Average internal rate of return with interval arithmetic*

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1 Extended abstract

The internal-rate-of-return investment choice method is widely used to evaluate and compare projects; the decision is based on the ranking of the Internal Rate of Return (IRR) projects and finally on the choice of the higher one. Many contributions in the last eight decades have been devoted to the corrections of the method that can reserve many technical difficulties, misunderstandings **and incompatibility with the net-present-value rule**.

The use of a rate of return for investment decisions under uncertainty can be approached in several ways. The debate about the use of the theory of probability to model uncertainty in economics has a long history and in investment decisions many weak points have been focused.

The interval and the fuzzy arithmetics represent alternative ways to model uncertainty in investment decisions: here we summarize the results achieved in some contributions.

The studies in [7] go into the direction **of studying** the possible ways to model uncertainty in economics; in particular, they show that robustness of capital budgeting techniques depends on the statistical estimation of cash amounts and interest rates, exhibiting often hard difficulties that can be captured by fuzzy numbers. In [12] methods for integrating probability and possibility distributions are discussed and a computer simulation used for an investment project risk assessment shows that it is possible to model some parameters with fuzzy numbers and others with a probability distribution.

Kuchta in [8] proposes generalized fuzzy equivalents in order to incorporate uncertainty into the most commonly used techniques for capital budgeting. These fuzzy equivalents allow to evaluate projects whose cash flows and/or duration are not known precisely, but given in the form of fuzzy numbers. In [4] we follow the Kuchta approach with particular attention to the parametric

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representation of fuzzy numbers. In a second paper (details in [9]) Kuchta assumes a fuzzy Net Present Value for some projects and shows that the common realization of them may allow savings in the resource utilization.

In [1] the concept of possibilistic mean and variance is extended to adaptive fuzzy numbers and applied to the computation of the fuzzy net present value of future cash flows.

Also in [6] the underlying hypothesis is that fuzzy variables can reflect uncertainty of investment outlays, annual net cash flows and investment capital; a new mean-variance model based on credibility measure is proposed for optimal capital allocation.

The potentialities of the fuzzy approach are highlighted also in [13], fewer assumptions about the data distribution and market behavior are enough to describe fluctuations that go beyond the probability model. A measure of the risk associated with each investment opportunity and an estimate of the projects' robustness towards market uncertainty is then derived.

A fuzzy logic system in [15] is adopted to extend the classical discounted cash flow model in order to take into account the uncertain information intrinsic in the value of a company's financial asset.

In [10] a fuzzy binomial approach for project valuation under uncertainty is proposed and a method to compute the mean value of a project's fuzzy Net Present Value is provided.

The approach we choose in order to manage uncertainty in investment decision making is the definition of variables as intervals.

The arithmetic of data represented with intervals is well established and formally consistent; in particular it is possible to manage the four operations through the extension principle in order to build an arithmetic based on the extended operations.

The key aspect is, in fact, the concept of equation that has to be defined in a rigorous way; some misunderstanding can be eventually produced by the semilinearity structure of the intervals space. The analysis must be much careful when dealing with the uncertainty propagation that is an implicit feature in the intervals arithmetic.

In order to give an overview on the meaning of the uncertainty in intervals arithmetic we can think about the fact that the length of the sum of two intervals is equal to the sum of each interval itself. So the equation $A + X = B$ admits solution only when the length of A is not bigger than the length of B and it is not equivalent to $A = B - X$. The non equivalence holds in terms of the way in which the uncertainty propagates: the position of X at the left or at the right changes the propagation.

In order to deeper analyze the true meaning of the equation $A+X = B$ when the variables are intervals, we have to distinguish between the input data and the variables to be determined; it follows that the equation may have solution only in the arithmetic form $A + X = B$ or $A = B - X$ where the two equations are equivalent if and only if A and B are real numbers, in other words only when there is no uncertainty on the values.

The same reasoning can be applied to the multiplication and to more general equations, involving two operations, like $Ax + B = C$.

Any time we want to define a variable varying in an interval in terms of other variables by mean of the arithmetic is like to define a variable X such that $A + X = B$ and to interpret the left side $A + X$ as it is the same as B. However to find such a variable X has a different meaning if we think about the interval that has to be equal to $B - A$

In the intervals arithmetic it is possible to write:

$$X = B \ominus_{gH} A$$

where $\ominus_{gH}$ is the generalized Hukuhara difference, whose properties are established in [14], defined as:

$$X = B \ominus_{gH} A \Leftrightarrow \begin{cases} A = B + X \\ or \\ B = A + (-1)X \end{cases}$$

The crucial aspect requires that the applications based on the intervals arithmetic have to be studied contextually to an epistemic analysis on the meaning of the relations between variables

When managing with relations having a financial or economic nature, the uncertainty propagation becomes an urgent analysis.

In this paper we follow the new perspective of the **Average Internal Rate of Return (AIRR)**, introduced in [11], **where the IRR is dismissed and replaced by a mean of the project's one-period rates, weighed by the outstanding capitals invested in each period. We generalize [11]** by modeling the uncertainty of some key variables **by means of** intervals. The interval arithmetic and the interval ordering we apply are introduced in [5].

The basic variables for a valuation project are now defined in their interval-valued form. The temporal sequence is expressed in the classical form as $t \in \{0, 1, 2, ..., T\}$, the cash flow at time t is defined as $x_t = \left[x_t^-, x_t^+\right]$, the interest rate is $r = [r^-, r^+]$ and the discount rate is $v = [1 + r^-, 1 + r^+] = [v^-, v^+]$ under the condition $1 + r^- > 0$. Given the above definitions, the following identity is verified:

$$\left(\frac{1}{1+r}\right)^t = \left[\frac{1}{(1+r^+)^t}, \frac{1}{(1+r^-)^t}\right] = \left[\frac{1}{(v^+)^t}, \frac{1}{(v^-)^t}\right]$$

and the Net Present Value (NPV) of the project, with values varying in an interval, is

$$PV\left(\underline{x} \mid r\right) = \sum_{t=0}^{T} x_t \left(1+r\right)^{-t} = \left[PV^-\left(\underline{x} \mid r\right), PV^+\left(\underline{x} \mid r\right)\right]$$

where, in order to obtain $PV^-\left(\underline{x} \mid r\right) = \sum\limits_{t=0}^{T} \frac{x_t^-}{(1+r^+)^t}$ and $PV^+\left(\underline{x} \mid r\right) = \sum\limits_{t=0}^{T} \frac{x_t^+}{(1+r^-)^t}$, the use of the extension principle is recommended in a rigorous way, producing:

$$PV\left(\underline{x} \mid r\right) = \left[\min_{\substack{\xi_t \in x_t \\ s \in r}} \sum_{t=0}^{T} \frac{\xi_t}{(1+s)^t}, \max_{\substack{\xi_t \in x_t \\ s \in r}} \sum_{t=0}^{T} \frac{\xi_t}{(1+s)^t} \right]$$

The Net Future Value (NFV) is the future value of NPV and is defined as:

$$PV_t\left(\underline{x} \mid r\right) = (1+r)^t\, PV\left(\underline{x} \mid r\right) \qquad t \geq 1.$$

The **NPV rule** rule implies that **a project is worth undertaking if and only if** $PV\left(\underline{x} \mid r\right) > 0$ that is equivalent to $PV_t\left(\underline{x} \mid r\right)$ for every t. The internal rate of return (IRR) of the project is a value k that satisfies: $PV_T\left(\underline{x} \mid k\right) = 0$. **However, the latter is never defined in the interval case. By contrast, the AIRR is well defined even in the interval case and consistent with the (interval) NPV.**

Definition 1 *Let* $c_t = \left[c_t^-, c_t^+\right]$ *be the interval-valued invested capital (or borrowed capital) in period* $[t-1, t]$ $t = 0, 1, ..., T$. *Assume* $c_0 = -x_0 = \left[-x_0^+, -x_0^-\right]$ *(also assume, by definition,* $c_T = 0 = [0, 0]$. *The return* R_t *for period* $[t, t+1]$ *is defined to satisfy, formally,:*

$$c_{t-1} + R_t = c_t + x_t \tag{1}$$

that is equivalent to:
1. $c_{t-1} = (c_t + x_t) \ominus_{gH} R_t$
2. $R_t = (c_t + x_t) \ominus_{gH} c_{t-1}$
3. $c_t = (c_{t-1} + R_t) \ominus_{gH} x_t$
4. $x_t = (c_{t-1} + R_t) \ominus_{gH} c_t$
depending on the value that requires to be computed and on the values that are known (with the corresponding uncertainties).

The uncertainty propagation produced by (1) is the key feature of the fuzzy version of the new **AIRR model**; **from this more general perspective an interval average rate of return may be derived, which enables investors to deal with uncertainty in a rather sophisticated way.**

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