but they did not have any machine to trace it. This specific activity constitutes the core of the paper.

At the end of the group exploration, the students were asked to write an individual report. A collective discussion followed, in which the teacher could share elements of the students' exploration processes and ask about the relationships between the explored machine and the drawn curve.

From the methodological point of view, the eight students attending the course were divided into two groups (A and B). One student per group had the role of observer. Our analysis is based on the students' reports and the notes by the observer.

TRANSFER OF LEARNING AND UTILIZATION SCHEMES

In this section, we present sketches of the work of group B, through different phases. The situation can be considered as a learning situation: the students were dealing with a new experience in which they were implicitly required by the nature of the task to use their previous experience with drawers. In fact, the task specifically demanded for thinking of the construction of a new machine. The novelty of the machine lies in the request to draw a new curve: a hyperbola, not in the kind of machine, which still has to use a tightened thread.

We focus on the strategy that group B adopts to face the task, using written texts and drawings produced, as well as the notes taken by a student-observer that followed the entire work of the group. The students are referred to as B1, B2 (observer), B3 and B4. Four phases can be distinguished in their work, the first three depending on the tools with which the students work (paper and pencil; a wooden plan with two pins and a thread; a rod).

1) The students began by stating the metric definition of the hyperbola, writing down the relationship between foci and the generic point, or the equation. From the observer, we know that they made some considerations, like: "Surely, the machine draws only one branch", "The distance between the foci is constant, so the pins will be fixed". Since the beginning, the group thinks of the functioning of a machine that keeps the difference constant, what is present in the metric definition of hyperbola, and considers some components of the machine, like the pins.

This is where the previous experience with other machines comes into play. First, the students look for "a machine initially similar to the ellipsograph, but that keeps the difference constant" (B3). In so doing, they first join the point with the foci and draw a certain configuration depending on the definition of hyperbola (Figure 4A). Then, they explore the situation recalling the tightened thread of the ellypsograph (Figure 4B). The final tracing of the hyperbola by means of the potential machine (Figure 4C) does not suggest any way to keep the difference constant, so the configuration is abandoned (the rectangles mark the flat surface of the machine, as in Figure 4B).

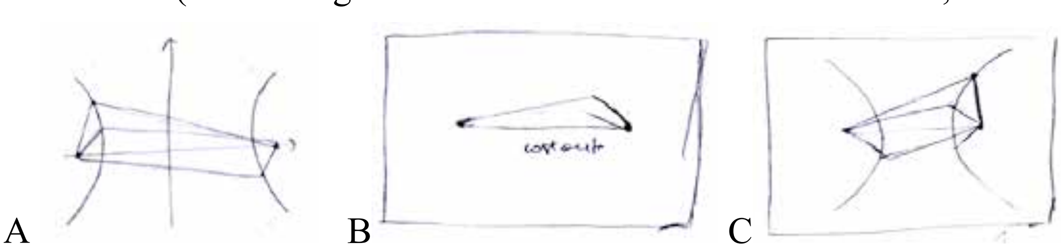


Figure 4. B4's sketches

The students go back to previous experience again, recalling the parabola drawer and the rod. Many sketches are produced in which the position of the rod is varied or two rods are considered, one for each focus, in the search for the suitable model for the machine (see Figure 5A).

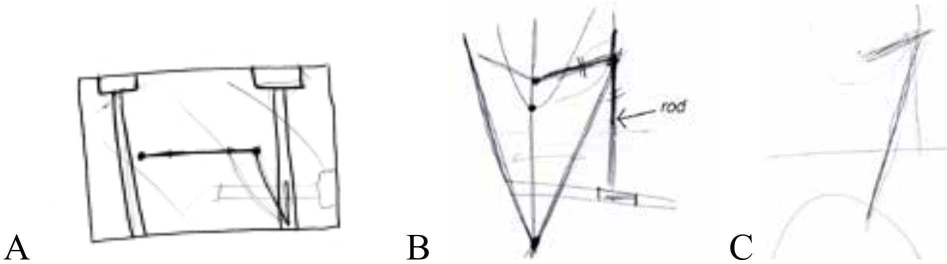


Figure 5. A) B3's sketch; B and C) B1's sketches

This is a phase of instrumental genesis for the rod (a new utilization scheme, a rod added with respect to the parabola drawer) and for the runner (for instance, different positions with respect to the case of the parabola drawer), aimed to the solution of the task, that is, to drawing the hyperbola. The students transfer a component of the artefact (the rod) from experience with the parabola drawer, and activate utilization schemes consistent with the functioning of that machine. For example, the tightened thread with respect to a point of the curve. Figure 5A instead marks the side of the machine as the runner with a rod for each focus. In Figure 5B, the rod is sketched together with the runner it is supposed to move. This constraint changes when the second focus is considered (Figure 5C). Unfortunately, this phase is not successful.

2) The teacher asked the students if they needed some material components, recognizing their presence in the students' discussion and drawings (for instance, pins and thread are also there; see Figure 5A). A wooden plan with two pins and a thread were given to each group. Once the length of the thread is chosen, student B2 tries to put it around the pins, falling in the configuration of the ellipsograph. The trial is soon abandoned since "the shape of the thread does not trace the hyperbola. It is like in the parabola drawer". This reflection is crucial: even if the drawing of the metric definition of the hyperbola is the same as that of the ellipse, the students confirms in a material way that the use of the thread and its related movement has to change. So, the students first try to tie the tread to the pins with the help of a pencil as a rod, then they experiment new movements. Seeking to create a machine that can draw a hyperbola, after a while, they transform the parabola drawer into a new artefact: "Observed that the two foci ordered the distance to keep constant, we removed the runner and put a second point below" (Figure 6).

Figure 6. B1's report

The arrow between the two graphical representations in Figure 6 emphasizes the passage, while the removal of the runner is marked by the dotted line.

In the group discussion, the students highlight that it is not relevant to have rods in a chosen configuration, but the rods must be able to turn. Thus, using pencils as rods, the students can explore new movements.

3) Students' drawings and speech referred to other components of the machine (like the rods in Figure 5A, 5B and Figure 6). Aware of this, the teacher asked student if they need some other materials. As soon as a rod is furnished, two issues became relevant: 1) the way the thread has to be tied, and 2) the position of the pencil. For the latter, the teacher recalls the relationship between the positions of the pen to tighten the thread and the points of the curve. Then, the students are able to overcome troubles and to find a way to proceed when the rod is based on one of the two pins with the ends of the thread "anchored to the two pins, which we have interpreted as foci of the hyperbola, and keeping the thread tightened along the rod, we have tried to represent a figure that looks like a hyperbola" (from B3's written report).

In this way, at the end, the students are able to trace a branch's arc of hyperbola.

4) The curve traced, the students want to justify that it is effectively a hyperbola.

Figure 7. B4's sketch

After some doubts, they first try to use algebraic calculation and then they recover on the machine all the useful parameters, from the length of the rod and of the thread and the constant difference (see Figure 7). However, the students do not make any reference to the fundamental elements of the curve drawer, especially to the position of the pencil.

CONCLUDING REMARKS

Our interests in this paper are on the way previous experience with a parabola drawer and an ellipsograph with tightened thread becomes part of a new experience. This experience in particular asks for the creation of a hyperbola drawer. The task differs from the previous ones, in which a machine was the starting point not the final one. In particular, we have observed the work of one group, B, and we could see that the process of constructing the new drawer implies both technological and scientific aspects (following Weisser, 2005). Technological aspects concern having a machine and the ways it can be used (like in "so the pins will be fixed" or "it has to be able to turn", and in drawings the sketch of a rectangle, rods, etc.). Scientific aspects mainly regard the mathematical constraints that have to be satisfied for drawing a precise curve (like in "the distance between the foci is constant" or "it keeps the difference constant", and in drawings reference to constant difference). The teacher is aware of these two dimensions and of their connections. Sometimes, in fact, she furnishes material objects to the students, so that they can test possibilities of functioning for the machine that before they only could imagine and reproduce by diagrams.

At least for group B, the interplay of technological and scientific aspects is affected by past experience with the other drawers. A lived experience that enters in the new situation bringing components of those machines that the students already used. The interesting thing is that these tools are part of the process in two manners. First, the new machine is being based on variations of the previous machines, recalled at different times (that is, instrumentalizations of those machines). Second, the "ways of use" have to furnish different configurations. In fact, it is when the students discard a configuration that the utilization schemes of the previous drawers clearly appear.

We may possibly speak of *gestures of usage* (or *gestures of ways of use*) that entail movements of the components, these movements recalling or not past experiences (and, in the second case, being similar to thought experiments). The image of gesture implies action. In effect, the instrumented activity is fundamental here; it is the basis for movement and dynamicity, not only of the machine, but also of the students' thinking processes. It is in this sense that we see both continuity and extension of analogous thinking in a way similar to Kepler's. So, when the students say, for example, "like in the parabola drawer", we can interpret that they are thinking as Kepler. In the same sense, we look at the entire situation as an example of transfer of learning. First, it is a learning situation. Second, learning is not acquired by the automatic activation of previous schemes. Instead, it is shaped by the continuous back and forth between technological and scientific aspects entailed by the task that constrains the actors (learners) to constantly review and vary their ideas.

NOTES

1. In Italian secondary school, the conics sections are mainly studied in analytical geometry, starting from their definitions as loci of points.

2. http://www.macchinematematiche.org/index.php?option=com_content&view=article&id=76&Itemid=153

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