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Monic Bivariate Polynomials on Quadratic and q-Quadratic Lattices

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Abstract. Monic families of bivariate Askey–Wilson polynomials and q-Racah polynomials are explicitly given. Monic families of bivariate orthogonal polynomials, both in quadratic and q-quadratic lattices, are also explicitly given using appropriate limit relations.

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1. Introduction

Univariate Askey–Wilson polynomials can be explicitly defined as [13, page 415]

$$\begin{aligned} p_n(x;a,b,c,d|q) &= \frac{(ab,ac,ad;\,q)_n}{a^n}\,{}_4\phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}\,\middle|\, q;q\right), \\ & x=\cos\theta, \end{aligned} \tag{1.1}$$

where the q-Pochhammer symbol is defined as

$$(a;\,q)_n = \prod_{j=0}^{n-1} 1-aq^j, \quad (a;\,q)_0 = 1,$$

and the basic hypergeometric series is defined as

$${}_r\phi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots b_s \end{matrix}\,\middle|\, q;z\right) = \sum_{k=0}^{\infty} \frac{(a_1,\dots,a_r;\,q)_k}{(b_1,\dots,b_s;\,q)_k}(-1)^{(1+s-r)k} q^{(1+s-r)\binom{k}{2}} \frac{z^k}{(q;\,q)_k},$$

with

$$(a_1,\dots,a_r;\,q)_k = (a_1;\,q)_k \cdots (a_r;\,q)_k.$$

The polynomial $p_n(x; a, b, c, d|q)$ is a polynomial of degree n in the q-quadratic lattice [7,22]

$$x(s) = \cos\theta = \frac{q^s + q^{-s}}{2}, \quad q^s = e^{i\theta}. \tag{1.2}$$

Let us introduce the divided-difference operators $\mathbb{D}_x$ and $\mathbb{S}_x$ [20,21,31]

$$\mathbb{D}_x f(s) = \frac{f(s+1/2) - f(s-1/2)}{x(s+1/2) - x(s-1/2)}, \quad \mathbb{S}_x f(s) = \frac{f(s+1/2) + f(s-1/2)}{2}. \tag{1.3}$$

The above operators transform polynomials of degree n in the lattice $x(s)$ into polynomials of, respectively, degree $n-1$ and n in the same variable $x(s)$. Then, univariate Askey–Wilson polynomials satisfy the second-order linear divided-difference equation [9]

$$\phi_1(x(s))\mathbb{D}_x^2 p_n(s) + \tau_1(x(s))\mathbb{S}_x\mathbb{D}_x p_n(s) + \lambda_n^1 p_n(s) = 0, \tag{1.4}$$

where ϕ_1 is a polynomial of degree two in the lattice $x(s)$ given by

$$\begin{aligned}\phi_1(x(s)) = {} & 2(abcd+1)x(s)^2 - (a+b+c+d+abc+abd+acd+bcd)x(s) \\ & +ab+ac+ad+bc+bd+cd-abcd-1,\end{aligned}$$

τ_1 is a polynomial of degree one in the lattice $x(s)$

$$\tau_1(x(s)) = \frac{4(abcd-1)q^{1/2}}{q-1}x(s) + \frac{2(a+b+c+d-abc-abd-acd-bcd)q^{1/2}}{q-1},$$

and

$$\lambda_n^1 = -4\,\frac{(q^n-1)\,q^{\frac{1}{2}}\,(abcdq^n - q)}{(q-1)^2\,q^n}.$$

By introducing

$$P_n(z) = \frac{(ab, ac, ad;\, q)_n}{a^n}\,{}_4\phi_3\left(\left.\begin{matrix} q^{-n}, abcdq^{n-1}, az, az^{-1} \\ ab, ac, ad \end{matrix}\,\right|\, q; q\right),$$

it is possible to rewrite the q-difference equation in the form [13, page 418]

$$\begin{aligned}q^{-n}(1-q^n)(1-abcdq^{n-1})P_n(z) = {} & A(z)P_n(qz) - [A(z) + A(z^{-1})]P_n(z) \\ & +A(z^{-1})P_n(q^{-1}z),\end{aligned}$$

where

$$A(z) = \frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}$$

is a rational function.

Bivariate Askey–Wilson polynomials have been introduced by Gasper and Rahman [14] as

$$P_{n,m}(s,t;a,b,c,d,e_2|q) = p_n(x(s);a,b,e_2q^t,e_2q^{-t}|q)\,p_m(y(t);ae_2q^n,be_2q^n,c,d|q), \tag{1.5}$$

$$\tilde{P}_{n,m}(s,t;a,b,c,d,e_2|q) = p_n(x(s);ce_2q^m,de_2q^m,a,b|q)\,p_m(y(t);c,d,e_2q^s,e_2q^{-s}|q). \tag{1.6}$$

Both families (1.5) and (1.6) are polynomials of total degree $n+m$ in the variables $x(s)$ and $y(t)=x(t)$. Iliev [17] obtained a partial divided-difference equation for the Askey–Wilson polynomials defined by Gasper–Rahman (in arbitrary dimension) containing rational coefficients. The equation can be rewritten in terms of a divided-difference equation of hypergeometric type with polynomial coefficients, but it is required to consider a fourth-order equation [28]. Moreover, in [28], a monic family of bivariate Askey–Wilson polynomials is introduced, from the three-term recurrence relations they satisfy, but the explicit expression has not been provided yet in the literature, to the best of our knowledge. Monic bivariate orthogonal polynomials as eigenfunctions of partial differential (difference, q-difference or divided-difference) operators have been analyzed in detail in [1–3, 5, 26]. The q-analogues, i.e., bivariate and multivariate orthogonal polynomials on q-quadratic lattices have similar properties in terms of eigenfunctions of the fourth-order divided-difference equations, and representation as products of univariate orthogonal polynomials on q-quadratic lattices.

Let us introduce some other bivariate extensions of classical univariate orthogonal polynomials. Let us first consider the q-Racah polynomials defined for $n=0,1,\ldots,N$ by [6, 13, 15]

$$r_n(x;a,b,c,N;q)=(aq,bcq,q^{-N};q)_n(q^N/c)^{n/2}\,{}_4\phi_3\left(\begin{matrix}q^{-n},abq^{n+1},q^{-x},cq^{x-N}\\ aq,bcq,q^{-N}\end{matrix}\,\middle|\,q;q\right),\tag{1.7}$$

where the factor $(q^N/c)^{n/2}$ were chosen, so that certain symmetry properties of the q-Racah polynomials are satisfied. Note that $r_n(x;a,b,c,N;q)$ is a polynomial of degree n in the q-quadratic lattice

$$\lambda(x)=q^{-x}+cq^{x-N}.\tag{1.8}$$

Gasper and Rahman [15] defined the multivariate q-Racah polynomials from which we deduce the bivariate q-Racah polynomials given by

$$\begin{aligned}R_{n,m}(x,y;a_1,a_2,a_3,b,N;q)&=r_n\Big(x;b,a_2/q,a_1q^y,y;q\Big)r_m\Big(y\\&\quad -n;ba_2q^{2n},a_3/q,a_1a_2q^{N+n},N-n;q\Big).\end{aligned}\tag{1.9}$$

We would like to notice that $R_{n,m}(x,y;a_1,a_2,a_3,b,N;q)$ is a polynomial of total degree $n+m$ in the variables $\mu(x)$ and $\nu(y)$, where

$$\mu(x)=q^{-x}+a_1q^x,\quad \nu(y)=q^{-y}+a_1a_2q^y.\tag{1.10}$$

As for the quadratic cases, let [13, page 190]

$$\begin{aligned}r_n(\alpha,\beta,\gamma,\delta;s)=r_n(s)&=(\alpha+1)_n\,(\beta+\delta+1)_n\,(\gamma+1)_n\\&\quad\times{}_4F_3\left(\begin{matrix}-n,n+\alpha+\beta+1,-s,s+\gamma+\delta+1\\ \alpha+1,\beta+\delta+1,\gamma+1\end{matrix}\,\middle|\,1\right),\\&\quad n=0,1,\ldots,N,\end{aligned}\tag{1.11}$$

be the univariate Racah polynomials, where $(A)_n=A(A+1)\cdots(A+n-1)$ with $(A)_0=1$ denotes the Pochhammer symbol. The polynomial $r_n(\alpha,\beta,$

$\gamma, \delta; s)$ is a polynomial of degree $2n$ in s and of degree n in the quadratic lattice [7,22]

$$\eta(s) = s(s+\gamma+\delta+1). \tag{1.12}$$

Univariate Racah polynomials satisfy the second-order linear divided-difference equation [9]

$$\phi(\eta(s))\mathbb{D}_\eta^2 r_n(s) + \tau(\eta(s))\mathbb{S}_\eta\mathbb{D}_\eta r_n(s) + \lambda_n r_n(s) = 0, \tag{1.13}$$

where ϕ is a polynomial of degree two in the lattice $\eta(s)$

$$\begin{aligned}\phi(\eta(s)) = -(\eta(s))^2 &+ \frac{1}{2}(-\alpha(2\beta+\delta+\gamma+3)+\beta(\delta-\gamma-3)\\ &-2(\delta\gamma+\delta+\gamma+2))\eta(s)\\ &-\frac{1}{2}(\alpha+1)(\gamma+1)(\beta+\delta+1)(\delta+\gamma+1),\end{aligned}$$

τ is a polynomial of degree one in the lattice $\eta(s)$

$$\tau(\eta(s)) = -(\alpha+\beta+2)\eta(s) - (\alpha+1)(\gamma+1)(\beta+\delta+1),$$

and

$$\lambda_n = n(\alpha+\beta+n+1).$$

Equation (1.13) can be also written in many other forms, e.g., [13, Eq. (9.2.5)]

$$n(n+\alpha+\beta+1)r_n(s) = B(s)r_n(s+1) - (B(s)+D(s))r_n(s) + D(s)r_n(s-1),$$

where $B(s)$ and $D(s)$ are the rational functions given by

$$\begin{aligned}B(s) &= \frac{(\alpha+s+1)(\gamma+s+1)(\beta+\delta+s+1)(\delta+\gamma+s+1)}{(\delta+\gamma+2s+1)(\delta+\gamma+2s+2)},\\ D(s) &= \frac{s(\delta+s)(-\beta+\gamma+s)(-\alpha+\delta+\gamma+s)}{(\delta+\gamma+2s)(\delta+\gamma+2s+1)}.\end{aligned}$$

The extension to the multivariable situation was given by Tratnik in [30] and later analyzed by Geronimo and Iliev in [11]. In the two-dimensional situation, the bivariate Racah polynomials given in [11] are defined in terms of univariate Racah polynomials (1.11) as

$$\begin{aligned}R_{n,m}(s,t;\beta_0,\beta_1,\beta_2,\beta_3,N) = r_n&(\beta_1-\beta_0-1,\beta_2-\beta_1-1,-t-1,\beta_1+t;s)\\ \times r_m&(2n+\beta_2-\beta_0-1,\beta_3\\ -\beta_2&-1,n-N-1,n+\beta_2+N;t-n),\end{aligned} \tag{1.14}$$

which are polynomials in the lattices $x(s) = s(s+\beta_1)$ and $y(t) = t(t+\beta_2)$. If we consider the substitutions

$$\begin{aligned}&\beta_0 = a_1-\eta-1, \quad \beta_1 = a_1, \quad \beta_2 = a_1+a_2, \quad \beta_3 = a_1+a_2+a_3,\\ &\text{and} \quad N = -\gamma-1,\end{aligned} \tag{1.15}$$

then the above polynomials exactly coincide with the bivariate Racah polynomials of parameters a_1, a_2, a_3, γ, and η introduced by Tratnik [30, Eq. (2.1)]. In [11], an equation for bivariate Racah polynomials is given, which involves 9 rational coefficients. This equation can be rewritten as divided-difference equation of hypergeometric type, but again the equation must be of fourth order [27]. Moreover, as in the q-quadratic case, in [27] a monic family of bivariate Racah polynomials is introduced, from the three-term recurrence relations they satisfy, but again the explicit expression has not been provided.

The main aim of this paper is to give explicitly a representation of bivariate monic Askey–Wilson polynomials—see Sect. 2. Monic bivariate q-Racah polynomials are also explicitly given in Sect. 3. Using appropriate limit relations, families of monic bivariate orthogonal polynomials on q-quadratic lattices are introduced in Sect. 4. Monic bivariate Racah polynomials are explicitly represented in Sect. 5. Finally, some other families of monic bivariate orthogonal polynomials on quadratic lattices are deduced in Sect. 6.

2. Monic Bivariate Askey–Wilson Polynomials

In a very interesting contribution, Iliev [17] obtained a partial divided-difference equation for the Askey–Wilson polynomials defined by Gasper–Rahman (in arbitrary dimension). This equation, containing rational coefficients, can be rewritten as a fourth-order divided-difference equation of hypergeometric type [28].

Theorem 2.1. *Let*

$$x(s)=\frac{q^s+q^{-s}}{2}=\cos\theta_1,\quad y(t)=\frac{q^t+q^{-t}}{2}=\cos\theta_2.$$

The bivariate Askey–Wilson polynomials defined in (1.5) are solution of the following fourth-order linear partial divided-difference equation:

$$\begin{aligned}
&f_1(x(s),y(t))\mathbb{D}_x^2\mathbb{D}_y^2P_{n,m}(s,t)+f_2(x(s),y(t))\mathbb{S}_x\mathbb{D}_x\mathbb{D}_y^2P_{n,m}(s,t)\\
&\quad+f_3(x(s),y(t))\mathbb{S}_y\mathbb{D}_y\mathbb{D}_x^2P_{n,m}(s,t)\\
&\quad+f_4(x(s),y(t))\mathbb{S}_x\mathbb{D}_x\mathbb{S}_y\mathbb{D}_yP_{n,m}(s,t)+f_5(x(s))\mathbb{D}_x^2P_{n,m}(s,t)\\
&\quad+f_6(y(t))\mathbb{D}_y^2P_{n,m}(s,t)\\
&\quad+f_7(x(s))\mathbb{S}_x\mathbb{D}_xP_{n,m}(s,t)+f_8(y(t))\mathbb{S}_y\mathbb{D}_yP_{n,m}(s,t)+\lambda_{n,m}P_{n,m}(s,t)=0,
\end{aligned}\tag{2.1}$$

where

$$\lambda_{n,m}=16q^{-(m+n)+3}(1-q^{m+n})(1-abcde_2^2q^{m+n-1}),$$

$P_{n,m}(s,t)$ *stands for* $P_{n,m}(s,t;a,b,c,d,e_2|q)$*, with*

$$\begin{aligned}
f_8(y(t))=&\,8q^2(q-1)\big(2(1-abcde_2^2)y(t)+(d+c)(abe_2^2-1)\\
&+(cd-1)(a+b)e_2\big),\\
f_7(x(s))=&\,8q^2(q-1)\big(2(1-abcde_2^2)x(s)\\
&+(cde_2^2-1)(a+b)+(d+c)(ab-1)e_2\big),\\
f_6(y(t))=&\,4q^{3/2}(q-1)^2\Big(-2(abcde_2^2+1)(y(t))^2
\end{aligned}$$

$$
\begin{aligned}
&+\Big(\big(d+c\big)\big(abe_2^2+1\big)+\big(cd+1\big)\big(a+b\big)e_2\Big)y(t)+\big(cd-1\big)\\
&\times\big(abe_2^2-1\big)-\big(d+c\big)\big(a+b\big)e_2\Big),\\
f_5(x(s)) =\,& 4q^{3/2}\big(q-1\big)^2\Big(-2\big(abcde_2^2+1\big)\big(x(s)\big)^2+\Big(\big(cde_2^2+1\big)\big(a+b\big)\\
&+\big(d+c\big)\big(ab+1\big)e_2\Big)x(s)+\big(cde_2^2-1\big)\big(ab-1\big)\\
&-\big(d+c\big)\big(a+b\big)e_2\Big),\\
f_4(x(s),y(t)) =\,& 4q\big(q-1\big)^2\Big(-4\big(abcdqe_2^2+1\big)x(s)y(t)+2\big(d+c\big)\big(abqe_2^2+1\big)x(s)\\
&+2\big(cdqe_2^2+1\big)\big(a+b\big)y(t)-\big(d+c\big)\big(a+b\big)\big(qe_2^2+1\big)\\
&+\big(q+1\big)\big(cd-1\big)\big(ab-1\big)e_2\Big),\\
f_3(x(s),y(t)) =\,& 2\sqrt{q}\big(q-1\big)^3\Big(4\big(1-abcdqe_2^2\big)\big(x(s)\big)^2y(t)+2\big(d+c\big)\big(abqe_2^2-1\big)\\
&\times\big(x(s)\big)^2+2\big(cdqe_2^2-1\big)\big(a+b\big)x(s)y(t)\\
&+\big(-\big(d+c\big)\big(a+b\big)\big(qe_2^2-1\big)\\
&+\big(q+1\big)\big(cd-1\big)\big(ab+1\big)e_2\big)x(s)\\
&+2\big(cdqe_2^2+1\big)\big(ab-1\big)y(t)\\
&-\big(d+c\big)\big(ab-1\big)\big(qe_2^2+1\big)-\big(q+1\big)\big(cd-1\big)\big(a+b\big)e_2\Big),\\
f_2(x(s),y(t)) =\,& 2\sqrt{q}\big(q-1\big)^3\Big(-4\big(abcdqe_2^2-1\big)\big(y(t)\big)^2x(s)+2\big(cdqe_2^2-1\big)\\
&\times\big(a+b\big)\big(y(t)\big)^2+2\big(d+c\big)\big(abqe_2^2-1\big)x(s)y(t)\\
&+2\big(cd-1\big)\big(abqe_2^2+1\big)x(s)\\
&+\big(-\big(d+c\big)\big(a+b\big)\big(qe_2^2-1\big)\\
&+\big(q+1\big)\big(cd+1\big)\big(ab-1\big)e_2\big)y(t)\\
&-\big(cd-1\big)\big(a+b\big)\big(qe_2^2+1\big)\\
&-\big(q+1\big)\big(d+c\big)\big(ab-1\big)e_2\Big),\\
f_1(x(s),y(t)) =\,& \big(q-1\big)^4\Big(-4\big(abcdqe_2^2+1\big)\big(x(s)\big)^2\big(y(t)\big)^2+2\big(d+c\big)\\
&\times\big(abqe_2^2+1\big)\big(x(s)\big)^2y(t)+2\big(cdqe_2^2+1\big)\big(a+b\big)x(s)\big(y(t)\big)^2\\
&+2\big(cd-1\big)\big(abqe_2^2-1\big)\big(x(s)\big)^2+2\big(cdqe_2^2-1\big)\big(ab-1\big)\big(y(t)\big)^2\\
&+\Big(-\big(d+c\big)\big(a+b\big)\big(qe_2^2+1\big)\\
&+\big(q+1\big)\big(cd+1\big)\big(ab+1\big)e_2\Big)x(s)y(t)\\
&-\Big(\big(cd-1\big)\big(a+b\big)\big(qe_2^2-1\big)+\big(q+1\big)\big(d+c\big)\\
&\times\big(ab+1\big)e_2\Big)x(s)-\big(\big(d+c\big)\big(ab-1\big)\big(qe_2^2-1\big)\\
&+\big(q+1\big)\big(cd+1\big)\big(a+b\big)e_2\big)y(t)\\
&-\big(cd-1\big)\big(ab-1\big)\big(qe_2^2+1\big)\\
&+\big(q+1\big)\big(d+c\big)\big(a+b\big)e_2\Big).
\end{aligned}
$$

Our first contribution is to present an explicit monic solution of the above fourth-order linear partial divided-difference equation. In doing so, it is extremely important to choose appropriate bases for both variables s and t. While the first variable involves just a classical quadratic basis $(aq^s, aq^{-s};\, q)_k$, the basis for the second variable in the summation contains also the summation index of the first variable, i.e., $(ae_2q^{k+t}, ae_2q^{k-t};\, q)_p$.

Theorem 2.2. *The fourth-order divided-difference equation* (2.1) *has the following monic solution:*

$$\hat{P}_{n,m}(s,t) = \sum_{k=0}^{n}\sum_{p=0}^{m}(-1)^{k+p-n-m} q^{p(n-k)+\frac{1}{2}(n+m-k-p-1)(n+m-k-p)} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} m \\ p \end{bmatrix}_q$$
$$\times \frac{(abq^k;\, q)_{n-k}(ace_2q^{k+p}, ade_2q^{k+p};\, q)_{m+n-k-p}(abe_2^2q^{n+k+p};\, q)_{m-p}}{(abcde_2^2q^{n+m+k+p-1};\, q)_{n+m-k-p}}$$
$$\times (aq^s, aq^{-s};\, q)_k (ae_2q^{k+t}, ae_2q^{k-t};\, q)_p. \tag{2.2}$$

Proof. Let

$$\mathcal{B}_k^1(s) = (aq^s, aq^{-s};\, q)_k, \quad \mathcal{B}_p^2(t) = (ae_2q^{k+t}, ae_2q^{k-t};\, q)_p.$$

Using the definitions of the divided-difference operators as well as the latter definitions, the following relations hold true:

$$\begin{aligned}
x(s)\mathcal{B}_k^1(s) &= \frac{q^{-k}\left(a^2q^{2k}+1\right)\mathcal{B}_k^1(s)}{2a} - \frac{q^{-k}\mathcal{B}_{k+1}^1(s)}{2a},\\
y(t)\mathcal{B}_p^2(t) &= \frac{q^{-k-p}\left(a^2e_2^2q^{2k+2p}+1\right)\mathcal{B}_p^2(t)}{2ae_2} - \frac{q^{-k-p}\mathcal{B}_{p+1}^2(t)}{2ae_2},\\
\mathbb{S}_x\mathbb{D}_x\mathcal{B}_k^1(s) &= \frac{2a\left(1-q^k\right)\left(\frac{1}{2}\left(1-q^{1-k}\right)\left(1-a^2q^{2(k-1)}\right)\mathcal{B}_{k-1}^1(s) + \left(\frac{q^{1-k}}{2}+\frac{1}{2}\right)\mathcal{B}_k^1(s)\right)}{(q-1)\mathcal{B}_1^1(s)},\\
\mathbb{S}_y\mathbb{D}_y\mathcal{B}_p^2(t) &= \frac{2ae_2q^k\left(1-q^p\right)\left(\frac{1}{2}\left(1-q^{1-p}\right)\left(1-a^2e_2^2q^{2k+2(p-1)}\right)\mathcal{B}_{p-1}^2(t) + \left(\frac{q^{1-p}}{2}+\frac{1}{2}\right)\mathcal{B}_p^2(t)\right)}{(q-1)\mathcal{B}_1^2(t)},\\
\mathbb{D}_x^2\mathcal{B}_k^1(s) &= \frac{4a^2\sqrt{q}\left(1-q^{k-1}\right)\left(1-q^k\right)\mathcal{B}_{k-1}^1(s)}{(q-1)^2\mathcal{B}_1^1(s)}, \quad \mathbb{D}_y^2\mathcal{B}_p^2(t)\\
&= \frac{4a^2e_2^2q^{2k+\frac{1}{2}}\left(1-q^{p-1}\right)\left(1-q^p\right)\mathcal{B}_{p-1}^2(t))}{(q-1)^2\mathcal{B}_1^2(t)}.
\end{aligned}$$

If we apply the fourth-order divided-difference Eqs. (2.1) to (2.2), the result follows using the latter properties. □

Remark 1. Let

$$\mathbb{P}_n = \left(P_{n-k,k}(s,t;a,b,c,d,e_2|q)\right)_{k=0}^{n}$$

be the column vector of bivariate Askey–Wilson polynomials, where $P_{n,m}$ $(s,t;a,b,c,d,e_2|q)$ are defined in (1.5). Let us also introduce the column vector of monic bivariate Askey–Wilson polynomials

$$\hat{\mathbb{P}}_n = \left(\hat{P}_{n-k,k}(s,t)\right)_{k=0}^{n},$$

where $\hat{P}_{n,m}(s,t)$ are defined in (2.2). Then, by computing the leading coefficients of the bivariate Askey–Wilson polynomials $P_{n,m}(s,t;a,b,c,d,e_2|q)$ defined in (1.5), it yields

$$\hat{\mathbb{P}}_n = \mathbb{W}_n\mathbb{P}_n, \quad n = 0, 1, 2, \ldots, \tag{2.3}$$

where the upper triangular matrix $\mathbb{W}_n$ of size $(n+1)\times(n+1)$ is defined as

$$w_{k,l}(n) = (-1)^n \begin{bmatrix} n-k \\ n-l \end{bmatrix}_{\mathrm{q}} q^{\binom{n}{2}} \frac{a^n e_2^l (1-abe_2^2 q^{2n-2l-1})(abq^{n-l};\, q)_{l-k}}{(abe_2^2 q^{n-l-1};\, q)_{n+1-k}(abcde_2^2 q^{2n-l-1};\, q)_l}.$$

3. Monic Bivariate *q*-Racah Polynomials

Theorem 3.1. *The bivariate q-Racah polynomials are solution of a fourth-order linear partial divided-difference equation of the form*

$$\begin{aligned} &f_1(\mu(x),\nu(y))\mathbb{D}_\mu^2\mathbb{D}_\nu^2 R_{n,m}(x,y;q) + f_2(\mu(x),\nu(y))\mathbb{S}_\mu\mathbb{D}_\mu\mathbb{D}_\nu^2 R_{n,m}(x,y;q)\\ &\quad + f_3(\mu(x),\nu(y))\mathbb{S}_\nu\mathbb{D}_\nu\mathbb{D}_\mu^2 R_{n,m}(x,y;q)\\ &\quad + f_4(\mu(x),\nu(y))\mathbb{S}_\mu\mathbb{D}_\mu\mathbb{S}_\nu\mathbb{D}_\nu R_{n,m}(x,y;q)\\ &\quad + f_5(\mu(x))\mathbb{D}_\mu^2 R_{n,m}(x,y;q) + f_6(\nu(y))\mathbb{D}_\nu^2 R_{n,m}(x,y;q)\\ &\quad + f_7(\mu(x))\mathbb{S}_\mu\mathbb{D}_\mu R_{n,m}(x,y;q)\\ &\quad + f_8(\nu(y))\mathbb{S}_\nu\mathbb{D}_\nu R_{n,m}(x,y;q) + 4\,q^{-m-n+3}\left(q^{m+n}-1\right)\\ &\quad \left(ba_2a_3q^{m+n}-1\right) R_{n,m}(x,y;q) = 0, \end{aligned} \tag{3.1}$$

where $R_{n,m}(x,y;a_1,a_2,a_3,b,N;q) := R_{n,m}(x,y;q)$ *and*

$$\begin{aligned} f_8(\nu(y)) =& -4\,q^2\,(1-q)\,(1-ba_2a_3q)\,\nu(y)\\ &+ 4\,q^2\,(-1+q)\Big((ba_2q-1)(a_1a_2a_3q^N + q^{-N})\\ &+ a_2\,(a_3-1)\,(bq+a_1)\Big),\\ f_7(\mu(x)) =& -4\,q^2\,(1-q)\,(1-ba_2a_3q)\,\mu(x)\\ &+ 4\,q^2\,(-1+q)\Big(\,(bq-1)\,(a_3a_1a_2q^N + q^{-N})\\ &+ (a_3a_2-1)\,(bq+a_1)\Big),\\ f_6(\nu(y)) =& (1-q)^2\Big[-2\,(ba_2a_3q+1)\,q^{3/2}\nu(y)^2\\ &+ 2q^{\frac{3}{2}}\Big(\,(ba_2q+1)\,(a_1a_2a_3q^N + q^{-N}) + a_2\\ &\times(a_3+1)\,(bq+a_1)\Big)\nu(y) - 4\,a_2q^{3/2}\Big((bq+a_1)\,(a_1a_3a_2q^N + q^{-N})\\ &-a_1\,(1-a_3)\,(1-ba_2q)\Big)\Big],\\ f_5(\mu(x)) =& (1-q)^2\Big[-2\,(ba_2a_3q+1)\,q^{3/2}(\mu(x))^2\\ &+ 2\,q^{3/2}\Big(\,(1+bq)\,(a_3a_1a_2q^N + q^{-N})\\ &+ (1+a_3a_2)\,(bq+a_1)\Big)\mu(x) - 4\,q^{3/2}\Big((bq+a_1)\,(a_1a_3a_2q^N + q^{-N})\\ &-a_1\,(bq-1)\,(a_3a_2-1)\Big)\Big],\\ f_4(\mu(x),\nu(y)) =& (1-q)^2\Big[-4\,\left(ba_2a_3q^2+1\right)q\,\mu(x)\nu(y)\\ &+ 4\,q\,\left(bq^2a_2+1\right)\left(a_3a_1a_2q^N + q^{-N}\right)\,\mu(x) \end{aligned}$$

$$
\begin{aligned}
&+ 4\,(a_2a_3q+1)\,(bq+a_1)\,q\,\nu(y)\\
&- 4\,q\Big(\,(qa_2+1)\,(bq+a_1)\,(a_1a_2a_3q^N+q^{-N})\\
&- a_1a_2\,(q+1)\,(bq-1)\,(-1+a_3)\,\Big)\Big],\\
f_3(\mu(x),\nu(y)) = &(-1+q)^3\Big[-2\,\left(ba_2a_3q^2-1\right)\sqrt{q}\,\mu(x)^2\nu(y)\\
&+ 2\sqrt{q}\left(bq^2a_2-1\right)\mu(x)^2\\
&\times\left(a_3a_1a_2q^N+q^{-N}\right)+2\,(a_2a_3q-1)\,(bq+a_1)\,\sqrt{q}\mu(x)\nu(y)\\
&+ 4\,(bq-1)\,(a_2a_3q+1)\,\sqrt{q}a_1\nu(y)\\
&- 2\sqrt{q}\,\Big((qa_2-1)\,(bq+a_1)\,(a_1a_2a_3q^N+q^{-N})\\
&-a_1a_2\,(q+1)\,(1+bq)\\
&(-1+a_3))\,\mu(x)-4\sqrt{q}a_1\,\Big((bq-1)\,(qa_2+1)\,(a_1a_2a_3q^N+q^{-N})\\
&+a_2\,(q+1)\,(-1+a_3)\,(bq+a_1))\Big],\\
f_2(\mu(x),\nu(y)) = &(-1+q)^3\Big[-2\,\left(ba_2a_3q^2-1\right)\sqrt{q}\,\mu(x)\nu(y)^2\\
&+ 2\,(a_2a_3q-1)\,(bq+a_1)\,\sqrt{q}\,\nu(y)^2\\
&+ 2\,\sqrt{q}\left(bq^2a_2-1\right)\left(a_3a_1a_2q^N+q^{-N}\right)\,\mu(x)\nu(y)\\
&+ 4\,(-1+a_3)\left(bq^2a_2+1\right)a_2a_1\sqrt{q}\,\mu(x)\\
&- 2\,\sqrt{q}\,\Big((-1+qa_2)\,(bq+a_1)\,(a_1a_2a_3q^N+q^{-N})\\
&-a_1a_2\,(q+1)\,(bq-1)\,(a_3+1))\,\nu(y)\\
&- 4\,a_2a_1\sqrt{q}\,\Big((q+1)\,(bq-1)\,(a_3a_1a_2q^N+q^{-N})\\
&+ (-1+a_3)\,(qa_2+1)\,(bq+a_1))\Big],\\
f_1(\mu(x),\nu(y)) = &(1-q)^4\Big[-\left(ba_2a_3q^2+1\right)\mu(x)^2\nu(y)^2\\
&+ \left(bq^2a_2+1\right)\left(a_3a_1a_2q^N+q^{-N}\right)\\
&\times\mu(x)^2\nu(y)+(qa_3a_2+1)\,(bq+a_1)\,\mu(x)(\nu(y))^2\\
&+ 2a_1a_2\,(a_3-1)\left(bq^2a_2-1\right)\mu(x)^2\\
&+ 2a_1\,(bq-1)\,(qa_3a_2-1)\,\nu(y)^2\\
&- \Big((qa_2+1)\,(bq+a_1)\,(a_1a_2a_3q^N+q^{-N})\\
&-a_1a_2\,(q+1)\,(bq+1)\,(a_3+1))\,\mu(x)\nu(y)\\
&- 2a_1a_2\,\Big((q+1)\,(bq+1)\,(a_3a_1a_2q^N+q^{-N})\\
&+ (-1+a_3)\,(-1+qa_2)\,(bq+a_1))\,\mu(x)\\
&- 2\,a_1\,\Big((bq-1)\,(-1+qa_2)\,(a_1a_2a_3q^N+q^{-N})\\
&+a_2\,(q+1)\,(a_3+1)\,(bq+a_1))\,\nu(y)\\
&+ 4a_1a_2\,\Big((q+1)\,(bq+a_1)\,(a_1a_3a_2q^N+q^{-N})
\end{aligned}
$$

$$-a_1\,(bq-1)\,(-1+a_3)\,(qa_2+1))\Big].$$

Similarly as in the Askey–Wilson case, it is possible to obtain a explicit expression for the monic bivariate q-Racah polynomials

Theorem 3.2. *The fourth-order divided-difference Eq. (3.1) has the following monic solution*

$$\begin{aligned}\hat{R}_{n,m}(x,y) &= \sum_{k=0}^{n}\sum_{p=0}^{m}(-1)^{n+m-k-p}q^{p+\frac{1}{2}(k+1-n)(k-2m-n)+\frac{1}{2}(m-p)(m-p+1)}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}m\\p\end{bmatrix}_q\\ &\times\frac{(bq^{k+1};\,q)_{n-k}(a_2bq^{n+p+k+1};\,q)_{m-p}(a_1a_2a_3q^{N+p+k},q^{-N+p+k};\,q)_{n+m-k-p}}{(a_2a_3bq^{n+m+k+p};\,q)_{n+m-k-p}}\\ &\times(q^{-x},a_1q^{x};\,q)_k(q^{k-y},a_1a_2q^{k+y};\,q)_p.\end{aligned}\tag{3.2}$$

Remark 2. Let

$$\mathbb{R}_n=\left(R_{n-k,k}(s,t;a_1,a_2,a_3,b,N,q)\right)_{k=0}^{n}$$

be the column vector of bivariate q-Racah polynomials, where $R_{n,m}(s,t;a_1,a_2,a_3,b,N,q)$ are defined in (1.9). Let us also introduce the column vector of monic bivariate q-Racah polynomials

$$\hat{\mathbb{R}}_n=\left(\hat{R}_{n-k,k}(s,t)\right)_{k=0}^{n},$$

where $\hat{R}_{n,m}(s,t)$ are defined in (3.2). Then, by computing the leading coefficients of the bivariate q-Racah polynomials $R_{n,m}(s,t;a_1,a_2,a_3,b,N,q)$ defined in (1.9), it yields

$$\hat{\mathbb{R}}_n=\mathbb{T}_n\mathbb{R}_n,\quad n=0,1,2,\ldots,\tag{3.3}$$

where the upper triangular matrix $\mathbb{T}_n$ of size $(n+1)\times(n+1)$ is defined as

$$t_{k,l}(n)=(-1)^n\begin{bmatrix}n-k\\n-l\end{bmatrix}_q q^{\binom{n}{2}}\frac{a_1^{\frac{n}{2}}a_2^{\frac{l}{2}}(1-a_2bq^{2n-2l})(bq^{n-l+1};\,q)_{l-k}}{(a_2bq^{n-l};\,q)_{n+1-k}(a_2a_3bq^{2n-l};\,q)_l}.$$

4. Limit Transitions on q-Quadratic Lattices

4.1. Bivariate Monic Dual q-Hahn Polynomials

The family of bivariate monic dual q-Hahn polynomials follows by taking the limit as $b\to 0$ from (3.2):

$$\begin{aligned}\hat{D}_{n,m}(x,y;q) &= \hat{D}_{n,m}(x,y;a_1,a_2,a_3,N;q)\\ &= \sum_{k=0}^{n}\sum_{p=0}^{m}(-1)^{n+m-k-p}q^{p+\frac{1}{2}(k+1-n)(k-2m-n)+\frac{1}{2}(m-p)(m-p+1)}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}m\\p\end{bmatrix}_q\\ &\times(a_1a_2a_3q^{N+p+k},q^{-N+p+k};\,q)_{n+m-k-p}(q^{-x},a_1q^{x};\,q)_k(q^{k-y},a_1a_2q^{k+y};\,q)_p.\end{aligned}$$

4.2. Bivariate Monic q-Hahn Polynomials

If we take the limit $a_1\to 0$ and replace $b\leftarrow a_1$, $a_2\leftarrow a_2q$, $a_3\leftarrow a_3q$ in (3.2), this yields the bivariate monic q-Hahn polynomials

$$\begin{aligned}\hat{H}_{n,m}(x,y;q) &= \hat{H}_{n,m}(x,y;a_1,a_2,a_3,N;q)\\ &= \sum_{k=0}^{n}\sum_{p=0}^{m}(-1)^{n+m-k-p}q^{p+\frac{1}{2}(k+1-n)(k-2m-n)+\frac{1}{2}(m-p)(m-p+1)}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}m\\p\end{bmatrix}_q\end{aligned}$$

$$\times \frac{(a_1 q^{k+1};\, q)_{n-k} (a_1 a_2 q^{n+p+k+2};\, q)_{m-p} (q^{-N+p+k};\, q)_{n+m-k-p}}{(a_1 a_2 a_3 q^{n+m+k+p+2};\, q)_{n+m-k-p}}$$
$$\times (q^{-x};\, q)_k (q^{k-y};\, q)_p.$$

4.3. Bivariate Monic q-Krawtchouk Polynomials

If we let $a_1 \to \infty$ in $\hat{H}_{n,m}(x, y; q)$ and replace $a_2 \leftarrow a_1$, $a_3 \leftarrow a_2$, we obtain the bivariate monic q-Krawtchouk polynomials

$$\hat{K}_{n,m}(x, y; q) = \hat{K}_{n,m}(x, y; a_1, a_2, N; q)$$
$$= \sum_{k=0}^{n} \sum_{p=0}^{m} (-1)^{n+m-k-p} q^{\frac{1}{2}\left[k^2 - m + k(3+2m) - 3n - (n+m)^2 + p + p^2\right]} \begin{bmatrix} n \\ k \end{bmatrix}_{\mathrm{q}} \begin{bmatrix} m \\ p \end{bmatrix}_{\mathrm{q}}$$
$$\times \frac{(q^{-N+p+k};\, q)_{n+m-k-p}}{a_1^{n-k} a_2^{n+m-k-p}} (q^{-x};\, q)_k (q^{k-y};\, q)_p.$$

4.4. Bivariate Monic q-Meixner Polynomials

In the bivariate monic q-Racah polynomials, if we substitute $q^{-N} \leftarrow \beta$, take the limits $b \to \infty$ and $a_1 \to 0$ and replace $a_2 \leftarrow a_1$, $a_3 \leftarrow a_2$, we obtain the bivariate monic q-Meixner polynomials given by

$$\hat{\mathcal{M}}_{n,m}(x, y; q) = \hat{\mathcal{M}}_{n,m}(x, y; a_1, a_2, \beta; q)$$
$$= \sum_{k=0}^{n} \sum_{p=0}^{m} (-1)^{n+m-k-p} q^{\frac{1}{2}\left[k^2 + k(2m+1) - (n+m-p)(n+m+p-1)\right]}$$
$$\times - n \begin{bmatrix} n \\ k \end{bmatrix}_{\mathrm{q}} \begin{bmatrix} m \\ p \end{bmatrix}_{\mathrm{q}} \frac{(\beta q^{p+k};\, q)_{n+m-k-p}}{a_1^{n-k} a_2^{n+m-k-p}} (q^{-x};\, q)_k (q^{k-y};\, q)_p.$$

4.5. Bivariate Monic q-Charlier Polynomials

The $\beta \to 0$ limit of the bivariate monic q-Meixner polynomials gives the bivariate monic q-Charlier polynomials

$$\hat{\mathcal{C}}_{n,m}(x, y; q) = \hat{\mathcal{C}}_{n,m}(x, y; a_1, a_2; q)$$
$$= \sum_{k=0}^{n} \sum_{p=0}^{m} (-1)^{n+m-k-p} \begin{bmatrix} n \\ k \end{bmatrix}_{\mathrm{q}} \begin{bmatrix} m \\ p \end{bmatrix}_{\mathrm{q}} \frac{q^{\frac{1}{2}\left[k^2 + k(2m+1) - (n+m-p)(n+m+p-1)\right] - n}}{a_1^{n-k} a_2^{n+m-k-p}}$$
$$\times (q^{-x};\, q)_k (q^{k-y};\, q)_p.$$

5. Monic Bivariate Racah Polynomials

Let

$$x(s) = s(s + \beta_1), \quad y(t) = t(t + \beta_2). \tag{5.1}$$

In [11], a difference equation for multivariate Racah polynomials was giving, involving rational coefficients. In the bivariate case, the equation was rewritten using appropriate divided-difference operators, in terms of a fourth-order linear partial divided-difference equation with polynomial coefficients in [27].

Theorem 5.1. *The bivariate Racah polynomials defined in (1.14) are solution of the following fourth-order linear partial divided-difference equation:*

$$\begin{aligned} & f_1(x(s),y(t))\mathbb{D}_x^2\mathbb{D}_y^2 R_{n,m}(s,t) + f_2(x(s),y(t))\mathbb{S}_x\mathbb{D}_x\mathbb{D}_y^2 R_{n,m}(s,t) \\ & \quad + f_3(x(s),y(t))\mathbb{S}_y\mathbb{D}_y\mathbb{D}_x^2 R_{n,m}(s,t) + f_4(x(s),y(t))\mathbb{S}_x\mathbb{D}_x\mathbb{S}_y\mathbb{D}_y R_{n,m}(s,t) \\ & \quad + f_5(x(s))\mathbb{D}_x^2 R_{n,m}(s,t) + f_6(y(t))\mathbb{D}_y^2 R_{n,m}(s,t) \\ & \quad + f_7(x(s))\mathbb{S}_x\mathbb{D}_x R_{n,m}(s,t) \\ & \quad + f_8(y(t))\mathbb{S}_y\mathbb{D}_y R_{n,m}(s,t) + (m+n)(\beta_3-\beta_0+m+n-1)R_{n,m}(s,t) = 0, \end{aligned} \tag{5.2}$$

where $R_{n,m}(s,t) := R_{n,m}(s,t;\beta_0,\beta_1,\beta_2,\beta_3,N)$, *and the coefficients* f_i, $i=1,\ldots,8$ *are polynomials in the lattices* $x(s)$ *and* $y(t)$ *defined in (5.1) given by*

$$\begin{aligned} f_8(y(t)) &= (\beta_0-\beta_3)y(t) - N(\beta_0-\beta_2)(\beta_3+N), \\ f_7(x(s)) &= (\beta_0-\beta_3)x(s) - N(\beta_0-\beta_1)(\beta_3+N), \\ f_6(y(t)) &= -(y(t))^2 + \frac{1}{2}(2N^2+2\beta_3(\beta_0+N)-\beta_2(\beta_3+\beta_0))y(t) \\ &\quad - \frac{1}{2}N\beta_2(\beta_0-\beta_2)(\beta_3+N), \\ f_5(x(s)) &= -(x(s))^2 + \frac{1}{2}(2\beta_3(N+\beta_0)+2N^2-\beta_1(\beta_3+\beta_0))x(s) \\ &\quad - \frac{1}{2}N\beta_1(\beta_0-\beta_1)(\beta_3+N), \\ f_4(x(s),y(t)) &= -2x(s)y(t) + (2N^2+\beta_2(1-\beta_0)+\beta_3(\beta_0-1+2N))x(s) \\ &\quad + (\beta_0-\beta_1)(\beta_3+1)y(t) - N(\beta_0-\beta_1)(\beta_2+1)(\beta_3+N), \\ f_3(x(s),y(t)) &= (\beta_2-\beta_3)(x(s))^2 + x(s)\Big(-(1+\beta_1+\beta_3-2\beta_0)y(t) \\ &\quad + (1+\beta_1-2\,\beta_0+\beta_2)\,N^2 \\ &\quad - \beta_3\,(-\beta_1-\beta_2-1+2\,\beta_0)\,N + \frac{1}{2}\,(\beta_2-\beta_3)\,(\beta_1\beta_0-2\,\beta_0+\beta_1)\Big) \\ &\quad + \frac{1}{2}\,\beta_1\,(\beta_3+1)\,(\beta_0-\beta_1)\,y(t) - \frac{1}{2}\,\beta_1 N\,(\beta_2+1)\,(\beta_3+N)\,(\beta_0-\beta_1)\,, \\ f_2(x(s),y(t)) &= (\beta_0-\beta_1)(y(t))^2 + x(s)\Big((\beta_0+\beta_2-2\beta_3-1)y(t) + (1-\beta_0+\beta_2)\,N^2 \\ &\quad - \beta_3\,(-1+\beta_0-\beta_2)\,N + \frac{1}{2}\,\beta_2\,(\beta_2-\beta_3)\,(\beta_0-1)\Big) \\ &\quad - \frac{1}{2}\,(\beta_0-\beta_1)\,\big(2\,\beta_3 N - \beta_3\beta_2 + 2\,N^2 - 2\,\beta_3 + \beta_2\big)\,y(t) \\ &\quad - \frac{1}{2}\,(\beta_0-\beta_1)\,N\beta_2\,(\beta_2+1)\,(\beta_3+N)\,, \\ f_1(x(s),y(t)) &= -(x(s))^2 y(t) - x(s)(y(t))^2 + (N^2+\beta_3 N - \frac{1}{2}\,\beta_2\,(\beta_2-\beta_3))(x(s))^2 \\ &\quad + \frac{1}{2}\,\beta_1\,(\beta_0-\beta_1)\,(y(t))^2 \\ &\quad + \left(\left(\frac{1}{2}\,\beta_1 + \frac{1}{2} - \frac{1}{2}\,\beta_3 - \beta_0\right)\beta_2 - \beta_3 - \frac{1}{2}\,\beta_1 + 2\,\beta_0\beta_3\right. \\ &\quad \left. + \beta_0 + N^2 - \beta_1\beta_3 + \beta_3 N - \frac{1}{2}\,\beta_1\beta_0\right)x(s)y(t) \\ &\quad + \left(\left(\frac{1}{2}\,\beta_1\beta_0 + \frac{1}{2}\,{\beta_2}^2 + \frac{1}{2}\,\beta_1\beta_2 + \frac{1}{2}\,\beta_2 - \beta_0\beta_2 + \frac{1}{2}\,\beta_1 - \beta_0\right)N^2\right. \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\,\beta_3\left(\beta_1\beta_0+{\beta_2}^2+\beta_1\beta_2+\beta_2-2\,\beta_0\beta_2+\beta_1-2\,\beta_0\right)N\\
&-\frac{1}{4}\,\beta_2\left(\beta_2-\beta_3\right)\left(\beta_1\beta_0+\beta_1-2\,\beta_0\right))x(s)\\
&-\frac{1}{4}\,\beta_1\left(\beta_2-2\,\beta_3+2\,\beta_3N+2\,N^2-\beta_2\beta_3\right)\left(\beta_0-\beta_1\right)y(t)\\
&-\frac{1}{4}\,N\beta_1\beta_2\left(\beta_2+1\right)\left(\beta_0-\beta_1\right)\left(\beta_3+N\right).
\end{aligned}
$$

Next, we present the monic solution of the above fourth-order linear partial divided-difference equation. In doing so, as it happens in the q-quadratic case, it is extremely important to choose appropriate bases for both variables s and t. Similarly, the first variable involves just a classical quadratic basis $\{(-s)_j(s+\beta_1)_j\}_{j\geq 0}$, and the basis for the second variable in the summation contains also the summation index of the first variable, i.e., $\{(j-t)_r(j+t+\beta_2)_r\}_{r\geq 0}$.

Theorem 5.2. *The fourth-order divided-difference Eq.* (5.2) *has the following monic solution:*

$$
\begin{aligned}
\hat{P}_{n,m}(s,t) = \sum_{j=0}^{n}\sum_{r=0}^{m}\binom{n}{j}\binom{m}{r}&\frac{(-s)_j(s+\beta_1)_j(j-t)_r(j+t+\beta_2)_r}{(-2m-2n+\beta_0-\beta_3+2)_{-j+m+n-r}}\\
&\times(\beta_0-\beta_1-n+1)_{n-j}\,(\beta_0\\
&-\beta_2-j-m-n+1)_{m-r}\\
&\times(N-m-n+1)_{m+n-j-r}(j+N+r+\beta_3)_{m+n-j-r}.
\end{aligned}
\tag{5.3}
$$

In the particular case $n=0$, the above expression reduces to

$$
\hat{P}_{n,0}(s,t) = \sum_{j=0}^{n}\binom{n}{j}\frac{(-s)_j(\beta_0-\beta_1-n+1)_{n-j}(s+\beta_1)_j(N-n+1)_{n-j}(j+N+N)_{n-j}}{(\beta_0-N-2n++2)_{n-j}},
$$

which can be expressed as

$$
\begin{aligned}
\hat{P}_{n,0}(s,t) &= \frac{\Gamma(\beta_0-\beta_1+1)\Gamma(N+1)\Gamma(\beta_0-\beta_3-2n+2)\Gamma(n+N+\beta_3)}{\Gamma(\beta_0-\beta_1-n+1)\Gamma(\beta_0-\beta_3-n+2)\Gamma(N+\beta_3)\Gamma(N-n+1)}\\
&\quad\times{}_4F_3\left(\left.\begin{matrix}-n,-s,s+\beta_1,n-\beta_0+\beta_3-1\\ -N,\beta_1-\beta_0,N+\beta_3\end{matrix}\,\right|1\right).
\end{aligned}
$$

Similarly, in the case $m=0$, we have

$$
\hat{P}_{0,m}(s,t) = \sum_{r=0}^{m}\binom{m}{r}\frac{(-t)_r(\beta_0-\beta_2-m+1)_{m-r}(t+\beta_2)_r(N-m+1)_{m-r}(N+r+N)_{m-r}}{(\beta_0-N-2m+2)_{m-r}},
$$

or equivalently

$$
\begin{aligned}
\hat{P}_{0,m}(s,t) &= \frac{\Gamma(\beta_0-\beta_2+1)\Gamma(N+1)\Gamma(\beta_0-\beta_3-2m+2)\Gamma(m+N+\beta_3)}{\Gamma(\beta_0-\beta_2-m+1)\Gamma(\beta_0-\beta_3-m+2)\Gamma(N+\beta_3)\Gamma(N-m+1)}\\
&\quad\times{}_4F_3\left(\left.\begin{matrix}-m,-t,t+\beta_2,m-\beta_0+\beta_3-1\\ -N,\beta_2-\beta_0,N+\beta_3\end{matrix}\,\right|1\right).
\end{aligned}
$$

Proof. Let

$$
\begin{aligned}
\vartheta_j^1(s) &= (-s)_j(s+\beta_1)_j,\\
\vartheta_m^2(t) &= (-t+j)_m(t+\beta_2+j)_m.
\end{aligned}
$$

Using the definitions of the divided-difference operators as well as the latter definitions, the following relations hold true:

$$
x(s)\vartheta_n^1(s) = -\vartheta_{n+1}^1(s)+n(n+\beta_1)\vartheta_n^1(s),
$$

$$
\begin{aligned}
y(t)\vartheta_n^2(t) &= -\vartheta_{n+1}^2(t) + (n+j)(n+j+\beta_2)\vartheta_n^2(t),\\
\mathbb{S}_x\mathbb{D}_x\vartheta_n^1(s) &= -\frac{2n\vartheta_n^1(s) - n(n-1)(2n+\beta_1-2)\vartheta_{n-1}^1(s)}{2\vartheta_1^1(s)},\\
\mathbb{S}_y\mathbb{D}_y\vartheta_n^2(t) &= \left(-\frac{n(2\vartheta_1^2(t) + (\beta_2+2j)(n-1))}{2\vartheta_1^2(t)}\right)\vartheta_{n-1}^2(t),\\
\mathbb{D}_x^2\vartheta_n^1(s) &= \frac{n(n-1)}{\vartheta_1^1(s)}\vartheta_{n-1}^1(s), \quad \mathbb{D}_y^2\vartheta_n^2(t) = \frac{n(n-1)}{\vartheta_1^2(t)}\vartheta_{n-1}^2(t).
\end{aligned}
$$

If we apply the fourth-order divided-difference Eqs. (5.2)–(5.3), the result follows using the latter properties. □

Remark 3. Let

$$\mathbb{R}_n = \left(R_{n-k,k}(s,t;\beta_0,\beta_1,\beta_2,\beta_3,N)\right)_{k=0}^n$$

be the column vector of bivariate Racah polynomials, where $R_{n,m}(s,t;\beta_0,\beta_1,\beta_2,\beta_3,N)$ are defined in (1.14), assuming the substitutions (1.15). Let us also introduce the column vector of monic bivariate Racah polynomials

$$\hat{\mathbb{R}}_n = \left(\hat{P}_{n-k,k}(s,t)\right)_{k=0}^n,$$

where $\hat{P}_{n,m}(s,t)$ are defined in (5.3). Then, by computing the leading coefficients of the bivariate Racah polynomials $R_{n,m}(s,t;\beta_0,\beta_1,\beta_2,\beta_3,N)$ are defined in (1.14), it yields

$$\hat{\mathbb{R}}_n = \mathbb{U}_n\mathbb{R}_n, \quad n = 0,1,2,\ldots, \tag{5.4}$$

where the upper triangular matrix $\mathbb{U}_n$ of size $(n+1)\times(n+1)$ is defined as

$$u_{i,j}(n) = (-1)^{n+1}\binom{n-i+1}{j-i}\frac{(\eta+\alpha_2+2(n-j+1))(\eta+n-j+2)_{j-i}}{(\alpha_2+\eta-j+n+1)_{n-i+2}\,(\alpha_2+\alpha_3+\eta-j+2n+1)_{j-1}}.$$

6. Limit Transitions on Quadratic Lattices

6.1. Bivariate Monic Wilson Polynomials

Let [11, p. 443]

$$V_0 := \begin{cases} \beta_0 = a-b, \quad \beta_1 = 2a, \quad \beta_2 = 2a+2e_2, \quad \beta_3 = 2a+2e_2+c+d,\\ s = -a+ix, \quad t = -a-e_2+iy, \quad N = -a-d-e_2. \end{cases} \tag{6.1}$$

Under the above change of variables monic bivariate Racah polynomials (1.14) transform into the monic bivariate Wilson polynomials (in a similar way as in the univariate case [13, p. 196])

$$
\begin{aligned}
&\hat{W}_{n,m}(x,y;a,b,c,d;e_2)\\
&= \sum_{j=0}^{n}\sum_{r=0}^{m}\binom{n}{j}\binom{m}{r}\frac{(a-ix)_j(a+ix)_j(-a-b-n+1)_{n-j}}{(-a-b-c-d-2m-2n-2e_2+2)_{m+n-j-r}}\\
&\quad\times(a+j-iy+e_2)_r(a+j+iy+e_2)_r(-a-b-j-m-n-2e_2+1)_{m-r}\\
&\quad\times(a+c+j+r+e_2)_{m+n-j-r}(-a-d-m-n-e_2+1)_{m+n-j-r}.
\end{aligned} \tag{6.2}
$$

6.2. Bivariate Monic Continuous Dual Hahn Polynomials

If we take the limit $b \to \infty$ in (6.2) (after redefining $c \to b$, $d \to c$), we obtain the monic bivariate continuous dual Hahn polynomials [29]

$$\begin{aligned}&\hat{D}_{n,m}(a,b,c,e_2;x,y)\\&\quad=\sum_{j=0}^{n}\sum_{r=0}^{m}\binom{n}{j}\binom{m}{r}(a-ix)_j(a+ix)_j(a+j-iy+e_2)_r(a+j+iy+e_2)_r\\&\qquad\times(-1)^{j+m+n+r}(a+b+j+r+e_2)_{m+n-j-r}(a+c+j+r+e_2)_{m+n-j-r}.\end{aligned} \tag{6.3}$$

6.3. Bivariate Monic Continuous Hahn Polynomials

Let in (6.2) [29]

$$a\to a+\frac{1}{2}i\epsilon,\quad b\to b-\frac{1}{2}i\epsilon,\quad c\to c+\frac{1}{2}i\epsilon,\quad d\to d-\frac{1}{2}i\epsilon,\quad x\to x-\frac{1}{2}\epsilon,$$
$$y\to y-\frac{1}{2}\epsilon.$$

If we take the limit as $\epsilon \to \infty$, the resulting polynomials are the bivariate monic continuous Hahn polynomials

$$\begin{aligned}&\hat{h}_{n,m}(a,b,c,d|x)\\&\quad=\sum_{j=0}^{n}\sum_{r=0}^{m}(-1)^{n+m+j+r}\binom{n}{j}\binom{m}{r}(ix+a_1)_j(j+iy+a_1+e_2)_r\\&\qquad\times\frac{(j+a_1+b_1)_{n-j}(j+r+a_1+a_3+e_2)_{m+n-j-r}(j+n+r+a_1+b_1+2e_2)_{m-r}}{(j+m+n+r+a_1+a_3+b_1+b_3+2e_2-1)_{m+n-j-r}}.\end{aligned} \tag{6.4}$$

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References

[1] Area, I., Atakishiyev, N.M., Godoy, E., Rodal, J.: Linear partial q-difference equations on q-linear lattices and their bivariate q-orthogonal polynomial solutions. Appl. Math. Comput. **223**, 520–536 (2013)

[2] Area, I., Godoy, E.: On limit relations between some families of bivariate hypergeometric orthogonal polynomials. J. Phys. A Math. Theor. **46**(035202), 11 (2013)

[3] Area, I., Godoy, E., Rodal, J.: On a class of bivariate second-order linear partial difference equations and their monic orthogonal polynomial solutions. J. Math. Anal. Appl. **389**, 165–178 (2012)

[4] Area, I., Godoy, E., Ronveaux, A., Zarzo, A.: Corrigendum: minimal recurrence relations for connection coefficients between classical orthogonal polynomials: discrete case. J. Comput. Appl. Math. **89**(2), 309–325 (1998)

[5] Area, I., Godoy, E., Ronveaux, A., Zarzo, A.: Bivariate second-order linear partial differential equations and orthogonal polynomial solutions. J. Math. Anal. Appl. **387**(2), 1188–1208 (2012)

[6] Askey, R., Wilson, J.: Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials. Mem. Am. Math. Soc. **54**, iv+55 (1985)

[7] Atakishiyev, N.M., Rahman, M., Suslov, S.K.: On classical orthogonal polynomials. Constr. Approx. **11**(2), 181–226 (1995)

[8] Dunkl, C.F., Xu, Y.: Orthogonal Polynomials of Several Variables, Volume 81 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge (2001)

[9] Foupouagnigni, M.: On difference equations for orthogonal polynomials on nonuniform lattices. J. Differ. Equ. Appl. **14**, 127–174 (2008)

[10] Foupouagnigni, M., Koepf, W., Kenfack-Nangho, M., Mboutngam, S.: On solutions of holonomic divided-difference equations on nonuniform lattices. Axioms **2**, 404–434 (2013)

[11] Geronimo, J.S., Iliev, P.: Bispectrality of multivariable Racah–Wilson polynomials. Constr. Approx. **31**, 417–457 (2010)

[12] Ismail, M.E.H., Stanton, D.: Some combinatorial and analytical identities. Ann. Comb. **16**, 755–771 (2012)

[13] Koekoek, R., Lesky, P.A., Swarttouw, R.F.: Hypergeometric Orthogonal Polynomials and Their q-Analogues, Springer Monographs in Mathematics. Springer, Berlin (2010)

[14] Gasper, G., Rahman, M.: Some systems of multivariable orthogonal Askey–Wilson polynomials, in theory and applications of special functions. Dev. Math. **13**, 209–219 (2005)

[15] Gasper, G., Rahman, M.: Some systems of multivariable orthogonal q-Racah polynomials. Ramanujan J. **13**, 389–405 (2007)

[16] Godoy, E., Ronveaux, A., Zarzo, A., Area, I.: Minimal recurrence relations for connection coefficients between classical orthogonal polynomials: continuous case. J. Comput. Appl. Math. **84**(2), 257–275 (1997)

[17] Iliev, P.: Bispectral commuting difference operators for multivariable Askey–Wilson polynomials. Trans. Am. Math. Soc. **363**(3), 1577–1598 (2011)

[18] Lewanowicz, S., Woźny, P.: Two-variable orthogonal polynomials of big q-Jacobi type. J. Comput. Appl. Math. **233**, 1554–1561 (2010)

[19] Lewanowicz, S., Woźny, P., Nowak, R.: Structure relations for the bivariate big q-Jacobi polynomials. App. Math. Comput. **219**, 8790–8802 (2013)

[20] Magnus, A.P.: Associated Askey–Wilson polynomials as Laguerre–Hahn orthogonal polynomials, In: Orthogonal polynomials and their applications (Segovia, 1986), Lecture Notes in Mathematics, vol. 1329, pp. 261–278. Springer, Berlin (1988)

[21] Magnus, A.P.: Special nonuniform lattice (SNUL) orthogonal polynomials on discrete dense sets of points, In: Proceedings of the International Conference on Orthogonality, Moment Problems and Continued Fractions (Delft, 1994), vol. 65, pp. 253–265 (1995)

[22] Nikiforov, A.F., Suslov, S.K., Uvarov, V.B.: Classical Orthogonal Polynomials of a Discrete Variable. Springer Series in Computational Physics. Springer, Berlin (1991)

[23] Njionou Sadjang, P., Koepf, W., Foupouagnigni, M.: On structure formulas for Wilson polynomials. Int. Transf. Spec. Funct. **26**(12), 1000–1014 (2015)

[24] Olver, F.W.J., Lozier, D.W., Boisvert, R.F., Clark, C.W.: NIST Handbook of Mathematical Functions. Department of Commerce and Cambridge University Press, National Institute of Standards and Technology U.S (2010)

[25] Post, S.: Racah polynomials and recoupling schemes of $\mathfrak{su}(1,1)$. SIGMA **11**, 057 (2015)

[26] Rodal, J., Area, I., Godoy, E.: Orthogonal polynomials of two discrete variables on the simplex. Integral Transforms Spec Funct **16**, 263–280 (2005)

[27] Tcheutia, D.D., Guemo Tefo, Y., Foupouagnigni, M., Godoy, E., Area, I.: Linear partial divided-difference equation satisfied by multivariate orthogonal polynomials on quadratic lattices. Math. Model. Nat. Phenom. **12**(3), 14–43 (2017)

[28] Tcheutia, D.D., Foupouagnigni, M., Guemo Tefo, Y., Area, I.: Divided-difference equation and three-term recurrence relations of some systems of bivariate q-orthogonal polynomials. J. Differ. Equ. Appl. **23**(12), 2004–2036 (2017)

[29] Tratnik, M.V.: Some multivariable orthogonal polynomials of the Askey tableau–continuous families. J. Math. Phys. **32**, 2065–2073 (1991)

[30] Tratnik, M.V.: Some multivariable orthogonal polynomials of the Askey tableau–discrete families. J. Math. Phys. **32**, 2337–2342 (1991)

[31] Witte, N.S.: Semi-classical orthogonal polynomial systems on non-uniform lattices, deformations of the Askey table and analogs of isomonodromy. http://arxiv.org/abs/1204.2328 (2011)

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