Guiding-like functions for semilinear evolution equations with retarded nonlinearities

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Abstract

The paper deals with a semilinear evolution equation in a reflexive and separable Banach space. The non-linear term is multivalued, upper Carathéodory and it depends on a retarded argument. The existence of global almost exact, i.e. classical, solutions is investigated. The results are based on a continuation principle for condensing multifields and the required transversalities derive from the introduction of suitable generalized guiding functions. As a consequence, the equation has a bounded globally viable set. The results are new also in the lack of retard and in the single valued case. A brief discussion of a non-local diffusion model completes this investigation.

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1 Introduction

The paper deals with the multivalued evolution equation

$$x'(t) + A(t)x(t) \in F(t, x_t), \qquad t \in [a, b] \tag{1.1}$$

depending on a retarded argument and satisfying the initial condition

$$x(t) = \varphi(t - a), \qquad t \in [a - h, a]. \tag{1.2}$$

We assume that the state space E is a reflexive Banach space; when it is needed we take E also separable. $A: [a, b] \to \mathcal{L}(E)$ where $\mathcal{L}(E)$ is the Banach space of bounded linear operators $E \to E$ and the multivalued map (multimap) $F: [a, b] \times C^0\big([-h, 0]; E\big) \multimap E$ has convex, compact, nonempty values. The function $\varphi \in C^0\big([-h, 0]; E\big)$, where $C^0\big(J; E\big)$ stands

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for the Banach space of continuous functions on the closed interval J. As usual, for each $x \in C^0\big([a-h,b];E\big)$ and $t \in [a,b]$, x_t denotes the function $s \mapsto x(t+s)$, $s \in [-h,0]$. We investigate the existence of global almost exact, i.e. classical, solutions of (1.1)-(1.2). By a classical global solution we mean a function $x\colon [a-h,b] \to E$ such that $x \in C^0\big([a-h,b];E\big) \cap AC^0\big([a,b];E\big)$, $x(t) = \varphi(t-a)$ for $t \in [a-h,a]$ and $x'(t)+A(t)x(t) = f(t)$ with $f(t) \in F(t,x_t)$ for a.a. $t \in [a,b]$. $AC^0\big([a,b];E\big)$ is the subspace of $C^0\big([a,b];E\big)$ given by absolutely continuous functions.

We recall that in a Banach space with the Radon-Nikodym property, in particular in a reflexive Banach space, any absolutely continuous function $x(\cdot)$ is almost everywhere differentiable and it satisfies the classical integral formula.

Definition 1.1. *We say that a multivalued function $\mathcal{F}\colon I \times A \multimap E_2$, where I is a closed real interval, $A \subseteq E_1$ and E_1, E_2 are Banach spaces, is integrably bounded on every bounded set if, for every bounded subset $\Omega \subset A$ there exists $\mu_\Omega^{\mathcal{F}} \in L^1_+(I)$ such that $\|y\|_{E_2} \le \mu_\Omega^{\mathcal{F}}(t)$, for a.a. $t \in I$, all $x \in \Omega$ and $y \in \mathcal{F}(t,x)$.*

We assume the following conditions

(A) The function $A\colon [a,b] \to \mathcal{L}(E)$ is Bochner integrable.

(F1) $F(\cdot,x)$ has a strongly measurable selection for every $x \in C^0\big([-h,0];E\big)$.

(F2) $F(t,\cdot)$ is upper semicontinuous (u.s.c.) for a.a. $t \in [a,b]$.

(F3) F is integrably bounded on every bounded set.

(F4) There exists $k^F \in L^1_+\big([a,b]\big)$ such that, for a.a. $t \in [a,b]$, $\chi\big(F(t,\Omega)\big) \le k^F(t) \sup\limits_{s\in[-h,0]} \chi(\Omega(s))$, for any bounded set $\Omega \subset C^0\big([-h,0];E\big)$,

where (F4) the set $\Omega(s) := \{x(s) : x \in \Omega\} \subset E$ and the function χ is the Hausdorff *measure of noncompactness*(m.n.c. for short). Section 2 contains a brief introduction of m.n.c. and condensing multimaps. We remark that a strongly measurable function is also measurable and when E is separable then also the converse is true.

A multimap which satisfies (F1) and (F2) is said to be an *upper-Carathéodory* (*u-Carathéodory* for short) multifunction. A multimap between Banach spaces $\mathcal{F}:\ E_1 \multimap E_2$ is said to be *quasi-compact* if it maps compact subsets onto relatively compact ones.

Definition 1.2. *Let X and Y be normed spaces, Ω an open subset of X and $\mathcal{L}(X,Y)$ the Banach space of linear, bounded operators from X to Y. A function $f:\Omega \to Y$ is said to be* Gateaux differentiable *in $x_0 \in \Omega$ if there exists $A \in \mathcal{L}(X,Y)$ such that*

$$\lim_{t\to 0} \left\| \frac{f(x_0+tv)-f(x_0)}{t} - Av \right\|_Y = 0, \quad \textit{for all } v \in X.$$

The operator A is said to be the Gateaux differential *of f in x_0 and we will use the notation $A = f^G_{x_0}$ and $Av = f^G_{x_0}(v)$.*

The existence of local solutions, i.e. defined on some $[a,h] \subseteq [a,b]$, for problem (1.1)-(1.2) without delay was recently investigated both in [11], under conditions (A)-(F1-4), and in [7] where the u.s.c. in (F2) is intended in the sense of the usual weak topologies. The results depend on suitable fixed point theorems.
When the nonlinearity F is sublinear, i.e. when condition (F3) is replaced by

(F3′) There exists $\alpha \in L^1_+\big([a,b]\big)$ such that for all $x \in C^0\big([-h,0];E\big)$

$$\|F(t,x)\|_E \le \alpha(t)\left(1 + \|x\|_{C^0\left([-h,0];E\right)}\right), \quad \text{a.e. } t \in [a,b]$$

then the following result was obtained, which is a special case of [9, Theorem 3.1],

Theorem 1.1. *Problem* (1.1)-(1.2) *is solvable, under conditions (A), (F1,2,4) and (F3′).*

Nonlinearities F satisfying (F3) may also be superlinear in x and the use of a fixed point approach does not seem appropriate for the investigation of (1.1)-(1.2). In this paper we show that a continuation principle for condensing multifields (see Theorem 3.1 in Section 3) can be used, in alternative, and we introduce suitable generalized guiding functions in order to prove its *transversality condition*, i.e. condition *(d)* in Theorem 3.1.

Definition 1.3. *Let $K \subset E$ be nonempty open and bounded. A Gateaux differentiable function $V\colon E \to \mathbb{R}$ is said to be a generalized guiding function on ∂K for* (1.1)-(1.2) *if it satisfies the following conditions:*

(i) there exists $\delta > 0$ satisfying $\|V_x^G\|_{\mathcal{L}(E;\mathbb{R})} \ge \delta$, for all $x \in \partial K$;

(ii) $V_x^G(-A(t)x + w) \le 0$ for a.a. $t \in (a,b]$, all $x \in \partial K$ and every $w \in F(t,\vartheta)$, with $\vartheta \in C^0\big([-h,0];\overline{K}\big)$ and $\vartheta(0) = x$.

We further assume that

(V) $V(x) \le 0$ for all $x \in \overline{K}$ and $V(x) \equiv 0$ on ∂K.

If there exists a set of functions parametrized by $x \in \partial K$ and with *similar* properties as V (see e.g. [25]) we say that (1.1)-(1.2) has a family of bounding functions. The notions of guiding and bounding function were first introduced by Gaines and Mawhin in [18] for the investigation of single-valued equations in Euclidean spaces and we refer to [22] for recent results and several references in this context. Important contributions to the theory of bounding functions were given by Zanolin (see e.g. [25] and references therein). The theory of guiding and bounding functions was then generalized in [2, 3, 4, 5] for multivalued equations of first and second order also in infinite dimensional Banach spaces.

Given an arbitrary Banach space X and a positive value r, we denote with $\mathbb{B}(x,r)$ the open ball of X with ray r centered in the point $x \in X$ and we simply write $\mathbb{B}$ in the case of open unit ball with center 0. The symbol τ stands for the Lebesgue measure on the real line.

When the state space is separable and we assume the existence of a guiding function, we obtain the following existence result for problem (1.1)-(1.2) which is the main result of this paper. As far as we know, it is new also when the nonlinear part is single-valued and in the non-retarded case.

Theorem 1.2. *Consider the initial value problem* (1.1)-(1.2) *in a reflexive, separable Banach space E under conditions (A), (F1-4). Assume the existence of an open, bounded, convex subset $K \subset E$ and of a generalized guiding function on ∂K, $V\colon E \to \mathbb{R}$, satisfying (V); let $\kappa > 0$ be such that V_x^G is Lipschitzian on $\partial K + \kappa\mathbb{B}$. If $\varphi \in C^0([-h,0];K)$, then problem* (1.1)-(1.2) *has at least one solution, $x \in C^0\big([a-h,b];E\big) \cap AC^0\big([a,b];E\big)$, satisfying $x(t) \in \overline{K}$ for all $t \in [a,b]$.*

The detailed proof is in Section 7 and it is divided into some parts. Thanks to a Scorza-Dragoni type result discussed in Section 3, we introduce a sequence of related initial value problems (P_m). Their solvability depends on some preliminary existence results in Section 6, the continuation principle in Section 3 combined with the generalized guiding function V. The compactness and regularity properties that we need are respectively discussed in Sections 4 and 5. The conclusion then follows from a limiting process. We remark that Theorem 1.2 asserts that K is a viable set (see e.g. Section 8) for the semilinear multivalued equation (1.1).

An alternative approach for the study of viability problems can be found in [14] where the tangency conditions depend on some new tangent sets. Section 8 contains some concrete applications of Theorem 1.2 to viability theory in the case when the norm in E is sufficiently regular.

Remark 1.1. *Let L^p be the usual Lebesgue space $L^p(\Omega,\Sigma,\mu)$, for some measure space (Ω,Σ,μ), with the canonical norm $\|\cdot\|_p$. Very frequently, when* (1.1) *is the standard abstract formulation of some partial differential equation, $E = L^p$ (see e.g. [8] for some concrete examples). It is known (see e.g. [15, Theorem V.1.1]) that, when $p \in (1,+\infty)$ and it is not an integer, the function $\Upsilon\colon L^p(\Omega,\Sigma,\mu) \to \mathbb{R}$ defined by $\Upsilon(x) = \|x\|_p^p$ is at least $C^{[p]}$-smooth and its derivative of order $[p]$, i.e. $\Upsilon^{([p])}$ is Hölder continuous. The symbol $[p]$ denotes the integer part of p. If p is an integer, then $\Upsilon(x)$ is at least C^{p-1}-smooth with lipschitzian derivative of order $p-1$. Therefore, for each $R > 0$, the function $\Upsilon(x) - R^p$ is Fréchet differentiable, with a Lipschitzian Fréchet derivative, provided that $p \geq 2$. Notice moreover that*

$$\lim_{\lambda \to 0} \frac{\|x + \lambda \frac{x}{\|x\|}\|_p^p - \|x\|_p^p}{\lambda} = \|x\|_p^{p-1}, \quad \text{for all } x \in L^p \setminus \{0\}. \tag{1.3}$$

Consequently, when K is the ball centered in the origin and radius R, then $\Upsilon(x) - R^p$ satisfies (V) and conditions (i) *in Definition 1.3; hence it is a good candidate for the construction of a generalized guiding function. Of course $p = 2$ is a special case, since L^2 is an Hilbert space, and all previous computations can be obtained directly.*

It was recently showed that, in some cases, nonlocal conditions of the type $x(a) + \sum_{k=1}^{p} c_k x(t_k) = x_0$ with $c_k \in \mathbb{R} \setminus \{0\}$ and $x_0 \in E$ can better describe the evolution of some non-retarded processes (see [10] and references therein). Impulses can be very naturally included in semilinear evolution dynamics such as (1.1) (see e.g. [9, 12]). We think that a generalized guiding function approach can be fruitful in both cases as well as in the investigation of half-linear equations such as those in [16].

Given $x \in C^0\big([-h,0];E\big)$, we denote with $x_{(\cdot)}$ the map $[a,b] \to C^0\big([-h,0];E\big), t \mapsto x_t$ and put $x_{n,(\cdot)}$ when indices are present.

Given a multimap $F: [a,b] \times C^0\big([-h,0];E\big) \multimap E$ and $q \in C^0\big([a-h,b];E\big)$ we can define $\Phi(t) := F(t,q_t)$ on the whole interval $[a,b]$. If Φ has measurable selections we say that F is superpositionally measurable and denote with $\mathcal{S}^1_{F(\cdot,q_{(\cdot)})}$ the collection of all measurable selections. It is known that, whenever F satisfies (F1-2), then it is superpositionally measurable (see e.g. [19, Theorem 1.3.5]).

2 Measures of noncompactness and condensing multimaps

A m.n.c. is a function $\beta: \mathcal{P}(E) \to \mathcal{N}$ defined on the collection of all nonempty subsets of a Banach space E and taking values in a partially ordered set $\mathcal{N}$ such that for all $\Omega \subset E$, $\beta(\overline{co}\Omega) = \beta(\Omega)$.
A m.n.c. may enjoy the following properties:

(i) *monotonicity*: $\beta(\Omega_1) \leq \beta(\Omega_2)$ whenever $\Omega_1 \subset \Omega_2 \subset E$.

(ii) *nonsingularity*: $\beta(\{x\} \cup \Omega) = \beta(\Omega)$, for every $x \in E, \Omega \subset E$.

If, in addition, $\mathcal{N}$ is a cone in a Banach space a m.n.c. may also satisfy the property of

(iii) *algrebraic subadditivity*: $\beta(\Omega_1 + \Omega_2) \leq \beta(\Omega_1) + \beta(\Omega_2)$, for all $\Omega_1, \Omega_2 \subset E$.

(iv) *regularity*: $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.

(v) *semi-homogeneity*: $\beta(t\Omega) = |t|\beta(\Omega)$, for $t \in \mathbb{R}$ and $\Omega \subset E$.

Definition 2.1. *Let $X \subseteq E$. A multimap $F: X \multimap E$ or a family of multimaps $G: [0,1] \times X \multimap E$ with compact values is called* condensing*, with respect to a m.n.c. β (β-condensing for short) if*

$$\beta(F(\Omega)) \geq \beta(\Omega), \quad \text{or} \quad \beta(G([0,1] \times \Omega)) \geq \beta(\Omega)$$

only when $\Omega \subseteq X$ is relatively compact.

A well-known example of m.n.c. satisfying all the above properties is the Hausdorff m.n.c. χ, defined by

$$\chi(\Omega) = \inf\{\epsilon > 0 : \exists N \in \mathbb{N}, \exists x_i \in E, 1 \leq i \leq N : \Omega \subset \bigcup_{i=1}^{N} \mathbb{B}(x_i,\epsilon)\}.$$

In the following we also need to consider m.n.c. defined on spaces of continuous functions. Given a nonnegative constant L and a bounded subset $\Omega \subset C^0([\alpha,\beta];E)$, the function

$$\gamma(\Omega) = \sup_{t \in [\alpha,\beta]} e^{-L(t-\alpha)} \chi\big(\Omega(t)\big)$$

is a m.n.c. on $C^0([\alpha,\beta];E)$. However, if Ω is not equicontinuous, the m.n.c. γ is not regular. For this reason we always pair it with the following m.n.c.

$$\mathrm{mod}_C(\Omega) = \lim_{\delta\to 0} \sup_{\omega\in\Omega} \max_{|t_1-t_2|\le\delta} \|\omega(t_1)-\omega(t_2)\|_E\,,$$

and define the m.n.c.

$$\nu(\Omega) = \max_{\{w_n : n\in\mathbb{N}\}\subseteq\Omega} (\gamma(\{w_n : n\in\mathbb{N}\}), \mathrm{mod}_C(\{w_n : n\in\mathbb{N}\})), \tag{2.1}$$

where the maximum is carried out with respect to the ordering of $\mathbb{R}^2$ induced by the positive cone. It can be showed that ν is well-defined and it turns out to be a monotone, nonsingular, regular m.n.c. defined on $C^0([\alpha,\beta];E)$ (see e.g. [19, Ex. 2.1.4]). We finally remark that when $\Omega \subset C^0([\alpha,\beta];E)$ is relatively compact, it is also equicontinuous and so, if $\gamma(\Omega)=0$, then $\nu(\Omega)=(0,0)$.

3 Continuation principle and Scorza-Dragoni type result

We propose now the continuation principle which is the basis for our investigation. We sketch here its proof as a consequence of the theory of relative topological degree for convex-valued multifields, developed in [19] but we remark that it can be also derived from the topological index introduced in [1].

Theorem 3.1. *Let Q be a closed, convex subset of a Banach space Y with nonempty interior and $H\colon Q\times[0,1]\multimap Y$ be such that*

(a) H is nonempty convex valued and it has closed graph;

(b) H is quasi-compact and μ-condensing with respect to a monotone, nonsingular m.n.c. defined on Y;

(c) $H(Q,0)\subset Q$;

(d) $H(\cdot,\lambda)$ is fixed points free on the boundary of Q for all $\lambda\in[0,1)$.

Then there exists $y\in Q$ such that $y\in H(y,1)$.

Proof. If $H(\cdot,1)$ has a fixed point in the boundary ∂Q of Q the proof is finished. Hence, we can assume that $H(\cdot,\lambda)$ is fixed points free on ∂Q for all $\lambda\in[0,1]$. Since H has closed graph and it is quasi-compact, then H has compact values and hence it is u.s.c. (see e.g. [19, Theorem 1.1.12]). Under these assumptions, all the multimaps $H(\cdot,0)$, $H(\cdot,1)$ and H are completely, fundamentally restrictible. Therefore the relative topological degree can be defined (see e.g. [19]), for the corresponding vector-fields $\Phi_0 := Id - H(\cdot,0)$ and $\Phi_1 := Id - H(\cdot,1)$ with $\Phi_i\colon Q\multimap Y$ for $i=0,1$. In addition, $H(\cdot,0)$ has a fixed point in Q, implying that $deg_Y(\Phi_0,Q)=1$. Finally, since H is a homotopy from Φ_0 to Φ_1 and H is fixed points free on ∂Q, we obtain that $deg_Y(\Phi_0,Q)=deg_Y(\Phi_1,Q)=1$; hence $H(\cdot,1)$ has a fixed point in Q and the proof is complete. □

In Frechét spaces transversality condition *(d)* in previous theorem needs to be replaced by the stronger so called *pushing condition*. A guiding function approach was proposed, in [8], for getting pushing condition. It is useful for treating problems with regularities in (F2) given by the weak topologies and also for the investigation of boundary value problems on unbounded intervals such as those in [6, 20, 21].

We propose now a Scorza-Dragoni type result for the multimap F when the space E is separable. Its proof is a direct consequence of [23, Theorem 1].

Theorem 3.2. *Assume that E is separable and that $F\colon [a,b] \times C^0\big([-h,0];E\big) \multimap E$ is quasi-compact and satisfies (F1-2). Then there is a multimap $F_0\colon [a,b] \times C^0\big([-h,0];E\big) \multimap E$ with convex, compact, possibly empty values satisfying the following conditions:*

(i) $F_0(t,x) \subseteq F(t,x)$ *for a.a.* $t \in [a,b]$ *and all* $x \in E$;

(ii) *if* $u\colon [a,b] \to C^0\big([-h,0];E\big)$ *and* $v\colon [a,b] \to E$ *are measurable and* $v(t) \in F(t,u(t))$ *a.e. in* $[a,b]$, *then* $v(t) \in F_0(t,u(t))$ *a.e. in* $[a,b]$;

(iii) *for every* $\varepsilon > 0$, *there is* $A_\varepsilon \subset [a,b]$ *closed and with* $\tau\left([a,b] \setminus A_\varepsilon\right) < \varepsilon$ *and* F_0 *is nonempty valued and u.s.c. on* $A_\varepsilon \times C^0\big([-h,0];E\big)$.

Proof. When E is separable, the space $C^0\big([-h,0];E\big)$ is separable too, hence we can apply [23, Theorem 1] and find $\overline{F}\colon [a,b] \times C^0\big([-h,0];E\big) \multimap E$ with closed, possibly empty values satisfying conditions valid for F_0 in *(i)* and *(ii)* and such that

(iii′) for every $\varepsilon > 0$, there is $A_\varepsilon \subset [a,b]$ closed with $\tau\left([a,b] \setminus A_\varepsilon\right) < \varepsilon$ such that the graph of $\overline{F}$ is closed on $A_\varepsilon \times C^0\big([-h,0];E\big)$.

Let $\{x_n : n \in \mathbb{N}\} \subset C^0\big([-h,0];E\big)$ be such that $\overline{\{x_n : n \in \mathbb{N}\}} = C^0\big([-h,0];E\big)$. According to (F1), and the property *(ii)* applied to $\overline{F}$, it is possible to find $N \subset [a,b]$ with $\tau(N) = 0$ and $\overline{F}(t,x_n) \neq \emptyset$ for all $t \notin N$ and all n. Now fix $\varepsilon > 0$ and put $B_\varepsilon = A_\varepsilon \setminus N$ with $\overline{F}$ satisfying *(iii)* on $A_\varepsilon \times C^0\big([-h,0];E\big)$. It is clear that $\tau\left([a,b] \setminus B_\varepsilon\right) < \varepsilon$. Given $x \in C^0\big([-h,0];E\big)$ and $t \in B_\varepsilon$, let $x_{n_p} \to x$ and $y_{n_p} \in \overline{F}(t,x_{n_p}) \subseteq F(t,x_{n_p})$ for all p. Since F is quasi-compact, it is possible to find $y \in E$ and a subsequence, denoted as the sequence, such that $y_{n_p} \to y \in E$. Since $\overline{F}$ restricted to $B_\varepsilon \times C^0\big([-h,0];E\big)$ has closed graph, then $y \in \overline{F}(t,x)$ and $\overline{F}$ is nonempty valued on $B_\varepsilon \times C^0\big([-h,0];E\big)$. Since F is quasi-compact, then $\overline{F}$ restricted to $B_\varepsilon \times C^0\big([-h,0];E\big)$ is u.s.c. (see e.g. [19, Theorem 1.1.12]). Define $F_0(t,x) = \overline{\mathrm{co}}\overline{F}(t,x)$ for all $(t,x) \in [a,b] \times C^0\big([-h,0];E\big)$. Hence F_0 has convex, compact values and it satisfies properties *(i)*, *(ii)* and *(iii)*. □

4 Compactness properties of evolution operators

When assuming condition (A) the linear operators $\{A(t) : t \in [a,b]\}$ give rise to a strongly continuous evolution operator (see e.g. [17]) $U\colon \Delta \to \mathfrak{L}(E)$ with $\Delta := \{(t,s) \in [a,b] \times [a,b] : a \le s \le t \le b\}$ and it is always true that

$$\|U(t,s)\|_{\mathfrak{L}(E)} \le e^{\int_s^t \|A(\sigma)\|_{\mathfrak{L}(E)} d\sigma}. \tag{4.1}$$

Consequently

$$\|U(t,s)\|_{\mathfrak{L}(E)} \leq e^{\int_a^b \|A(\sigma)\|_{\mathfrak{L}(E)} d\sigma} =: D, \qquad \text{for } (t,s) \in \Delta. \tag{4.2}$$

In the study of (1.1)-(1.2) we show that it is useful to introduce a parameterized family of linear operators $\{\lambda A(t) : t \in [a,b]\}$ depending on the real value λ; their corresponding evolution operators will be denoted by $U_\lambda : \Delta \to \mathfrak{L}(E)$. The following two lemmas show some compactness properties of $\{U_\lambda : \lambda \in \mathbb{R}\}$.

Lemma 4.1. *Let $A\colon [a,b] \to \mathfrak{L}(E)$ satisfy (A). If $\lambda_n \to \lambda$ for some $\lambda \in \mathbb{R}$, then $U_{\lambda_n}(t,s) \to U_\lambda(t,s)$ in $\mathfrak{L}(E)$, uniformly in Δ.*

Proof. Take $\lambda \in \mathbb{R}$ and $(t,s) \in \Delta$. Notice that the operators $U_\lambda(t,s)$ are strongly differentiable with respect to t and s and it follows that

$$\frac{\partial U_\lambda(t,s)}{\partial t} = \lambda A(t)U(t,s) \quad \text{and} \quad \frac{\partial U_\lambda(t,s)}{\partial s} = -\lambda U(t,s)A(s).$$

Consequently, for any $\lambda, \mu \in \mathbb{R}$,

$$\begin{aligned} \tfrac{\partial}{\partial \tau}\big(U_\lambda(t,\tau)U_\mu(\tau,s)\big) &= -\lambda U_\lambda(t,\tau)A(\tau)U_\mu(\tau,s) + \mu U_\lambda(t,\tau)A(\tau)U_\mu(\tau,s) \\ &= (\mu - \lambda)U_\lambda(t,\tau)A(\tau)U_\mu(\tau,s). \end{aligned}$$

Integrating the previous relation we have that

$$U_\mu(t,s) - U_\lambda(t,s) = (\mu - \lambda)\int_s^t U_\lambda(t,\tau)A(\tau)U_\mu(\tau,s)d\tau$$

and then

$$\begin{aligned} &\|U_\mu(t,s)w - U_\lambda(t,s)w\|_E \leq \\ &\qquad \leq \|w\|_E\,|\mu - \lambda| \int_s^t \|U_\lambda(t,\tau)\|_{\mathfrak{L}(E)}\,\|A(\tau)\|_{\mathfrak{L}(E)}\,\|U_\mu(\tau,s)\|_{\mathfrak{L}(E)}\,d\tau. \end{aligned}$$

According to (4.1) and (4.2) we obtain that

$$\|U_\lambda(t,s)w - U_\mu(t,s)w\|_E \leq \|w\|_E\,|\mu - \lambda| D^{|\lambda|+|\mu|}\ln D.$$

The stated result follows easily, when observing that any convergent sequence is bounded. □

Remark 4.1. *Let $\{\lambda_n : n \in \mathbb{N}\} \subset [0,1]$ and $(t,s) \in \Delta$. As a consequence of the previous lemma the set $\{U_{\lambda_n}(t,s) : n \in \mathbb{N}\}$ is relatively compact in $\mathfrak{L}(E)$ implying the relative compactness in E of $\{U_{\lambda_n}(t,s)x_0 : n \in \mathbb{N}\}$ for every fixed $x_0 \in E$.*

Lemma 4.2. *Let $\{\lambda_n : n \in \mathbb{N}\} \subset [0,1]$, $\{f_n : n \in \mathbb{N}\} \subset L^1\big([a,b];E\big)$ and $(t,s) \in \Delta$. It follows that*

$$\chi\big(\{U_{\lambda_n}(t,s)f_n(s) : n \in \mathbb{N}\}\big) \leq e^{\int_s^t \|A(\sigma)\|_{\mathfrak{L}(E)} d\sigma}\chi\big(\{f_n(s) : n \in \mathbb{N}\}.$$

Proof. Fix $(t,s) \in \Delta$ and $\alpha > 0$. If $r := \chi\left(\{f_n(s) : n \in \mathbb{N}\}\right)$, there exist $p \geq 1$ and $e_1, ..., e_p \in E$ satisfying

$$\{f_n(s) : n \in \mathbb{N}\} \subset \bigcup_{j=1}^{p} \mathbb{B}(e_j, r + \frac{\alpha}{2}).$$

According to Remark 4.1, the set $\{U_{\lambda_n}(t,s) : n \in \mathbb{N}\}$ is relatively compact in $\mathfrak{L}(E)$. Therefore, given

$$H := \max\{\|e_j\|_E\,,\ j = 1, ..., p\}, \qquad \text{and} \qquad \varepsilon < \frac{\alpha}{2H} e^{\int_s^t \|A(\sigma)\|_{\mathfrak{L}(E)} d\sigma}$$

there exist $q \geq 1$ and $U_1, ..., U_q \in \mathfrak{L}(E)$ satisfying

$$\{U_{\lambda_n}(t,s) : n \in \mathbb{N}\} \subset \bigcup_{i=1}^{q} \mathbb{B}(U_i, \varepsilon).$$

Fix $n \in \mathbb{N}$ and take $U_i \in \mathfrak{L}(E)$ and $e_j \in E$ such that $U_{\lambda_n}(t,s) \in \mathbb{B}(U_i, \epsilon)$ and $f_n(s) \in \mathbb{B}(e_j, r + \frac{\alpha}{2})$. It follows that

$$\|U_{\lambda_n}(t,s)f_n(s) - U_i e_j\|_E \leq \|U_{\lambda_n}(t,s)f_n(s) - U_{\lambda_n}(t,s)e_j\|_E + \|U_{\lambda_n}(t,s)e_j - U_i e_j\|_E\,.$$

Therefore, according to (4.2), we obtain that

$$\|U_{\lambda_n}(t,s)f_n(s) - U_i e_j\|_E \leq \left(r + \frac{\alpha}{2}\right) e^{\int_s^t \|A(\sigma)\|_{\mathfrak{L}(E)} d\sigma} + \varepsilon H \leq (r+\alpha) e^{\int_s^t \|A(\sigma)\|_{\mathfrak{L}(E)} d\sigma},$$

implying that

$$\{U_{\lambda_n}(t,s)f_n(s) : n \in \mathbb{N}\} \subset \bigcup_{\substack{i = 1, ..., q \\ j = 1, ..., p}} \mathbb{B}\left(U_i e_j, (r+\alpha) e^{\int_s^t \|A(\sigma)\|_{\mathfrak{L}(E)} d\sigma}\right).$$

The arbitrariness of α implies the estimate. □

Definition 4.2. *We say that a sequence $\{f_n : n \in \mathbb{N}\} \subset L^1\big([a,b]; E\big)$ is semicompact if it is integrably bounded, i.e. $\|f_n(t)\|_E \leq \omega(t)$ for some $\omega \in L^1_+\big([a,b]\big)$ and a.a. $t \in [a,b]$, and the set $\{f_n(t) : n \in \mathbb{N}\}$ is relatively compact for a.a. $t \in [a,b]$.*

Every semicompact sequence $\{f_n : n \in \mathbb{N}\} \subset L^1\big([a,b]; E\big)$ is weakly compact in $L^1\big([a,b]; E\big)$ (see e.g. [19, Proposition 4.2.1]) and the following convergence result can be proved, as a straightforward consequence of [11, Theorem 2] and [19, Theorem 5.1.1].

Theorem 4.1. *Let $\{f_n : n \in \mathbb{N}\} \subset L^1\big([a,b]; E\big)$ be a semicompact sequence and $U : \Delta \to \mathfrak{L}(E)$ a strongly continuous evolution operator. Then the sequence $\left\{t \mapsto \int_a^t U(t,s)f_n(s)ds,\, t \in [a,b] : n \in \mathbb{N}\right\}$ is relatively compact in $C^0\big([a,b]; E\big)$. Moreover, if $f_n \rightharpoonup f$ weakly in $L^1\big([a,b]; E\big)$, then*

$$\int_a^t U(t,s)f_n(s)ds \to \int_a^t U(t,s)f(s)ds$$

uniformly in $C^0\big([a,b]; E\big)$.

5 The solution multi-operator

We embed now the linearized version of (1.1) into a family of equations depending on a real parameter and we study the main features of the solution multi-operator T of the parameterized family. Precisely, given $q \in C^0\big([a-h,b];E\big)$ and $\lambda \in [0,1]$, we consider the multivalued equation

$$x'(t) + \lambda A(t)x(t) \in G(t, q_t, \lambda), \qquad t \in [a,b] \tag{5.1}$$

with $G : [a,b] \times C^0\big([-h,0];E\big) \times [0,1] \multimap E$ having convex, compact, nonempty values and satisfying:

(G1) $G(\cdot, x, \lambda)$ has a strongly measurable selection for every $(x,\lambda) \in C^0\big([-h,0];E\big) \times [0,1]$;

(G2) $G(t,\cdot,\cdot)$ is u.s.c. for a.a. $t \in [a,b]$;

(G3) G is integrably bounded on every bounded set;

(G4) $\chi\big(G(t,\Omega,\lambda)\big) \le k^F(t) \sup_{s\in[-h,0]} \chi(\Omega(s))$, for any bounded set $\Omega \subset C^0\big([-h,0];E\big)$, a.a. $t \in [a,b]$ and all $\lambda \in [0,1]$ with the same k^F as in (F4);

(G5) $G(t,x,1) \subseteq F(t,x)$ for all $(t,x) \in [0,1] \times C^0\big([-h,0];E\big)$

and we denote by $T(q,\lambda)$ the set of all solutions of problem (5.1)-(1.2).
Hence $T : C^0\big([a-h,b];E\big) \times [0,1] \multimap C^0\big([a-h,b];E\big)$ and the following two propositions show its main features. We always assume conditions (A) and (F1-4).

Remark 5.1. *If $G(t,x,\lambda) := \lambda F(t,x)$, for $(t,x,\lambda) \in [a,b] \times C^0\big([-h,0];E\big) \times [0,1]$, then it is easy to prove that (G1-5) are satisfied.*

Proposition 5.1. *The solution multi-operator $T : C^0\big([a-h,b];E\big) \times [0,1] \multimap C^0\big([a-h,b];E\big)$ is quasi-compact, with nonempty, convex values and it has closed graph.*

Proof. According to (G1-2), the multimap $G(\cdot,\cdot,\lambda)$ is superpositionally measurable for all λ and hence $\mathcal{S}^1_{G(\cdot,q_{(\cdot)},\lambda)}$ is nonempty, implying that $T(q,\lambda) \neq \emptyset$ (see e.g. [17]); thus T is well-defined. It is easy to see that T is also convex-valued, since G is such.
Let $\hat{\Omega} \subset C^0\big([a-h,b];E\big)$ be compact and consider $\{y_n : n \in \mathbb{N}\} \subset T(\hat{\Omega} \times [0,1])$. Let $\{q_n : n \in \mathbb{N}\} \subseteq \hat{\Omega}$, $\{\lambda_n : n \in \mathbb{N}\} \subset [0,1]$ and $\{g_n : n \in \mathbb{N}\}$, $g_n \in \mathcal{S}^1_{G(\cdot,q_{n,(\cdot)},\lambda_n)}$ for all n be such that

$$y_n(t) = U_{\lambda_n}(t,a)\varphi(0) + \int_a^t U_{\lambda_n}(t,s)g_n(s)\,ds, \quad t \in [a,b], \tag{5.2}$$

with the evolution operators U_{λ_n} introduced in Section 4. The compactness of $\hat{\Omega}$ and $[0,1]$ guarantee the existence of a subsequence, that we continue to denote as the related sequence, such that $(q_n,\lambda_n) \to (q,\lambda) \in \hat{\Omega} \times [0,1]$ and from now on we restrict our attention to it. Let us re-write (5.2) as follows

$$y_n(t) = U_{\lambda_n}(t,a)\varphi(0) + \int_a^t U_\lambda(t,s)g_n(s)ds + \int_a^t \left[U_{\lambda_n}(t,s) - U_\lambda(t,s)\right] g_n(s)ds, \quad t \in [a,b]. \tag{5.3}$$

For $t \in [a,b]$ put $\Theta(t) := \{q_{n,t} : n \in \mathbb{N}\}$ and let $\Theta := \cup_{t\in[a,b]}\Theta(t)$. Since $\{q_n : n \in \mathbb{N}\}$ is bounded in $C^0\big([a-h,b];E\big)$, according to (G3) there is $\mu_\Theta^G \in L^1_+\big([a,b]\big)$ such that $\|g_n(t)\|_E \le \mu_\Theta^G(t)$ for a.a. $t \in [a,b]$ and all $n \in \mathbb{N}$. Moreover, since for all $t \in [a,b]$ the set $\Theta(t)$ is relatively compact in $C^0\big([-h,0];E\big)$, according to (G4) we have

$$\chi\left(\{g_n(t) : n \in \mathbb{N}\}\right) \le \chi\left(\{G(t,\Theta(t),\lambda_n) : n \in \mathbb{N}\}\right) \le k^F(t) \sup_{s\in[-h,0]} \chi\left(\{q_n(t+s) : n \in \mathbb{N}\}\right) = 0$$

for a.a. $t \in [a,b]$.
The sequence $\{g_n : n \in \mathbb{N}\}$ is then semicompact (see Definition 4.2) and so it is weakly compact in $L^1\big([a,b];E\big)$. Hence there exists a subsequence, again denoted as the sequence, and a function $g \in L^1\big([a,b];E\big)$ satisfying $g_n \rightharpoonup g$ in $L^1\big([a,b];E\big)$. According to Theorem 4.1 we also obtain that

$$\int_a^t U_\lambda(t,s)g_n(s)ds \to \int_a^t U_\lambda(t,s)g(s)ds$$

in $C^0\big([a,b];E\big)$. Moreover

$$\left\|\int_a^t \left[U_{\lambda_n}(t,s) - U_\lambda(t,s)\right] g_n(s)ds\right\|_E \le \sup_{(t,s)\in\Delta} \|U_{\lambda_n}(t,s) - U_\lambda(t,s)\|_{\mathcal{L}(E)} \cdot \|\mu_\Theta\|_1,\ t \in [a,b].$$

Therefore, since $\lambda_n \to \lambda$ and $U_{\lambda_n}(t,s) \to U_\lambda(t,s)$ in $\mathcal{L}(E)$ uniformly in Δ as proved in Lemma 4.1 we obtain that the first addendum in (5.3) converges to $U_\lambda(t,a)\varphi(0)$ and the third one converges to zero both of them in $C^0\big([a,b];E\big)$. We have then proved that T is quasi-compact.

Let now $\{q_n : n \in \mathbb{N}\} \subset C^0\big([a-h,b];E\big)$, $\{\lambda_n : n \in \mathbb{N}\} \subset [0,1]$ and $\{y_n : n \in \mathbb{N}\} \subset AC^0\big([a,b];E\big)$, with $y_n \in T(q_n,\lambda_n)$ for all n, be such that $q_n \to q$, $\lambda_n \to \lambda$ and $y_n \to y$, each one with respect to the corresponding topology. Consequently there exists $g_n \in \mathcal{S}^1_{G\left(\cdot,q_{n,(\cdot)},\lambda_n\right)}$ such that y_n satisfies (5.3) and $y_n(t) = \varphi(t-a)$ on $[a-h,a]$ for all n.
As showed in the first part of this proof, the sequence $\{g_n : n \in \mathbb{N}\}$ is semicompact and there exist $g \in L^1\big([a,b];E\big)$ and a subsequence of $\{y_n : n \in \mathbb{N}\}$, again denoted as the sequence, which converges in $C^0\big([a,b];E\big)$ to the function $t \mapsto p(t) := U_\lambda(t,a)\varphi(0) + \int_a^t U_\lambda(t,s)g(s)ds$. The uniqueness of the limit implies $y = p$. Applying Mazur's Lemma, we may assume the existence of a sequence $\{\tilde{g}_n : n \in \mathbb{N}\}$ such that $\tilde{g}_n$ is a finite convex combination of $\{g_i : i \ge n\}$ and $\tilde{g}_n \to g$ in $L^1\big([a,b];E\big)$. Passing to a subsequence, denoted as the sequence, we obtain that $\tilde{g}_n(t) \to g(t)$ for a.a. $t \in [a,b]$.

Let $M \subseteq [a,b]$ be such that $G(t,\cdot,\cdot)$ is u.s.c., $g_n(t) \in G(t,q_{n,t},\lambda_n)$ and $\tilde{g}_n(t) \to g(t)$ for all $t \in M$ and $n \in \mathbb{N}$. According to (G2) and the definition of $\mathcal{S}^1_G$, the set $[a,b] \setminus M$ has null Lebesgue measure. Let $t \in M$ be fixed. According to (G2), for each $\varepsilon > 0$ there exists $\delta > 0$ such that $G(t,\vartheta,\mu) \subset G(t,q_t,\lambda) + \varepsilon\mathbb{B}$ for all $(\vartheta,\mu) \in C^0\big([-h,0];E\big) \times [0,1]$ with $\|\vartheta - q_t\|_{C^0\left([-h,0];E\right)} \le \delta$ and $|\mu - \lambda| \le \delta$. Since $q_{n,t} \to q_t$ in $C^0\big([-h,0];E\big)$ and $\lambda_n \to \lambda$, there exists $\overline{n}$ such that $\|q_{n,t} - q_t\|_{C^0\left([-h,0];E\right)} \le \delta$ and $|\lambda_n - \lambda| \le \delta$ implying $g_n(t) \in G(t,q_t,\lambda) + \varepsilon\mathbb{B}$ for all $n > \overline{n}$.
Since $G(t,q_t,\lambda) + \varepsilon\mathbb{B}$ is convex, we also have that $\tilde{g}_n \in G(t,q_t,\lambda) + \varepsilon\mathbb{B}$ for all $n > \overline{n}$. The arbitrariness of ε and the closure of $G(t,q_t,\lambda)$ imply $g(t) \in G(t,q_t,\lambda)$. Therefore T has closed graph. □

Proposition 5.2. *Let $Q \subset C^0\big([a-h,b];E\big)$ be bounded. The solution multi-operator $T : Q \times [0,1] \multimap C^0\big([a-h,b];E\big)$ is condensing, with respect to the m.n.c. on $C^0\big([a-h,b];E\big)$ defined as in* (2.1)*, provided that L satisfies*

$$\sup_{t\in[a,b]} 2D\int_a^t e^{-L(t-s)}k^F(s)ds < 1. \tag{5.4}$$

Proof. Let $\Omega \subseteq Q$ be such that $\nu\left(T\left(\Omega\times[0,1]\right)\right) \geq \nu(\Omega)$ and let $\{y_n : n\in\mathbb{N}\} \subset T\left(\Omega\times[0,1]\right)$ satisfying

$$\nu\left(T\left(\Omega\times[0,1]\right)\right) = \left(\gamma\left(\{y_n : n\in\mathbb{N}\}\right), \operatorname{mod}_C\left(\{y_n : n\in\mathbb{N}\}\right)\right)$$

with ν defined in (2.1). Hence we can find $\{q_n : n\in\mathbb{N}\} \subseteq \Omega$, $\{\lambda_n : n\in\mathbb{N}\} \subset [0,1]$ and $\{g_n : n\in\mathbb{N}\}$ such that $g_n \in \mathcal{S}^1_{G(\cdot,q_{n,(\cdot)},\lambda_n)}$ and y_n satisfies (5.2) for all n. According to Remark 4.1, the set $\{U_{\lambda_n}(t,a)\varphi(0) : n\in\mathbb{N}\}$ is relatively compact in E for every $t\in[a,b]$. Since moreover $y_n(t)\equiv\varphi(t-a)$, $t\in[a-h,a]$ and according to the algebraic subadditivity of the Hausdorff m.n.c. we obtain

$$\gamma\left(\{y_n : n\in\mathbb{N}\}\right) \leq \sup_{t\in[a,b]} e^{-L(t-a+h)}\chi\left(\left\{\int_a^t U_{\lambda_n}(t,s)g_n(s)ds : n\in\mathbb{N}\right\}\right). \tag{5.5}$$

As a consequence of (4.2) and Lemma 4.2, for all $(t,s)\in\Delta$ we have that

$$\begin{aligned}\chi\left(\{U_{\lambda_n}(t,s)g_n(s) : n\in\mathbb{N}\}\right) \leq\ & e^{\int_s^t\|A(\sigma)\|_{\mathcal{L}(E)}d\sigma}\chi\left(\{g_n(s) : n\in\mathbb{N}\}\right)\\ \leq\ & D\chi\left(\{g_n(s) : n\in\mathbb{N}\}\right).\end{aligned} \tag{5.6}$$

According to (G4) for a.a. $t\in[a,b]$ we obtain that

$$\begin{aligned}\chi\left(\{g_n(t) : n\in\mathbb{N}\}\right) \leq\ & k^F(t)\sup_{s\in[-h,0]}\chi\big(\{q_n(t+s) : n\in\mathbb{N}\}\big)\\ \leq\ & k^F(t)e^{L(t-a+h)}\sup_{s\in[-h,0]}e^{-L(t+s-a+h)}\chi\big(\{q_n(t+s) : n\in\mathbb{N}\}\big)\\ \leq\ & k^F(t)e^{L(t-a+h)}\sup_{t\in[a-h,b]}e^{-L(t-a+h)}\chi\big(\{q_n(t) : n\in\mathbb{N}\}\big)\\ =\ & k^F(t)e^{L(t-a+h)}\gamma\big(\{q_n : n\in\mathbb{N}\}\big).\end{aligned}$$

Let $\Theta := \{q_t : q\in\Omega, t\in[a,b]\} \subset C^0\big([-h,0];E\big)$. Since Q is bounded then also Θ is bounded and so there exists $\mu_\Theta \in L^1_+\big([a,b]\big)$ satisfying

$$\|g_n(t)\|_E \leq \mu_\Theta(t), \qquad \text{for } n\in\mathbb{N} \text{ and a.a. } t\in[a,b]. \tag{5.7}$$

Fix $t\in[a,b]$. According to (4.2) and (G3), we obtain that $\|U_{\lambda_n}(t,s)g_n(s)\|_E \leq D\mu_{\Omega'}(s)$ for a.a. $s\in[a,t]$ and $n\in\mathbb{N}$. Moreover, we obtain from (5.6)

$$\chi\left(\{U_{\lambda_n}(t,s)g_n(s) : n\in\mathbb{N}\}\right) \leq D\chi\left(\{g_n(s) : n\in\mathbb{N}\}\right) \leq Dk^F(s)e^{L(s-a+h)}\gamma\left(\{q_n : n\in\mathbb{N}\}\right),$$

for a.a. $s\in[a,t]$. Applying a standard property of the Hausdorff m.n.c. (see e.g. [19, Corollary 4.2.5]) we have that

$$\chi\big(\left\{\int_a^t U_{\lambda_n}(t,s)g_n(s)ds : n\in\mathbb{N}\right\}\big) \leq 2D\gamma\big(\{q_n : n\in\mathbb{N}\}\big)\int_a^t e^{L(s-a+h)}k^F(s)ds$$

As a consequence of (5.5) we have that

$$\begin{aligned}\gamma\big(\{y_n : n \in \mathbb{N}\}\big) &\le \sup_{t\in[a,b]} 2D\gamma\big(\{q_n : n \in \mathbb{N}\}\big)e^{-L(t-a+h)}\int_a^t e^{L(s-a+h)}k^F(s)ds \\ &= \gamma\big(\{q_n : n \in \mathbb{N}\}\big)\sup_{t\in[a,b]} 2D\int_a^t e^{-L(t-s)}k^F(s)ds.\end{aligned}$$

So, according to (5.4) and the definition of the sequence $\{y_n : n \in \mathbb{N}\}$, if $\gamma\big(\{y_n : n \in \mathbb{N}\}\big) > 0$ we obtain the contradictory conclusion

$$\gamma\big(\{q_n : n \in \mathbb{N}\}\big) \le \gamma\big(\{y_n : n \in \mathbb{N}\}\big) < \gamma\big(\{q_n : n \in \mathbb{N}\}\big).$$

Hence $\gamma\big(\{q_n : n \in \mathbb{N}\}\big) = 0$ implying $\chi\big(\{q_n(t) : n \in \mathbb{N}\}\big) = 0$ for all $t \in [a-h, b]$. According to (G4) we obtain that $\{g_n(t) : n \in \mathbb{N}\}$ is relatively compact for a.a. $t \in [a, b]$. Therefore, according to (5.7), $\{g_n : n \in \mathbb{N}\}$ is semicompact. Let $\{\lambda_{n_p} : p \in \mathbb{N}\}$ be a convergent subsequence. With a similar reasoning as in the proof of Proposition 5.1 we can show the existence of a convergent subsequence $\{y_{n_p} : p \in \mathbb{N}\}$; this proves that the set $\{y_n : n \in \mathbb{N}\}$ is relatively compact in $C^0\big([a-h, b]; E\big)$. As stated in Section 2, this implies that $(0, 0) = \nu\big(\{y_n : n \in \mathbb{N}\}\big) = \nu\,(T(\Omega \times [0, 1]) \ge \nu(\Omega)$; since ν is a regular m.n.c. we obtain that Ω is relatively compact and thus T turns out to be $\nu-$condensing. □

6 Preliminary existence results

We state now an existence result for problem (1.1)-(1.2) which is valid in an arbitrary reflexive Banach space E and it is based on the parameterizations given by (5.1). Combining it with the Scorza-Dragoni type argument given in Theorem 3.2 and assuming that E is also separable, in next Section we will prove Theorem 1.2. Let $K \subset E$ be nonempty, open and bounded and $V : E \to \mathbb{R}$ a locally Lipschitzian function on ∂K such that for a.a. $t_0 \in (a, b]$ and all $(\vartheta_0, \lambda) \in C^0([-h, 0]; \overline{K}) \times (0, 1)$ with $\vartheta_0(0) \in \partial K$, there exists $\delta = \delta(t_0, \vartheta_0, \lambda)$ satisfying

$$\liminf_{\ell\to 0^-} \frac{V(\vartheta(0) - \ell\lambda A(t)\vartheta(0) + \ell w) - V(\vartheta(0))}{\ell} < 0 \tag{6.1}$$

for a.a. t with $|t - t_0| < \delta$, all $\vartheta \in C^0([-h, 0]; \overline{K})$ with $\|\vartheta - \vartheta_0\|_{C^0\big([-h,0];E\big)} < \delta$ and $w \in G(t, \vartheta, \lambda)$.

We remark that, when V is Gateaux differentiable, the estimate (6.1) reduces itself to $V^G_{\vartheta(0)}(-\lambda A(t)\vartheta(0) + w) < 0$.

Theorem 6.1. *Consider problem* (1.1)-(1.2) *under conditions (A). Assume the existence of* $G : [a, b] \times C^0\big([-h, 0]; E\big) \times [0, 1] \multimap E$ *with convex, compact, nonempty values satisfying (G1-5). Let* $K \subset E$ *be nonempty, open, bounded and convex and* $V : E \to \mathbb{R}$ *locally Lipschitzian in* ∂K *and satisfying (V) and* (6.1)*. Assume* $\varphi \in C^0\,([-h, 0]; K)$*. If every function* β *in* $\mathcal{S}^1_{G(\cdot,q_{(\cdot)},0)}$ *with* $q \in C^0([a-h, b]; \overline{K})$ *is such that* $\varphi(0) + \int_a^t \beta(s)ds \in K$ *for all* $t \in [a, b]$*, then* (1.1)-(1.2) *has at least one solution* $x \in C^0\big([a-h, b]; E\big) \cap AC^0\big([a, b]; E\big)$*, satisfying* $x(t) \in \overline{K}$ *for all* $t \in [a, b]$.

Proof. Let T be the solution multi-operator defined in Section 5, i.e. associated to the problem (5.1)-(1.2) and put $Q := C^0\left([a-h,b];\overline{K}\right)$. If we are able to prove that $T : Q\times[0,1] \multimap C^0\big([a-h,b];E\big)$ satisfies all the assumptions of the continuation principle given in Theorem 3.1, then $T(\cdot,1)$ has a fixed point which is, according to (G5), a solution of (1.1)-(1.2) and it satisfies $x(t)\in\overline{K}$ for all $t\in[a,b]$.

Property (a) in Theorem 3.1 derives from Propositions 5.1; since K is bounded, implying that also Q is bounded, property (b) in Theorem 3.1 comes from Propositions 5.1 and 5.2.

Since, by assumption, $T(Q,0)\subset \operatorname{int} Q$, then property (c) in Theorem 3.1 is true and it remains to prove the transversality condition (d) only for $\lambda\in(0,1)$. Let $x\in Q$ be a fixed point of $T(\cdot,\lambda)$ with $\lambda\in(0,1)$; hence there is $\beta_\lambda\in \mathcal{S}^1_{G(\cdot,x_{(\cdot)},\lambda)}$ such that

$$x'(t)+\lambda A(t)x(t)=\beta_\lambda(t), \qquad \text{for a.a. } t\in[a,b] \tag{6.2}$$

and define $g(t):=V(x(t))$ for $t\in[a,b]$. Assume further that $x\in\partial Q$; hence there is $t_0\in[a-h,b]$ satisfying $x(t_0)\in\partial K$. According to the properties of φ we have that $t_0\in(a,b]$. Since V is locally Lipschitzian in ∂K, there exist an open set $U\subseteq E$ with $x(t_0)\in U$ and a constant L such that, when restricted to U, V is $L-$Lipschitzian. Let $0<\hat{h}<\min\{t_0-a,\,\delta\}$ be such that $x(t)\in U$ and $\|x_t-x_{t_0}\|_{C^0\left([-h,0];E\right)}\le\delta$ for all $t\in[t_0-\hat{h},t_0]$ with $\delta=\delta(t_0,x_{t_0},\lambda)$ as in (6.1). It is easy to see that g is absolutely continuous in $[t_0-\hat{h},t_0]$. If we further prove that

$$g'(t)<0 \qquad \text{for a.a. } t\in(t_0-\hat{h},t_0) \tag{6.3}$$

then we arrive to the contradictory conclusion

$$-V(x(t_0-\hat{h}))=g(t_0)-g(t_0-\hat{h})=\int_{t_0-\hat{h}}^{t_0} g'(s)ds<0.$$

In fact, since without loss of generality we can take $x(t)\in K$ for $t\in[a-h,t_0)$, from (V) we obtain the contradictory conclusion $V(x(t_0-\hat{h}))>0$; it implies condition (d) for all $\lambda\in(0,1)$. So, it remains to show condition (6.3). Let $t\in(t_0-\hat{h},t_0)$ be fixed and such that conditions (6.1) and (6.2) are valid and there is $g'(t)$. Take $h\in(t_0-t-\hat{h},0)$ with $h=h(t)$ sufficiently small so that also $x(t)+hx'(t)\in U$. Since

$$\frac{g(t+h)-g(t)}{h}=\frac{V(x(t)+hx'(t))-V(x(t))}{h}+\Delta(h)$$

where $\Delta:=\frac{V(x(t+h))-V(x(t)+hx'(t))}{h}$, according to the Lipschitzianity of V in U we have that $\Delta(h)\to0$ as $h\to0$. According to (6.1) and (6.2)

$$\begin{aligned} g'(t) &= \lim_{h\to0^-}\frac{g(t+h)-g(t)}{h}=\liminf_{h\to0^-}\frac{V\left(x(t)+hx'(t)\right)-V(x(t))}{h}\\ &= \liminf_{h\to0^-}\frac{V\left(x(t)-h\lambda A(t)x(t)+h\beta_\lambda(t)\right)-V(x(t))}{h}<0 \end{aligned}$$

hence (6.3) is satisfied. □

Note that for the fully linearized parametrization $G(t,x,\lambda) = \lambda F(t,x)$ condition (6.1) can be replaced by the following simpler one, independent on λ

$$\liminf_{\ell\to 0^-} \frac{V(\vartheta(0) - \ell A(t)\vartheta(0) + \ell w) - V(\vartheta(0))}{\ell} < 0. \tag{6.4}$$

Indeed, let (ϑ,λ) as in (6.1), take $t \in (a,b]$ for which (6.4) is true and $w' \in \lambda F(t,\vartheta)$. Since $w' = \lambda w$ with $w \in F(t,\vartheta)$ it follows that

$$\liminf_{\ell'\to 0^-} \frac{V(\vartheta(0) - \ell'\lambda A(t)\vartheta(0) + \ell' w') - V(\vartheta(0))}{\ell'} = \liminf_{\ell\to 0^-} \lambda \frac{V(\vartheta(0) - \ell A(t)\vartheta(0) + \ell w) - V(\vartheta(0))}{\ell} < 0.$$

Moreover, $T(Q,0) = \{x_0\}$ where $x_0(t) = \varphi(t-a)$ for $t \in [a-h,a]$ while $x_0 \equiv \varphi(0)$ for $t \in (a,b]$. So the following result is an easy consequence of previous theorem

Corollary 6.1. *Consider problem* (1.1)-(1.2)*, under conditions (A), (F1-4), (V) and* (6.4) *with K as in Theorem 6.1. Whenever $\varphi \in C^0([-h,0];K)$ the problem has at least one solution $x \in C^0\big([a-h,b];E\big) \cap AC^0\big([a,b];E\big)$, satisfying $x(t) \in \overline{K}$ for all $t \in [a,b]$.*

Looking at the proof of Theorem 6.1 it is easy to see that $x(t_0) \in \partial K$ for some solution x of (5.1)-(1.2) and $t_0 \in (a,b]$ leads to a contradiction. This is to say that K is a positively invariant set for (5.1) for every $\lambda \in (0,1)$. Consequently, the proof of Theorem 6.1 can be derived from Theorem 1.1 with no need to introduce a continuation principle. Indeed, consider $\mathcal{H} := C^0([-h,0];\overline{K})$ and let $\mathcal{K} := \{x \in C^0\big([-h,0];E\big) : \|x\| \geq 2M\}$ where $M := \sup_{k\in\overline{K}} \|k\|_E$. Since $\mathcal{H}$ and $\mathcal{K}$ are closed, disjoint sets we can find a continuous function $\mu\colon C^0\big([-h,0];E\big) \to [0,1]$ satisfying $\mu(\mathcal{H}) \equiv 1$ and $\mu(\mathcal{K}) \equiv 0$. Its existence is guaranteed by Urishon Lemma. Fix $\lambda \in (0,1)$ and let $G_\lambda : [a,b] \times C^0\big([-h,0];E\big) \multimap E$ be given by $G_\lambda(t,x) := \mu(x)G(t,x,\lambda)$. It is clear that G_λ satisfies (F1-2). Morever, if $\Omega \subset C^0\big([-h,0];E\big)$ is bounded and $t \in [a,b]$, we have that $\chi\left(G_\lambda(t,\Omega)\right) = \chi\left(\{\mu(x)G(t,x,\lambda) : x \in \Omega\}\right) \leq \chi\left(\cup_{\mu\in[0,1]}\mu G(t,\Omega,\lambda)\right)$. Since $\chi\left(\cup_{\mu\in[0,1]}\mu G(t,\Omega,\lambda)\right) = \chi\left(G(t,\Omega,\lambda)\right)$ (see e.g. [5, (2.5)]), if t is such that G satisfies (G4), we obtain that

$$\chi\left(G_\lambda(t,\Omega)\right) \leq k^F(t) \sup_{s\in[-h,0]} \chi\left(\Omega(s)\right).$$

Therefore $G_\lambda(\cdot,\cdot)$ satisfies also (F4). Consider now the bounded set $\Omega := \{x \in C^0\big([-h,0];E\big) : \|x\| \leq 2M\}$; applying condition (G3) to the multimap G we obtain a function $\mu^G_\Omega \in L^1_+\big([a,b]\big)$ satisfying $\|G(t,x,\lambda)\|_E \leq \mu^G_\Omega(t)$ for a.a. $t \in [a,b]$ and all $x \in \Omega$. Therefore $\|G_\lambda(t,x)\|_E \leq \mu^G_\Omega(t)$ for a.a. $t \in [a,b]$ and all $x \in C^0\big([-h,0];E\big)$. This shows that $G_\lambda(\cdot,\cdot)$ is integrably bounded, hence it satisfies the sublinear growth condition (F3′). Consider a sequence $\{\lambda_n : n \in \mathbb{N}\}$ with $\lambda_n \to 1^-$ and denote with y_n a corresponding sequence of solutions of the equation

$$x'(t) + \lambda A(t)x(t) \in G_\lambda(t,x_t), \qquad t \in [a,b]. \tag{6.5}$$

satisfying the initial condition (1.2). Their existence can be guaranteed by Theorem 1.1. Since, moreover, we also assume condition (6.1) and it implies that $y_n(t) \in K$ for all $t \in [a,b]$, then each y_n is indeed a solution (5.1) with corresponding parameter. Let $g_n \in \mathcal{S}^1_{G(\cdot,y_{n,(\cdot)},\lambda_n)}$ be such that y_n satisfies (5.2) and consider the m.n.c. ν on $C^0\big([a-h,b];E\big)$ defined in (2.1) with L satisfying (5.4). Then $\Theta := \cup_{t\in[a,b]}\{y_{n,t} : n \in \mathbb{N}\}$ is bounded in $C^0\big([-h,0];E\big)$. Hence there is $\mu_\Theta^G \in L^1_+\big([a,b]\big)$ such that $\|g_n(t)\|_E \leq \mu_\Theta^G(t)$ for a.a. $t \in [a,b]$. Notice that $\{y_n : n \in \mathbb{N}\} \subseteq T\left(\{y_n : n \in \mathbb{N}\} \times [0,1]\right)$, implying that $\nu\left(T\left(\{y_n : n \in \mathbb{N}\} \times [0,1]\right)\right) \geq \nu\left(\{y_n : n \in \mathbb{N}\}\right)$; since T is ν-condensing (see Proposition 5.2), this implies that $\{y_n : n \in \mathbb{N}\}$ is relatively compact. Consider a subsequence, denoted as the sequence, such that $y_n \to y \in C^0\big([a-h,b];E\big)$. According to (G4), it implies that $\{g_n : n \in \mathbb{N}\}$ is semicompact and hence there is $g \in L^1\big([a,b];E\big)$ and a subsequence, again denoted as the sequence, such that $g_n \rightharpoonup g$ in $L^1\big([a,b];E\big)$. Finally, with a similar reasoning as in the proof of Proposition 5.1, we obtain that $y(t) := U(t,a)\varphi(0) + \int_a^t U(t,s)g(s)ds$ for $t \in [a,b]$ and $y(t) = \varphi(t-a)$ on $[a-h,a]$ with $g \in \mathcal{S}^1_{G(\cdot,y_{(\cdot)},1)}$. Therefore, according to (G5), y is a solution of (1.1)-(1.2).

If we assume that V is a generalized guiding function on ∂K and it satisfies (V), instead of taking condition (6.1) (or (6.4) in the special case) K does no longer become a positively invariant set for any λ (see e.g. [13, Example 3.1]). For this reason our main result, i.e. Theorem 1.2, can not be derived from Theorem 1.1.

7 Proof of the main result

We need the following technical lemma which is a straightforward generalization of [13, Theorem 2.2] to the case of a function V only Gateaux differentiable. Hence we omit its proof, which is very technical.

Lemma 7.1. *Let E be a Banach space and $K \subset E$ be nonempty, bounded, open and convex. Assume that $V : E \to \mathbb{R}$ is a generalized guiding function on ∂K (see Definition 1.3) satisfying (V). Let $\kappa > 0$ be such that V_x^G is Lipschitzian on $\partial K + \kappa\mathbb{B}$. Then it is possible to find $\varepsilon \in (0,\kappa)$ and a Lipschitzian function $\phi : \partial K + \varepsilon\mathbb{B} \to E$ such that $V_x^G(\phi(x)) \equiv 1$.*

Proof of Theorem 1.2 Take ε as in the statement of Lemma 7.1. According to Urishon Lemma we can find a continuous map $\mu\colon E \to [0,1]$ such that $\mu \equiv 1$ on $\partial K + \frac{\varepsilon}{2}\mathbb{B}$ and $\mu \equiv 0$ on $E \setminus (\partial K + \varepsilon\mathbb{B})$. Consider $\tilde{\phi}\colon E \to E$ defined by

$$\tilde{\phi}(x) = \begin{cases} \mu(x)\|V_x^G\|_{\mathcal{L}(E;\mathbb{R})}\phi(x) & x \in \partial K + \varepsilon\mathbb{B} \\ 0 & \text{otherwise} \end{cases}$$

where ϕ was introduced in Lemma 7.1. It is easy to see that $\tilde{\phi}$ is well-defined. Since the functions: $x \longmapsto V_x^G$ and $x \longmapsto \phi(x)$ are Lipschitzian on $\partial K + \varepsilon\mathbb{B}$ (see Lemma 7.1) and $\tilde{\phi} \equiv 0$ on $E \setminus (\partial K + \varepsilon\mathbb{B})$, then $\tilde{\phi}$ is continuous and bounded on its whole domain and let $M_{\tilde{\phi}} > 0$ be such that $\left\|\tilde{\phi}(x)\right\|_E \leq M_{\tilde{\phi}}$, for all $x \in E$. For $t \in [a,b]$, put

$$p(t) := \mu^F_{\Omega_\varepsilon}(t) + \|A(t)\|_{\mathcal{L}(E)}\left(\|\partial K\| + \frac{\varepsilon}{2}\mathbb{B}\right) + 1$$

where $\Omega_\varepsilon = \left\{\vartheta \in C^0\big([-h,0];\overline{K}\big) : \vartheta(0) \in (\partial K + \frac{\varepsilon}{2}\mathbb{B})\right\}$; $\mu^F_{\Omega_\varepsilon}$ is the function given by (F3) and $\|\partial K\| := \sup\limits_{k\in\partial K}\|k\| < +\infty$ since K is bounded. Now the proof splits into some parts.

1. Introduction of a sequence of associated problems Since p is measurable in $[a,b]$ and $(t,x) \longmapsto -A(t)x$ is a Carathéodory function on $[a,b] \times E$, they both are almost continuous on their domains. Moreover, according to (F4), F is quasi-compact and so it satisfies the assumptions of Theorem 3.2, i.e. it has the Scorza-Dragoni property. Therefore, we can find a decreasing sequence $\{J_m : m \in \mathbb{N}\}$ of sets $J_m \subset [a,b]$ with $\tau(J_m) < \frac{1}{m}$ and a multifunction $F_0\colon [a,b] \times C^0\big([a-h,b];E\big) \multimap E$, with convex, compact values such that: $[a,b] \setminus J_m$ is closed for all m and the map $(t,x) \longmapsto -A(t)x + F_0(t,x)$ is u.s.c. on $([a,b] \setminus J_m) \times C^0\big([a-h,b];E\big)$ and p is continuous on $[a,b] \setminus J_m$. Moreover, if $J = \cap_{m=1}^{\infty} J_m$, then $\tau(J) = 0$ and we have that $F_0(t,x) \neq \emptyset$ and u.s.c. for all $(t,x) \in ([a,b] \setminus J) \times E$ and p is continuous on $[a,b] \setminus J$.

For each $m \in \mathbb{N}$ we consider the convex, compact, nonempty valued multimap $F_m(t,x)\colon [a,b] \times C^0\big([-h,0];E\big) \multimap E$ defined by

$$F_m(t,x) = \begin{cases} F_0(t,x) - p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\tilde{\phi}(x(0)) & (t,x) \in ([a,b] \setminus J) \times C^0\big([-h,0];E\big) \\ -p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\tilde{\phi}(x(0)) & (t,x) \in J \times C^0\big([-h,0];E\big) \end{cases}$$

and consider the multivalued equation

$$x'(t) + A(t)x(t) \in F_m(t, x_t), \qquad t \in [a,b]. \tag{7.1}$$

2. Solvability of the sequence of problems (7.1)-(1.2) Fix $m \in \mathbb{N}$. Now we show that, for every sufficiently large m, the initial value problem (7.1)-(1.2) satisfies all the assumptions of Theorem 6.1 and hence, it is solvable.

Notice that F_m satisfies (F1) and, since $\tilde{\phi}$ is continuous and bounded, also (F2) and (F3) are respectively true. Let us introduce, now, the convex, compact, nonempty valued multimap $G_m(t,x)\colon [a,b] \times C^0\big([-h,0];E\big) \times [0,1] \multimap E$ given by

$$G_m(t,x,\lambda) = \begin{cases} \lambda F_0(t,x) - p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\tilde{\phi}(x(0)) & (t,x,\lambda) \in ([a,b] \setminus J) \times C^0\big([-h,0];E\big) \times [0,1] \\ -p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\tilde{\phi}(x(0)) & (t,x,\lambda) \in J \times C^0\big([-h,0];E\big) \times [0,1]. \end{cases}$$

It is easy to see that G_m satisfies (G1-3) and (G5). Now we prove (G4). Let $\Omega \subset C^0\big([-h,0];E\big)$ bounded. For $t \notin J$ we have that

$$\chi(G_m(t,\Omega,\lambda)) \leq \lambda \chi\left(F_0(t,\Omega)\right) + p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\chi\left(\tilde{\phi}(\Omega(0))\right). \tag{7.2}$$

The Lipschitzianity of V_x^G implies the existence of $M_V > 0$ such that $\|V_x^G\|_{\mathcal{L}(E;\mathbb{R})} \leq M_V$ for all $x \in \partial K + \varepsilon\mathbb{B}$. Moreover, since $\tilde{\phi} \equiv 0$ outside $\partial K + \varepsilon\mathbb{B}$, we have that

$$\chi\left(\tilde{\phi}(\Omega(0))\right) = \chi\left(\left[\tilde{\phi}(\Omega(0)) \cap (\partial K + \varepsilon\mathbb{B})\right] \cup \{0\}\right) = \chi\left(\tilde{\phi}(\Omega(0)) \cap (\partial K + \varepsilon\mathbb{B})\right).$$

Therefore,

$$\begin{aligned} \chi\left(\tilde{\phi}(\Omega(0))\right) &= M_V \chi\left(\left\{\mu(y)\frac{\|V_y^G\|_{\mathcal{L}(E;\mathbb{R})}}{M_V}\phi(y) \,:\, y \in \Omega(0) \cap (\partial K + \varepsilon\mathbb{B})\right\}\right) \\ &\leq M_V \chi\left(\cup_{\lambda \in [0,1]}\left\{\lambda\phi(y) \,:\, y \in \Omega(0) \cap (\partial K + \varepsilon\mathbb{B})\right\}\right). \end{aligned}$$

According to a property of the Hausdorff m.n.c. (see e.g. (2.5) in [5]) we obtain that

$$\chi\left(\tilde{\phi}(\Omega(0))\right) \leq M_V \chi\left(\{\phi(y) \,:\, y \in \Omega(0) \cap (\partial K + \varepsilon\mathbb{B})\}\right).$$

Hence we have that

$$\chi\left(\tilde{\phi}(\Omega(0))\right) \le M_V L \chi\left(\Omega(0) \cap (\partial K + \varepsilon\mathbb{B})\right),$$

where L is the Lipschitz constant of ϕ. Combining with (7.2) and according to (F4) we obtain that

$$\chi(G_m(t,\Omega,\lambda) \le \left(k^F(t) + p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right) M_V L\right) \sup_{s\in[-h,0]} \chi\left(\Omega(s)\right).$$

It proves, at the same time, that F_m and G_m respectively satisfy (F4) and (G4).

We investigate now the transversality condition (6.1). First of all we take $(t,\vartheta,\lambda) \in J_m \times \Omega_\varepsilon \times (0,1)$ with $t \neq a$ and $t \notin J$. If $w_m \in G_m(t,\varphi,\lambda)$, then $w_m = \lambda w_0 - \frac{(m+1)p(t)}{m}\tilde{\phi}(\vartheta(0))$ and we obtain that $V^G_{\vartheta(0)}\left(-\lambda A(t)\vartheta(0) + w_m\right) = V^G_{\vartheta(0)}\left(-\lambda A(t)\vartheta(0) + \lambda w_0\right) - \frac{(m+1)p(t)}{m}\|V^G_{\vartheta(0)}\|_{\mathcal{L}(E;\mathbb{R})} V^G_{\vartheta(0)}\left(\phi(\vartheta(0))\right)$. According to Lemma 7.1 and the definition of p we have that

$$\begin{aligned} V^G_{\vartheta(0)}\left(-\lambda A(t)\vartheta(0) + w_m\right) = &\ V^G_{\vartheta(0)}\left(-\lambda A(t)\vartheta(0) + \lambda w_0\right) - \tfrac{(m+1)p(t)}{m}\|V^G_{\vartheta(0)}\|_{\mathcal{L}(E;\mathbb{R})} \\ \le &\ \left|V^G_{\vartheta(0)}\left(-\lambda A(t)\vartheta(0) + \lambda w_0\right)\right| - \tfrac{(m+1)p(t)}{m}\|V^G_{\vartheta(0)}\|_{\mathcal{L}(E;\mathbb{R})} \\ \le &\ \|V^G_{\vartheta(0)}\|_{\mathcal{L}(E;\mathbb{R})}\left(\mu^F_{\Omega_\varepsilon}(t) + \|A(t)\|_{\mathcal{L}(E)}\left(\|\partial K\| + \tfrac{\varepsilon}{2}\right) - \tfrac{(m+1)p(t)}{m}\right) \\ < &\ 0. \end{aligned} \tag{7.3}$$

Consider now $(t,\lambda) \in ((a,b] \setminus J_m) \times (0,1)$ and take $\vartheta \in C^0\left([-h,0];\overline{K}\right)$ with $\vartheta(0) \in \partial K$. With w_m and w_0 as before, we have that $V^G_{\vartheta(0)}\left(-\lambda A(t)\vartheta(0) + w_m\right) = V^G_{\vartheta(0)}\left(-\lambda A(t)\vartheta(0) + \lambda w_0\right) - \frac{p(t)}{m}\|V^G_{\vartheta(0)}\|_{\mathcal{L}(E;\mathbb{R})} V^G_{\vartheta(0)}\left(\phi(\vartheta(0))\right)$. Since V is a generalized guiding function on ∂K (see Definition 1.3) and $p(t) \ge 1$, for a.a. t we obtain that

$$\begin{aligned} V^G_{\vartheta(0)}\left(-\lambda A(t)\vartheta(0) + w_m\right) = &\ V^G_{\vartheta(0)}\left(-\lambda A(t)\vartheta(0) + \lambda w_0\right) - \tfrac{p(t)}{m}\|V^G_{\vartheta(0)}\|_{\mathcal{L}(E;\mathbb{R})} \\ \le &\ -\tfrac{p(t)}{m}\|V^G_{\vartheta(0)}\|_{\mathcal{L}(E;\mathbb{R})} \le -\tfrac{\delta}{m}. \end{aligned}$$

Notice that the multimap $(t,\vartheta,\lambda) \multimap -\lambda A(t)\vartheta(0) + \lambda F_0(t,\vartheta) - \frac{p(t)}{m}\tilde{\phi}(\vartheta(0))$ is u.s.c. on $([a,b] \setminus J_m) \times C^0\big([-h,0];E\big) \times [0,1]$ and according to the Lipschitzianity of $V^G_{(\cdot)}$ also the multimap

$$\begin{aligned} \Phi\colon\ ([a,b] \setminus J_m) \times \Omega_\varepsilon \times [0,1] &\multimap \mathbb{R} \\ (t,\vartheta,\lambda) &\longmapsto V^G_{\vartheta(0)}\left(-\lambda A(t)\vartheta(0) + \lambda F_0(t,\vartheta) - \tfrac{p(t)}{m}\tilde{\phi}(\vartheta(0))\right) \end{aligned}$$

is u.s.c. Therefore, we can find $\delta = \delta(t,\vartheta,\lambda)$ such that, if $|t'-t| < \delta$ with $t' \in [a,b] \setminus J_m$ and $\|\varphi' - \vartheta\|_{C^0\left([-h,0];E\right)} < \delta$ with $\varphi' \in \Omega_\varepsilon$, then $\Phi(t',\varphi',\lambda) < -\frac{\delta}{2m}$. This is to say that $V^G_{\varphi'(0)}\left(-\lambda A(t')\varphi'(0) + \lambda w_0 - \frac{p(t)}{m}\tilde{\phi}(\varphi'(0))\right) < -\frac{\delta}{2m} < 0$ for all $w_0 \in F_0(t',\varphi')$. Together with (7.3) it proves condition (6.1). It remains to investigate the case when $\lambda = 0$. So, let $q \in C^0\left([a-h,b];\overline{K}\right)$ and x^0_m be a solution of the initial value problem

$$x^{0\,\prime}_m(t) = -p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\tilde{\phi}(q(t)), \quad t \in [a,b], \qquad x^0_m(a) = \varphi(0).$$

If $\eta > 0$ is such that $\varphi(0) + \eta\mathbb{B} \subset K$, then

$$\begin{aligned}\|x_m^0(t) - \varphi(0)\|_E \leq & \int_{[a,t]\setminus J_m} \|x_m^{0\prime}\|_E(s)ds + \int_{J_m} \|x_m^{0\prime}\|_E(s)ds \\ \leq & \frac{M_{\tilde{\phi}}}{m}\int_{[a,t]\setminus J_m} p(s)ds + 2M_{\tilde{\phi}}\int_{J_m} p(s)ds \leq \frac{M_{\tilde{\phi}}}{m}\int_a^b p(s)ds + 2M_{\tilde{\phi}}\int_{J_m} p(s)ds.\end{aligned}$$

Thus it is clear that $\|x_m^0(t) - \varphi(0)\|_E \leq \eta$, implying that $x_m^0(t) \in K$, for all $t \in [a, b]$ and every sufficiently large m.

All the assumptions of Theorem 6.1 are then satisfied and hence the initial value problem (7.1)-(1.2) has a solution, denoted x_m, for all m sufficiently large and it follows that $x_m(t) \in \overline{K}$ for all $t \in [a, b]$.

3. Existence and localization of a solution Consider the sequence $\{x_m : m \in \mathbb{N}\} \subset C^0\big([a-h, b]; E\big)$ obtained in previous part. There exists $\{f_m : m \in \mathbb{N}\} \subset L^1\big([a, b]; E\big)$ with $f_m \in \mathcal{S}^1_{F(\cdot, x_{m,(\cdot)})}$ such that $x_m(t) = U(t, a)\varphi(0) + \int_a^t U(t, s)h_m(s)ds$, with $h_m(t) := f_m(t) - p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\tilde{\phi}(x_m(t))$ for a.a. $t \in [a, b]$ and according to (F4), we obtain

$$\chi\left(\{f_m(t) : m \in \mathbb{N}\}\right) \leq k^F(t) \sup_{s\in[-h,0]} \chi\left(\{x_m(t+s) : m \in \mathbb{N}\}\right), \quad \text{for a.a. } t \in [a, b].$$

With a similar computation as in the proof of Proposition 5.2 we have that

$$\chi\left(\{f_m(t) : m \in \mathbb{N}\}\right) \leq k^F(t)e^{L(t-a+h)}\gamma\left(\{x_m : m \in \mathbb{N}\}\right), \quad \text{for a.a. } t \in [a, b], \tag{7.4}$$

where γ is the m.n.c. on $C^0\big([a-h, b]; E\big)$ defined in (5.5) and with L satisfying condition (5.4). Notice that, if $t \notin J$ there exists m_0 such that $t \notin J_m$ for all $m \geq m_0$; since $\tilde{\phi}$ is bounded and $\tau(J) = 0$, it is not difficult to see that $p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\tilde{\phi}(x_m(t)) \longrightarrow 0$ for a.a. $t \in [a, b]$. Consequently, we have that $\chi\left(\{h_m(t) : m \in \mathbb{N}\}\right) = \chi\left(\{f_m(t) : m \in \mathbb{N}\}\right)$ for a.a. $t \in [a, b]$. According to (4.2) and (7.4) and applying a classical result about Hausdorff m.n.c. we obtain that

$$\chi\left(\{U(t, s)h_m(s) : m \in \mathbb{N}\}\right) \leq Dk^F(s)e^{L(s-a+h}\gamma\left(\{x_m : m \in \mathbb{N}\}\right)$$

for all $a \leq s \leq t \leq b$. Moreover, according to (F3) we have that $\|h_m(t)\|_E \leq \mu^F_{C^0\left([-h,0];\overline{K}\right)}(t) + 2p(t)M_{\tilde{\Phi}}$ for a.a. $t \in [a, b]$. It implies that

$$\chi\left(\{x_m(t) : m \in \mathbb{N}\}\right) = D\gamma\left(\{x_m : m \in \mathbb{N}\}\right)\int_a^t k^F(s)e^{L(s-a+h)}ds$$

and then

$$\begin{aligned}\gamma\left(\{x_m : m \in \mathbb{N}\}\right) = & \sup_{t\in[a,b]} e^{-L((t-a+h)}\chi\left(\{x_m(t) : m \in \mathbb{N}\}\right) \\ \leq & \gamma\left(\{x_m : m \in \mathbb{N}\}\right) \sup_{t\in[a,b]} D\int_a^t e^{-L(t-s)}k^F(s)ds.\end{aligned}$$

It implies that $\gamma\left(\{x_m : m \in \mathbb{N}\}\right) = 0$ and hence $\chi\left(\{x_m(t) : m \in \mathbb{N}\}\right) = 0$ for all $t \in [a, b]$. According to (F4) we obtain that

$$\chi\left(\{h_m(t) : m \in \mathbb{N}\}\right) = \chi\left(\{f_m(t) : m \in \mathbb{N}\}\right) \leq k^F(t) \sup_{s\in[-h,0]} \chi\left(\{x_m(t+s) : m \in \mathbb{N}\}\right) = 0$$

for a.a. $t \in [a,b]$; since the sequence $\{h_m : m \in \mathbb{N}\}$ is integrably bounded, it is weakly compact. Therefore, we can find $h \in L^1\big([a,b];E\big)$ such that, passing to a subsequence that we denote as the sequence, we obtain that $h_m \rightharpoonup h$ weakly in $L^1\big([a,b];E\big)$; therefore, according to Theorem 4.1, we have that

$$x_m(t) \to x(t) := U(t,a)\varphi(0) + \int_a^t U(t,s)h(s)ds$$

in $C^0\big([a,b];E\big)$, implying $x(t) \in \overline{K}$ for all $t \in [a,b]$. With an argument based on Mazur's Lemma very similar to the one in the proof of Proposition 5.1 we can show that $h \in \mathcal{S}^1_{F(\cdot,x_{(\cdot)})}$ and the proof is complete.

8 An application to viability theory

As an application of Theorem 1.2 we obtain now a viability result (see Theorem 8.2) for the semilinear evolution equation (1.1). Indeed, in order to eliminate some technicalities, we restrict to the case when the nonlinearity F does not contain delays, i.e. we consider

$$x'(t) + A(t)x(t) \in F(t,x(t)), \quad t \in [a,b] \tag{8.1}$$

with $F\colon [a,b] \times E \multimap E$. In this case $C^0\big([-h,0];E\big)$ needs to be replaced by E in conditions (F1-3) while (F4′) becomes

(F4′) There exists $k^F \in L^1_+\big([a,b]\big)$ such that, for a.a. $t \in [a,b]$, $\chi\big(F(t,\Omega)\big) \le k^F(t)\chi(\Omega(t)$, for any bounded set $\Omega \subset E$.

First we briefly recall the notion of viable set. Given the subset $\mathcal{K} \subseteq [a,b] \times E$ and the multimap $F\colon \mathcal{K} \multimap E$, consider

$$x'(t) \in F(t,x(t)), \quad t \in [a,b], \tag{8.2}$$

an almost exact, i.e. a classical, solution $x : [\tau,T] \to \mathcal{K}$ of (8.2) satisfying $x(\tau) = \xi$, with $(\tau,\xi) \in \mathcal{K}$, is said to be global if

$$T = T_{\mathcal{K}} = \sup\{t \in \mathbb{R} \,:\, \text{there exists } \eta \in E \text{ with } (t,\eta) \in \mathcal{K}\}.$$

Definition 8.1. *The set $\mathcal{K}$ is said to be almost exact globally viable for the multivalued equation* (8.2) *if for each $(\tau,\xi) \in \mathcal{K}$ there exists $T \in \mathbb{R}$, $T > \tau$ and an almost exact solution $x\colon [\tau,T] \to \mathcal{K}$ of* (8.2) *which is global and it satisfies $x(\tau) = \xi$.*

Necessary and even necessary and sufficient conditions in order that $\mathcal{K}$ is a viable set where recently obtained in [14] in an arbitrary Banach space. They involve new notions of tangent set and quasi-tangent set, which are more general than the classical Boulingand tangent vector and deal with globally u.s.c and positively sublinear terms F (see e.g. Theorem 8.2 and Theorem 9.2 in [14]). We need the state space E to be reflexive, separable and with a sufficiently regular norm; the case when $E = L^p$ with $2 \le p < +\infty$ is included. However, our analysis extends to any Bochner integrable $A(t)$ and u-Carathéodory nonlinearity F satisfying (F4′) which is integrably bounded on bounded set.

Theorem 8.2. *Let E be a reflexive and separable Banach space. Assume that the function $x \longmapsto \|x\|_E^p$, $p > 0$ is Gateaux differentiable and has a Lipschitzian derivative. Consider* (8.1) *under conditions (A), (F1-3) and (F4′). If there exists $R > 0$ such that*

$$\lim_{h\to 0} \frac{\|x - hA(t)x + hw\|_E^p - \|x\|_E^p}{h} \le 0 \tag{8.3}$$

for a.a. $t \in (a, b]$, all x with $\|x\|_E = R$ and every $w \in F(t, x)$, then the set $R\mathbb{B}$ is almost exact globally viable.

Proof. Let $V\colon E \to \mathbb{R}$ be given by $V(x) = \|x\|_E^p - R^p$. Hence V satisfies (V) with $K = R\mathbb{B}$. Consider the initial condition

$$x(\tau) = \xi \tag{8.4}$$

with $(\tau, \xi) \in [a, b] \times R\mathbb{B}$ fixed. According to (8.3) and since the estimate (1.3) is indeed valid in any Banach space, it is not difficult to show that V is a generalized guiding function for (8.1)-(8.4) on $\{x \in E : \|x\|_E = R\}$. The conclusion then follows from Theorem 1.2. □

Let E be a separable Hilbert space with scalar product $\langle\cdot,\, \cdot\rangle$ and $V(x) = \frac{1}{2}\left(\langle x,\, x\rangle - R^2\right)$ for some positive R. It is easy to see that V is Fréchet differentiable with $V_x^F(h) = \langle x, h\rangle$ and also that $\|V_x^F\|_{\mathcal{L}(E;\mathbb{R})} = \|x\|_E$. Moreover $\|V_x^F - V_y^F\|_{\mathcal{L}(E;\mathbb{R})} = \|x - y\|_E$ for each $x, y \in E$, implying that V_x^F is Lipschitzian and we obtain the following consequence of previous result

Corollary 8.1. *In a separable Hilbert space consider* (8.1) *under conditions (A), (F1-3) and (F4′). If there exists $R > 0$ such that*

$$\langle x,\, -A(t)x + w\rangle \le 0$$

for a.a. $t \in (a, b]$, all x with $\langle x,\, x\rangle = R^2$ and every $w \in F(t, x)$, then the set $R\mathbb{B}$ is almost exact globally viable.

Example 8.1. *The integro-differential equation*

$$u_t(t, x) + a(t)\int_{\mathbb{R}} g(x - y)u(t, y)dy = \left(1 - p\left(t,\, \int_{\mathbb{R}} \varphi(x)u(t, x)\, dx\right)\right) u(t, x), \quad t \in [a, b],\ x \in \mathbb{R} \tag{8.5}$$

with $\varphi\colon \mathbb{R} \to \mathbb{R}$ and $p\colon [a, b] \times \mathbb{R} \to \mathbb{R}$ is a model for studying biological invasions and disease spread and the integral in the left hand, which is a convolution product, takes into account the long-distance dispersal (see e.g. [24]). We assume that

- *(a) $g(-x) = g(x) \ge 0$ for all $x \in \mathbb{R}$; $a(t)$ and $g(t)$ in $L_+^1(\mathbb{R})$ with $\|g\|_1 = 1$;*
- *(b) $\varphi \in L^2(\mathbb{R})$ with $\|\varphi\|_1 = 1$;*
- *(c) (c1) $p(t, r) \ge 0$ for all $(t, r) \in [a, b] \times \mathbb{R}$;*
  - *(c2) $p(\cdot, r)$ is measurable for all $r \in \mathbb{R}$;*
  - *(c3) there exist $r_1 < r_2 < \ldots < r_n$ such that, for a.a. $t \in [a, b]$, $p(t, \cdot)$ is continuous for $r \ne r_i$ and $p(t, r_i)$ has jump discontinuities for $i = 1, \ldots, n$.*

Since the function p *contains some discontinuities, a solution of* (8.5) *satisfying a given initial condition*

$$u(a, x) = u_0(x), \quad x \in \mathbb{R} \tag{8.6}$$

will be interpreted in the sense of Filippov. More precisely, given $t \in [a, b]$, *consider*

$$P(t, r) = \begin{cases} 1 - p(t, r) & \text{if } r \neq r_i, \\ \\ 1 - \left[\min\{p(t, r_i),\, p(t, r_i^-),\, p(t, r_i^+)\},\ \max\{p(t, r_i),\, p(t, r_i^-),\, p(t, r_i^+)\}\right] & \text{if } r = r_i, \\ & i = 1, 2, ..., n \end{cases}$$

where $p(t, r_i^{\mp}) := \lim_{r \to r_i^{\mp}} p(t, r)$. *A function* $u \in C([a, b]; L^2(\mathbb{R}))$ *is said to be a solution of* (8.5)-(8.6) *if it is a solution of the multivalued equation*

$$u_t(t, x) + a(t) \int_{\mathbb{R}} g(x - y) u(t, y) dy \in P\left(t, \int_{\mathbb{R}} \varphi(x) u(t, x)\, dx\right) u(t, x), \quad t \in [a, b] \tag{8.7}$$

and it satisfies (8.6). *If we further assume the existence of* $R > 0$ *such that*

$$\min_{r \in [-R, R]} \min\{p(t, r),\, p(t, r_i^-),\, p(t, r_i^+)\} \geq 1 + a(t), \quad \text{for a.a. } t \in [a, b] \tag{8.8}$$

and $\|u_0\|_2 < R$, *then the set* $[a, b] \times \{y \in L^2(\mathbb{R}) \, : \, \|y\|_2 < R\}$ *is globally viable for* (8.7) *and hence problem* (8.5)-(8.6) *has a solution satisfying* $\|u(t, \cdot)\|_2 \leq R$ *for a.a.* $t \in [a, b]$.

In fact, problem (8.7)-(8.6) *can then be transformed in abstract setting*

$$\begin{cases} y'(t) + A(t) y(t) \in F(t, y(t)), & t \in [a, b] \\ y(a) = u_0 \end{cases}$$

where $y(t) := u(t, \cdot) \in L^2(\mathbb{R})$, $A(t)\colon\ L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *is such that* $y \longmapsto a(t) \int_{\mathbb{R}} g(x - \xi) y(\xi) d\xi$ *and* $F\colon\ [a, b] \times L^2(\mathbb{R}) \multimap L^2(\mathbb{R})$ *is defined by*

$$F(t, y) = \left\{ py \, : \, p \in P\left(t, \int_{\mathbb{R}} \varphi(x) y(x)\, dx\right)\right\}.$$

According to (a) and (b), both A *and* F *are well defined and it is not difficult to show that* A *satisfies (A) and* F *is nonempty, convex compact valued and satisfies (F1-3). If* $\Omega \subset L^2(\mathbb{R})$ *is bounded, according to (a) we have that* $\int_{\mathbb{R}} \varphi(x) y(x)\, dx \in [-C, C]$ *for some* $C > 0$ *and all* $y \in \Omega$. *Put*

$$k^F(t) = 1 + \max_{r \in [-C, C]} \max\left\{p(t, r),\, p(t, r_i^-),\, p(t, r_i^+)\right\}$$

which exists according to (c). According to the well-known properties of then Hausdorff m.n.c. (see e.g. Section 2) and [5, (2.5)], we have that

$$\begin{aligned} \chi\left(F(t, \Omega)\right) = & \ \chi\left(\{py \, : \, p \in P(t, \textstyle\int_{\mathbb{R}} \varphi(x) y(x) dx),\, y \in \Omega\}\right) \\ \leq & \ k^F(t) \chi\left(\{\tfrac{p}{k^F(t)} y \, : \, p \in P(t, \textstyle\int_{\mathbb{R}} \varphi(x) y(x) dx,\, y \in \Omega\}\right) \\ \leq & \ 2 k^F(t) \chi\left(\{\alpha y \, : \, \alpha \in [0, 1],\, y \in \Omega\}\right) \leq 2 k^F(t) \chi(\Omega) \end{aligned}$$

and also condition (F4′) is satisfied.
Now take $y \in L^2(\mathbb{R})$ *with* $\|y\|_2 = R$*; if* $w \in F(t,y)$ *then* $w = \pi y$ *with* $\pi \in P(t,y)$ *and it holds that*

$$\begin{aligned}\langle y, -A(t)y + \pi y\rangle = &\ \int_{\mathbb{R}} y(x)\left[-a(t)\int_{\mathbb{R}} g(x-\xi)y(\xi)d\xi + \pi y(x)\right]dx \\ = &\ \pi R^2 - a(t)\int_{\mathbb{R}} g(x-\xi)y(\xi)d\xi \le \pi R^2 + a(t)R^2.\end{aligned}$$

It is easy to see that $\pi \le 1 - \min_{r\in[-R,R]} \min\left\{p(t,r),\, p(t,r_i^-),\, p(t,r_i^+)\right\}$*. According to condition* (8.8) *we then derive that* $\langle y, -A(t)y + \pi y\rangle \le 0$ *for a.a.* $t \in [a,b]$*. Corollary 8.1 can then be applied. It implies the global viability of the set* $R\mathbb{B} \subset L^2(\mathbb{R})$ *for problem* (8.7)-(8.6) *and hence the solvability, in the sense of Filippov, of* (8.5)-(8.6) *provided that* $\|u_0\|_2 < R$*.*

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