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The natural pseudo-distance as a quotient pseudo-metric, and applications

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Abstract. The natural pseudo-distance is a similarity measure conceived for the purpose of comparing shapes. In this paper we revisit this pseudo-metric from the point of view of quotients. In particular, we show that the natural pseudo-distance coincides with the quotient pseudo-metric on the space of continuous functions on a compact manifold, endowed with the uniform convergence metric, modulo self-homeomorphisms of the manifold. As applications of this result, the natural pseudo-distance is shown to be actually a metric on a number of function subspaces such as the space of topological embeddings, of isometries, and of simple Morse functions on surfaces.

Keywords. Function space, Morse function, homeomorphism group action, isometry, shape comparison.

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1 Introduction

The natural pseudo-distance was introduced by Frosini and Mulazzani in [13], and further studied in [8–10, 12], as a measure of similarity that behaves nicely when invariance to deformations or different poses of the compared objects is a key requirement. These contributions stemmed from the actual need for these sort of similarity measures in pattern recognition to cope with the matching of natural or articulated objects. In general, such measures are reckoned to be beneficial for the organization of the huge collections of digital models produced nowadays through massive data acquisitions and shape modeling. In recent years, the development and study of topology-invariant metrics with stability properties has widely increased, as the numerous studies on similarity of non-rigid shapes testify (cf., e.g., [2, 6]).

The natural pseudo-distance is usually defined on the space $\mathcal{C}(M, \mathbb{R}^n)$ of $\mathbb{R}^n$-valued continuous functions with domain a compact manifold M in the fol-

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lowing way. Assuming $\mathbb{R}^n$ endowed with the usual maximum norm

$$\|(x_1, x_2, \ldots, x_n)\|_\infty = \max_{1 \le i \le n} |x_i|,$$

for $f, g \in \mathcal{C}(M, \mathbb{R}^n)$,

$$\delta(f, g) = \inf_{h \in \mathcal{H}(M)} \max_{p \in M} \| f(p) - g \circ h(p)\|_\infty,$$

$\mathcal{H}(M)$ being the set of self-homeomorphisms of M.

In the shape comparison problem, M is conceptualized as the object under study (up to homeomorphisms), and f, g as measures of some shape properties of the object (e.g., height, depth, curvature, or color) which are relevant in a specific context [1]. Then, the dissimilarity between shapes is measured by the natural pseudo-distance as the variation in these properties when we move from one shape to the other through homeomorphisms.

In practice, one wishes the natural pseudo-distance to be zero only when computed on objects that share the same shape properties with respect to the chosen functions, i.e. when $g = f \circ h$ for some homeomorphism h. In general, this is not the case, but we can ask ourselves whether it is true at least for some subspace of functions, possibly generic, since counterexamples usually involve non-generic functions (see, for example, [5, 9]).

Starting from an idea presented in [4], the aim of this paper is to put the natural pseudo-distance in context with the classical notion of quotient pseudo-metric.

This link between the natural pseudo-distance and quotients is not only interesting per se, but also enables to positively answer the aforementioned question. Indeed, as a further contribution of this paper, we show that, for an open dense subspace of smooth functions on a surface, the space of simple Morse functions, up to homeomorphisms, the natural pseudo-distance is actually a metric, thus distinguishing surfaces with different shape properties (Section 4.3). For the case of curves, the analogue of this result was obtained in [5] using a constructive technique, while it is proved here for surfaces by indirect arguments based on properties of quotient pseudo-metrics.

Besides this result, we also prove that the natural pseudo-distance turns to a metric when defined on subspaces of $\mathcal{C}(M, \mathbb{R}^n)$ such as the spaces of embeddings and immersions (Section 4.1), and on quotient spaces induced by compact subgroups of $\mathcal{H}(M)$ (Section 4.2). As a special case of this type, we consider another space of functions quite common in applications, i.e. simplicial functions on compact simplicial manifolds.

In perspective, we hope that other well-known properties of quotient pseudo-metrics will turn useful for the study of the natural pseudo-distance.

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2 Quotients of pseudo-metric spaces

In this section we review the notion of a quotient pseudo-metric. Further details can be found in [3] (where the quotient pseudo-metric is called the quotient semi-metric).

Let us recall that a pseudo-metric space is a pair (X, d) where X is a set and d is a pseudo-metric (also known as pseudo-distance), i.e. a function with all the axioms of a metric except the requirement that, for $x, y \in X$, $d(x, y) = 0$ implies $x = y$.

Let us consider the category **PMet** of pseudo-metric spaces and non-expansive maps, i.e. functions between pseudo-metric spaces that do not increase metrics: A map $h : (Z, d) \to (Z', d')$ is *non-expansive* if, for all $z_1, z_2 \in Z$,

$$d'(h(z_1), h(z_2)) \leq d(z_1, z_2).$$

A map h between pseudo-metric spaces is an isometry if and only if it is non-expansive, it is a bijection, and its inverse is also non-expansive. In particular, since non-expansive maps are always continuous, any isometry in **PMet** is a homeomorphism.

Non-expansive maps are the suitable maps between pseudo-metric spaces to pass to quotients. Indeed, if (Z, d) is a pseudo-metric space and $\sim$ is an equivalence relation on Z, the quotient set $Z/_\sim$ can be endowed with the following pseudo-metric: Given two equivalence classes $[z]$ and $[y]$, the *quotient pseudo-metric* is defined by

$$d_\sim([z], [y]) = \inf\left\{\sum_{i=1}^{n} d(z_i, y_i)\right\}$$

where the infimum above is taken over all finite sequences $(z_1, z_2, \dots, z_n)$ and $(y_1, y_2, \dots, y_n)$ with $[z_1] = [z], [y_{i-1}] = [z_i], \dots, [y_n] = [y]$, $i = 2, \dots, n$ (see [3, Definition 3.1.12]).

The quotient pseudo-metric $d_\sim$ is characterized by the following universal property. If $h : (Z, d) \to (Z', d')$ is a non-expansive map between pseudo-metric spaces such that $h(z) = h(y)$ whenever $z \sim y$, then the induced quotient map $h_\sim : (Z/_\sim, d_\sim) \to (Z', d')$ is non-expansive, that is the following diagram commutes in **PMet**:

$$\begin{array}{ccc} Z & \xrightarrow{\pi} & Z/_\sim \\ & \searrow^{h} & \downarrow h_\sim \\ & & Z'. \end{array}$$

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In general the quotient topology induced by $\sim$ is different from the topology induced by the quotient pseudo-metric $d_\sim$. However, when the equivalence classes are the orbits of the action of a group of isometries, the following result holds.

Theorem 2.1. *If (Z, d) is a pseudo-metric space endowed with an equivalence relation where the equivalence classes are the orbits of the action of a group of isometries on (Z, d), then:*

(i) $d_\sim([z],[y]) = \inf\{d(z', y') : z' \sim z,\ y' \sim y\}$.

(ii) *The topology induced by the quotient pseudo-metric coincides with the quotient topology.*

(iii) *$d_\sim$ is a metric if and only if the topology it induces is T_0.*

(iv) *$d_\sim$ is a metric if and only if the orbits of the action are closed.*

Proof. Statements (i) and (ii) have been proved in [14, Theorem 4], while statement (iii) can be found in [20, p. 85].

It remains to verify (iv). From (iii), $(Z/\sim, d_\sim)$ is a metric space if and only if the topology it generates is T_0. Let us observe that a pseudo-metric space is T_0 if and only if is T_1. Therefore $(Z/\sim, d_\sim)$ is a metric space if and only if its points are closed, i.e. if and only if the equivalence classes $[z] \in Z/\sim$ are closed. Moreover, since from (ii) the projection $\pi : (Z, d) \to (Z/\sim, d_\sim)$ is a topological quotient, $[z] \in Z/\sim$ is closed if and only if $\pi^{-1}([z])$, that is the orbit of z in Z induced by the action, is closed. □

3 The natural pseudo-distance as a quotient pseudo-metric

In this section we show that the natural pseudo-distance is a quotient pseudo-metric. In order to see this, we endow the space of continuous functions $\mathcal{C}(M, \mathbb{R}^n)$ with the uniform convergence metric d, $d(f, g) = \max_{p\in M} \|f(p) - g(p)\|_\infty$, which induces the compact open topology. In this way, the space $\mathcal{C}(M, \mathbb{R}^n)$ belongs to **PMet**.

Next we quotient $\mathcal{C}(M, \mathbb{R}^n)$ by the following equivalence relation: denoting by $\mathcal{H}(M)$ the set of self-homeomorphisms of M, for $f, f' \in \mathcal{C}(M, \mathbb{R}^n)$, $f \sim f'$ if and only if there exists an $h \in \mathcal{H}(M)$ such that $f' = f \circ h$. In other words, the equivalence classes of $\sim$ coincide with the orbits induced by the action of $\mathcal{H}(M)$ on $\mathcal{C}(M, \mathbb{R}^n)$ given by

$$\circ : \mathcal{H}(M) \times \mathcal{C}(M, \mathbb{R}^n) \to \mathcal{C}(M, \mathbb{R}^n),$$
$$(h, f) \mapsto f \circ h.$$

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By definition, the natural pseudo-distance is trivial on the orbits of this action: $\delta(f, f \circ h) = 0$ for every $f \in \mathcal{C}(M, \mathbb{R}^n)$ and every $h \in \mathcal{H}(M)$. Therefore it is natural to identify functions in the same orbit and consider the natural pseudo-distance on $\mathcal{C}(M, \mathbb{R}^n)/_\sim$ rather than on $\mathcal{C}(M, \mathbb{R}^n)$:

Definition 3.1. The *natural pseudo-distance*

$$\delta : \mathcal{C}(M, \mathbb{R}^n)/_\sim \times \mathcal{C}(M, \mathbb{R}^n)/_\sim \to \mathbb{R}$$

is defined by setting

$$\delta([f], [g]) = \inf_{h \in \mathcal{H}(\mathcal{M})} \max_{p \in M} \| f(p) - g \circ h(p) \|_\infty .$$

Clearly this definition does not depend on the choice of the representatives f, g.

Let us observe that δ is not in general a metric, even when we define it on $\mathcal{C}(M, \mathbb{R}^n)/_\sim$ rather than on the space $\mathcal{C}(M, \mathbb{R}^n)$. Indeed, there may exist functions $f, g \in \mathcal{C}(M, \mathbb{R}^n)$ such that $\delta([f], [g]) = 0$, but with no $h \in \mathcal{H}(M)$ for which $f = g \circ h$. Some examples of this fact can be found in [5, Section 2].

Now, as a corollary of Theorem 2.1, we get the result below.

Corollary 3.2. *The following statements hold:*

(i) *The natural pseudo-distance δ is the quotient pseudo-metric induced by the action of $\mathcal{H}(M)$ on $(\mathcal{C}(M, \mathbb{R}^n), d)$: $\delta = d_\sim$.*

(ii) *The topology induced on $\mathcal{C}(M, \mathbb{R}^n)/_\sim$ by δ coincides with the quotient topology.*

(iii) *$(\mathcal{C}(M, \mathbb{R}^n)/_\sim, \delta)$ is a metric space if and only if the topology induced by δ is T_0.*

(iv) *$(\mathcal{C}(M, \mathbb{R}^n)/_\sim, \delta)$ is a metric space if and only if each orbit induced by the action of $\mathcal{H}(M)$ on $\mathcal{C}(M, \mathbb{R}^n)$ is closed.*

Proof. We observe that

$$\begin{aligned} \delta([f], [g]) &= \inf\{\max_{p \in M} \| f'(p) - g'(p) \|_\infty : f' \in [f], g' \in [g]\} \\ &= \inf\{d(f', g') : f' \sim f, g' \sim g\}. \end{aligned}$$

Furthermore, any self-homeomorphism of M induces an isometry on the metric space $(\mathcal{C}(M, \mathbb{R}^n), d)$: for every $h \in \mathcal{H}(M)$,

$$\max_{p \in M} \| f(p) - g(p) \|_\infty = \max_{p \in M} \| f \circ h(p) - g \circ h(p) \|_\infty .$$

Hence, it is sufficient to apply Theorem 2.1 (i) to obtain that $\delta = d_\sim$ and Theorem 2.1 (ii–iv) to obtain the other three claims. □

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We end the section by considering the case when the natural pseudo-distance is defined using only homeomorphisms in a subgroup $\mathcal{K}(M)$ of $\mathcal{H}(M)$.

Definition 3.3. Let $\mathcal{K}(M)$ be a subgroup of $\mathcal{H}(M)$, and let us consider the action of $\mathcal{K}(M)$ on $\mathcal{C}(M,\mathbb{R}^n)$: $f \sim_{\mathcal{K}} f'$ if and only if $f' = f \circ k$, for some $k \in \mathcal{K}(M)$. We define $\delta_{\mathcal{K}} : \mathcal{C}(M,\mathbb{R}^n)/_{\sim_{\mathcal{K}}} \times \mathcal{C}(M,\mathbb{R}^n)/_{\sim_{\mathcal{K}}} \to \mathbb{R}$ by

$$\delta_{\mathcal{K}}([f]_{\mathcal{K}}, [g]_{\mathcal{K}}) = \inf_{k \in \mathcal{K}(\mathcal{M})} \max_{p \in M} \|f(p) - g \circ k(p)\|_\infty.$$

Proposition 3.4. *The following statements hold:*

(1) *$\delta_{\mathcal{K}}$ is a quotient pseudo-metric induced by the action of $\mathcal{K}(M)$ on the metric space $(\mathcal{C}(M,\mathbb{R}^n), d)$.*

(2) *The topology induced on $\mathcal{C}(M,\mathbb{R}^n)/_{\sim_{\mathcal{K}}}$ by $\delta_{\mathcal{K}}$ coincides with the quotient topology.*

(3) *If the subgroup $\mathcal{K}(M)$ is compact in $\mathcal{H}(M)$ with the compact open topology, then $(\mathcal{C}(M,\mathbb{R}^n), \delta_{\mathcal{K}})$ is a metric space.*

Proof. The proofs of (1) and (2) follow immediately from Theorem 2.1 (i–ii). As for (3), by Theorem 2.1 (iv) it is sufficient to show that each orbit induced by $\mathcal{K}(M)$ on $\mathcal{C}(M,\mathbb{R}^n)$ is closed. Let $[\bar{f}]_{\mathcal{K}} \in (\mathcal{C}(M,\mathbb{R}^n)/_{\sim_{\mathcal{K}}}, \delta_{\mathcal{K}})$, and let (f_i) be a sequence such that $d(f_i, f) \to_{i\to\infty} 0$ for some $f \in \mathcal{C}(M,\mathbb{R}^n)$, and $f_i \in [\bar{f}]_{\mathcal{K}}$ for every i. Since $f_i = \bar{f} \circ k_i$, with $k_i \in \mathcal{K}(M)$, for every i, and $\mathcal{K}(M)$ is compact, there exists a subsequence (k_{i_j}) of (k_i) converging to a certain $k \in \mathcal{K}(M)$. Then we can take the subsequence (f_{i_j}) of (f_i), with $f_{i_j} = \bar{f} \circ k_{i_j}$ for every j. As composition is continuous with the compact open topology [11, Theorem 2.2], it follows that $\bar{f} \circ k_{i_j}$ converges to $\bar{f} \circ k$, and hence, $f = \bar{f} \circ k$. This proves that $f \in [\bar{f}]_{\mathcal{K}}$, i.e. that the orbit $\pi^{-1}([\bar{f}]_{\mathcal{K}})$ is closed, being π a topological quotient from (2). □

4 Applications

This section concerns some applications of Corollary 3.2 and Proposition 3.4 to subspaces of $\mathcal{C}(M,\mathbb{R}^n)$ under the action of $\mathcal{H}(M)$ or its subgroups $\mathcal{K}(M)$. In particular, in Section 4.1 we show that the natural pseudo-distance is a metric when induced by the action of $\mathcal{H}(M)$ on the space $\mathcal{E}(M,\mathbb{R}^n)$ of topological embeddings, and of the group $\mathcal{D}(M)$ of diffeomorphisms on the space $\mathcal{I}(M,\mathbb{R}^n)$ of immersions; Section 4.2 provides some examples of compact subgroups $\mathcal{K}(M)$ whose action on $\mathcal{C}(M,\mathbb{R}^n)$ induces a metric $\delta_{\mathcal{K}}$; Section 4.3 is devoted to prove that the natural pseudo-distance is a metric when we consider the space of simple Morse functions on surfaces under the action of $\mathcal{C}^2$-diffeomorphism.

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4.1 Embeddings in $\mathbb{R}^n$ under the action of $\mathcal{H}(M)$

Let us consider the space $(\mathcal{E}(M,\mathbb{R}^n), d)$ of topological embeddings of M in $\mathbb{R}^n$ (i.e. of homeomorphisms onto their image) endowed with the uniform convergence metric d. If K is a compact subset of M, and U is an open subset of $\mathbb{R}^n$, then $V(K,U) = \{f \in \mathcal{E}(M,\mathbb{R}^n) : f(K) \subset U\}$ is an open set in the compact open topology induced by d.

Let us pass to the quotient by considering the action of $\mathcal{H}(M)$ on $\mathcal{E}(M,\mathbb{R}^n)$, and take $\mathbb{R}^n$ endowed with the maximum norm. The following result, whose proof is inspired by [18, Lemma 13.9], holds.

Proposition 4.1. $(\mathcal{E}(M,\mathbb{R}^n)/_\sim, \delta)$ *is a metric space.*

Proof. Since $(\mathcal{E}(M,\mathbb{R}^n)/_\sim, \delta)$ is pseudo-metric, from Corollary 3.2 (iii), it is sufficient to verify that it is T_0. Let $[f],[g] \in \mathcal{E}(M,\mathbb{R}^n)/_\sim$, $[f] \neq [g]$. We want to show that there exists an open set containing $[f]$ and not $[g]$ or viceversa.

Observe that $[f] \neq [g]$ implies $f(M) \neq g(M)$. Indeed, if $f(M) = g(M)$, then we have $g = f \circ (f^{-1} \circ g)$, with $f^{-1} \circ g \in \mathcal{H}(M)$, i.e. $[f] = [g]$. Therefore, since $[f] \neq [g]$, without loss of generality, we can assume that there exists a point $y \in f(M) \setminus g(M)$. Since $g(M)$ is compact as the image of a compact space through a continuous function, an open set $U \subset \mathbb{R}^n$ can be found such that $g(M) \subset U$, $y \notin U$. Then, considering the open subset $V(M,U)$ of $\mathcal{E}(M,\mathbb{R}^n)$, we have $g \in V(M,U)$, while $f \notin V(M,U)$. Now, let us observe that $\pi(V(M,U))$ is open if and only if $\pi^{-1} \circ \pi(V(M,U))$ is open. Given that

$$\pi^{-1} \circ \pi(V(M,U)) = V(M,U)$$

which is open, then $\pi(V(M,U))$ is open. Clearly, the class $[f]$ cannot belong to $\pi(V(M,U))$, while $[g]$ does. Hence, $\mathcal{E}(M,\mathbb{R}^n)/_\sim$ is T_0. □

The above result can be generalized by considering the action of the group $\mathcal{D}(M)$ of diffeomorphisms of class $\mathcal{C}^1$ of M on the space $(\mathcal{I}(M,\mathbb{R}^n), d)$ of $\mathcal{C}^1$-immersions of M in $\mathbb{R}^n$ (i.e. of diffeomorphisms whose differential is injective) without points of self-tangency. Let us recall that an immersion $f : M \to \mathbb{R}^n$ has no points of self-tangency if whenever p, p' are distinct points of M with $f(p) = f(p')$, then $\operatorname{im} df_p \neq \operatorname{im} df_{p'}$.

To prove that $\mathcal{I}(M,\mathbb{R}^n)/_{\sim_\mathcal{D}}$ is T_0, let us consider $f, g \in \mathcal{I}(M,\mathbb{R}^n)$ such that $[f]_\mathcal{D} \neq [g]_\mathcal{D}$ and show that this implies $f(M) \neq g(M)$. The rest of the proof can be obtained using the same arguments as in the case of embeddings.

Let us assume $f(M) = g(M)$ and prove the existence of $h \in \mathcal{D}(M)$ such that $f = g \circ h$, i.e. $[f]_\mathcal{D} = [g]_\mathcal{D}$. Let $\mathcal{P}_f, \mathcal{P}_g \subset M$ be the sets of preimages of multiple points of f and g. Since $f, g \in \mathcal{I}(M,\mathbb{R}^n)$, it follows that $f_{|M \setminus \mathcal{P}_f}$, $g_{|M \setminus \mathcal{P}_g}$

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are $\mathcal{C}^1$-diffeomorphisms between $M \setminus \mathcal{P}_f$ and $f(M \setminus \mathcal{P}_f)$, and between $M \setminus \mathcal{P}_g$ and $g(M \setminus \mathcal{P}_g)$, respectively.

Moreover, as $f(M) = g(M)$, for any $p \in M$, the set $g^{-1}(f(p))$ is not empty. Seeing that in particular $f(\mathcal{P}_f) = g(\mathcal{P}_g)$, if $p \in M \setminus \mathcal{P}_f$, the set $g^{-1}(f(p))$ contains only one point p' and we can define $h(p) = p'$. If $p \in \mathcal{P}_f$, then we have $g^{-1}(f(p)) \subset \mathcal{P}_g$. In this case, there is just one point $p' \in g^{-1}(f(p))$ verifying $\operatorname{im} dg_{p'} = \operatorname{im} df_p$ because multiple points of g are without self-tangency. Thus, we can define $h(p) = p'$. Because of its definition, the function h verifies the equality $g \circ h = f$. Let us show that $h \in \mathcal{D}(M)$. Recalling that $f(M) = g(M)$, the definition of h implies that h is injective and surjective. Furthermore, for each point $p \in M$, there exist an open neighborhood $U(p)$ of p in M such that $f_{|U(p)}$ is a $\mathcal{C}^1$-diffeomorphism, a point $p' \in M$ for which $g(p') = f(p)$, and an open neighborhood $U'(p')$ of p' in M such that $g_{|U'(p')}$ is a $\mathcal{C}^1$-diffeomorphism and $g(U'(p')) = f(U(p))$. Hence, the restriction $h_{|U(p)}$ equals the $\mathcal{C}^1$-diffeomorphism $g^{-1}_{|U'(p')} \circ f_{|U(p)}$. This proves that $[f]_{\mathcal{D}} = [g]_{\mathcal{D}}$.

4.2 $\mathcal{C}(M, \mathbb{R}^n)$ under the action of compact groups

As an application of Proposition 3.4 (3), let us consider the space $(\mathcal{C}(M, \mathbb{R}^n), d)$ with M a submanifold of $\mathbb{R}^n$ verifying one of the following properties:

- M is of revolution,
- M is invariant with respect to a rotation of $\frac{2\pi}{n}$.

In the previous cases or in each combination of them we can consider respectively $\mathcal{K}(M) \cong S^1$, $\mathcal{K}(M) \cong \mathbb{Z}_n$ or the corresponding product of these compact groups. Then the orbits induced by the action of $\mathcal{K}(M)$ on $\mathcal{C}(M, \mathbb{R}^n)$ are closed, so that $\delta_{\mathcal{K}}$ is a metric on $\mathcal{C}(M, \mathbb{R}^n)/_{\sim_{\mathcal{K}}}$.

More in general, we can consider the action of the group $\mathfrak{I}(M)$ of isometries on M (i.e. of metric preserving self-homeomorphisms of M). Indeed, as stated in [15, Theorem 1.2], in the case M is a compact manifold, the group $\mathfrak{I}(M)$ results to be compact in the compact open topology. Consequently, because of Proposition 3.4 (3), we obtain that $(\mathcal{C}(M, \mathbb{R}^n)/_{\sim_{\mathfrak{I}}}, \delta_{\mathfrak{I}})$ is always a metric space.

As another application of Proposition 3.4, we consider the space $(\mathcal{S}(M, \mathbb{R}), d)$ of simplicial maps on a compact simplicial manifold M, under the action of the subgroup $\mathcal{S}\mathcal{H}(M)$ of $\mathcal{H}(M)$ of simplicial homeomorphisms of M. This subgroup is finite, and therefore compact, since each $h \in \mathcal{S}\mathcal{H}(M)$ is defined by the natural extension from its value at the finitely many vertices of M to all the simplices of M. Hence, Proposition 3.4 (3) allows to conclude that $(\mathcal{S}(M, \mathbb{R}), \delta_{\mathcal{S}\mathcal{H}})$ is a metric space.

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4.3 The Morse functions space under the action of the group of diffeomorphisms

In what follows, M will denote a 2-dimensional smooth compact connected manifold without boundary, and $(\mathcal{M}_0(M,\mathbb{R}),d)$ the space of simple Morse functions on M endowed with the uniform convergence metric d. Let us recall the following facts: $f : M \to \mathbb{R}$ is a Morse function if it is of class $\mathcal{C}^2$ and all its critical points are non-degenerate (i.e. the Hessian matrix at each critical point is non-singular); the number of negative eigenvalues of the Hessian matrix at a critical point is called the index of f at the critical point; as a consequence of the compactness of M and the property of being non-degenerate, Morse functions' critical points are isolated [19]. Moreover, the Morse function f is said to be simple if each of its critical values corresponds to a different critical point. Accordingly, it makes sense to use the terminology index of a critical value c to indicate the index of f at the only critical point whose value is c.

Given $f \in \mathcal{M}_0(M,\mathbb{R})$, we denote by $K(f)$ the set of its critical points, and by f^a the set $f^{-1}((-\infty,a])$, $a \in \mathbb{R}$.

We want to show that, under the action of the group $\mathcal{D}(M)$ of $\mathcal{C}^2$-diffeomorphisms on M, the natural pseudo-distance $\delta_{\mathcal{D}}$ turns out to be a metric:

Theorem 4.2. *$(\mathcal{M}_0(M,\mathbb{R})/_{\sim_{\mathcal{D}}},\delta_{\mathcal{D}})$ is a metric space.*

By virtue of Corollary 3.2 (iv), the proof of Theorem 4.2 will be provided showing that any orbit in $\mathcal{M}_0(M,\mathbb{R})$ induced by the action of the group $\mathcal{D}(M)$ is closed. To be more precise, we will prove that, if (f_i) is a converging sequence of simple Morse functions with $d(f_i,f) \to_{i\to\infty} 0$ for some $f \in \mathcal{M}_0(M,\mathbb{R})$, and such that $f_i \in [\bar{f}]_{\mathcal{D}} = \{f' \in \mathcal{M}_0(M,\mathbb{R}) : f' = \bar{f} \circ h, h \in \mathcal{D}(M)\}$ for every i, then $f \in [\bar{f}]_{\mathcal{D}}$. All these notations will be maintained throughout the section.

The main tool we will use is a result by Kudryavsteva [16, Lemma 1], rewritten here in Lemma 4.11, that works only in the case of surfaces. It states that two Morse functions sharing the same collection of critical points, the same graph in the sense of Definition 4.9, and the same values at critical points, belong to the same equivalence class under the action of $\mathcal{D}(M)$.

The proof of Theorem 4.2 is by steps. Firstly, we prove that f and $\bar{f}$ share the same set of critical values with the same indices (Proposition 4.5); secondly, we show that each converging sequence of critical points of (f_i) corresponding to a certain critical value converges to the critical point of f corresponding to the same critical value (Corollary 4.7); thirdly, we demonstrate the existence of a function $f' \in [\bar{f}]_{\mathcal{D}}$ with the same collection of critical points, the same values at critical points as f (Proposition 4.8), and the same graph as the one of f (Remark 4.10). In this way, applying Lemma 4.11 to f and f', Theorem 4.2 is proved.

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The following two lemmas will be used to prove that f and the functions in $[\bar{f}]_{\mathcal{D}}$ have the same critical values with the same indices (Proposition 4.5).

Lemma 4.3. ([17, Lemma 4.1]) *Let $X_1, X_2, X_3, X_1', X_2', X_3'$ be topological spaces such that $X_1 \subseteq X_2 \subseteq X_3 \subseteq X_1' \subseteq X_2' \subseteq X_3'$. Assume that $H_k(X_3, X_1) = 0$ and $H_k(X_3', X_1') = 0$ for every $k \in \mathbb{Z}$. Then the homomorphism induced by inclusion $H_k(X_1', X_1) \to H_k(X_2', X_2)$ is injective for every $k \in \mathbb{Z}$.*

Lemma 4.4 ([7, Theorem A.3]). *Let $g \in \mathcal{M}_0(M, \mathbb{R})$, and let c be a critical value of index k of g. Then there exists a real number $\eta(g, c) > 0$ such that each function $g' \in \mathcal{M}_0(M, \mathbb{R})$ verifying $d(g, g') \le \eta(g, c)$ admits at least one critical value c' of index k for which $|c - c'| \le d(g, g')$.*

Proposition 4.5. *The functions f, $\bar{f}$, and all f_i have the same critical values with the same indices.*

Proof. Since $f_i \in [\bar{f}]_{\mathcal{D}}$, i.e. $f_i = \bar{f} \circ h_i$ for some $h_i \in \mathcal{D}(M)$, and critical values are preserved under diffeomorphisms, f_i and $\bar{f}$ share the same set of critical values with the same indices for every i.

Let us prove the claim for f_i and f. By Lemma 4.4, if c is a critical value of index k of f, then there exists a real number $\eta(f, c) > 0$ such that each f_i verifying $d(f_i, f) \le \eta(f, c)$ admits at least one critical value c' of index k, with $|c - c'| \le d(f_i, f)$. Let us underline that c' does not depend on the index i as seen at the beginning of the proof. Letting i tend to infinity, we obtain $c = c'$. This proves that the set of critical values of f is contained in the set of critical values of f_i for every i.

To show that this inclusion cannot be proper, let us assume, by contradiction, that there exists a $c \in \mathbb{R}$ that is a critical value for some, and hence all, f_i, and it is regular for f. Since $f \in \mathcal{M}_0(M, \mathbb{R})$, there exists a real number $\eta(f, c) > 0$ such that $[c - \eta(f, c), c + \eta(f, c)]$ does not contain any critical value of f. If we consider i large enough that $d(f_i, f) \le \eta(f, c)$, Lemma 4.4 implies the existence of at least one critical value of f distant less than $\eta(f, c)$ from c. This gives a contradiction. □

The result below shows that there exists a subsequence of critical points of (f_i) corresponding to a certain critical value which converges to the critical point of f corresponding to the same critical value.

Proposition 4.6. *Let c be a critical value of $\bar{f}$, and hence of f and all f_i. Let $q_i \in K(f_i) \cap f_i^{-1}(c)$ for every i. Then $q \in K(f) \cap f^{-1}(c)$ if and only if there exists a subsequence of (q_i) converging to q.*

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Proof. Let us begin by proving that, if $q \in K(f) \cap f^{-1}(c)$, then there exists a subsequence of (q_i) converging to q.

Let us assume by contradiction that no subsequences of (q_i) converging to q exist. Then a sufficiently small neighborhood $U \subset M$ of q can be found such that U contains neither any other critical point of f besides q, nor critical points of f_i for any i. Moreover, since $f \in \mathcal{M}_0(M, \mathbb{R})$ and we are assuming $f(q) = c$, there exists a real number $\eta(f, c) > 0$ such that $[c - 3 \cdot \eta(f, c), c + 3 \cdot \eta(f, c)]$ does not contain any critical value of f besides c.

Let us consider an index $\bar{\imath}$ so large that $\eta' = d(f_{\bar{\imath}}, f) < \eta(f, c)$, and write

$$f^{c-3\cdot\eta'} \cap U = X_1, \quad f_{\bar{\imath}}^{c-2\cdot\eta'} \cap U = X_2, \quad f^{c-\eta'} \cap U = X_3,$$
$$f^{c+\eta'} \cap U = X_1', \quad f_{\bar{\imath}}^{c+2\cdot\eta'} \cap U = X_2', \quad f^{c+3\cdot\eta'} \cap U = X_3'.$$

Since $X_1 \subseteq X_2 \subseteq X_3 \subseteq X_1' \subseteq X_2' \subseteq X_3'$, and both $H_k(X_3, X_1)$ and $H_k(X_3', X_1')$ are trivial for every $k \in \mathbb{Z}$, we can apply Lemma 4.3 to obtain that the homomorphism $H_k(X_1', X_1) \to H_k(X_2', X_2)$ induced by inclusion is injective for every $k \in \mathbb{Z}$. But assuming that c is a critical value of index $\bar{k}$ of f, $H_{\bar{k}}(X_1', X_1)$ is not trivial because the critical point $q \in f^{-1}([c - 3 \cdot \eta', c + \eta']) \cap U$, while $H_{\bar{k}}(X_2', X_2) = 0$ because $f_{\bar{\imath}}^{-1}([c - 2 \cdot \eta', c + 2 \cdot \eta']) \cap U$ does not contain any critical point of $f_{\bar{\imath}}$. This implies a contradiction.

Let us prove now that if there exists a subsequence of (q_i) converging to q, then $q \in K(f) \cap f^{-1}(c)$.

Let us denote again by (q_i) the subsequence converging to q. The fact that $d(f_i, f) \to_{i\to\infty} 0$, with $f_i(q_i) = c$ for every i, immediately implies $f(q) = c$. By contradiction, let us assume that q is a regular value of f. Since f is a simple Morse function, an arbitrarily small neighborhood $U \subset M$ of q can be found such that U does not contain critical points of f. Because of the convergence of (q_i), U contains the critical points q_i of f_i for every $i > \hat{\imath}$, for a certain index $\hat{\imath}$. Moreover, since $f_i \in \mathcal{M}_0(M, \mathbb{R})$ for every i, it is not restrictive to assume that U does not contain any other critical point of f_i besides q_i for every i large enough. Hence, under the assumption $f_i(q_i) = c$, a real number $\eta(f_i, c) > 0$ can be chosen such that $[c - 3 \cdot \eta(f_i, c), c + 3 \cdot \eta(f_i, c)]$ does not contain any critical value of f_i besides c.

Fixed an index $\bar{\imath}$ large enough, we obtain again a contradiction using the same arguments as in the first part of the proof with the roles of f and $f_{\bar{\imath}}$ exchanged. □

Corollary 4.7. *Let c be a critical value of $\bar{f}$, and hence of f and all f_i. Let $q_i \in K(f_i) \cap f_i^{-1}(c)$ for every i. Every converging subsequence of (q_i) converges to $q \in K(f) \cap f^{-1}(c)$.*

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Proof. It is sufficient to observe that, if two different converging subsequences of (q_i) converged to two different points q and q', respectively, by Proposition 4.6 we would have that both q and q' are critical points of f, with $f(q) = f(q')$, against the assumption that f is simple. □

The following result shows the existence of a function in $[\bar{f}]_{\mathcal{D}}$ having $K(f)$ as the set of its critical points, as well as the same values at critical points as f.

Proposition 4.8. *There exists an $f' \in [\bar{f}]_{\mathcal{D}}$ such that f and f' have the same collection of critical points, the same index and the same value at each of them.*

Proof. Let $K(f_i) = \{q_i^1, \dots, q_i^n\}$ and $K(f) = \{q^1, \dots, q^n\}$, and let us assume that $f_i(q_i^j) = f(q^j)$ for every $j = 1, \dots, n$. We apply iteratively Corollary 4.7 to extract a subsequence of (f_i), say again (f_i), such that, for every $j = 1, \dots, n$, the sequences of critical points (q_i^j) converge to $q^j \in K(f)$. Moreover, for every $j = 1, \dots, n$, let (U^j, ψ^j) be a local chart centered at q^j, and fix an index i large enough that $q_i^j \in U^j$.

Let $h : M \to M$ be a $\mathcal{C}^2$-diffeomorphism such that

$$h(p) = \begin{cases} p, & p \in M \setminus \bigcup_{j=1}^n U^j, \\ (\psi^j)^{-1} \circ h_i^j \circ \psi^j(p), & p \in U^j,\ j = 1, \dots, n \end{cases}$$

where, denoting by D^2 the unit 2-disk in $\mathbb{R}^2$, $h_i^j : D^2 \to D^2$ is a diffeomorphism which takes $\psi^j(q^j)$ to $\psi^j(q_i^j)$, and is the identity in a neighborhood of ∂D^2. Then we can define the function $f' : M \to \mathbb{R}$ as $f' = f_i \circ h$.

We observe that f' is a simple Morse function because obtained from f_i by composition with a diffeomorphism. Hence $f' \in [\bar{f}]_{\mathcal{D}}$. By Proposition 4.5, f' and f have the same critical values with the same indices. Furthermore, by construction, $h(q^j) = q_i^j$, and therefore, $f'(q^j) = f_i(h(q^j)) = f_i(q_i^j) = f(q^j)$. Since diffeomorphisms take critical points to critical points, $q^j \in K(f')$. This shows that $K(f') \subset K(f)$. On the other side, $K(f')$ and $K(f)$ have the same cardinality, so $K(f') = K(f)$. In conclusion, f and f' have the same set of critical points $\{q^1, \dots, q^n\}$, and the same values at them since $f'(q^j) = f(q^j)$ for every $j = 1, \dots, n$. □

Let us recall the following concept introduced in [16].

Definition 4.9. Let $g : M \to \mathbb{R}$ denote a Morse function with r saddle points $p_1, \dots, p_r$. The graph G_g associated with g is the graph obtained from the set $g^{-1}\{g(p_1), \dots, g(p_r)\}$ by removing all connected components containing no saddle critical points.

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Note that the graph G_g in Definition 4.9 has r vertices (which are the saddle points $p_1, \dots, p_r$); the degree of each vertex is equal to 4 and hence the graph has $2r$ edges.

Remark 4.10. The functions f, f' considered in Proposition 4.8 have the same associated graph in virtue on the fact that they are both simple Morse functions.

To prove Theorem 4.2, we use the following Lemma 4.11.

Lemma 4.11 ([16, Lemma 1]). *Let $g, g' : M \to \mathbb{R}$ be Morse functions with the same collection of critical points, the same graph $G_g = G_{g'}$, and the same values at critical points. Then*

$$g = g' \circ h$$

for some $h \in \mathcal{D}(M)$ homotopic to id_M.

Proof of Theorem 4.2. Observe that Proposition 4.8 and Remark 4.10 allow to apply Lemma 4.11 to f and f'. This proves that $[f]_{\mathcal{D}} = [f']_{\mathcal{D}}$, and therefore that the orbits induced by $\mathcal{D}(M)$ on $\mathcal{M}_0(M, \mathbb{R})$ are closed. Eventually, applying Corollary 3.2 (iv), the claim follows. □

Bibliography

[1] S. Biasotti, L. De Floriani, B. Falcidieno, P. Frosini, D. Giorgi, C. Landi, L. Papaleo and M. Spagnuolo, Describing shapes by geometrical-topological properties of real functions, *ACM Comput. Surv.* **40** (2008), 1–87.

[2] A. M. Bronstein, M. M. Bronstein and R. Kimmel, Topology-invariant similarity of nonrigid shapes, *Int. J. Comput. Vis.* **81** (2009), 281–301.

[3] D. Burago, Y. Burago and S. Ivanov, *A Course in Metric Geometry*, Grad. Stud. Math. 33, American Mathematical Society, Providence, 2001.

[4] F. Cagliari, Natural pseudodistance for shape comparison and quotients of metric spaces, *Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia* **57** (2010), 63–67.

[5] A. Cerri and B. Di Fabio, On certain optimal diffeomorphisms between closed curves, *Forum Math.* (2011), DOI 10.1515/form.2011.172.

[6] F. Chazal, D. Cohen-Steiner, L. J. Guibas, F. Mémoli and S. Y. Oudot, Gromov–Hausdorff stable signatures for shapes using persistence, *Computer Graphics Forum (proc. SGP 2009)* (2009), 1393–1403.

[7] B. Di Fabio and C. Landi, Reeb graphs of curves are stable under function perturbations, *Math. Methods Appl. Sci.* **35** (2012), 1456–1471.

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[8] P. Donatini and P. Frosini, Natural pseudodistances between closed manifolds, *Forum Math.* **16** (2004), 695–715.

[9] P. Donatini and P. Frosini, Natural pseudodistances between closed surfaces, *J. Eur. Math. Soc. (JEMS)* **9** (2007), 231–253.

[10] P. Donatini and P. Frosini, Natural pseudo-distances between closed curves, *Forum Math.* **21** (2009), 981–999.

[11] J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1966.

[12] P. Frosini and C. Landi, No embedding of the automorphisms of a topological space into a compact metric space endows them with a composition that passes to the limit, *Appl. Math. Lett.* **24** (2011), 1654–1657.

[13] P. Frosini and M. Mulazzani, Size homotopy groups for computation of natural size distances, *Bull. Belg. Math. Soc. Simon Stevin* **6** (1999), 455–464.

[14] C. J. Himmelberg, Pseudo-metrizability of quotient spaces, *Fund. Math.* **63** (1968), 1–6.

[15] S. Kobayashi, *Transformation Groups in Differential Geometry*, Springer-Verlag, Berlin, 1972.

[16] E. A. Kudryavtseva, Uniform Morse lemma and isotopy criterion for Morse functions on surfaces, *Moscow Univ. Math. Bull.* **64** (2009), 150–158.

[17] A. Marino and G. Prodi, Metodi perturbativi nella teoria di Morse, *Boll. Unione Mat. Ital. (4)* **11** (1975), 1–32.

[18] P. W. Michor, *Manifolds of Differentiable Mappings*, Shiva, Orpington, 1980.

[19] J. Milnor, *Morse Theory*, Based on lecture notes by M. Spivak and R. Wells, Ann. of Math. Stud. 51, Princeton University Press, Princeton, 1963.

[20] S. Willard, *General Topology*, Addison-Wesley, Reading, 1970.

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