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A class of singular first order differential equations with applications in reaction-diffusion

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Abstract

We study positive solutions $y(u)$ for the first order differential equation

$$y' = q(c\,y^{\frac{1}{p}} - f(u))$$

where $c > 0$ is a parameter, $p > 1$ and $q > 1$ are conjugate numbers and f is a continuous function in $[0, 1]$ such that $f(0) = 0 = f(1)$. We shall be particularly concerned with positive solutions $y(u)$ such that $y(0) = 0 = y(1)$. Our motivation lies in the fact that this problem provides a model for the existence of travelling wave solutions for analogues of the FKPP equation in one spacial dimension, where diffusion is represented by the p-Laplacian operator. We obtain a theory of admissible velocities and some other features that generalize classical and recent results, established for $p = 2$.

Key words: p-Laplacian, FKPP equation, heteroclinic, travelling wave, critical speed, sharp solution.

AMS Subject Classification: 34B18, 34C37, 35K57.

1 Introduction

In this paper we study some features of positive solutions to ordinary differential equations of the form

$$y' = q(c\,{y_+}^{\frac{1}{p}} - f(u)), \quad 0 \le u \le 1 \tag{1.1}$$

where $y_+ = \max(y, 0)$. We look for certain positive solutions $y = y(u)$ of (1.1) that vanish at one or both endpoints of the interval $[0, 1]$. Here p, q are positive numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1,$$

$c > 0$ is a parameter, and $f : [0, 1] \to \mathbb{R}$ is a continuous function of types A, B or C, by which we mean

(Type A) $f(0) = f(1) = 0$ and $f(u) > 0$ if $u \in (0, 1)$.

(Type B) $f(0) = f(1) = 0$ and there exists $\alpha \in (0, 1)$ such that $f(u) = 0$ if $u \in [0, \alpha]$ and $f(u) > 0$ if $u \in (\alpha, 1)$.

(Type C) $f(0) = f(1) = 0$ and there exists $\alpha \in (0, 1)$ such that $f(u) < 0$ if $u \in (0, \alpha)$ and $f(u) > 0$ if $u \in (\alpha, 1)$.

This terminology was introduced by Berestycki and Nirenberg ([4]).

The motivation for considering (1.1) is the following. Consider the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[D(u)\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right] + g(u), \tag{1.2}$$

which, in case $p = 2$, provides a model for a large variety of biological and chemical phenomena. We refer the reader to [3], [4],[11], [17], just to cite a few, and their references. In this equation g is a reaction term, while the first term in the right-hand side represents density dependent nonlinear diffusion in one-dimensional space. In the case $p = 2$ this is the well known FKPP equation (without convection term). Recently, models where the p-Laplacian operator replaces the usual Laplacian have been considered in the literature (e.g. [15], [8]).

When g is of one of the types A, B or C, $u = 0$ and $u = 1$ are two equilibrium solutions. An important problem related to this equation is that of finding travelling wave solutions, that is, solutions of the form $u(t, x) = U(x - ct)$ for some $c > 0$. Here c is the propagation speed of the wave. It is in addition required that the wave front $U(s)$ is defined in $(-\infty, +\infty)$ and satisfies $U(-\infty) = 1$, $U(+\infty) = 0$. This amounts to look for the solutions of the second order ordinary differential equation

$$(D(u)|u'|^{p-2}u')' + cu' + g(u) = 0 \tag{1.3}$$

satisfying the limit conditions

$$u(-\infty) = 1, \; u(+\infty) = 0. \tag{1.4}$$

For certain values of the parameter c, such solutions are in addition monotone, with $u' < 0$ in their whole domain. Now if we set

$$-v := D(u)|u'|^{p-2}u',$$

for such monotone decreasing solutions, v may be seen as function of u. For simplicity, assume that $D(u) > 0$ for all $u \in (0,1)$. Then a simple calculation shows that $v = v(u)$ must satisfy

$$\frac{1}{qD(u)^{q-1}}\frac{d}{du}v^q - c\left(\frac{v}{D(u)}\right)^{q-1} + g(u) = 0$$

and therefore if we define

$$y(u) = v(u)^q$$

the function y will solve (1.1) with $f(u) = D(u)^{q-1}g(u)$.

Moreover, the conditions (1.4) for monotone solutions defined in the real line imply that

$$u'(-\infty) = 0, \ \ u'(+\infty) = 0.$$

This fact is well known in case $p = 2$ (see for instance [5]) and the argument easily carries out to the general case, as we show later for completeness. In terms of y this translates into

$$y(0) = 0, \quad y(1) = 0, \tag{1.5}$$

thus motivating the study of the existence of solutions of (1.1)-(1.5).

The study of admissible speeds for the problem (1.3) in case $p = 2$ has a long and rich history, starting with the seminal paper by Kolmogorov, Petrovski and Piscounov [10], including the in-depth approach by Aronson and Weinberger [3], and many recent contributions that the reader may find in the references. In Gilding and Kersner [6] and Malaguti and Marcelli [13] a singular integral equation technique has been used in the investigation of (1.3) and analogue equations for $p = 2$.

In this paper we propose, alternatively, to study the singular differential equation (1.1), thus constructing a first order model for admissible speeds and asymptotic behaviour (a method already used in [5]). This in turn provides information for (1.3) that is the counterpart of classic and recent results which have been obtained along years by many authors in case $p = 2$. In particular, we consider the differences between problems with functions of type A, on one hand, and with types B and C, on the other [3, 4]; we acknowledge the occurrence of sharp solutions, which were found in [16] and later systhematized in [14]; we deal with sign change in diffusion density [11, 12] and with negative density diffusion [17].

The first order theory is developed in sections 2 to 4 and the applications to second order equations are given in section 6.

Notation. Let us introduce some notation and basic conditions to be used in the next sections. For $c > 0$, we consider the function $\phi_c : [0, \infty) \to \mathbb{R}$ defined by

$$\phi_c(z) = cz^{1/p} - z. \tag{1.6}$$

We remark that ϕ_c vanishes at the two points 0 and c^q, is positive if and only if $0 < z < c^q$ and ϕ_c attains its absolute maximum M_c at a point $\omega_c \in (0, c^q)$. Namely: $\omega_c = (c/p)^q$, $M_c = \omega_c(p-1)$. Now for any $x \in [0, M_c)$, the function ϕ_c takes the value x at exactly two points, let us say $\omega_c^-(x) \in [0, \omega_c)$ and $\omega_c^+(x) \in (\omega_c, c^q]$: in particular, we set

$$J_c(x) = [\omega_c^-(x), \omega_c^+(x)].$$

If $x \geq M_c$ we set, by definition, $\omega_c^-(x) = \omega_c^+(x) = \omega_c$.

Also, we shall be dealing with functions f of type A satisfying

$$\sup_{u\in(0,1)} \frac{f(u)}{u^{q-1}} = \mu < +\infty \tag{1.7}$$

or the stronger property

$$\lim_{u\to 0^+} \frac{f(u)}{u^{q-1}} = \lambda < +\infty. \tag{1.8}$$

2 Functions of type A: existence of solutions and admissible speeds

Consider the boundary value problem

$$\begin{cases} y'(u) = q(c\, y_+(u)^{\frac{1}{p}} - f(u)), \quad 0 \leq u \leq 1 \\ y(0) = y(1) = 0, \end{cases} \tag{2.1}$$

for which we look for positive solutions in the interval $(0,1)$. By a solution we mean a function $y \in C^1[0,1]$ satisfying the equation above and the boundary conditions. In what follows *positive solution* means a solution y such that $y(u) > 0\ \forall u \in (0,1)$.

The following proposition allows us to conclude that the existence of a positive lower solution satisfying a strict inequality in the interval $(0,1]$ is enough to get existence and uniqueness for (2.1).

Proposition 2.1. *Let f be of type A. Suppose that $s(u)$ is a C^1-function in $[0,1]$ such that $s(0) = 0$, $s(u) > 0$ if $u \in (0,1)$ and for all $u \in (0,1]$,*

$$s'(u) \leq q\left(c\, s(u)^{\frac{1}{p}} - f(u)\right). \tag{2.2}$$

Then (2.1) *has a unique positive solution.*

Proof. By a well known argument, (2.2) implies that there exists a solution $y(u)$ of the differential equation in (2.1) with $y(0) = 0$ such that $s(u) \leq y(u)$. Now consider the solution $\bar{y}$ of the initial value problem

$$\bar{y}' = q\left(c\, \bar{y}_+^{\frac{1}{p}} - f(u)\right), \quad \bar{y}(1) = 0. \tag{2.3}$$

(In fact this problem enjoys uniqueness in $[0,1]$ because the right-hand side of the equation is nondecreasing in the dependent variable.) It is easy to see that $\bar{y} \geq 0$ in $[0,1]$. Moreover $0 < \bar{y}(u) < y(u)$ for all $u \in (0,1)$. For, if u_0 is a zero of $\bar{y}$ in $(0,1)$, then the differential equation implies $\bar{y}'(u_0) < 0$, which is a contradiction with the fact that $\bar{y} \geq 0$; and if there exists $u_1 \in (0,1)$ such that $\bar{y}(u_1) = y(u_1)$, then by uniqueness of solution we would have $\bar{y} = y$, which contradicts the fact that $\bar{y}(1) = 0$. By continuity we have $\bar{y}(0) = 0$. The fact that the solution is unique is a direct consequence of uniqueness for (2.3). □

Remark 2.2. One could also have invoked the fact that that the functions s and 0 are lower and upper solutions with respect to the periodic problem for (1.1) in $[0, 1]$.

Proposition 2.3. *Assume that f is a function of type A in $[0, 1]$ satisfying (1.7). Then there exists a constant $c^* > 0$ (depending on f and p) such that (2.1) admits a unique positive solution if and only if $c \geq c^*$. Moreover we have the estimate $c^* \leq q^{\frac{1}{q}} p^{\frac{1}{p}} \mu^{\frac{1}{q}}$.*

Proof. It is obvious that for c large enough, the inequality $\phi_c(\beta) \geq \mu$ has positive solutions β if and only if $\omega_c(p - 1) \geq \mu$, that is,

$$c \geq q^{\frac{1}{q}} p^{\frac{1}{p}} \mu^{\frac{1}{q}}.$$

For one such β, let $s(u) = \beta u^q$. Then, for all $u \in (0, 1]$, we have

$$s'(u) = q\beta\, u^{q-1} \leq \left(c\,\beta^{\frac{1}{p}} - \mu\right) q\, u^{q-1} \leq q\left(c\, s(u)^{\frac{1}{p}} - f(u)\right).$$

The previous proposition allows us to conclude that for such value c, the boundary value problem (2.1) has a unique positive solution.

Now let c^* be the infimum of the values $c > 0$ such that problem (2.1) has a unique positive solution. The estimate for c^* follows from what we have just seen. Let us prove that for all $c > c^*$, problem (2.1) has a solution. Given $c_1 > c^*$, let us consider a value $\tilde{c}$ such that (2.1) has a positive solution $y_{\tilde{c}}$ and $c^* < \tilde{c} < c_1$. For all $u \in (0, 1]$ we have

$$y_{\tilde{c}}' = q\left(\tilde{c}y_{\tilde{c}}^{\frac{1}{p}} - f(u)\right) < q\left(c_1 y_{\tilde{c}}^{\frac{1}{p}} - f(u)\right),$$

so $y_{\tilde{c}}$ is a lower solution for the problem with $c = c_1$ and by the previous proposition, we conclude the solvability for (2.1) with $c = c_1$.

To prove the solvability for $c = c^*$, consider a decreasing sequence c_n tending to c^* and the correspondent positive solutions y_n. First, the argument we used at the end of the proof of Proposition 2.1 shows that $y_n \leq y_1\ \forall n$. It is also easy to conclude that the sequence (y_n) is uniformly bounded in $C^1([0, 1])$. By the Ascoli-Arzelà theorem, there exists a continuous function y^* such that $y_n \to y^*$ uniformly in $[0, 1]$. It turns out that y^* satisfies (2.1) with $c = c^*$ and $y^* > 0$ in $(0, 1)$.

Let us now prove that $c^* > 0$. If $c^* = 0$, using the above notation we would obtain $y^*(u) = -q\int_0^u f(s)\,ds$ meaning that $y^* < 0$ in $(0, 1]$, a contradiction. □

3 Behaviour of the solutions near the origin

Throughout this section we suppose, as in the previous one, that f is a type A function, and try to get more detailed information about the behaviour of the solutions of equation (1.1) near 0, under the only condition $y(0) = 0$. To this end, we need conditions (1.7) or (1.8) on f. We are going to show that, whenever (1.8) holds:

$$(a)\ \lim_{u\to 0} \frac{y(u)}{u^q} = \omega_c^-(\lambda) \quad \text{or} \quad (b)\ \lim_{u\to 0} \frac{y(u)}{u^q} = \omega_c^+(\lambda). \tag{3.1}$$

In particular we point out that, if the stronger condition

$$\sup_{u\in(0,1)} \frac{f(u)}{u^q} < +\infty \tag{3.2}$$

holds, then $\lambda = 0$, so that $\omega_c^-(\lambda) = 0$. In this case, actually, we can say more than (3.1a), namely:

$$\sup_{0<u\le 1} \frac{y(u)}{u^{pq}} < +\infty. \tag{3.3}$$

In order to show the properties above, we need some preliminary results. To this end, let $y \in \mathcal{S}_c$ be fixed, and put

$$\gamma(u) = \frac{y'(u)}{qu^{q-1}}, \quad z(u) = \frac{y(u)}{u^q}, \quad \lambda(u) = \frac{f(u)}{u^{q-1}}. \tag{3.4}$$

Let us denote respectively by γ^-, l^- and λ^- the lower limits of $\gamma(u)$, $z(u)$ and $\lambda(u)$ as $u \to 0$. Similarly, the scripts γ^+, l^+ and λ^+ will stand for the corresponding upper limits. From

$$y' = q(c\, y^{\frac{1}{p}} - f(u)) \tag{3.5}$$

it is easy to see that $z(u)$, at least for small values of u, solves the following differential equation:

$$z' = \frac{q}{u}(\phi_c(z) - \lambda(u)). \tag{3.6}$$

Lemma 3.1. *Let f satisfy (1.7), $y \in \mathcal{S}_c$, $\lambda^\pm$ and $l^\pm$ be defined as above. Then:*

(a) $M_c \ge \lambda^-$, that is: $c \ge (\lambda^- q)^{1/q} p^{1/p}$. In particular: $c^* \ge (\lambda^- q)^{1/q} p^{1/p}$.

(b) $l^+ \in J_c(\lambda^-)$, $l^- \notin J_c(\lambda^+)^\circ$.

(c) $\lambda^+ < M_c \;\Rightarrow\; \omega_c \notin (l^-, l^+)$.

(d) (1.8) $\Rightarrow$ (3.1).

Proof. (a) We remark that $\gamma^- \le l^- \le l^+ \le \gamma^+$, as follows easily from Cauchy's theorem. Furthermore the function y, as long as it is positive, solves equation (3.5). If we divide both its sides by qu^{q-1}, and recall that $(q-1)p = q$, we get:

$$\gamma(u) = cz(u)^{1/p} - \lambda(u). \tag{3.7}$$

But now (3.7) implies that $\gamma^+ \le c(l^+)^{1/p} - \lambda^-$, as we can argue on taking the upper limit of both sides as $u \to 0$. On the other hand, since $\gamma^+ \ge l^+$, we actually get $\lambda^- \le \phi_c(l^+) \le M_c$.

(b) It is enough to remark that $\phi_c(l^+) \ge \lambda^-$ and $\phi_c(l^-) \le \lambda^+$. Indeed, the first inequality was shown in the previous step. As regards the latter, it can be achieved in a similar way, when taking in (3.7) a lower limit instead than an upper limit.

(c) Suppose, by contradiction, $l^- < \omega_c < l^+$: in particular, the maximum among the three values λ^+, $\phi_c(l^-)$, $\phi_c(l^+)$, say h, is less than M_c. Now, let $j, m \in (h, M_c)$ such that $j < m$, and put $I = [\rho^-, \rho^+] := \phi_c^{-1}([m, M_c])$, so that $I \subseteq (l^-, l^+)$. According to the definitions of λ^+, l^- and l^+ we can find $\delta > 0$ such that $\lambda(u) \le j$ for $0 < u \le \delta$, and two points u^- and $u^+ \in (0, \delta]$ such that $u^- < u^+$, $z(u^-) = \rho^+$, $z(u^+) = \rho^-$ and

$\rho^+ \le z(u) \le \rho^-$ for $u^- \le u \le u^+$. In particular, the interval $[u^-, u^+]$ must contain a point θ at which $z' < 0$. On the other hand $\phi_c(z(\theta)) \ge m$, so that (3.6) would yield the contradiction $z'(\theta) \ge (q/\theta)(m-j) > 0$. Hence, actually, $\omega_c \notin (l^-, l^+)$.

(d) Since $\lambda^- = \lambda^+ = \lambda$, from (a) we get $\lambda \le M_c$, and applying claim (c) we infer that l^- and l^+ lie on the same side with respect to ω_c. On the other hand, let us replace λ^- and λ^+ in claim (b) by their common value λ: according to whether, respectively, l^- and l^+ lie to the left or to the right of ω_c, we infer what follows: either $l^- \le \omega_c^-(\lambda) \le l^+ \le \omega_c$ or $\omega_c \le l^- \le \omega_c^+(\lambda) \le l^+$. In both cases it is enough to show that $l^- = l^+$. As regards the former, let us suppose, by contradiction, that $l^- < l^+$: then both values l^- and l^+ can be approximated along a sequence of local extrema of z, which are, in particular, critical values. More precisely, we can find points a_i and b_i ($i \in \mathbf{Z}^+$) at which z' vanishes, in such a way that $a_i \to 0$, $b_i \to 0$ and the sequences $(z(a_i))_i$ and $(z(b_i))_i$ converge respectively to l^- and l^+. Since $z'(a_i) = z'(b_i) = 0$, (3.6) entails $\phi_c(z(a_i)) = \lambda(a_i)$ and $\phi_c(z(b_i)) = \lambda(b_i)$. Now, let us first suppose $l^+ < \omega_c$: then both equalities $z(a_i) = \omega_c^-(\lambda(a_i))$ and $z(b_i) = \omega_c^-(\lambda(b_i))$ hold for large values of i: since ω_c^- is continuous, and $\lambda(u) \to \lambda$ as $u \to 0$, from the previous relations we get, as $i \to +\infty$, the contradiction $l^- = l^+ = \omega_c^-(\lambda)$. Now assume let $l^+ = \omega_c$: then possibly $z(b_i) = \omega_c^+(\lambda(b_i))$ for infinitely many values of i: in this case, however, $\omega_c = l^+ = \omega_c^+(\lambda)$, so that, actually, $\lambda = M_c$. Then we can write again $l^+ = \omega_c^-(\lambda)$, and get the same contradiction as before. Finally in the case $\omega_c \le l^- \le \omega_c^+(\lambda) \le l^+$ the conclusion is straightforward by virtue of (b). □

Corollary 3.2. *If (1.7), (1.8) hold and $\mu = \lambda$, then $c^* = q^{\frac{1}{q}} p^{\frac{1}{p}} \lambda^{\frac{1}{q}}$.*

Proof. It sufices to combine the Proposition 2.3 with Lemma 3.1 (a). □

This generalizes the well known result for the case $p = 2$, where $\lambda = f'(0)$, for which $M = f'(0)$ implies $c^* = 2\sqrt{f'(0)}$.

Now, let $r, A, c > 0$ be fixed. For any function $y \in C([0, r])$ we denote by $N(y)$ the supremum of $|y(u)|/u^q$ for $0 < u \le r$: then it is easy to check that the subspace V of $C([0, r])$ where $N(y) < +\infty$ is a Banach space with respect to the norm $\|y\| := N(y)$. Now we define a closed subset E of V and a map $T : E \to V$ as follows:

$$E = \{y \in V : y(u) \ge Au^q, \ 0 \le u \le r\}, \tag{3.8}$$

$$[T(y)](u) = q \int_0^u (cy(s)^{1/p} - f(s)) ds, \quad y \in E, \quad 0 \le u \le r. \tag{3.9}$$

Lemma 3.3. *Let f fulfil (1.8), $\nu = sup\{f(u)/u^{q-1};\ 0 < u \le r\}$. Then the following properties hold.*

(a) $T(E) \subseteq V$.

(b) If $A > \omega_c$, $T : E \to V$ is a contraction with respect to $\|\cdot\|$.

(c) If $\phi_c(A) \ge \nu$, then $T(E) \subseteq E$. In particular, if $\omega_c < A \le \omega_c^+(\nu)$, T is a contraction of E into itself.

Proof. (a) If $y \in E$, then obvously $w := T(y) \in C([0,r])$. In order to prove that $\|w\| < +\infty$ we only need to divide both sides of the following inequality by u^q, and take the supremum for $0 < u \le r$.

$$w(u) \le qc \int_0^u y(s)^{1/p} ds \le qc \int_0^u (\|y\| s^q)^{1/p} ds = c\|y\|^{1/p} u^q. \tag{3.10}$$

(b) We notice that, for any $\alpha > 0$, the function $y^{1/p}$ admits, on the half-line $[\alpha, +\infty)$, the Lipschitz constant $L(\alpha) = (p\alpha^{1/q})^{-1}$. Now, for $i = 1, 2$, let $y_i \in E$, $w_i = T(y_i)$. Then:

$$\begin{aligned} |w_2(u) - w_1(u)| &\le cq \int_0^u |y_2(s)^{1/p} - y_1(s)^{1/p}| ds \le \\ &\le cq \int_0^u L(As^q) |y_2(s) - y_1(s)| ds \le \\ &\le \frac{cq}{p} \int_0^u (As^q)^{-1/q} \|y_2 - y_1\| s^q ds = \frac{c}{p} A^{-1/q} u^q \|y_2 - y_1\|. \end{aligned} \tag{3.11}$$

Also here we can divide the extreme sides by u^q and take the supremum for $0 < u \le r$, so as to infer that $k = (c/p)A^{-1/q}$ is a Lipschitz constant for T with respect to $\|\cdot\|$. But the condition $A > \omega_c$ is just equivalent to $k < 1$.

(c) If $y \in E$ and $w = T(y)$, then $w(u) \ge q \int_0^u [c(As^q)^{1/p} - \nu s^{q-1}] ds$, where the right-hand side is precisely $(cA^{1/p} - \nu)u^q$. Hence $w(u) \ge Au^q$ if and only if $\phi_c(A) \ge \nu$. As regards the last claim, it is enough to remark that the two conditions $\phi_c(A) \ge \nu$ and $A > \omega_c$ hold together if and only if $\omega_c < A \le \omega_c^+(\nu)$. □

Remark 3.4. The condition $\phi_c(A) \ge \nu$ in Lemma 3.3 is equivalent to state that the function Au^q is a subsolution of (1.1) on $[0, r]$.

Proposition 3.5. *Let f satisfy (1.8), set $\bar{c} := (\lambda q)^{1/q} p^{1/p}$, and assume $c > \bar{c}$. Then the following properties hold true.*

- *(a) $\mathcal{S}_c$ contains exactly one function y which verifies (3.1b), say $y =: \psi_c$.*
- *(b) If $y \in \mathcal{S}_c$, $y \ne \psi_c$, then (3.1a) holds.*
- *(c) If $y \in \mathcal{S}_c$, $y \ne \psi_c$, then $\psi_c(u) > y(u)$ for any $u \in (0, 1]$.*
- *(d) If w is a subsolution of (1.1) and $w(0) = 0$, then $\psi_c \ge w$ on $[0, 1]$.*
- *(e) If $\theta > c$ then $\psi_\theta(u) > \psi_c(u)$ for any $u \in (0, 1]$.*
- *(f) $\sup \{ |\psi_\theta(u) - \psi_c(u)|/u^q;\ u \in (0, 1] \} \to 0$ as $\theta \to c$.*
- *(g) If $y \in \mathcal{S}_c$, $y \ne \psi_c$ and (3.2) holds, then y satisfies (3.3).*

Proof. (a) From our condition on c and Lemma 3.1-(a) we get $M_c > \lambda$: therefore, if $r > 0$ is suitably small, the number ν which appears in Lemma 3.3 lies below M_c as well, that is $\omega_c < \omega_c^+(\nu)$. So, let $\omega_c < A < \omega_c^+(\nu)$, and define E and T as in (3.8), (3.9). Since E is a closed subset of the Banach space $(V, \|\cdot\|)$, Lemma 3.2-(c) and Banach's contraction principle ensure that T admits a unique fixed point y, which obviously solves (1.1) on $[0,r]$ and fulfils the condition $y(0) = 0$. In particular, the extension of y to the whole interval $[0,1]$ (as a solution of (1.1)) belongs to $\mathcal{S}_c$. On the other hand, since $y(u) \geq Au^q$ on $[0,r]$, of (3.1a) and (3.1b) only the latter can hold. As regards uniqueness, let $\tilde{y} \in \mathcal{S}_c$ fulfil (3.1b): then $\tilde{y}$ belongs to the same space E as before, and is again a fixed point for T, so that, necessarily, $\tilde{y} = y$.

(b) It follows at once from Lemma 3.1-(d).

(c) Since y and ψ_c satisfy respectively (3.1(a)) and (3.1(b)), and $\omega_c^-(\lambda) < \omega_c^+(\lambda)$, the inequality $\psi_c(u) > y(u)$ surely holds in a right neighbourhood of 0, say $(0,\rho]$. By contradiction, let $\sigma \in (\rho, 1]$ be the first point at which the function $z = \psi_c - y$ vanishes: since $z' \geq 0$ on $[0,\sigma]$ and $z(0) = z(\sigma) = 0$, we should get the contradiction $\psi_c \equiv y$ on $[0,\sigma]$.

(d) By virtue of the previous claim, the inequality $\psi_c \geq y$ holds true for any $y \in \mathcal{S}_c$. On the other hand, since w is a subsolution of (1.1), we can find $y \geq w$ such that $y(0) = 0$ and (1.1) holds: then $w \leq y \leq \psi_c$.

(e) If $\theta > c$, then $\phi_\theta > \phi_c$ and therefore $\omega_c^+ < \omega_\theta^+$. Hence $\psi_c < \psi_\theta$ in a right neighborhood of 0, by virtue of (3.1b). Then the inequality in fact holds in $(0,1]$.

(f) Let r, ν and A be again as in the previous steps. Since M_c, ω_c and $\omega_c^+(\nu)$ depend continuously on c, let $\alpha \in (\bar{c}, c)$, $\beta > c$ such that $M_\alpha > \nu$ and $\omega_\beta < A < \omega_\alpha^+(\nu)$. For any $\theta \in U := (\alpha, \beta)$ put $c = \theta$ in (3.9), denote by T_θ the corresponding map and by ψ_θ^r the restriction of ψ_θ to $[0,r]$, which can be characterized as the unique fixed point of T_θ. We point out that the maps T_θ, for $\theta \in U$, are defined on the same set E we introduced in the proof of claim (a), a set which does not depend on θ. Furthermore, the map $(\theta, y) \mapsto T_\theta(y)$ is continuous, and $k = (\beta/p)A^{-1/q} < 1$ is a Lipschitz constant, with respect to the norm of V, for all maps T_θ, $\theta \in U$. Then it is easy to show that the fixed point of T_θ depends continuously on θ. More precisely: the map $\theta \mapsto \psi_\theta^r$ is continuous from U to $(E, \|\cdot\|)$, and the same we can say, as a consequence, for the map $\theta \mapsto \psi_\theta(r)$ from U to $\mathbb{R}$. Then well-known results about the dependence on initial data of the solution of a Cauchy problem entail that, as $\theta \to c$, $\psi_\theta \to \psi_c$, uniformly on $[r,1]$. Now, let us put $\Delta(\theta) = \|\psi_\theta^r - \psi_c^r\|$, and denote by $S(\theta)$ the supremum of $|\psi_\theta - \psi_c|$ over $[r,1]$: according to the previous arguments, both $\Delta(\theta)$ and $S(\theta)$ converge to 0 as $\theta \to c$. On the other hand, the supremum which appears in our claim does not exceed $\max(\Delta(\theta), S(\theta)/r^q)$.

(g) Let $K < +\infty$ be the supremum in (3.2): in particular, as we already pointed out, (1.8) holds true with $\lambda = 0$. Since we are dealing with a function y which does not fulfil (3.1b), and $\omega_c^-(0) = 0$, from (3.1a) we argue that $y(u)/u^q$ converges to 0 as $u \to 0$, and the same we can say of $y(u)^{1/q}/u$. Now, let us suppose, by contradiction, that (3.3) is not satisfied: actually, in this case, the ratio $y(u)/u^{pq}$ is not bounded from above on any right neighbourhood of 0, and the same we can say of $y(u)^{1/p}/u^q$. By combining the two previous remarks, we easily find $\varepsilon > 0$ such that

$$\frac{y(\varepsilon)^{1/p}}{\varepsilon^q}\left(c - p\,\frac{y(\varepsilon)^{1/q}}{\varepsilon}\right) \geq K. \tag{3.12}$$

Let $h = y(\varepsilon)/\varepsilon^{pq}$: then the function $w(u) = hu^{pq}$ satisfies the conditions

$$(a)\ w(\varepsilon) = y(\varepsilon), \quad (b)\ w'(u) \leq q(cw(u)^{1/p} - f(u)),\ 0 \leq u \leq \varepsilon. \tag{3.13}$$

Indeed, (3.13a) is obviously due to our choice of h, while (3.13b) can be proved as follows:

$$\begin{aligned} f(u) &\leq Ku^q \leq [c(y(\varepsilon)^{1/p}/\varepsilon^q) - p\varepsilon^{p-1}(y(\varepsilon)/\varepsilon^{pq})]u^q = \\ &= (ch^{1/p} - ph\varepsilon^{p-1})u^q \leq ch^{1/p}u^q - phu^{pq-1} = cw(u)^{1/p} - (w'(u)/q). \end{aligned} \tag{3.14}$$

In particular: the second inequality in (3.14) follows from (3.12). The inequality of the second line of (3.14) comes from $u \leq \varepsilon$ and $p + q = pq$. Then (3.13b) holds true as well, so that w is a subsolution of (1.1), and (3.13a) implies $y \leq w$ on $[0, \varepsilon]$. But now, from the expression of w, we conclude that (3.3) holds, in contrast with our initial assumption. □

Remark 3.6. Let us consider the estimate from below which is given by Lemma 3.1(a) on the critical value c^*, and combine it with the final claim of Prop. 2.3: under the assumption (1.8), and according to the notation we introduced in Proposition 3.5, we can write $\bar{c} = q^{\frac{1}{q}} p^{\frac{1}{p}} \lambda^{\frac{1}{q}} \leq c^* \leq q^{\frac{1}{q}} p^{\frac{1}{p}} \mu^{\frac{1}{q}}$. We also point out that, in the limit case $c = \bar{c}$, the maximum value M_c of (1.6) is λ, so that $\omega_c^-(\lambda) = \omega_c^+(\lambda) = \omega_c$: hence (3.1) has no meaning for $c = \bar{c}$.

Theorem 3.7. *Let f satisfy (1.8), $c \geq c^*$ and, according to Proposition 2.3, let y be the only solution of (1.1) such that $y(0) = y(1) = 0$. Then:*

(a) $c > c^* \;\Rightarrow\; y$ satisfies (3.1a).

(b) $c = c^* \;\Rightarrow\; y$ satisfies (3.1b).

Proof. (a) Let $c^* < \theta < c$, put $c = \theta$ in (1.1) and call $\tilde{y}$ the solution of the corresponding equation such that $\tilde{y}(0) = \tilde{y}(1) = 0$. Suppose, by contradiction, that (3.1a) does not hold, and exchange the roles of θ and c in Prop. 3.5e, so as to get $y = \psi_c > \psi_\theta \geq \tilde{y}$ on $(0, 1]$. In particular, we get the contradiction $0 = y(1) > \tilde{y}(1)$.

(b) It is enough to prove that (3.1a) $\Rightarrow c > c^*$. So, let us suppose that $y \in \mathcal{S}_c$, $y \neq \psi_c$: then, from Prop. 3.3c, we infer that $\psi_c(1) > y(1) = 0$. On the other hand, Prop. 3.5f ensures, in particular, that the map $c \mapsto \psi_c(1)$ is continuous, so that $\psi_\theta(1) > 0$ for some $\theta < c$. Now, let us put $c = \theta$ in (1.1), and denote by $\tilde{y}$ the solution of the corresponding equation which fulfils the condition $\tilde{y}(1) = 0$. By the same arguments as in the previous section, we get $\tilde{y}(0) = 0$ as well: hence $\theta \geq c^*$, so that $c > c^*$. □

The reader may find related and complementary results in [9] and in [1].

4 Functions of types B and C: existence of solutions

Let us now consider the cases where f is a type B or type C function.

Lemma 4.1. *Assume f is continuous in $[0, 1]$, $f(0) = 0$ and*

$$\liminf_{u \to 0} \frac{f(u)}{u^{q-1}} > -\infty. \tag{4.1}$$

Then any solution of

$$y' = q(c\, y_+^{\frac{1}{p}} - f(u)), \quad y(0) = 0, \tag{4.2}$$

positive in a neighborhood of 0*, satisfies*

$$\sup_{u\in(0,1]} \frac{y(u)}{u^q} < +\infty.$$

Proof.
Claim: Given $k > 0$, there exists $M > 0$ such that any solution of

$$z' = q(c\, z_+{}^{\frac{1}{p}} + ku^{q-1}), \quad z(0) = 0 \tag{4.3}$$

positive in a neighborhood of 0*, satisfies $z(u) \le Mu^q$, $0 \le u \le 1$.*

If we set $w(u) = z(u)^{1/q}$, we have $w' = c + k(\frac{u}{w})^{q-1}$, $w(0) = 0$. Defining

$$l = \liminf_{u\to 0} \frac{w(u)}{u}, \quad L = \limsup_{u\to 0} \frac{w(u)}{u}$$

and

$$l' = \liminf_{u\to 0} w'(u), \quad L' = \limsup_{u\to 0} w'(u)$$

we obtain $c \le l' \le l \le L \le L' = c + \frac{k}{l^{q-1}} < +\infty$ and our Claim follows.

Now choose $k > 0$ so that $-f(u) < ku^{q-1}$ if $0 < u \le 1$. For each $h > 0$ consider the solution z_h of

$$z_h' = q(c\, z_{h+}{}^{\frac{1}{p}} + ku^{q-1}), \quad z_h(0) = h. \tag{4.4}$$

Then z_h converges, as $h \to 0$, to the maximal solution of (4.3). On the other hand if we pick a solution y of (4.2) it is clear that $y < z_h$. Using the Claim, we obtain the conclusion of the lemma.

Theorem 4.2. *Let f be a type B or a type C function. In the latter case assume $\int_0^1 f(s)\, ds > 0$ and (4.1) holds. Then there exists a number $\hat{c} > 0$ such that the boundary value problem $y' = q(c\, y_+{}^{\frac{1}{p}} - f(u)), \quad 0 \le u \le 1, \quad y(0) = y(1) = 0$ has a positive solution if and only if $c = \hat{c}$.*

Proof. By the hypothesis there exists $\alpha \in (0, 1)$ so that $f > 0$ in $(\alpha, 1)$ and either $f \equiv 0$ or $f < 0$ in $(0, \alpha)$. For $c \ge 0$ consider the Cauchy problem

$$y' = q(c\, y_+{}^{\frac{1}{p}} - f(u)), \quad y(1) = 0, \tag{4.5}$$

which, as we have already remarked, has a unique solution y_c in $[0, 1]$. Also, the usual compactness argument shows that y_c depends continuously on $c \in [0, +\infty)$ in the norm of $C([0, 1])$. Clearly, $y_c(u) \ge 0$ at least for $u \in (\alpha, 1)$. In particular, by our assumptions, $y_0(u) = q \int_u^1 f(s)\, ds > 0$ for all $u \in [0, 1)$.

Step 1: solutions decrease with c. Given $c_1 < c_2$, the corresponding solutions $y_1 \equiv y_{c_1}$ and $y_2 \equiv y_{c_2}$ are such that $y_1(u) > y_2(u)$ whenever $y_1(u) > 0$. In fact we cannot have $y_1 < y_2$ in any open subinterval of $(\alpha, 1)$, otherwise $y_1 - y_2$ would be decreasing in that interval, contradicting the fact that it must reach the value 0.

Set $\hat{c} = \sup\{c > 0 \,|\, y_c(u) > 0\, \forall u \in (0, 1)\}$.

Step 2: $0 < \hat{c} < +\infty$. It is obvious that $\hat{c} > 0$. If $\hat{c} = +\infty$, there exists $c_n \to +\infty$ with $y_n \equiv y_{c_n} > 0$ in $(0, 1)$. If f is type B, then

$$y_n(u)^{\frac{1}{q}} = y_n(\alpha)^{\frac{1}{q}} - c_n(\alpha - u) \tag{4.6}$$

for $u \in (0, \alpha)$. Note also that $y_n(\alpha) \le q \int_\alpha^1 f(s)\, ds$. Hence y_n must become negative for n large, a contradiction. If f is type C the same argument applies because then the solution in $(0, \alpha)$ must stay below the function given by the expression in the righthandside of (4.6).

Step 3: $y_{\hat{c}}(0) = 0$ and $y_{\hat{c}}(u) > 0\ \forall u \in (0, 1)$. By definition of $\hat{c}$ and continuous dependence on c, $y_{\hat{c}}$ must vanish in $[0, \alpha]$. Let $\gamma \in [0, \alpha]$ be its largest zero. If $\gamma > 0$, then for $c < \hat{c}$ and $u \le \gamma$

$$y_c(u)^{\frac{1}{q}} \le y_c(\gamma)^{\frac{1}{q}} - c(\gamma - u)$$

and since $y_c(\gamma) \to 0$ as $c \to \hat{c}$, if $\hat{c} - c$ is sufficiently small y_c must vanish in $(0, \gamma)$, contradicting the definition of $\hat{c}$.

Step 4: If $c > \hat{c}$ and f is of type B, then $y_c \equiv 0$ in some interval $[0, \gamma]$, $0 < \gamma \le \alpha$. By Step 1, $0 < y_c(\alpha) < y_{\hat{c}}(\alpha)$. The graph of y_c cannot meet the graph of $y_{\hat{c}}$ in $[0, \alpha]$ if $y_c > 0$ in $(0, \alpha)$. Hence there exists $\gamma \in (0, \alpha]$ such that $y_c(\gamma) = 0$ and the claim follows.

Step 5: If $c > \hat{c}$ and f is of type C, then $y_c(0) < 0$. As in the previous step, $0 < y_c(\alpha) < y_{\hat{c}}(\alpha)$. Since $f > 0$ in $(0, \alpha)$ we easily obtain $y_c(u) < y_{\hat{c}}(u)\ \forall u \in [0, \alpha]$.

Step 6: If $c < \hat{c}$, then $y_c(0) > 0$. Suppose to the contrary that $y_c(0) = 0$. By the previous arguments $y_c > y_{\hat{c}}$ on $(0, 1)$.

Case 1: f is of type B. By separation of variables, the only solution of $y' = q\, d\, y_+{}^{\frac{1}{p}}$ satisfying $y(0) = 0$ and positive in a neighborhood of zero is the function $y_0(u) \equiv d^q u^q$. Hence we obtain $c^q u^q > \hat{c}^q u^q$ in $[0, \alpha]$, a contradiction.

Case 2: f is of type C. By a lower solution argument, (4.2) with $c = \hat{c}$ has a solution $z(u)$ such that $z(0) = 0$ and $z > y_{\hat{c}}$ in $[0, \alpha]$. But we now show that such solutions must coincide, obtaining a contradiction. Let z, w be two solutions of (4.2). If $z \ne w$ it is easily seen that they are ordered, say $z < w$ in $(0, \alpha)$. By the preceeding Lemma there exists a constant $M > 0$ so that, with a computation similar to that of (3.11),

$$0 \le w(u) - z(u) \le qc \int_0^u \frac{w(s) - z(s)}{p(c^q s^q)^{1/q}}\, ds \le \frac{M}{p} u^q, \quad 0 \le u < \alpha.$$

Iterating this argument we obtain

$$0 \le w(u) - z(u) \le \frac{M}{p^2} u^q, \quad 0 \le u < \alpha$$

and in fact

$$0 \le w(u) - z(u) \le \frac{M}{p^k} u^q, \quad 0 \le u < \alpha$$

for all integers $k \in \mathbb{N}$. We conclude that $z \equiv w$ in $[0, \alpha]$.

□

Let f be a type A function such that $\sup_{0<u<1} \frac{f(u)}{u^{q-1}} < +\infty$. It is easy to see that there exists a decreasing sequence of positive values ϵ_n tending to zero such that the corresponding sequence of type B functions

$$f_n(u) = \begin{cases} 0, & u \in [0, \epsilon_{n+1}] \\ \min\left(l_n(u), f(u)\right), & u \in [\epsilon_{n+1}, \epsilon_n] \\ f(u), & u \in [\epsilon_n, 1], \end{cases}$$

where $l_n(u) = f(\epsilon_n)\frac{u-\epsilon_{n+1}}{\epsilon_n-\epsilon_{n+1}}$, is increasing and tends uniformly to $f(u)$. Let $\hat{c}(f_n)$ be the unique value such that the boundary value problem (2.1) with $f(u) = f_n(u)$ has a positive solution. The following theorem uses this fact to give a new characterization of the critical speed c^* introduced in section 2. Results of this type may be also found in [4, 9].

Theorem 4.3. *Consider a type A function f with $\sup_{u\in(0,1)} \frac{f(u)}{u^{q-1}} < +\infty$ and a sequence of type B functions f_n in the conditions mentioned above. Then $\hat{c}(f_n)$ is an increasing sequence and $\lim \hat{c}(f_n) = c^*$ where c^* is associated to f in Proposition 2.3.*

Proof. Consider two arbitrary consecutive elements f_n and f_{n+1} of the sequence of type B functions considered above. These two functions are different in some interval $(\epsilon_{n+2}, b) \subset (0, \epsilon_n)$, where $f_n(u) < f_{n+1}(u)$, having the same values outside the interval $(\epsilon_{n+2}, \epsilon_n)$. Consider the problem

$$y' = q(c\,{y_+}^{\frac{1}{p}} - f_{n+1}(u)), \quad y(0) = 0, \tag{4.7}$$

and let $\hat{c}(f_{n+1})$ be the unique value such that there exists a solution of (4.7) satisfying $y(1) = 0$. It is easy to see that this solution $y_{n+1}(u)$ satisties

$$y_{n+1}{}'(u) \leq q(\hat{c}(f_{n+1})\,(y_{n+1})_+(u)^{\frac{1}{p}} - f_n(u)), \quad \forall u \in [0, 1]$$

with strict inequality for $u \in (\epsilon_{n+2}, b)$. so the equation

$$y'(u) = q(\hat{c}(f_{n+1})\,y_+(u)^{\frac{1}{p}} - f_n(u))$$

has a solution $z_n(u)$ such that $z_n(0) = 0$, $z_n(u) > y_{n+1}(u)$ for $u \in (\epsilon_{n+2}, 1]$ and in fact $z_n(u) > y_{n+1}(u)$ for $u \in (0, 1]$ (since we see that $z_n - y_{n+1}$ increases). Since the positive solution starting from $(0, 0)$ is unique, the solution $w_n(u)$ of the same equation with $w_n(1) = 0$ must vanish at some point $\gamma_n \in (0, \epsilon_n]$ (see the argument in Step 4 of the preceeding proof). Hence by the construction of $\hat{c}_n$ we have $\hat{c}(f_n) < \hat{c}(f_{n+1})$. This allows us to conclude that $\hat{c}(f_n)$ is an increasing sequence.

Now let $c \geq c^*$ and consider the unique solution $z(t)$ of (2.1). Then we have $z' < q(c\,{z_+}^{\frac{1}{p}} - f_n(u))$ for $u \in (0, \epsilon_{n+1})$ and $z' \leq q(c\,{z_+}^{\frac{1}{p}} - f_n(u))$ for $u \in [0, 1]$, The same argument as above allows us to conclude that $\hat{c}(f_n) < c$ and consequently, the sequence is bounded from above by c^*. A simple application of Ascoli-Arzelà's lemma allows us to conclude that the solutions y_n tend to a solution of (2.1). This implies that $\hat{c}(f_n) \to l \geq c^*$ and consequently we conclude that $\hat{c}(f_n) \nearrow c^*$ □

5 Behaviour near $u = 1$

Lemma 5.1. *Let $c > 0$, and let y be a positive solution of (1.1) in some interval $(a, 1)$ such that $y(1) = 0$. Suppose that*

$$m := \lim_{u \to 1^-} \frac{f(u)}{(1-u)^{q-1}} < \infty \quad \textit{exists.} \tag{5.1}$$

Then $\lim_{u \to 1^-} \frac{y(u)}{(1-u)^q}$ exists and is the root α of $\alpha + c\alpha^{1/p} = m$.

Proof. Claim: $\sup_{u \in (a,1)} \frac{y(u)}{(1-u)^q} < \infty$. Just take $m_1 > m$ and integrate the inequality $y'(u) \geq -qm_1(1-u)^{q-1}$ in a suitable interval of the form $(b, 1)$.

Let us complete the proof, setting $y(u) = z(u)(1-u)^q$. Then z is a solution of the differential equation

$$z' = \frac{q}{1-u}(z + cz^{\frac{1}{p}} - \mu(u)), \tag{5.2}$$

where $\mu(u) = \frac{f(u)}{(1-u)^{q-1}}$. Arguing as in the proof of Lemma 3.1-(d) it is easy to see that $\liminf_{u \to 1} z(u)$ and $\limsup_{u \to 1} z(u)$ must be equal and must coincide with the root of $\alpha + c\alpha^{1/p} = m$.

6 Some applications to the second order problem

In this section we consider the problem (1.3)-(1.4) with several assumptions on D and g.

Lemma 6.1. *Let g be a function of type A. The derivative of a nonincreasing solution u of (1.3) with $0 < u(x) < 1$ does not vanish. If the interval where u is defined extends to $+\infty$, we have $\lim_{t \to +\infty} D(u)|u'|^{p-1} = 0$. A similar statement holds with $-\infty$ replacing $+\infty$.*

Proof. If there exists x_0 such that $u'(x_0) = 0$ and $0 < u(x_0) < 1$, using the differential equation we would have $\left(D(u)\,|u'|^{p-2}\,u'\right)'|_{(x=x_0)} < 0$, which contradicts the fact that $D(u)\,|u'|^{p-2}\,u'$ attains a maximum at $x = x_0$.

Concerning the statement on the limit we will only consider $+\infty$, the case of $-\infty$ being similar. Suppose towards a contradiction that $\liminf_{x \to +\infty} D(u)|u'|^{p-1} = \delta > 0$. We can take two sequences t_n and s_n tending to $+\infty$ such that $u'(t_n) \to 0$ and $D(u(s_n))|u'|^{p-1}(s_n) \to \delta$. Integrating the differential equation in $[0, t_n]$, we easily conclude that the sequence $\int_0^{t_n} g(u(x))\,dx$ is bounded and therefore $\int_0^{+\infty} g(u(x))\,dx$ is convergent. Consequently we have

$$0 = \int_{t_n}^{s_n} \left(D(u)\,|u'|^{p-2}\,u'\right)' + cu' + g(u)\,dx =$$

$$= D(u(s_n))\,|u'(s_n)|^{p-2}\,u'(s_n) - D(u(t_n))\,|u'(t_n)|^{p-2}\,u'(t_n) + c\,(u(s_n) - u(t_n)) + \int_{t_n}^{s_n} g(u)\,dx$$

and making $n \to \infty$ we would get the contradiction $-\delta = 0$. □

We set

$$f(u) = D(u)^{q-1} g(u), \quad u \in [0,1]. \tag{6.1}$$

and we assume

$(D1)$ $D \in C^1[0,1]$ and $D > 0$ in $(0,1)$.

$(G1)$ g is a function of type A.

Clearly, f given by (6.1) is of type A.

Proposition 6.2. *We have that $u(t)$ is a monotone solution of (1.3) in some interval (a,b) such that $0 < u(t) < 1$ $\forall t \in (a,b)$ and*

$$\lim_{t \to a^+} u(t) = 1, \quad \lim_{t \to b^-} u(t) = 0, \quad \lim_{t \to a^+} D(u)|u'|^{p-1} = \lim_{t \to b^-} D(u)|u'|^{p-1} = 0 \tag{6.2}$$

if and only if $y = v^q$ where $v = D(u)|u'|^{p-1}$ is a positive solution of (1.1)-(1.5).

Proof. The necessary condition was essentially proved in the introduction. Conversely, given a positive solution $y(u)$ of (1.1)-(1.5), we recover a solution of (1.3) by solving the Cauchy problem

$$u' = -\frac{y(u)^{\frac{1}{p}}}{D(u)^{q-1}}, \quad u(0) = \frac{1}{2}. \tag{6.3}$$

The solution of (6.3) exists in (t_-, t_+), where

$$t_- = -\int_{1/2}^{1} \frac{D(u)^{q-1}\, du}{y(u)^{\frac{1}{p}}}, \qquad t_+ = \int_0^{1/2} \frac{D(u)^{q-1}\, du}{y(u)^{\frac{1}{p}}}. \tag{6.4}$$

□

Consider the assumptions

$$\sup_{u \in (0,1)} \frac{g(u)}{u^{q-1}} < +\infty. \tag{6.5}$$

$$\sup_{u \in (0,1)} \frac{g(u)}{(1-u)^{p-1}} < +\infty. \tag{6.6}$$

and the following strenghtened form of $(D1)$

$(D1')$ $D \in C^1[0,1]$ and $D > 0$ in $[0,1]$.

Under the conditions $(D1)$, $(G1)$, (6.5), Proposition 2.3 is applicable to $f = D^{q-1} g$ and a positive number c^* is associated to f. This number plays a central role in the following theorem, which in case $p = 2$ corresponds to well known results, see [10, 3, 13] and references. Note that according to the results in sections 2 and 3 we have the estimate

$$q^{\frac{1}{q}} p^{\frac{1}{p}} \left(\liminf_{u\to 0} \frac{D(u)^{q-1} g(u)}{u^{q-1}} \right)^{\frac{1}{q}} \le c^* \le q^{\frac{1}{q}} p^{\frac{1}{p}} \left(\sup_{u\in(0,1)} \frac{D(u)^{q-1} g(u)}{u^{q-1}} \right)^{\frac{1}{q}}. \tag{6.7}$$

Theorem 6.3. *Suppose that* $(D1')$, $(G1)$, *(6.5) and (6.6) are satisfied and let* $1 < p \le 2$. *Then (1.3)-(1.4) has a decreasing solution* $u(t)$ *taking values in* $(0,1)$ *if and only if* $c \ge c^*$. *That solution is unique up to translation.*

If, further, $g^*(0) \equiv \lim_{u\to 0^+} \frac{g(u)}{u^{q-1}}$ *exists then*

$$\lim_{t\to+\infty} \frac{u'(t)}{u(t)^{q-1}} = \begin{cases} -\frac{\omega_c^- (D(0)^{q-1} g^*(0))^{1/p}}{D(0)^{q-1}}, & c > c^* \\ -\frac{\omega_{c^*}^+ (D(0)^{q-1} g^*(0))^{1/p}}{D(0)^{q-1}}, & c = c^* \end{cases}$$

Proof. Let $y(u)$ be a solution of (1.1)-(1.5) for some $c \ge c^*$. Consider the Cauchy problem (6.3). The solution of (6.3) exists in (t_-, t_+), given by (6.4). Since (1.1) implies $y' \le qcy^{1/p}$, it turns out that

$$\sup_{u\in(0,1)} \frac{y(u)}{u^q} < +\infty, \tag{6.8}$$

and it is clear that $t_+ = +\infty$, since $q \ge 2$. Similarly the estimate we get from (6.5) and $(D1')$ on

$$\frac{f(u)}{(1-u)^{p-1}}$$

implies $y(u) \le C(1-u)^p$ for some constant C and therefore $t_- = -\infty$. The solution of (6.3) satisfies (1.3)-(1.4).

On the other hand, since we can write $\frac{u'(t)}{u(t)^{q-1}} = -(\frac{y(u)}{u^q})^{\frac{1}{p}} \frac{1}{D(u)^{q-1}}$, the last statement follows easily from Theorem 3.7. □

Next we shall consider the case where D is "degenerate" in the sense that

$(D2)$ $D \in C^1[0,1]$, $D > 0$ in $(0,1]$, $D(0) = 0$ and $D'(0) > 0$.

The following theorem corresponds to results given in [16, 13] for $p = 2$.

Theorem 6.4. *Suppose that* $(D2)$, $(G1)$, *(6.5) and (6.6) are satisfied. Let* $1 < p \le 2$. *Then*

(i) Problem (1.3)-(1.4) has a decreasing solution $u(t)$ *taking values in* $(0,1)$ *if and only if* $c > c^*$.

(ii) If $c = c^*$ *(1.3) has a decreasing solution defined in* $(-\infty, 0]$ *with* $u(-\infty) = 1$, $u(0) = 0$ *and*

$$u'(0) = -\left(\frac{c^*}{D'(0)} \right)^{q-1}. \tag{6.9}$$

Those solutions are unique up to translation.

(iii) If $c < c^*$, *problem (1.3) has no decreasing solution in any interval* $(-\infty, b)$ *with* $\lim_{t\to-\infty} u(t) = 1$, $\lim_{t\to b^-} u(t) = 0$.

The solutions considered in (ii) are called *sharp solutions.*

Proof. Proceeding as in the preceeding proof, we consider (6.3). Under the assumptions of the theorem, we have $f^*(0) = 0$. To compute the limit of the right-hand side of (6.3) we write

$$\lim_{u\to 0^+} \frac{y(u)^{\frac{1}{p}}}{D(u)^{q-1}} = \lim_{u\to 0^+} \left(\frac{y(u)}{u^q}\right)^{\frac{1}{p}} \left(\frac{u}{D(u)}\right)^{q-1}. \tag{6.10}$$

Noting that

$$\omega_c^-(0) = 0, \quad \omega_c^+(0) = c^q$$

and using Theorem 3.7 we may conclude:

$$\lim_{u\to 0^+} \frac{y(u)^{\frac{1}{p}}}{D(u)^{q-1}} = \begin{cases} 0, & c > c^* \\ \left(\frac{c^*}{D'(0)}\right)^{q-1}, & c = c^* \end{cases} \tag{6.11}$$

Moreover if $c > c^*$ and since f satisfies (3.2b), Proposition 3.5-(g) implies $t_+ = +\infty$. The fact that $t_- = -\infty$ follows as in the proof of the preceeding theorem.

If $c = c^*$ it is clear that the solution of (6.3) can remain positive for $t \geq 0$ only in some finite interval $[0, b)$ so that $\lim_{t\to b^-} u(t) = 0$ and $\lim_{t\to b^-} u'(t) = -\left(\frac{c^*}{D'(0)}\right)^{q-1}$. □

We next consider a case of negative diffusion (see [17]), considering the assumption
$(D3)$ $\quad D(u) < 0\ \forall u \in (0, 1]$; $D(0) = 0$ and $D'(0) < 0$.

and introducing the conditions

$$\sup_{u\in(0,1)} \frac{g(u)}{(1-u)^{q-1}} < +\infty, \qquad \sup_{u\in(0,1)} \frac{g(u)}{u^p} < +\infty. \tag{6.12}$$

In [17] the authors consider (1.3)-(1.4) with $D < 0$ for $p = 2$ and reduce this problem to a non-singular system, assuming $(D3)$ and additional regularity assumptions.

It is easily seen that the change of variables

$$u(-t) = 1 - z(t)$$

yields an equivalence between (1.3)-(1.4) and

$$(E(z)|z'|^{p-2}z')' + cz' + h(z) = 0, \quad z(-\infty) = 1, \;\; z(+\infty) = 0 \tag{6.13}$$

where

$$E(z) = -D(1-z), \quad h(z) = g(1-z), \quad 0 \leq z \leq 1. \tag{6.14}$$

We have the following result that contains some statements made in Proposition 3 of [17] for $p = 2$.

Theorem 6.5. *Let $1 < p \le 2$. Suppose that $(D3)$, $(G1)$ and (6.12) are satisfied. Then there exists $c^* > 0$ such that (1.3)-(1.4) has a decreasing solution $u(t)$ taking values in $(0,1)$ if and only if $c \ge c^*$. Moreover*

$$q^{\frac{1}{q}} p^{\frac{1}{p}} \left(\liminf_{u \to 1} \frac{|D(u)|^{q-1} g(u)}{(1-u)^{q-1}} \right)^{\frac{1}{q}} \le c^* \le q^{\frac{1}{q}} p^{\frac{1}{p}} \left(\sup_{u \in (0,1)} \frac{|D(u)|^{q-1} g(u)}{(1-u)^{q-1}} \right)^{\frac{1}{q}}. \quad (6.15)$$

Those solutions are unique up to translation.

Proof. Let us start with problem (6.13). We consider the associated first order problem

$$y' = q(c\, y_+{}^{\frac{1}{p}} - E(z)^{q-1} h(z)), \quad 0 \le u \le 1, \quad y(0) = y(1) = 0. \quad (6.16)$$

There exists c^* such that this problem has a positive solution if and only if $c \ge c^*$ and c^* satisfies the desired estimates. We recover the solution of (6.13) via the differential equation

$$z' = -\frac{y(z)^{\frac{1}{p}}}{E(z)^{q-1}}, \quad z(0) = 1/2.$$

As in the proof of Theorem 6.3 we see, using the first condition (6.12), that the solution is defined in an interval (t_-, t_+) where $t_+ = +\infty$. The second condition (6.12) and (D3) imply, as is easily seen, that $y(z) \le K(1-z)^{p+q}$ for some constant K. Combining this with $(D3)$, it turns out that the integrand in the expression of t_- is bounded below by some multiple of $\frac{1}{1-u}$ and therefore $t_- = -\infty$. Setting $u(t) = 1 - z(-t)$ we obtain the desired solutions of (1.3). □

In a similar way we are able to deal with the analogue of a model considered in [18].

Theorem 6.6. *Let $1 < p \le 2$. Assume $-g$ is a function of type A, $(D2)$, (6.5) and (6.6) hold. Then there exists $-c^* < 0$ such that (1.3)-(1.4) has a decreasing solution $u(t)$ taking values in $(0,1)$ if and only if $c \le -c^*$. Moreover*

$$q^{\frac{1}{q}} p^{\frac{1}{p}} \left(\liminf_{u \to 1} \frac{D(u)^{q-1} |g(u)|}{(1-u)^{q-1}} \right)^{\frac{1}{q}} \le c^* \le q^{\frac{1}{q}} p^{\frac{1}{p}} \left(\sup_{u \in (0,1)} \frac{D(u)^{q-1} |g(u)|}{(1-u)^{q-1}} \right)^{\frac{1}{q}}. \quad (6.17)$$

Those solutions are unique up to translation.

Proof. We use the change of variables $u(-t) = 1 - z(t)$ again. Then the problem (1.3)-(1.4) turns into

$$(F(z)|z'|^{p-2} z')' - cz' + h(z) = 0, \quad z(-\infty) = 1, \;\; z(+\infty) = 0, \quad (6.18)$$

where

$$F(z) = D(1-z), \quad h(z) = -g(1-z), \quad 0 \le z \le 1. \quad (6.19)$$

□

Finally we examine a situation where both D and g change sign. A problem of this type has been studied in [11] in the case $p = 2$.

We introduce the assumptions:

$(G2)$ $g(0) = g(1) = 0$ and there exists $\alpha \in (0,1)$ such that $(u - \alpha)g(u) > 0$ $\forall u \in (0,1) \setminus \{\alpha\}$.

$(D4)$ There exists $\beta \in (0,1)$ such that $(u - \beta)D(u) < 0$ $\forall u \in (0,1) \setminus \{\beta\}$.

$(GD0)$

$$\liminf_{h \to 0} \frac{D(h)^{q-1} g(h)}{h^{q-1}} > -\infty, \qquad \int_0^\beta (\frac{D(h)}{h})^{q-1} \, dh = +\infty.$$

$(GD1)$

$$\limsup_{h \to 1} \frac{|D(h)|^{q-1} g(h)}{(1-h)^{q-1}} < +\infty, \qquad \int_\beta^1 (\frac{|D(h)|}{1-h})^{q-1} \, dh = +\infty.$$

$(GD2)$ $\alpha < \beta$ and $\int_0^\beta D(u)^{q-1} g(u) \, du > 0$.

Let us consider the problem in the interval $[0, \beta]$

$$y' = q(c\, y^{\frac{1}{p}} - D(u)^{q-1} g(u)), \quad 0 \le u \le \beta \tag{6.20}$$

Note that $D(u)^{q-1} g(u)$ is of type C on $[0, \beta]$. According to Theorem 4.2 we know that under the first condition in $(GD0)$ and $(GD2)$ there exists a (unique) number $\hat{c} > 0$ such that (6.20) has a positive solution satisfying

$$y(0) = 0 = y(\beta). \tag{6.21}$$

On the other hand, the problem in the interval $[\beta, 1]$

$$y' = q(c\, y^{\frac{1}{p}} - \tilde{D}(z)^{q-1} \tilde{g}(z)), \quad \beta \le z \le 1 \tag{6.22}$$

where

$$\tilde{D}(z) = |D(1 + \beta - z)|, \quad \tilde{g}(z) = g(1 + \beta - z); \quad \beta \le z \le 1 \tag{6.23}$$

involves a function of type A (in $[\beta, 1]$, of course) and, since the first condition in $(GD1)$ holds, there exists a number c^* such that (6.22) has a positive solution satisfying

$$y(\beta) = 0 = y(1) \tag{6.24}$$

if and only if $c \ge c^*$. We are now in a position to state the following

Theorem 6.7. *Let $1 < p \le 2$. Assume $(G2)$, $(D4)$, $(GD0)$, $(GD1)$, $(GD2)$, $D'(\beta) < 0$. Then if $\hat{c} \ge c^*$, the problem (1.3)-(1.4) has a solution (unique up to translation) for $c = \hat{c}$.*

Proof. *Step 1: connection between* 0 *and* β. We fix $c = \hat{c}$ and consider the positive solution of (6.20) satisfying (6.21). A corresponding solution of (1.3) is obtained via

$$u' = -\frac{y(u)^{\frac{1}{p}}}{D(u)^{q-1}}, \quad u(0) = \frac{\beta}{2}. \tag{6.25}$$

Using assumption $D'(\beta) < 0$ and Lemma 5.1 we easily compute

$$\lim_{u \to \beta^-} \frac{y(u)^{\frac{1}{p}}}{D(u)^{q-1}} = \frac{\alpha^{\frac{1}{p}}}{|D'(\beta)|^{q-1}} \tag{6.26}$$

where $\alpha + \hat{c}\alpha^{\frac{1}{p}} = |D'(\beta)|^{q-1} g(\beta)$. Using the first part of $(GD0)$ and Lemma 4.1 (where $f = |D|^{q-1}g$) we see that $\frac{y(u)}{u^q}$ is bounded in $(0, \beta)$. Hence we have obtained a solution of (1.3) that satisfies $0 < u(t) \leq \beta$ and (using the second part of $(GD0)$)

$$u(t_1) = \beta, \quad u'(t_1) = -\frac{\alpha^{\frac{1}{p}}}{|D'(\beta)|^{q-1}}, \quad u(+\infty) = 0 \tag{6.27}$$

for some $t_1 > -\infty$.

Step 2: connection between β *and* 1. The change of variable

$$u(t) = 1 + \beta - z(-t)$$

defines a new second order problem

$$(\tilde{D}(z)|z'|^{p-2}z')' + cz' + \tilde{g}(z) = 0 \tag{6.28}$$

with $\tilde{D}$ and $\tilde{g}$ given by (6.23), in such a way that $u(t)$ takes values in $(\beta, 1)$ and solves (1.3) if and only if $z(t)$ takes values in $(\beta, 1)$ and solves (6.28). Now since $\hat{c} \geq c^*$ the problem (6.22) with $c = \hat{c}$ has a positive solution $y(z)$ that satisfies (6.24). This originates a solution of

$$z' = -\frac{y(z)^{\frac{1}{p}}}{\tilde{D}(z)^{q-1}}, \quad z(0) = \frac{1-\beta}{2}. \tag{6.29}$$

Now $\lim_{z \to 1^-} \frac{\tilde{D}(z)^{q-1}\tilde{g}(z)}{(1-z)^{q-1}} = |D'(\beta)|^{q-1} g(\beta)$. hence, as in Step 1, we compute

$$\lim_{z \to 1^-} \frac{y(z)^{\frac{1}{p}}}{\tilde{D}(z)^{q-1}} = \frac{\alpha^{\frac{1}{p}}}{|D'(\beta)|^{q-1}} \tag{6.30}$$

with the same meaning of α. Now from Lemma 3.1-(d) (where $f = |\tilde{D}|^{q-1}\tilde{g}$, β playing the role of 0) and the first part of $(GD1)$ we know that $\frac{y(z)}{(z-\beta)^q}$ is bounded in $(\beta, 1)$. Taking also the second part of $(GD1)$ into account, we have shown that the solution of (6.29) is defined in some interval $(-t_2, +\infty)$ with $t_2 < +\infty$, $\beta < z(t) \leq 1$ and

$$z(-t_2) = 1, \quad z'(-t_2) = -\frac{\alpha^{\frac{1}{p}}}{|D'(\beta)|^{q-1}}, \quad z(+\infty) = \beta. \tag{6.31}$$

Accordingly, the function $u(t) = 1 + \beta - z(-t)$ satisfies $\beta < u(t) \leq 1$ and

$$u(t_2) = \beta, \quad u'(t_2) = -\frac{\alpha^{\frac{1}{p}}}{|D'(\beta)|^{q-1}}, \quad u(-\infty) = 1. \tag{6.32}$$

Step 3: conclusion. Comparing (6.27) with (6.32) we see that, after a translation of one of the solutions thus defined, we obtain the desired connection between 0 and 1. □

Remark 6.8. (a) As in [11] for $p = 2$, it can be shown that the solution of (1.3)-(1.4) exists only if $c = \hat{c}$.

(b) Under the conditions of Theorem 6.7 we must have

$$\hat{c} \geq \{\liminf_{h\to 0} \left[\left(\frac{|D(1-h)|}{h} \right)^{q-1} g(1-h) \right] q\}^{1/q} p^{1/p}.$$

Remark 6.9. If we modify condition $(D4)$ so as to extend the strict inequality to the endpoints 0 and 1, then the second part of assumptions $(GD0)$ and $(GD1)$ of Theorem 6.7 can obviously be dropped, since it follows from the inequality $q \geq 2$.

Remark 6.10. In the above theorems we have obtained heteroclinic solutions taking values strictly between 0 and 1. If we let $p > 2$ the same procedure yields heteroclinics that are possible *finite* in the sense that they become constant outside a finite interval.

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