Bifurcation and Stability for Nonlinear Schrödinger Equations with Double Well Potential in the Semiclassical

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Abstract We consider the stationary solutions for a class of Schrödinger equations with a symmetric double-well potential and a nonlinear perturbation. Here, in the semiclassical limit we prove that the reduction to a finite-mod approximation give the stationary solutions, up to an exponentially small term, and that symmetry-breaking bifurcation occurs at a given value for the strength of the nonlinear term. The kind of bifurcation picture only depends on the nonlinearity power. We then discuss the stability/instability properties of each branch of the stationary solutions. Finally, we consider an explicit one-dimensional toy model where the double well potential is given by means of a couple of attractive Dirac's delta pointwise interactions.

Keywords Nonlinear Schrödinger equation · Spontaneous symmetry breaking bifurcation · Orbital stability

1 Introduction

Here, we consider the stationary solutions of the nonlinear Schrödinger (hereafter NLS) equations

$$i\hbar \frac{\partial \psi}{\partial t} = H_0 \psi + \epsilon g(x) |\psi|^{2\sigma} \psi, \quad \| \psi(\cdot, t) \| = 1, \tag{1}$$

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where $\epsilon \in \mathbb{R}$ and $\| \cdot \|$ denotes the L^2 norm,

$$H_0 = -\frac{\hbar^2}{2m}\Delta + V, \quad \Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}, \tag{2}$$

is the linear Hamiltonian and $g(x)|\psi|^{2\sigma}$ is a nonlinear perturbation. For the sake of definiteness we assume the units such that $2m = 1$.

Atomic Bose-Einstein condensates (BECs) are described by means of nonlinear Schrödinger equations of the type (1) where H_0 represents the Hamiltonian of a single trapped atom and the nonlinear term $|\psi|^{2\sigma}$, $\sigma = 1, 2, \ldots$, is the $(\sigma + 1)$-body contact potential [24]. In fact, BECs strongly depend by interatomic forces and the binary coupling term $|\psi|^2\psi$ usually represents the dominant nonlinear term and equation (1) takes the form of the well-known Gross-Pitaevskii equation [30]. Even if in most of the applications the parameter σ takes only integer and positive values, here we take that σ can assume noninteger values too, as considered in [34]. It is worth mentioning also the fact that equation (1) with nonlinearity corresponding to the power-law $|\psi|^{2\sigma}$, where the parameter σ takes any positive real value, is used in other contexts, including semiconductors [28] and nonlinear optics [6, 35, 36]. We would also mention that NLS is also useful in order to describe classical behavior in quantum structures [22, 23].

In this paper we consider the case of symmetric potentials V with *double well* shape; the function $g(x)$ is a bounded regular function (in the following we assume, for argument's sake, that $g(x)$ has the same symmetric properties as $V(x)$).

If the nonlinear term is absent then the linear Hamiltonian H_0 has even-parity and odd-parity eigenstates: the d-dimensional linear Schrödinger equation with a symmetric double well potential has stationary states of a definit even φ_+ and odd-parity φ_-, with associate nondegenerate eigenvalues $\lambda_+ < \lambda_-$.

However, the introduction of a nonlinear term, which usually models in quantum mechanics an interacting many-particle system, may give rise to asymmetrical states related to spontaneous symmetry breaking phenomenon.

In NLS problems with double well potentials the *effective nonlinearity parameter* is given by the ratio between the strength ϵ of the nonlinear term and the splitting ω between the f rst two levels (as define in (5) below). The spontaneous symmetry breaking effect, and the associated localization phenomena, occurs when such a ratio is equal to a (finite critical value. This fact has been seen, for instance, in the study of the localization effect in a gas of pyramidal molecules as the ammonia ones [22, 23]. In these paper has been shown, in full agreement with the experimental data, that the inversion frequency of the ammonia gas depends on the ratio between the strength of the nonlinear term (which depends on the gas pressure) and the splitting between the f rst two levels; furthermore, when this ratio is equal to a critical value then localization occurs. For this reason we introduce an *effective nonlinearity parameter* η in (4) below and we investigate the spontaneous symmetry breaking effect for values of η in a f nite interval of values.

On the other side in this paper we also have to treat the problem of the validity of the two-level approximation, obtained by restricting our analysis to the two-dimensional space associated to the f rst two eigenvectors of the linear problem; in our approach we solve this problem considering the semiclassical limit of small $\hbar$. Since the splitting ω is not fi ed, but it is exponentially small when $\hbar$ goes to zero, then, in order to have a f nite value for η (if not then we simply have localization), we have to require that ϵ should be exponentially small, too. Hence, in our model we introduce the multi-scale limit (22) below in order to observe the bifurcation phenomena.

In the *semiclassical limit* and in the *two-level approximation* has been seen [33] that the symmetric/antisymmetric stable stationary state bifurcates when the adimensional nonlinear parameter η takes absolute value equal to the critical value

$$\eta^{\star} = 2^{\sigma}/\sigma. \tag{3}$$

The parameter η is associated with the coupling factor of the nonlinear perturbation by

$$\eta = c\epsilon/\omega \tag{4}$$

and it is the effective nonlinear coupling factor, where ω is the (half of the) splitting between the two levels

$$\omega = \frac{1}{2}(\lambda_{-} - \lambda_{+}) \tag{5}$$

and c is a constant define below in Sect. 2.2. In fact, in the semiclassical limit (or also for large distance between the two wells) the splitting ω is exponentially small, as $\hbar$ goes to zero. Furthermore, in [33] it has been also seen that for σ less than a critical value

$$\sigma_{threshold} = \frac{1}{2}[3 + \sqrt{13}]$$

then a supercritical pitchfork bifurcation occurs; on the other hand, for σ bigger than the critical value $\sigma_{threshold}$ a subcritical pitchfork bifurcation associated to the appearance on a couple of saddle node points occurs.

It is worth mentioning the fact that the main problem consists in proving the stability of the two-level approximation (which basically is a two-mode problem) with respect to the NLS equation (1). So far, the stability of the two-level approximation has been proved, in the semiclassical limit, only for times of the order of the beating period $T = 2\pi\hbar/\omega$ [32], or for exponentially large times (that is of the order e^{T}) under further assumptions as proved by [3]. In fact, our previous approach was rather eff cient in order to study the dynamics, but only give a partial result in order to look for the stationary solutions. Recently, Kirr, Kevrekidis, Shlizerman and Weinstein [25] has considered the stationary solution problem for the Cauchy problem (1) with $\hbar$ fi ed (i.e. $\hbar = 1$) in the limit of large barrier between the two wells, and in the case of cubic nonlinearities. In their seminal paper they make use of the Lyapunov-Schmidt reduction method to the two-level approximation equation for the stationary solutions. In such a way they overcome the limit of the method applied by [32] for the study of the stationary solutions. Furthermore, they also applied the same method in order to study the orbital stability of the obtained solutions.

In this paper we follow the ideas developed by [25], adapted to the semiclassical limit and considering the case of any positive and real nonlinearity power σ, in order to study the stationary solutions of (1) and their stability properties as function of the nonlinearity power σ. In particular we are able to prove that the result obtained by [33] for the two-level approximation, concerning the existence on the critical value $\sigma_{threshold}$, holds true for the whole Cauchy problem (1), too. To this end we prove the stability of the two-level approximation, when restricted to the stationary problem, and then count all the branches associated to the stationary solutions.

It is worth to mention the fact that the stability of the two-level approximation holds true in order to classify the stability/instability properties of the stationary solutions, too. In fact, stability/instability properties of the stationary solutions for the two level approximation are easily obtained since such an approximation has a f nite-dimensional Hamiltonian structure. On the other side, orbital stability/instability properties of the stationary solutions of the full nonlinear problem are much harder to obtain. However, in this paper, by making use of the methods developed by Grillakis, Shatah and Strauss [17–19], and successfully applied by [25] for double well problems with cubic nonlinearity, we prove the equivalence between the stability/instability properties when we restrict our problem to the case of attractive nonlinearity and when we restrict our analysis to the "ground state".

There are already many studies on the existence of stationary solutions and the stability of (1) in the semiclassical limit (e.g., [12, 17–19]). However, our aim is to understand what happens with double-well problem. When we consider the stationary problem with symmetric double-well and nonlinearity strength large enough, the bifurcation picture tells us that we have asymmetrical stationary solutions localized on just one well, as well as asymmetrical stationary solution delocalized between the two wells. The firs type of solution was obtained, but the second type of solution was not considered in [12], and it is identifie with the multi-bump stationary solution studied in, e.g., [10]. Also it would be important to understand the destruction of the beating motion in the framework of the dynamics (see [16] for related topics).

The paper is organized as follows. In Sect. 2 we recall some preliminary spectral results for Schrödinger operator with double well potential in the semiclassical limit, we introduce the main assumptions and we collect some general global well-posedness results for the Cauchy problem (1). In Sect. 3 we prove (Theorem 1) concerning the occurrence and the nature of spontaneous symmetry breaking phenomenon for (1) by applying, in the semiclassical limit, the Lyapunov-Schmidt reduction method to the two-level approximation. In Sect. 4 we consider the dynamical properties of the stationary solutions of the two-level approximation, which has Hamiltonian form. In Sect. 5 we consider the orbital stability properties of the ground state stationary solutions. Appendix is devoted to an application of all the arguments in the previous sections to an explicit one dimensional toy model where the double well potential is given by a couple of attractive Dirac's delta interactions.

1.1 Notations

Hereafter,

- $y = \tilde{O}(x)$, means that for any $0 < \alpha < 1$ there exists a positive constant $C := C_\alpha$ such that $|y| \le C_\alpha |x|^\alpha$. Here, as usual $y = O(x)$ means that there exists a positive constant C such that $|y| \le C|x|$, and $x \sim y$ means that $\lim_{\hbar\to 0} \frac{x}{y} = C$ for some $C \in \mathbb{R}$;
- $\|\cdot\|_p$ and $\|\cdot\|$ denote the norm of the spaces L^p and L^2, $\langle \phi, \varphi\rangle = \int \bar{\phi}\varphi$ denotes the scalar product in the Hilbert space L^2;
- C denotes any positive constant which value is independent of $\hbar$.

2 Main Assumptions and Preliminary Results

Here, we recall some preliminary results. Throughout the paper we always assume the Hypotheses below in this section.

2.1 Linear Operator

Here, we introduce the assumptions on the double-well potential V and we collect some well known results on the linear operator H_0.

Hypothesis 1 *The potential $V(x)$ is a bounded real valued function such that*:

i. *V is a symmetric potential. For the sake of definiteness we can always assume that, by means of a suitable choice of the coordinates, V is symmetric with respect to the spatial coordinate x_1, that is*

$$[\mathcal{S}, V] = 0 \tag{6}$$

where

$$[\mathcal{S}\psi](x_1, x_2, \ldots, x_d) = \psi(-x_1, x_2, \ldots, x_d).$$

Hence, the Hamiltonian H_0 is invariant under the space inversion: $[\mathcal{S}, H_0] = 0$.

ii. $V \in C^\infty(\mathbb{R}^d)$.

iii. *$V(x)$ admits two minima at $x = x_\pm$, where $x_- = \mathcal{S}x_+ \neq x_+$, such that*

$$V(x) > V_{min} = V(x_\pm), \quad \forall x \in \mathbb{R}^d, \ x \neq x_\pm. \tag{7}$$

For the sake of simplicity, we assume also that

$$\nabla V(x_\pm) = 0 \quad \textit{and} \quad \textit{Hess}\, V(x_\pm) > 0.$$

iv. *Finally we assume that the two minima are not degenerate*:

$$V_\infty^- = \liminf_{|x|\to\infty} V(x) > V_{min}. \tag{8}$$

Remark 1 In fact, some assumptions on V may be weakened. In particular, the case of degenerate minima, that is det$[\textit{Hess}\, V(x_\pm)] = 0$, could be treated in a similar way; however, we don't dwell here on such details. Furthermore, boundedness of V is assumed just for sake of definiteness if V is not bounded we could make use of the argument by [3] in order to prove the well-posedness of the Cauchy problem (1), under some assumptions of the behavior of the potential at infinit . For instance, we could assume that there exists a positive constant $0 < m \leq 2$ such that for large $|x|$

$$C\langle x\rangle^m \leq V(x) \leq C^{-1}\langle x\rangle^m, \quad \langle x\rangle = \left(1 + |x|^2\right)^{1/2},$$

for some $C > 0$, and

$$\left|\partial_{x_1}^{\alpha_1} \ldots \partial_{x_d}^{\alpha_d} V(x)\right| \leq C_\alpha \langle x\rangle^{m-|\alpha|}, \quad |\alpha| = \sum_{j=1}^d \alpha_j,$$

for any multi-index $\alpha \in \mathbb{N}^d$.

The operator H_0 formally define by (2) admits a self-adjoint realization (still denoted by H_0) on $H^2(\mathbb{R}^d)$ since V is a bounded potential.

Let $\sigma(H_0) = \sigma_d \cup \sigma_{ess}$ be the spectrum of the self-adjoint operator H_0, where σ_d denotes the discrete spectrum and σ_{ess} denotes the essential spectrum. It follows that

$$\sigma_d \subset \left(V_{min}, V_\infty^-\right) \quad \text{and} \quad \sigma_{\text{ess}} = [V_\infty^-, +\infty).$$

Furthermore, for any $\hbar \in (0, \hbar^\star)$, for some $\hbar^\star > 0$ fixed and small enough, it follows that σ_d is not empty and, in particular, it contains two eigenvalues at least λ_+^1 and λ_-^1 where $\lambda_+^1 < \lambda_-^1$ and

$$\inf_{\zeta \in \sigma(H_0)\setminus\{\lambda_\pm^1\}} \left[\zeta - \lambda_\pm^1\right] \geq C\hbar, \tag{9}$$

for some positive constant C independent of $\hbar$.

Remark 2 Actually, from Hypothesis 1 and for $\hbar$ small enough in general it follows that for some $E > V_{min}$ then

$$\sigma_d \cap (V_{min}, E)$$

is given by a sequence of couple of nondegenerate eigenvalues $\lambda_\pm^j$, $j = 1, 2, \ldots, n$ where $n \sim \hbar^{-1}$, such that $\lambda_+^j < \lambda_-^j$ and

$$\inf_{\zeta \in \sigma(H_0)\setminus\{\lambda_\pm^j\}} \left|\zeta - \lambda_\pm^j\right| \geq C\hbar \tag{10}$$

hold true. In fact, degeneracy may occur for some $j > 1$ only in special cases, for instance when other symmetry properties for the potential V are present (see, e.g., [20]). Hereafter, for the sake of definiteness we assume that degeneracy does not occur and that (10) holds true for any $j = 1, 2, \ldots, n$.

Let $\varphi_\pm^j$ be the normalized eigenvectors associated to $\lambda_\pm^j$, then $\varphi_\pm^j$ can be chosen to be real-valued functions such that

$$\mathcal{S}\varphi_\pm^j = \pm\varphi_\pm^j. \tag{11}$$

Furthermore

Lemma 1 *The eigenvectors $\varphi_\pm^j$ belong to the space $H^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ where*

$$2 \leq p \begin{cases} \leq +\infty & \text{if } d = 1, \\ < +\infty & \text{if } d = 2, \\ < 2d/(d-2) & \text{if } d > 2. \end{cases} \tag{12}$$

In particular, it follows that

$$\left\|\nabla\varphi_\pm^j\right\| \leq C_j \hbar^{-1/2} \quad \text{and} \quad \left\|\varphi_\pm^j\right\|_{H^2} \leq C_j \hbar^{-1} \tag{13}$$

and

$$\left\|\varphi_\pm^j\right\|_p \leq C_j \hbar^{-d\frac{p-2}{4p}}, \tag{14}$$

for some positive constant C_j, independent on $\hbar$.

Proof Indeed, $\varphi_\pm^j$ is normalized and it satisfie to the following eigenvalue equation $-\hbar^2\Delta\varphi_\pm^j = (\lambda_\pm^j - V)\varphi_\pm^j$, from which immediately follows that

$$\begin{aligned}\hbar^2\left\|\nabla\varphi_\pm^j\right\|^2 &= \left\langle\left(\lambda_\pm^j - V\right)\varphi_\pm^j, \varphi_\pm^j\right\rangle \\ &\leq \left\langle\left(\lambda_\pm^j - V\right)\varphi_\pm^j, \varphi_\pm^j\right\rangle_{L^2(\Omega_\pm^j)} \\ &\leq C_j\hbar\left\|\varphi_\pm^j\right\|^2\end{aligned}$$

where

$$\Omega_\pm^j = \left\{x \in \mathbb{R}^d \mid V(x) \leq \lambda_\pm^j\right\}$$

is such that $\lambda_\pm^j - V \geq \lambda_\pm^j - V_{min} \geq C_j\hbar$ for any fi ed j and $\hbar$ small enough. Similarly

$$\hbar^2\left\|\Delta\varphi_\pm^j\right\|^2 = \left\|\left(\lambda_\pm^j - V\right)\varphi_\pm^j\right\| \leq C_j\left\|\varphi_\pm^j\right\|.$$

Since V is a bounded potential. Estimate (14) follows by means of the Gagliardo-Nirenberg inequality:

$$\left\|\varphi_\pm^j\right\|_p \leq C\left\|\nabla\varphi_\pm^j\right\|^\delta\left\|\varphi_\pm^j\right\|^{1-\delta} \leq C\hbar^{-\delta/2}$$

where $\delta = \frac{p-2}{2p}d$. □

Remark 3 Actually, $\varphi_\pm^j \in L^p$ for any p and, by means of the Riesz-Thorin interpolation Theorem, inequality (14) holds true for any p independently on the dimension d (see, e.g., [32]). Indeed, by means of the semiclassical expression of φ_j it follows that $\|\varphi_j\|_\infty \leq C_j\hbar^{-d/4}$.

The *splitting* between the two eigenvalues

$$\omega^j = \frac{1}{2}\left(\lambda_-^j - \lambda_+^j\right) \tag{15}$$

vanishes as $\hbar$ goes to zero. In order to give a precise estimate of the splitting ω^j we make use of the fact that V is a symmetric double-well potential with nonzero barrier between the wells. That is, let j be fi ed and let

$$\rho = \inf_\gamma \int_\gamma \sqrt{\left[V(x) - V_{min}\right]_+}\, dx > 0, \tag{16}$$

be the Agmon distance between the two wells; where γ is any path connecting the two wells, that is $\gamma \in AC([0,1], \mathbb{R}^d)$ such that $\gamma(0) = x_-$ and $\gamma(1) = x_+$, and where $[\cdot]_+ = \max(\cdot, 0)$. From standard WKB arguments (see [20] for details) then it follows that the splitting is *exponentially small*, that is

$$\omega^j = \tilde{O}\left(e^{-\rho/\hbar}\right). \tag{17}$$

Let $\varphi_{R,L}^j$ be the normalized *single well states* associated to the linear eigenstates $\varphi_\pm^j$ by means of

$$\varphi_R^j = \left(\varphi_+^j + \varphi_-^j\right)/\sqrt{2} \tag{18}$$

and

$$\varphi_L^j = \left(\varphi_+^j - \varphi_-^j\right)/\sqrt{2}. \tag{19}$$

They are *localized on one well* in the sense that and for any $p \in [2, +\infty]$ then

$$\left\| \varphi_R^j \varphi_L^j \right\|_p = \tilde{O}\left(e^{-\rho/\hbar}\right). \tag{20}$$

More precisely, these functions are localized on only one of the two wells in the sense that for any $r > 0$ there exists $c := c(r) > 0$ such that

$$\int_{D_r(x_+)} \left|\varphi_R^j(x)\right|^2 dx = 1 + O\left(e^{-c/\hbar}\right)$$

and

$$\int_{D_r(x_-)} \left|\varphi_L^j(x)\right|^2 dx = 1 + O\left(e^{-c/\hbar}\right)$$

where $D_r(x_\pm)$ is the ball with center $x_\pm$ and radius r. For such a reason we call them *single-well* (normalized) states.

Remark 4 In the following, for the sake of definiteness we restrict ourselves to the couple of eigenvalues λ_+^1 and λ_-^1, corresponding to the lowest energies. Hereafter, we simply denote them by $\lambda_\pm$ dropping out the index 1, and $\varphi_\pm$ denote the associated eigenvectors. The symmetric solution φ_+ is the f rst eigenfunction of H_0, so it is positive. We remark that the existence of the stationary solutions for (1) and their dynamical stability still hold true when we consider all the unperturbed energy levels $\lambda_\pm^j$ provided that degeneracy does not occur as discussed in Remark 2.

2.2 Assumption on the Nonlinear Term

In order to obtain some a priori estimates of the wavefunction $|\psi|^{2\sigma}\psi$ we introduce the following assumption on the nonlinearity power σ.

Hypothesis 2 *We assume that*

$$0 < \sigma < \begin{cases} +\infty & \text{if } d = 1, 2, \\ \frac{1}{d-2} & \text{if } d > 2, \end{cases} \tag{21}$$

where d is the spatial dimension.

Let

$$C_R = \left\langle \varphi_R^{\sigma+1}, g\varphi_R^{\sigma+1} \right\rangle \quad \text{and} \quad C_L = \left\langle \varphi_L^{\sigma+1}, g\varphi_L^{\sigma+1} \right\rangle$$

where $C_R = C_L$ because of the symmetric properties of g and V. We assume also the following scaling limit.

Hypothesis 3 *Let $\omega = \frac{1}{2}(\lambda_- - \lambda_+)$ be the splitting (15) satisfying to the asymptotic estimate (17). We assume that the real-valued parameter ϵ depends on $\hbar$ in such a way*

$$|\eta| \leq C \quad \text{where } \eta = \frac{\epsilon c}{\omega}, \; c := C_R = C_L, \tag{22}$$

for some positive constant C, independent of $\hbar$. The parameter η plays the role of effective nonlinearity parameter. Hereafter, we assume that $g(x)$ has the same symmetry property (6) of the potential V and it is such that $\langle \varphi_R^{\sigma+1}, g\varphi_R^{\sigma+1}\rangle \neq 0$. In particular, for the sake of definiteness, let

$$\left\langle \varphi_R^{\sigma+1}, g\varphi_R^{\sigma+1}\right\rangle > 0. \tag{23}$$

2.3 Existence Results in H^1 and Conservation Laws

The results below follow from [5] and from the a priori estimate given by [32].

2.3.1 Local Existence in H^1

Let the initial state $\psi^0 \in H^1$, then there exists $T^\star > 0$ and a unique solution $\psi(x,t) \in C([0, T^\star), H^1) \cap C^1([0, T^\star), H^{-1})$ of (1), where $T^\star = +\infty$ or $\|\nabla\psi\| \to +\infty$ as $t \to T^\star - 0$. Furthermore, the conservation of the norm and of the energy hold true for $t \in [0, T^\star]$:

$$\left\|\psi(\cdot,t)\right\| = \left\|\psi^0(\cdot)\right\|$$

and

$$\tilde{\mathcal{H}}\big(\psi(\cdot,t)\big) = \tilde{\mathcal{H}}\big(\psi^0(\cdot)\big)$$

where

$$\tilde{\mathcal{H}}(\psi) = \langle \psi, H_0\psi\rangle + \frac{\epsilon}{\sigma+1}\left\langle \psi^{\sigma+1}, g\psi^{\sigma+1}\right\rangle$$

represents the energy functional.

2.3.2 Global Existence

The solution ψ of (1) globally exists, that is $T^\star = +\infty$, provided that the state is initially prepared on the f rst N states of the linear problem, for any N fi ed, and $\hbar$ is small enough. Indeed, this fact immediately follows from a priori estimate of the norm of the gradient of the wavefunction [32].

Remark 5 The solution $\psi(x,t)$ globally exists for both positive and negative values of the parameter ϵ, provided that $\hbar$ is small enough and ϵ satisfie Hypotesis 3. That is, because of the scaling assumptions, blow-up effect cannot occur.

3 Stationary Solutions and Bifurcation

Since the beating period $T = \frac{2\pi\hbar}{\omega}$ plays the role of the unit of time it is convenient to introduce the *slow time*

$$\tau = \frac{\omega t}{\hbar},$$

then (1) takes the form (here $'$ denotes the derivative with respect to τ and where, with abuse of notation, $\psi = \psi(\tau, x)$)

$$i\omega\psi' = H_0\psi + \epsilon g|\psi|^{2\sigma}\psi, \quad \|\psi(\cdot,\tau)\| = 1. \tag{24}$$

In order to study the stationary solution we set

$$\psi(x,\tau) = e^{-i\lambda\tau/\omega}\psi(x), \quad \|\psi(\cdot)\| = 1, \ \lambda = \Omega + \omega E,$$

where

$$\Omega = \frac{1}{2}[\lambda_+ + \lambda_-].$$

As specifie in Remark 4 we restrict ourselves, for the sake of definiteness to the f rst couple of energy level $\lambda^1_\pm$, where we simply denote them by $\lambda_\pm$ dropping out the index 1; similarly $\varphi_\pm$ denote the associated eigenvectors and $\varphi_{R,L}$ the associated single-well states.

Hence, (24) takes the form

$$\lambda\psi = H_0\psi + \epsilon g|\psi|^{2\sigma}\psi, \quad \|\psi(\cdot)\| = 1. \tag{25}$$

Now, let us set

$$\psi(x) = a_R\varphi_R(x) + a_L\varphi_L(x) + \psi_c(x), \tag{26}$$

where

$$\psi_c(x) = \Pi_c\psi(x)$$

and

$$a_R = \langle\varphi_R, \psi\rangle \quad \text{and} \quad a_L = \langle\varphi_L, \psi\rangle$$

are unknown complex-valued values. Here,

$$\Pi_c = 1 - \Pi, \quad \Pi = \left[\langle\varphi_+,\cdot\rangle\varphi_+ + \langle\varphi_-,\cdot\rangle\varphi_-\right]$$

denotes the projection operator onto the eigenspace orthogonal to the bi-dimensional space associated to the doublet $\{\lambda_\pm\}$.

Since

$$\begin{aligned} H_0\psi &= a_R H_0\varphi_R + a_L H_0\varphi_L + H_0\psi_c \\ &= a_R[\Omega\varphi_R - \omega\varphi_L] + a_L[-\omega\varphi_R + \Omega\varphi_L] + H_0\psi_c \end{aligned} \tag{27}$$

then, by substituting (26) in (25) and projecting the resulting equation onto the one-dimensional spaces spanned by the *single-well* states φ_R and φ_L, and on the space $\Pi_c L^2(\mathbb{R}^d)$ it follows that (25) takes the form

$$\begin{cases} Ea_R = -a_L + r_R, & r_R = r_R(a_R, a_L, \psi_c) = \frac{\epsilon}{\omega}\langle \varphi_R, g|\psi|^{2\sigma}\psi\rangle, \\ Ea_L = -a_R + r_L, & r_L = r_L(a_R, a_L, \psi_c) = \frac{\epsilon}{\omega}\langle \varphi_L, g|\psi|^{2\sigma}\psi\rangle, \\ E\psi_c = \frac{1}{\omega}[H_0 - \Omega]\psi_c + r_c, & r_c = r_c(a_R, a_L, \psi_c) = \frac{\epsilon}{\omega}\Pi_c g|\psi|^{2\sigma}\psi \end{cases} \tag{28}$$

with the normalization condition

$$|a_R|^2 + |a_L|^2 + \|\psi_c\|^2 = 1.$$

Remark 6 Since (28) has stationary solutions (26) define up to a phase term then we can always assume, for the sake of definiteness that the stationary solution of (25) is such that a_L is a real-valued positive constant: $a_R \in \mathbb{C}$ and $a_L \in \mathbb{R}^+$. Furthermore, we remark that $[H_0, \mathcal{S}] = 0$ and $[g, \mathcal{S}] = 0$; hence, if ψ is a stationary solution of (25) associated to a given value λ, then $\mathcal{S}\psi$ is a solution associated to the same level, too.

Then, collecting the results from Lemmata 3 and 4 (and the associated remarks) by [32] we have the following.

Lemma 2 *Let ρ be the Agmon distance between the two wells defined as in (16). It follows that*

$$r_{R,L}(a_R, a_L, \psi_c) = r_{R,L}(a_R, a_L, 0) + r^c_{R,L}(a_R, a_L, \psi_c)$$

where

(i)

$$r_{R,L}(a_R, a_L, 0) = \frac{\epsilon}{\omega} C_{R,L}|a_{R,L}|^{2\sigma} a_{R,L} + \tilde{O}\left(e^{-\rho/\hbar}\right) \tag{29}$$

and

$$C_{R,L} = \langle \varphi_{R,L}, g|\varphi_{R,L}|^{2\sigma}\varphi_{R,L}\rangle = \left\langle \varphi^{\sigma+1}_{R,L}, g\varphi^{\sigma+1}_{R,L}\right\rangle = O\left(\hbar^{-d\sigma/2}\right); \tag{30}$$

by the symmetry assumptions it turn out that

$$C_R = C_L.$$

(ii) *The remainder terms are estimated as follow*

$$|r^c_{R,L}| \le \frac{\epsilon}{\omega} C\hbar^{-d\sigma/2}\|\psi_c\|^{\gamma}$$

where

$$\gamma = \begin{cases} 1 & \text{if } d = 1, 2, \\ 1 + (2-d)\gamma & \text{if } d > 2. \end{cases} \tag{31}$$

Here we come with the existence result of stationary states for the nonlinear Schrödinger equation (25).

Theorem 1 *Let*

$$a_R = pe^{i\theta}, \qquad a_L = q \quad \text{and} \quad z = p^2 - q^2 \tag{32}$$

where $p, q \in [0, 1]$ *and* $\theta \in [0, 2\pi)$. *Let* $\hbar \in (0, \hbar^\star)$, *where* $\hbar^\star$ *is small enough, let* ρ *be the Agmon distance between the two wells and let* η *be the effective nonlinearity defined by* (22). *Then the stationary problem* (28) *always has*

- *a* ***symmetric*** *solution* ψ_E^s *such that*

  $$\theta^s = 0, \qquad z^s = 0,$$

  associated to

  $$E := -1 + \eta \frac{1}{2^\sigma} + \tilde{O}\left(e^{-\rho\gamma/\hbar}\right),$$

- *an* ***antisymmetric*** *solution* ψ_E^a *such that*

  $$\theta^a = \pi, \qquad z^a = 0,$$

  associated to

  $$E := +1 + \eta \frac{1}{2^\sigma} + \tilde{O}\left(e^{-\rho\gamma/\hbar}\right).$$

Furthermore, in the case of negative (resp. positive) η, *then asymmetrical solution* ψ_E^{as} *corresponding to* $\theta^{as} = 0$ *(resp.* $\theta^{as} = \pi$*) may appear as a result of spontaneous symmetry bifurcation phenomenon. That is:*

- *for* $\sigma \le \sigma_{threshold}$ *the symmetric (resp. antisymmetric) state corresponding to* $z^s = 0$ *bifurcates showing a pitchfork bifurcation when the adimensional nonlinear parameter* $|\eta|$ *is larger than the critical value* $\eta^\star$ *given by (see Fig. 1, panel (a))*

  $$\eta^\star = \frac{2^\sigma}{\sigma},$$

- *for* $\sigma > \sigma_{threshold}$ *two couples of new* ***asymmetrical*** *stationary states appear as saddle-node bifurcations when* $|\eta|$ *is equal to a given value* η^+ *such that* $\eta^+ < \eta^\star$; *then, for increasing values of* $|\eta|$ *two branches of the solutions disappear at* $|\eta| = \eta^\star$ *showing a subcritical pitchfork bifurcation (see Fig. 1, panel (b)). The critical value* η^+ *is given by* $\eta(z^+)$ *where*

  $$\eta(z) = \frac{2z}{\sqrt{1-z^2}} \left[\left(\frac{1+z}{2}\right)^\sigma - \left(\frac{1-z}{2}\right)^\sigma \right]^{-1} \tag{33}$$

  and $z^+ \in (0, 1)$ *is the nonzero solution of the equation* $\eta'(z) = 0$.

In all the cases, the remainder term ψ_c *of the stationary solutions is such that*

$$\|\psi_c\|_{H^2} = \tilde{O}\left(e^{-\rho/\hbar}\right). \tag{34}$$

The critical value $\sigma_{threshold}$ *is given by*

$$\sigma_{threshold} = \frac{1}{2}[3 + \sqrt{13}]$$

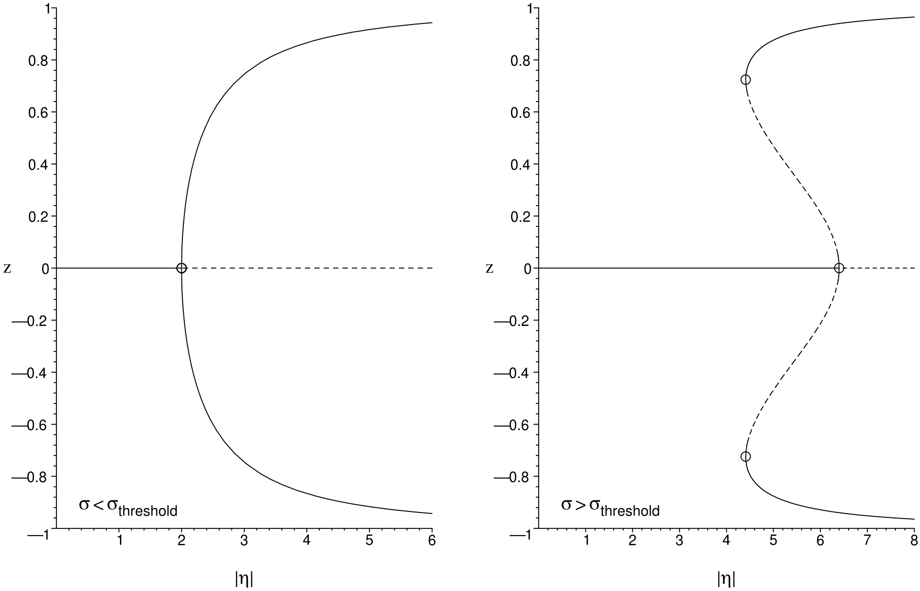


Fig. 1 In this figur we plot the graph of the stationary states of the nonlinear Schrödinger equation (25) as function of the nonlinearity parameter η for nonlinearity $\sigma = 1 < \sigma_{threshold}$ (panel (**a**)) and for nonlinearity $\sigma = 5 > \sigma_{threshold}$ (panel (**b**)); here $z = |a_R|^2 - |a_L|^2$ is the imbalance function. *Full lines* represent stable stationary states and *broken lines* represent unstable stationary states, where the notion of stability is referred to the dynamical stability associated to the Hamiltonian system given by the two-level approximation, as discussed in Sect. 4; and also to orbital stability, as discussed in Sect. 5 in the case of attractive nonlinear case (i.e. $\eta < 0$)

and it is an universal value in the sense that it does not depend on the shape of the double well potential as well as on the dimension d.

Remark 7 Because of the technical assumptions on σ, this critical value $\sigma_{threshold}$ makes sense for the nonlinear Schrödinger equation (25) only in dimensions 1 and 2. This is not the case when we restrict our analysis to the two-level approximation.

Remark 8 From Theorem 1 it appears that we have only two pictures, accordingly with the value of σ. In Fig. 1 (panel (a)) we consider the bifurcation scenario for the imbalance function $z = |a_R|^2 - |a_L|^2$ appearing when $\sigma \leq \sigma_{threshold}$. In Fig. 1 (panel (b)) we consider the bifurcation scenario appearing when $\sigma > \sigma_{threshold}$. The same picture has been previously obtained for the two-level approximation (see, e.g., [33]) where we have taken $\psi_c = 0$; in fact, ψ_c is exponentially small as proved in Theorem 1.

Remark 9 The stationary solutions $\psi := \psi_E$, associated to the level E, given in Theorem 1 are such that

$$\|\nabla \psi_E\| \leq C\sqrt{\Lambda} \tag{35}$$

and

$$\|\psi_E\|_p \leq C\Lambda^{d\frac{p-2}{4p}} \tag{36}$$

where p satisfie condition (12) and where

$$\Lambda = \frac{\mathcal{H}(\psi_E) - V_{min}}{\hbar^2} \sim \hbar^{-1}$$

and

$$\mathcal{H}(\psi) = \langle \psi, H_0 \psi \rangle + \frac{\epsilon}{\sigma + 1} \left\langle \psi^{\sigma+1}, g \psi^{\sigma+1} \right\rangle$$

is the energy functional define on $H^1(\mathbb{R}^d) \cap L^{2(\sigma+1)}(\mathbb{R}^d)$. Indeed, estimates (35) and (36) hold true for any vector ψ belonging to the space $\Pi(L^2)$ (see Theorem 2 in [32]). The results finall follow from this fact and since $\Pi_c \psi_E = \tilde{O}(e^{-\rho/\hbar})$.

3.1 Proof of Theorem 1

Here, we prove the existence of the stationary solutions by making use of the Lyapunov-Schmidt method and applying some results of the theory of numbers in order to count the number of stationary solutions of the equation coming from the two-level approximation. In this section, for argument's sake, we take $\eta > 0$; however, the same results still hold true also for $\eta < 0$.

Lemma 3 *We consider the following equation*

$$[H_0 - \Omega - \omega E]\psi_c + \epsilon \Pi_c g |\psi|^{2\sigma} \psi = 0, \tag{37}$$

where the nonlinearity power σ satisfies condition (21). For any fixed $C > 0$ let

$$D = \left\{ (a_R, a_L, E) \in \mathbb{C}^2 \times \mathbb{R} : |a_R|^2 + |a_L|^2 \leq 1,\ |\omega E| \leq C\hbar^2 \right\}.$$

There exists $\hbar^\star > 0$ small enough such that for any $\hbar \in (0, \hbar^\star)$ then there exists a unique solution $\psi_c \in H^2$ of (37) depending on a_R, a_L and E, and such that

$$\max_{(a_R, a_L, E) \in D} \|\psi_c\|_{H^2} = \tilde{O}\left(e^{-\rho/\hbar}\right), \quad \textit{as } \hbar \to 0. \tag{38}$$

Proof Recalling that

$$\psi = \varphi + \psi_c, \quad \text{where, } \varphi = a_R \varphi_R + a_L \varphi_L,$$

then (37) takes the form

$$\psi_c = F(\psi_c) \tag{39}$$

where

$$F(\psi_c) := F(\psi_c; a_R, a_L, E) = -\epsilon [H_0 - \Omega - \omega E]^{-1} \Pi_c g |\psi|^{2\sigma} \psi \tag{40}$$

and where

$$\left\| [H_0 - \Omega - \omega E]^{-1} \Pi_c \right\|_{\mathcal{L}(L^2 \to H^2)} \leq C_1 \hbar^{-1} \tag{41}$$

for some positive constant C_1 and for $\hbar$ small enough, since (9) and since $\omega E = O(\hbar^2)$. On the other side we have that

$$\left\| F(u) - F(v) \right\|_{H^2} \le \epsilon \frac{C_2}{\hbar} \left\| |f|^{2\sigma} f - |g|^{2\sigma} g \right\|$$
$$\le \epsilon \frac{C_2}{\hbar} \left\| \left(|f|^{2\sigma} + |g|^{2\sigma} \right) |f - g| \right\|$$
$$\le \epsilon \frac{C_2}{\hbar} \left(\|f\|_{H^1}^{2\sigma} + \|g\|_{H^1}^{2\sigma} \right) \|f - g\|_{H^1}$$

for some positive constant C_2, where we set

$$f = \varphi + u \quad \text{and} \quad g = \varphi + v,$$

with $|a_R|^2 + |a_L|^2 + \|u\|^2 = 1$, $|a_R|^2 + |a_L|^2 + \|v\|^2 = 1$. We have indeed made use of the Hölder inequality and of the Gagliardo-Nirenberg inequality with σ satisfying condition (21): if $2p\sigma < b$ and $2p/(p-2) < b$ where $b = +\infty$ if $d = 1, 2$ and $b = 2d/(d-2)$ if $d > 2$, i.e. σ satisfie (21). Finally, we get the wanted estimate

$$\left\| F(u) - F(v) \right\|_{H^2} \le \epsilon \frac{2^{2\sigma} C_2}{\hbar} \left\{ \max\left[\|\varphi + u\|_{H^2}, \|\varphi + v\|_{H^2} \right] \right\}^{2\sigma} \|u - v\|_{H^2} \tag{42}$$

provided that σ satisfie condition (21).

Now, let $C_3 = \max[C_1, 2^{2\sigma} C_2]$ and let

$$K = \left\{ u \in H^2 : \|u\|_{H^2} \le c(\hbar) \right\}, \quad c(\hbar) = \max\left\{ \left[\frac{\hbar}{2^{2\sigma+2} 3 C_3 \epsilon} \right]^{1/2\sigma}, \|\varphi\|_{H_2} \right\}.$$

Since $\|\varphi\|_{H_2} = O(\hbar^{-1})$, by Lemma 1, and $\epsilon = \tilde{O}(e^{-\rho/\hbar})$ then $c(\hbar) = [\frac{\hbar}{2^{2\sigma+2} 3 C_3 \epsilon}]^{1/2\sigma}$.

Then F is an operator from K to K; indeed, from (41) and (42) it follows that

$$\left\| F(u) \right\|_{H^2} \le \epsilon C_3 \hbar^{-1} \|u + \varphi\|_{H_2}^{2\sigma+1} \le \left[2\epsilon C_3 \hbar^{-1} (2c)^{2\sigma} \right] c(\hbar) = \frac{1}{2} c(\hbar) < c(\hbar).$$

Moreover, $F(u)$ is a contraction in K:

$$\left\| F(u) - F(v) \right\|_{H^2} \le C_3 \epsilon \hbar^{-1} \left[2c(\hbar) \right]^{2\sigma} \|u - v\|_{H^2} < \frac{1}{4} \|u - v\|_{H^2}.$$

Hence, equation

$$F(u) = u$$

admits a unique solution ψ_c in K for any $(a_R, a_L, E) \in D$ and any ϵ satisfying Hypotesis 3. This solution is given by the limit of the following sequence $\{u_n\}_{n=0}^{\infty}$ where

$$u_0 = 0 \quad \text{and} \quad u_{n+1} = F(u_n).$$

In particular (the convergence is in H^2)

$$\psi_c = \lim_{n \to +\infty} u_n = \sum_{j=1}^{+\infty} [u_{j+1} - u_j] = \sum_{j=1}^{+\infty} \left[F(u_j) - F(u_{j-1}) \right].$$

Since

$$\left\| F(u_{j+1}) - F(u_j) \right\|_{H^2} \le C_3 \epsilon \hbar^{-1} \left[2c(\hbar)\right]^{2\sigma} \left\| F(u_j) - F(u_{j-1}) \right\|_{H^2}$$
$$\le \left[C_3 \epsilon \hbar^{-1} \left[2c(\hbar)\right]^{2\sigma}\right]^{j+1} \left\| F(u_0) \right\|_{H^2}$$

then we have that

$$\begin{aligned} \|\psi_c\|_{H^2} &\le \frac{1}{1 - C_2 \epsilon \hbar^{-1} [2c(\hbar)]^{2\sigma}} \left\| F(u_0) \right\|_{H^2} \\ &\le \frac{1}{1 - C_3 \epsilon \hbar^{-1} [2c(\hbar)]^{2\sigma}} C_3 \epsilon \hbar^{-1} \|a_R \varphi_R + a_L \varphi_L\|_{H^2}^{2\sigma+1} \\ &= \tilde{O}\left(e^{-\rho/\hbar}\right). \end{aligned} \tag{43}$$

Since the constants C_1 and C_2 depend on a_R, a_L and E in such a way that

$$\max_{|\omega E| \le C\hbar^2} C_1 < +\infty$$

and

$$\max_{|a_R|^2 + |a_L|^2 \le 1} C_2 < +\infty$$

then the estimate (43) uniformly holds true on the set D. □

Remark 10 By means of the same arguments it follows that $\psi_c \in H^2$, as function on a_R, a_L and E, admits the f rst derivatives and in particular these derivatives satisfy estimate (38) in the sense that

$$\max_{(a_R, a_L, E) \in D} \left[\left\| \frac{\partial \psi_c}{\partial E} \right\|_{H^2}, \left\| \frac{\partial \psi_c}{\partial a_R} \right\|_{H^2}, \left\| \frac{\partial \psi_c}{\partial a_L} \right\|_{H^2} \right] = \tilde{O}\left(e^{-\rho/\hbar}\right), \quad \text{as } \hbar \to 0. \tag{44}$$

We can also give an estimate of the dependence of ψ_c on the parameter ϵ; this estimate will be given in Lemma 7.

Now, setting $\psi_c = \psi_c(a_R, a_L, E)$ in (28), let any $0 < \rho' < \rho$ fi ed, let

$$\nu = e^{-\rho' \gamma / \hbar} \tag{45}$$

where γ is define in (31), and making use of Lemma 2, then (28) takes the form

$$\begin{cases} E a_R = -a_L + \eta |a_R|^{2\sigma} a_R + \nu f_R(a_R, a_L, E), \\ E a_L = -a_R + \eta |a_L|^{2\sigma} a_L + \nu f_L(a_R, a_L, E), \\ \quad 1 = |a_R|^2 + |a_L|^2 \qquad\quad + \nu f_c(a_R, a_L, E), \end{cases} \tag{46}$$

where f_R, f_L and f_c are uniformly bounded on D with their f rst derivatives. Since Lemma 3 and Remark 10, and recalling that $\epsilon/\omega = \eta / \langle \varphi_R^{\sigma+1}, g \varphi_R^{\sigma+1} \rangle = O(\hbar^{-d\sigma/2})$. From (32) then (46) takes the form

$$\begin{cases} Ep = -q e^{-i\theta} + \eta p^{2\sigma+1} + \nu e^{-i\theta} f_R, \\ Eq = -p e^{i\theta} + \eta q^{2\sigma+1} + \nu f_L, \\ \quad 1 = p^2 \qquad + q^2 \qquad + \nu f_c. \end{cases}$$

By taking the real and imaginary part of the previous equations we obtain the following system

$$G(p,q,E,\theta;\nu)=0 \tag{47}$$

on

$$D'=\left\{(p,q,E,\theta)\in[0,1]^2\times\mathbb{R}\times[0,2\pi): p^2+q^2\leq 1,\ |\omega E|\leq C\hbar^2\right\}$$

and where $G=(G_1,G_2,G_3,G_4)$ are given by

$$\begin{aligned}
G_1 &= E-\frac{1}{1-\nu f_c}\left[-2pq\cos\theta+\eta\left(p^{2\sigma+2}+q^{2\sigma+2}\right)+\nu\Re\left(pe^{-i\theta}f_R+qf_L\right)\right]\\
&= E+2pq\cos\theta-\eta\left(p^{2\sigma+2}+q^{2\sigma+2}\right)+\nu f_1,\\
G_2 &= \left(p^2+q^2\right)\sin\theta+\nu\Im\left(e^{-i\theta}f_R-pf_L\right)=\left(p^2+q^2\right)\sin\theta+\nu f_2,\\
G_3 &= \left(p^2-q^2\right)\cos\theta+\eta pq\left(p^{2\sigma}-q^{2\sigma}\right)+\nu\Re\left(qe^{-i\theta}f_R-pf_L\right)\\
&= \left(p^2-q^2\right)\cos\theta+\eta pq\left(p^{2\sigma}-q^{2\sigma}\right)+\nu f_3,\\
G_4 &= p^2+q^2+\nu f_c-1=p^2+q^2-1+\nu f_4,
\end{aligned}$$

where f_j, $j=1,2,3,4$, are uniformly bounded on the set D' with their first derivatives.

From equations $G_2=0$ and $G_4=0$ we obtain that

$$p^2+q^2=1+O(\nu)\quad\text{and}\quad\theta=O(\nu),\ \theta=\pi+O(\nu).$$

From this fact and from equations $G_1=0$ and $G_3=0$ we finally obtain the equations

$$G_\pm+O(\nu)=0, \tag{48}$$

$$E_\pm=\mp 2pq+\eta\left(p^{2\sigma+2}+q^{2\sigma+2}\right)+O(\nu) \tag{49}$$

where the asymptotics is uniformly on D', the index $+$ corresponds to the choice $\theta=O(\nu)$, the index $-$ corresponds to the choice $\theta=\pi+O(\nu)$ and where

$$G_\pm=\pm\left[\left(p^2-q^2\right)\pm\eta pq\left(p^{2\sigma}-q^{2\sigma}\right)\right].$$

The imbalance function $z=p^2-q^2$ is such that

$$p=\sqrt{\frac{1+z}{2}}+O(\nu)\quad\text{and}\quad q=\sqrt{\frac{1-z}{2}}+O(\nu)$$

and thus (48) and (49) take the form

$$f_\pm(z,\eta)+O(\nu)=0, \tag{50}$$

$$E_\pm=\mp\sqrt{1-z^2}+\eta\left[\left(\frac{1+z}{2}\right)^{\sigma+1}+\left(\frac{1-z}{2}\right)^{\sigma+1}\right]+O(\nu) \tag{51}$$

where

$$f_\pm(z,\eta) = z \pm \eta \frac{\sqrt{1-z^2}}{2}\left[\left(\frac{1+z}{2}\right)^\sigma - \left(\frac{1-z}{2}\right)^\sigma\right]. \tag{52}$$

Since the asymptotic term $O(\nu)$ in (50), with its derivative with respect to z, is uniform with respect to $z \in [-1,+1]$ then it is enough to look for the solutions of equations $f_\pm(z,\eta) = 0$.

Of course, equation

$$f_\pm(0,\eta) = 0$$

holds true for any η; that is the symmetric stationary solution ($z = 0, \theta = 0$) which is positive and the antisymmetric stationary solution ($z = 0, \theta = \pi$) exist for the nonlinear problem (up to an exponentially small perturbation) as well as for the linear one.

Since we have assumed, for the sake of definiteness $\eta > 0$; then equation $f_+(z,\eta) = 0$ does not have nonzero solutions, indeed the derivative of f_+ with respect to z is given by

$$f_+'(z,\eta) = 2\frac{1+z^2}{[1-z^2]^{3/2}} + \frac{1}{2}\eta\sigma\left[\left(\frac{1+z}{2}\right)^{\sigma-1} + \left(\frac{1-z}{2}\right)^{\sigma-1}\right]$$

which is always positive for any $z \in [-1,+1]$ and for any $\eta > 0$. Thus, we have only to look for the nonzero solutions z of equation

$$f_-(z,\eta) = 0. \tag{53}$$

If $\eta < 0$ then there is an exchange between f_+ and f_- and the same arguments apply.

In order to compute the solutions of (53), we consider the function $\eta(z)$, def ned by (33), which satisfie the implicit equation

$$f_-\big[z,\eta(z)\big] = 0, \quad \forall z \in (0,1).$$

Thus, the inverse function $z = z(\eta)$ of $\eta(z)$ gives the solutions of (53); in order to count the branches of the inverse function $z = z(\eta)$ we compute the firs derivative

$$\eta'(z) = 2^{\sigma+1}\frac{g(z)-g(-z)}{[1-z^2]^{3/2}[(1+z)^\sigma-(1-z)^\sigma]^2},$$

where

$$g(z) = \big(\sigma z^2 - \sigma z + 1\big)(1+z)^\sigma.$$

Since

$$\lim_{z\to 0^+} \eta'(z) = 0$$

then a bifurcation of the stationary solution occurs at $z = 0$ for

$$\eta^\star = \lim_{z\to 0^+} \eta(z) = 2^\sigma/\sigma.$$

Furthermore, a straightforward calculation gives also that

$$\lim_{z\to 0^+} \eta''(z) = -\frac{2^\sigma}{3\sigma}\left(\sigma^2 - 3\sigma - 1\right)$$

and

$$\lim_{z\to 0} \eta''(z) \begin{cases} > 0 & \text{if } \sigma < \sigma_{threshold}, \\ = 0 & \text{if } \sigma = \sigma_{threshold}, \\ < 0 & \text{if } \sigma > \sigma_{threshold}, \end{cases} \tag{54}$$

where

$$\sigma_{threshold} = \frac{3+\sqrt{13}}{2}.$$

Hence, we can conclude that in the case $\sigma \leq \sigma_{threshold}$ then we have a supercritical pitchfork bifurcation at $z = 0$ (see Fig. 1, panel (a)), and for $\sigma > \sigma_{threshold}$ then we have a subcritical pitchfork bifurcation at $z = 0$ (see Fig. 1, panel (b)).

Finally, we only have to count the number of branches of the function $z(\eta)$ and thus we look for the number N of the solutions (counting multiplicity) of the equation

$$h(z) = 0, \quad h(z) = g(z) - g(-z), \tag{55}$$

for z in the interval $z \in (-1, +1)$.

Lemma 4 *Let N be the number of solutions z of the equation $h(z) = 0$ in the interval $[-1, +1]$, counting multiplicity. It follows that $z = 0$ is a solution with multiplicity 3 if $\sigma \neq \sigma_{threshold}$, and with multiplicity 5 if $\sigma = \sigma_{threshold}$. Furthermore, it also follows that*

$$N = \begin{cases} 3 & \text{if } \sigma < \sigma_{threshold}, \\ 5 & \text{if } \sigma \geq \sigma_{threshold}. \end{cases} \tag{56}$$

Proof We may remark that if $z^\star$ is such that $h(z^\star) = 0$ then $h(-z^\star) = 0$, too; furthermore $h(\pm 1) = \pm 2^\sigma \neq 0$. First of all we see that $z = 0$ is a solution of (55) with multiplicity 3 for any $\sigma \neq \sigma_{threshold}$; indeed, a straightforward calculation gives that

$$h(0) = h'(0) = h''(0) = 0 \quad \text{and} \quad h'''(0) = 4\sigma\left(-\sigma^2 + 3\sigma + 1\right).$$

Then $h'''(0) \neq 0$ if $\sigma \neq \sigma_{threshold}$. If $\sigma = \sigma_{threshold}$ then a straightforward calculation gives that $h'''(0) = h^{IV}(0) = 0$ and

$$\begin{aligned} h^V(0) &= -86\left(\sigma^4_{threshold} - 10\sigma^3_{threshold} + 20\sigma^2_{threshold} - 5\sigma_{threshold} - 6\right) \\ &= 24(3+\sqrt{13})(4+\sqrt{13}) > 0. \end{aligned}$$

Hence, it follows that

$$N \text{ is } \begin{cases} \geq 5 & \text{if } \sigma > \sigma_{threshold}, \\ = 5 \text{ or } \geq 9 & \text{if } \sigma = \sigma_{threshold}, \\ = 3 \text{ or } \geq 7 & \text{if } \sigma < \sigma_{threshold}, \end{cases}$$

where N is number of solutions, counting multiplicity, of equation $f_-(z, \eta) = 0$.

Indeed, we see that

$$\lim_{z\to\pm 1}\eta(z) = +\infty.$$

Then, in the case $\sigma > \sigma_{threshold}$ since $\lim_{z\to 0}\eta''(z) < 0$ then there exists two nonzero solutions of (55) in the interval $(-1,+1)$ at least; hence, the number N of solutions of (55), counting multiplicity, is $N \geq 5$.

In the opposite case $\sigma < \sigma_{threshold}$ it follows $\lim_{z\to 0}\eta''(z) > 0$, then we have two cases: either (55) does not have solutions $z \in (-1,+1)$, $z \neq 0$, and in this case $N = 3$; or (55), counting multiplicity, has other solutions $z \in (-1,+1)$, $z \neq 0$, and in this case the number of such a solutions is bigger than 4, thus $N \geq 7$.

Finally, in the case $\sigma = \sigma_{threshold}$ it follows that $\lim_{z\to 0}\eta''(z) = \lim_{z\to 0}\eta'''(z) = 0$ and

$$\lim_{z\to 0}\eta^{IV}(z) = \frac{6\cdot 2^{\sigma_{threshold}}(829\sqrt{13}+2989)}{5(649+180\sqrt{13})} > 0,$$

hence $N = 5$ or $N \geq 9$.

If we can prove $N \leq 5$ for any $\sigma > 0$ then the Lemma is completely proved.

To this end we set

$$y = \frac{1-z}{1+z}, \quad y \in (0,+\infty).$$

Hence, equation $h(z) = 0$ in the interval $(-1,1)$ reduces to the equation of the form $p_\sigma(y) = 0$ where

$$\begin{aligned} p_\sigma(y) &= y^\sigma\left(y^2+by+a\right) - \left(ay^2+by+1\right) \\ &= y^{\sigma+2} + by^{\sigma+1} + ay^\sigma - ay^2 - by - 1 \end{aligned}$$

and where

$$a = 1+2\sigma, \qquad b = 2-2\sigma.$$

We remark that if $N := N(p_\sigma)$ is the number of roots of the function $p_\sigma(y)$ for $y \in (0,+\infty)$ then the classical Rolle Theorem implies that

$$N(p_\sigma) \leq N\left(\frac{dp_\sigma}{dy}\right) + 1. \tag{57}$$

Hence

$$N := N(p_\sigma) \leq N\left(\frac{dp_\sigma}{dy}\right) + 1 \leq N\left(\frac{d^2p_\sigma}{dy^2}\right) + 2 \leq N\left(\frac{d^3p_\sigma}{dy^3}\right) + 3 \tag{58}$$

and thus we have only to estimate $N(\frac{d^3p_\sigma}{dy^3})$. Since

$$\frac{d^3p_\sigma}{dy^3} = \sigma y^{\sigma-3}\left[(\sigma+2)(\sigma+1)y^2 + b(\sigma+1)(\sigma-1)y + a(\sigma-2)(\sigma-1)\right]$$

then we can conclude that

$$N\left(\frac{d^3p_\sigma}{dy^3}\right) \leq 2,$$

from which

$$N := N(p_\sigma) \leq 5 \tag{59}$$

follows. □

In fact, we have proved, by means of perturbative techniques, that the stationary solutions (both symmetric and antisymmetrical) are such that

$$\theta^s = \tilde{O}\left(e^{-\rho\gamma/\hbar}\right), \qquad z^s = \tilde{O}\left(e^{-\rho\gamma/\hbar}\right), \tag{60}$$

and

$$\theta^a = \pi + \tilde{O}\left(e^{-\rho\gamma/\hbar}\right), \qquad z^a = \tilde{O}\left(e^{-\rho\gamma/\hbar}\right).$$

Similarly we obtained that the asymmetrical stationary solutions are such that

$$\theta^{as} = \tilde{O}\left(e^{-\rho\gamma/\hbar}\right) \quad \text{or} \quad \theta^{as} = \pi + \tilde{O}\left(e^{-\rho\gamma/\hbar}\right).$$

By means of symmetric properties we are able to prove now that such exponentially small errors are exactly zero. Indeed, concerning the symmetric solution $\psi_E^s = a_R^s\varphi_R + a_L^s\varphi_L + \psi_c$ we remark that the corresponding level E is nondegenerate in the sense that we have only this stationary solution corresponding to such value of E. On the other side, by means of a symmetrical argument, then $\mathcal{S}\psi_E^s = a_R^s\varphi_L + a_L^s\varphi_R + \mathcal{S}\psi_c$ is a solution associated to same level E, too. Hence, ψ_E^s and $\mathcal{S}\psi_E^s$ coincide, up to a phase factor. From this fact and from (60) it turns out that θ^s and z^s are **exactly** zero:

$$\theta^s = 0 \quad \text{and} \quad z^s = 0.$$

Similarly, it follows that

$$\theta^a = \pi \quad \text{and} \quad z^a = 0$$

and

$$\theta^{as} = 0 \quad \left(\text{respectively } \theta^{as} = \pi\right)$$

for negative value of η (resp. for positive value of η).

The proof of the theorem is so completed.

Remark 11 By means of a similar argument applied in the fina part of the proof of Theorem 1 we can also conclude that the stationary solution is, up to a phase term, a real valued function; indeed if ψ is a solution associated to a given level E, then $\bar{\psi}$ is a solution associated to the same value E, too.

Remark 12 From Lemma 4 it turns out that when $\sigma \leq \sigma_{threshold}$ then equation $\eta'(z) = 0$ has only solution $z = 0$ and therefore, under such condition on σ, we only observe a bifurcation of the stationary solution at $|\eta| = \eta^\star$. On the other side, when $\sigma > \sigma_{threshold}$ then the number of solutions (counting multiplicity) of equation $\eta'(z) = 0$ is 5, since the solution $z = 0$ has multiplicity 3 then the other 2 solutions are $\pm z^+$, where $z^+ \in (0, 1)$, and they are associated to saddle points appearing at $|\eta| = \eta^+$, where $\eta^+ = \eta(z^+)$.

Remark 13 We just point out that in the case of $\eta < 0$ then we can apply the same arguments; we only have to emphasize that for negative values of η then equation $f_-(z,\eta) = 0$ does not have nonzero solutions and that bifurcations come from equation $f_+(z,\eta) = 0$.

Remark 14 For large σ the roots $y < 1$ of the polynomial $p_\sigma(y)$ are asymptotically given by the roots of equation

$$(1+2\sigma)y^2 + (2-2\sigma)y + 1 = 0.$$

That is

$$y \sim \frac{1}{1+2\sigma} \quad \text{for } \sigma \gg 1.$$

Hence, the solution z^+ of equation $\eta'(z) = 0$ is asymptotically given by

$$z^+ \sim 1 - \frac{1}{\sigma} - \frac{1}{\sigma^2}$$

and we have that

$$\eta^+ = \sqrt{2e\sigma}\left[1 + O\left(\sigma^{-1}\right)\right]$$

in the limit of large σ.

Remark 15 The frequency λ of stationary solutions of (25) are thus given by

$$\lambda = \Omega + \omega E$$

where $E = E(z)$ is the multivalued function given by (51), where $z = z(\eta)$ are the roots of the equation $f_\pm(z) = 0$. For the graph of the functions $E(z)$, depending on η, we refer to Fig. 2, Fig. 3 and Fig. 4. We observe the following behaviors (where we assume $\eta < 0$ for argument's sake):

- When $-\eta^\star < \eta < 0$ for $\sigma \leq \sigma_{threshold}$, or $-\eta^+ < \eta < 0$ for $\sigma > \sigma_{threshold}$, then we only have the linear stationary states.
- When $\eta < -\eta^\star$ and $\sigma \leq \sigma_{threshold}$, then the symmetric solution bifurcates at $\eta = -\eta^\star$ and then we have 4 stationary solutions: the two linear stationary states and two new asymmetrical stationary states; a similar picture actually occurs also when $\sigma > \sigma_{threshold}$, but in this case the two new asymmetrical stationary solutions don't come by a bifurcation of the symmetric stationary solution, but they come from a branch of saddle points.
- When $-\eta^\star < \eta < -\eta^+$ and $\sigma > \sigma_{threshold}$, then a couple of saddle points occurs and thus we have 4 asymmetrical stationary solutions. Two of them, denoted as (as1), are much more localized on a single well than the ones denoted by (as2).

4 Dynamical Stability

The time-dependent equation (24), when projected on the one-dimensional spaces spanned by the single-well states φ_R and φ_L, and on the space $\Pi_c L^2(\mathbb{R}^d)$, takes the form

$$\begin{cases} i a_R' = -a_L + r_R, \\ i a_L' = -a_R + r_L, \\ i\psi_c' = \frac{1}{\omega}[H_0 - \Omega]\psi_c + r_c, \end{cases} \tag{61}$$

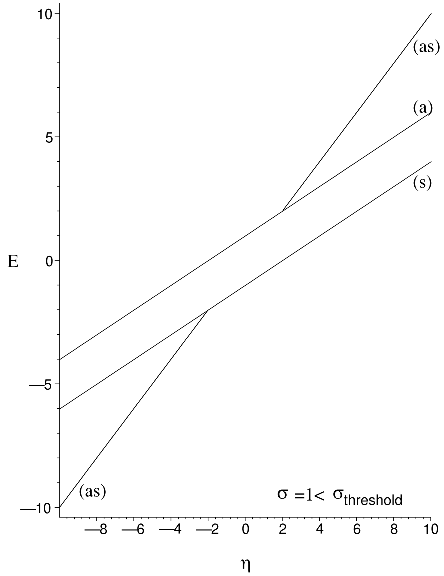


Fig. 2 In this figur we plot the graph of the values of the function E versus the nonlinearity parameter η for nonlinearity $\sigma = 1 < \sigma_{threshold}$. For $\eta = \pm\eta^\star$, $\eta^\star = 2$ for $\sigma = 1$, a bifurcation occurs and a new branch corresponding to the asymmetrical stationary state appears. *Line* (s) denotes the symmetric stationary solutions, *line* (a) denotes the antisymmetric stationary solutions, and (as) denote the asymmetrical stationary solutions

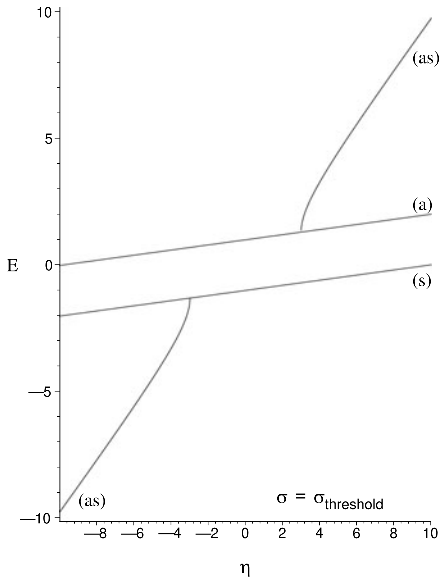
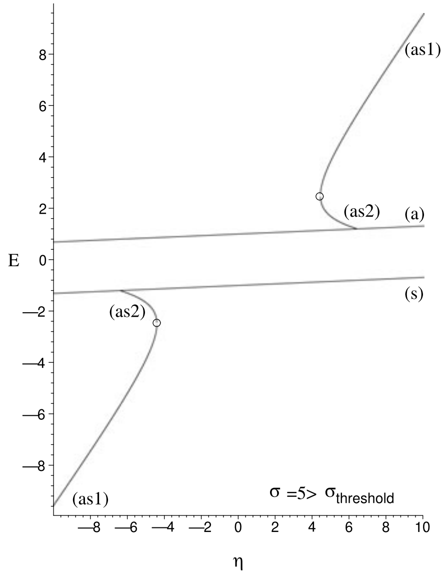


Fig. 3 In this figur we plot the graph of the values of E as function of the nonlinearity parameter η for critical nonlinearity $\sigma = \sigma_{threshold}$

where we have set $\psi \to e^{-i\Omega\tau/\omega}\psi(x,\tau)$. We call *two-level approximation* the system of differential equations coming from (61) taking $\psi_c = 0$ and neglecting the exponential remainder term in $r_{R,L}(a_R, a_L, 0)$ (see Lemma 2); in such a case the two-level approximation takes the form

$$\begin{cases} ia_R' = -a_L + \eta|a_R|^{2\sigma}a_R, \\ ia_L' = -a_R + \eta|a_L|^{2\sigma}a_L, \end{cases} \qquad |a_R|^2 + |a_L|^2 = 1. \tag{62}$$

Fig. 4 In this figur we plot the graph of the values of the function E versus the nonlinearity parameter η for nonlinearity $\sigma = 5 > \sigma_{threshold}$. At $|\eta| = \eta^+$, $\eta^+ \approx 4.41$ for $\sigma = 5$, a couple of saddle nodes appear, and the corresponding branches, denoted (*as1*) and (*as2*), are associated to asymmetrical stationary solutions; asymmetrical solution (*as2*) then disappears at $|\eta| = \eta^\star$, $\eta^\star = 6.4$ for $\sigma = 5$



We may remark that the two-level system (62) takes the Hamiltonian form

$$iA' = \partial_{\bar{A}}\mathcal{H}, \qquad A = (a_R, a_L),$$

with Hamiltonian function

$$\mathcal{H} = -\left[(\bar{a}_R a_L + \bar{a}_L a_R) - \frac{\eta}{\sigma+1}\left(|a_R|^{2(\sigma+1)} + |a_L|^{2(\sigma+1)}\right) \right] \tag{63}$$

corresponding to the energy functional restricted to the two-dimensional space spanned by the two single-well states. The stationary solutions of the two-level system (62) are associated to stationary points of the energy functional $\mathcal{H}$, then we can attribute them some stability/instability properties in the sense of the theory of dynamical system. In particular, let $\theta = \arg(a_R) - \arg(a_L)$ be the difference between the phases of a_R and a_L, and let $z = |a_R|^2 - |a_L|^2$ be the imbalance function, then system (62) takes the Hamiltonian form

$$\begin{cases} \dot{\theta} = \partial_z \mathcal{H}, \\ \dot{z} = -\partial_\theta \mathcal{H}, \end{cases} \tag{64}$$

where the Hamiltonian (63) takes now the form

$$\mathcal{H} = -\sqrt{1-z^2}\cos\theta + \frac{\eta}{\sigma+1}\left[\left(\frac{1+z}{2}\right)^{\sigma+1} + \left(\frac{1-z}{2}\right)^{\sigma+1} \right].$$

In order to study the stability properties of the stationary solutions of (64) we have to consider the matrix

$$Hess = \begin{pmatrix} \frac{\partial^2 \mathcal{H}}{\partial z \partial \theta} & \frac{\partial^2 \mathcal{H}}{\partial z^2} \\ -\frac{\partial^2 \mathcal{H}}{\partial \theta^2} & -\frac{\partial^2 \mathcal{H}}{\partial \theta \partial z} \end{pmatrix}$$

at the stationary points. Since the trace of $Hess$ is zero then we have that the stationary point is a circle if $\det Hess > 0$, and it is a saddle point if $\det Hess < 0$.

4.1 Dynamical Stability of the Symmetric and Antisymmetric Stationary States

We consider, at f rst, the symmetric and antisymmetric stationary states corresponding to $\theta = 0$ and $z = 0$ (symmetric), and $\theta = \pi$ and $z = 0$ (antisymmetric). A straightforward calculation gives that

$$\det Hess|_{\theta=0,\, z=0} = 1 + \eta \frac{\sigma}{2^\sigma} \quad \text{and} \quad \det Hess|_{\theta=\pi,\, z=0} = 1 - \eta \frac{\sigma}{2^\sigma}.$$

Then, it follows that the symmetric stationary solution is dynamically stable for any $\eta > -\eta^\star$, and the antisymmetric stationary solution is dynamically stable for any $\eta < \eta^\star$, where $\eta^\star = 2^\sigma/\sigma$.

4.2 Dynamical Stability of the Asymmetrical Stationary Solutions

For argument's sake let us assume $\eta < 0$. Then the symmetric stationary solution bifurcates and new asymmetrical solutions appear, they correspond to $\theta = 0$ and the values of z are the nonzero solutions of the equation $f_+(z, \eta) = 0$ (in fact, we have assumed $\eta < 0$; in the case of $\eta > 0$, as considered in Sect. 3 for the sake of definiteness then the stationary solutions corresponds to the roots z of equation $f_-(z, \eta) = 0$). A straightforward calculation gives that

$$\det Hess|_{\theta=0} = \sqrt{1-z^2}\left[\left(1-z^2\right)^{-3/2} + \frac{\eta\sigma}{4}\left(\left(\frac{1+z}{2}\right)^{\sigma-1} + \left(\frac{1-z}{2}\right)^{\sigma-1}\right)\right].$$

By the relation $\eta = \eta(z)$ implicitly define by the equation $f_+(z, \eta) = 0$ it follows that

$$\det Hess|_{\theta=0,\, \eta=\eta(z)} = \frac{g(z) - g(-z)}{(1-z^2)[(1+z)^\sigma - (1-z)^\sigma]}$$

where it has been already proved that the equation $g(z) - g(-z) = 0$ has a solution at $z = 0$ with multiplicity 3 (multiplicity 5 if $\sigma = \sigma_{threshold}$). Since this equation has no other solution for $\sigma \le \sigma_{threshold}$, since $q(z) = q(-z)$ and since

$$\lim_{z \to 1^-} \det Hess|_{\theta=0,\, \eta=\eta(z)} = +\infty$$

then

$$\det Hess|_{\theta=0,\, \eta=\eta(z)} > 0, \quad \forall z \neq 0.$$

Then, the asymmetrical solutions, if there, are stable. On the other side, for $\sigma > \sigma_{threshold}$ then the equation $g(z) - g(-z) = 0$ has three distinct solutions; hence, by means of the same arguments as before, it follows that the branch $(as2)$ is dynamically unstable and the branch $(as1)$ is dynamically stable.

We can collect all these results as follows (see also Fig. 1).

Theorem 2 *Let us consider the stationary solutions of the* ***two level approximation*** (62) *that coincide, up to an exponentially small term, with the solutions given in Theorem* 1. *The symmetric and antisymmetric solutions of the two-level approximation are such that*:

– *for any* $\sigma > 0$, *the symmetric stationary solution* (s) *is stable for any* $\eta \geq -\eta^\star$, *and it is unstable for any* $\eta < -\eta^\star$;
– *for any* $\sigma > 0$, *the antisymmetric stationary solution* (a) *is stable for any* $\eta \leq \eta^\star$, *and it is unstable for any* $\eta > \eta^\star$.

The asymmetrical solutions of the two-level approximation are such that:

– *for any* $\sigma \leq \sigma_{threshold}$ *the asymmetrical stationary solution* (as) *is stable*;
– *for any* $\sigma > \sigma_{threshold}$ *the branch* $(as2)$ *of the asymmetrical stationary solution there exists for any* $\eta^+ < |\eta| < \eta^\star$ *and it is unstable, the other branch* $(as1)$ *of the asymmetrical stationary solution there exists for any* $\eta^+ < |\eta|$ *and it is stable.*

5 Orbital Stability

In this section our aim is to study the orbital stability of the stationary solutions of the NLS (1). So far we have considered both cases of attractive and repulsive nonlinearity for any couple of eigenvalues $\lambda_\pm$. Hereafter we consider only the firs two eigenvalues and we assume to be in the attractive nonlinearity, that is:

Hypothesis 4 *Let* $\lambda_\pm$ *be the* ***first two eigenvalues*** *of* H_0. *Let* $\eta = \frac{\epsilon}{\omega}\langle \varphi_R^{\sigma+1}, g\varphi_R^{\sigma+1}\rangle$ *be the effective nonlinearity parameter in* (14) *where* $\langle \varphi_R^{\sigma+1}, g\varphi_R^{\sigma+1}\rangle > 0$; *we assume that*

$$\epsilon < 0 \quad \textit{that is} \quad \eta < 0.$$

If we rescale the solution ψ as $\phi = |\epsilon|^{1/2\sigma}\psi$, then (1) is equivalent to the equation

$$i\hbar\frac{\partial \phi}{\partial t} = H_0\phi - g|\phi|^{2\sigma}\phi, \quad \|\phi\| = |\epsilon|^{1/2\sigma}. \tag{65}$$

The stationary solutions of the equation

$$H_0\phi_{\lambda,\epsilon} - g|\phi_{\lambda,\epsilon}|^{2\sigma}\phi_{\lambda,\epsilon} - \lambda\phi_{\lambda,\epsilon} = 0, \quad \lambda = \Omega + \omega E, \tag{66}$$

are associated, by means of the scaling, to the stationary solutions ψ_E^s, ψ_E^a and ψ_E^{as} given in Theorem 1 where $E = E(\epsilon)$ is a multivalued function and where the stationary solutions are now denoted by

$\phi^s_{\lambda,\epsilon}$: symmetric stationary solution,
$\phi^a_{\lambda,\epsilon}$: antisymmetric stationary solution,
$\phi^{as}_{\lambda,\epsilon}$: asymmetrical stationary solution for $\sigma \leq \sigma_{threshold}$,
$\phi^{as1}_{\lambda,\epsilon}$ and $\phi^{as2}_{\lambda,\epsilon}$: asymmetrical stationary solutions for $\sigma > \sigma_{threshold}$.

If we consider a general stationary state, we denote the solution by $\phi_{\lambda,\epsilon}$ and ψ_E, but if we want to distinguish the branches, we insist, in such above way, by denoting s, a, as, $as1$ and $as2$, on each shoulder of solutions.

Here, we consider the orbital stability for the symmetric stationary solution $\phi^s_{\lambda,\epsilon}$ and for the asymmetrical stationary solutions $\phi^{as}_{\lambda,\epsilon}$ that bifurcate from the symmetric one.

Definition 1 The family of nonlinear bound states $\{e^{i\alpha}\phi_{\lambda,\epsilon}, \alpha \in \mathbb{R}\}$ is said to be orbitally stable in $H^1(\mathbb{R}^d)$ if for any $\kappa > 0$ there exists a $\delta > 0$ such that if ϕ_0 satisfie

$$\inf_{\alpha\in\mathbb{R}}\left\|\phi_0 - e^{i\alpha}\phi_{\lambda,\epsilon}\right\|_{H^1} < \delta, \tag{67}$$

then for all $t \geq 0$, the solution $\phi(t)$ of (65) with $\phi(0) = \phi_0$ exists and satisfie

$$\inf_{\alpha \in \mathbb{R}} \left\| \phi(\cdot, t) - e^{i\alpha} \phi_{\lambda,\epsilon} \right\|_{H^1} < \kappa.$$

Otherwise, it is said to be unstable in $H^1(\mathbb{R}^d)$.

The main result of this section is the following:

Theorem 3 *Fix any $\hbar > 0$ be sufficiently small such that $\hbar \in (0, \hbar_3)$ for some $\hbar_3 > 0$ small enough. Then, the following statements hold.*

- *Let $\sigma \leq \sigma_{threshold}$. The symmetric solution corresponding to $z^s = \tilde{O}(e^{-\rho/\hbar})$ is orbitally stable in H^1 for $|\eta| < \eta^\star$. At the bifurcation point $\eta = \eta^\star$, there is an exchange of stability, that is, for $|\eta| > \eta^\star$, the asymmetric solution is stable in H^1 and the symmetric solution is unstable.*
- *Let $\sigma > \sigma_{threshold}$. By Theorem 1, two couples of new asymmetric stationary states, denoted by ψ^{as1} and ψ^{as2} appears at $|\eta| = \eta^+$. For $|\eta| > \eta^+$, ψ^{as1} is orbitally stable in H^1, ψ^{as2} is unstable. On the other hand, the symmetric state is orbitally stable in H^1 for $|\eta| < \eta^\star$, and unstable for $|\eta| > \eta^\star$.*

As a standard method to prove the orbital stability of a stationary solution $\phi_{\lambda,\epsilon}$, the following proposition is well known. We f rst defin $L_+^{\lambda,\epsilon}$ and $L_-^{\lambda,\epsilon}$, which are respectively the real and the imaginary part of the linearized operators around a real valued stationary solution $\phi_{\lambda,\epsilon}$:

$$L_+^{\lambda,\epsilon} \equiv L_+[\phi_{\lambda,\epsilon}] = H_0 - \lambda - (2\sigma + 1) g |\phi_{\lambda,\epsilon}|^{2\sigma},$$

$$L_-^{\lambda,\epsilon} \equiv L_-[\phi_{\lambda,\epsilon}] = H_0 - \lambda - g |\phi_{\lambda,\epsilon}|^{2\sigma}.$$

It is clear that $L_-^{\lambda,\epsilon} \phi_{\lambda,\epsilon} = 0$ since $\phi_{\lambda,\epsilon}$ is a solution of (66). Moreover, $L_+^{\lambda,\epsilon}$ and $L_-^{\lambda,\epsilon}$ are self-adjoint operators on $L^2(\mathbb{R}^d)$ with domain $H^2(\mathbb{R}^d)$. The essential spectrum of these two operators coincides with the interval $[V_\infty^- - \lambda, \infty)$ with $V_\infty^- - \lambda > 0$, since $\phi_{\lambda,\epsilon}$ vanishes at infinity indeed, V is bounded, and we can apply the proof of Theorem 1 in [14], regarding the term $V\phi_{\lambda,\epsilon}$ of (66) as one of nonlinear parts. There are also finitel many of discrete spectrum and $\sigma_d(L_\pm^{\lambda,\epsilon}) \subset (-\infty, V_\infty - \lambda)$ (see [4]).

In order to prove the orbital stability we make use of the following criteria (see, e.g., [17] or Part I of [18, 19]).

Proposition 1 *Suppose that $L_-^{\lambda,\epsilon}$ is nonnegative. Let $F(\lambda) = \|\phi_{\lambda,\epsilon}\|^2$.*

(1) *If $L_+^{\lambda,\epsilon}$ has only one negative eigenvalue, and $dF/d\lambda < 0$, then, $\phi_{\lambda,\epsilon}$ is stable in $H^1(\mathbb{R}^d)$.*
(2) *If $L_+^{\lambda,\epsilon}$ has only one negative eigenvalue, and $dF/d\lambda > 0$, then, $\phi_{\lambda,\epsilon}$ is unstable in $H^1(\mathbb{R}^d)$.*
(3) *If $L_+^{\lambda,\epsilon}$ has at least two negative eigenvalues, then, $\phi_{\lambda,\epsilon}$ is unstable in $H^1(\mathbb{R}^d)$.*

Remark 16 For the instability (3), it is enough to fin a vector $p \in H^1$ such that

$$\left\langle L_+^{\lambda,\epsilon} p, p \right\rangle < 0, \quad p \perp \phi_{\lambda,\epsilon} \text{ in } L^2 \tag{68}$$

(see for e.g., [7, 17]). As we will see below, "$L_-^{\lambda,\epsilon}$ is nonnegative and $L_+^{\lambda,\epsilon}$ has two negative eigenvalues" occurs only for the symmetric stationary solution $\phi_{\lambda,\epsilon}^s$. In this case, we can fin the normalized antisymmetric solution $\frac{\phi_{\lambda,\epsilon}^a}{\|\phi_{\lambda,\epsilon}^a\|}$ as the vector p satisfying the property (68) for $\hbar$ small.

We shall therefore check the following properties:

- the number of negative eigenvalues of $L_+^{\lambda,\epsilon}$;
- $L_-^{\lambda,\epsilon}$ is a nonnegative operator;
- (Slope condition) the sign of the function $dF(\lambda)/d\lambda$.

5.1 Number of Negative Eigenvalues of $L_+^{\lambda,\epsilon}$

First we consider the number of negative eigenvalues of $L_+^{\lambda,\epsilon}$. We will prove that:

Lemma 5 *Let $\hbar^\star > 0$ small enough as in Theorem 1; there exists $\hbar_1 \in (0, \hbar^\star)$ such that for any $\hbar \in (0, \hbar_1)$ the following statements are satisfied.*

(i) *Let λ be the energy level associated to the symmetric stationary state $\phi_{\lambda,\epsilon} = \phi_{\lambda,\epsilon}^s$. Then, $L_+^{\lambda,\epsilon}$ admits only one negative eigenvalue provided that $|\eta| < \eta^\star$. On the other hand, $L_+^{\lambda,\epsilon}$ admits two negative eigenvalues provided that $|\eta| > \eta^\star$.*

(ii) *Let λ be the energy level associated to the asymmetrical stationary state $\phi_{\lambda,\epsilon} = \phi_{\lambda,\epsilon}^{as}$ if $\sigma \le \sigma_{threshold}$, and $\phi_{\lambda,\epsilon} = \phi_{\lambda,\epsilon}^{as1}$ and $\phi_{\lambda,\epsilon} = \phi_{\lambda,\epsilon}^{as2}$, if $\sigma > \sigma_{threshold}$. Then, $L_+^{\lambda,\epsilon}$ admits only one negative eigenvalue.*

Proof We set

$$\phi_{\lambda,\epsilon} = a_R^{\lambda,\epsilon}\varphi_R + a_L^{\lambda,\epsilon}\varphi_L + \phi_c^{\lambda,\epsilon}, \quad \left|a_R^{\lambda,\epsilon}\right|^2 + \left|a_L^{\lambda,\epsilon}\right|^2 + \left\|\phi_c^{\lambda,\epsilon}\right\|^2 = |\epsilon|^{1/\sigma},$$
$$\left\|\phi_c^{\lambda,\epsilon}\right\| = |\epsilon|^{1/2\sigma}\|\psi_c\|, \quad \psi_c = \psi_E - \left(a_R^\lambda\varphi_R + a_L^\lambda\varphi_L\right),$$

where ψ_E is a stationary solution obtained in Theorem 1.

We consider the eigenvalue problem $L_+^{\lambda,\epsilon}u = (\omega\mu)u$ with $u \in H^2(\mathbb{R}^d)$ and where

$$|\mu\omega| \le C\hbar^2. \tag{69}$$

By setting $u = a_R\varphi_R + a_L\varphi_L + u_c$ with $u_c \in \Pi_c L^2$, then the eigenvalue problem takes the following form

$$\begin{cases} \omega\mu a_R = a_R\Omega - a_L\omega - \lambda a_R - (2\sigma+1)\langle\varphi_R, g|\phi_{\lambda,\epsilon}|^{2\sigma}u\rangle, \\ \omega\mu a_L = a_L\Omega - a_R\omega - \lambda a_L - (2\sigma+1)\langle\varphi_L, g|\phi_{\lambda,\epsilon}|^{2\sigma}u\rangle, \\ \omega\mu u_c = (H_0-\lambda)u_c - \Pi_c(2\sigma+1)g|\phi_{\lambda,\epsilon}|^{2\sigma}u. \end{cases} \tag{70}$$

The last equation reads as

$$\left[I - [H_0-\lambda-\omega\mu]^{-1}\Pi_c(2\sigma+1)g|\phi_{\lambda,\epsilon}|^{2\sigma}\right]u_c$$
$$= (H_0-\lambda-\omega\mu)^{-1}\Pi_c(2\sigma+1)g|\phi_{\lambda,\epsilon}|^{2\sigma}(a_R\varphi_R + a_L\varphi_L).$$

Since $H_0 - \lambda \ge C\hbar$, when restricted to $\Pi_c L^2$, and since (69), then we have

$$\left\|(H_0-\lambda-\omega\mu)^{-1}\Pi_c\right\|_{\mathcal{L}(L^2\to H^2)} \le C_1\hbar^{-1}.$$

Here, we recall that, from (20) and (36),

$$\left\| g|\phi_{\lambda,\epsilon}|^{2\sigma} \right\| = |\epsilon| \left\| g|\psi_{E(\epsilon)}|^{2\sigma} \right\| \leq C|\epsilon|\hbar^{1-\alpha_0}$$

with $\alpha_0 = 1 + \frac{d(2\sigma-1)}{4}$. Thus, if $\omega\mu = O(\hbar^2)$, we get from (17), for sufficientl small $\hbar$,

$$\left\| (H_0 - \lambda - \omega\mu)^{-1} \Pi_c (2\sigma+1) g|\phi_{\lambda,\epsilon}|^{2\sigma} \right\|_{\mathcal{L}(L^2 \to H^2)} \leq C_2(2\sigma+1)|\epsilon|\hbar^{-\alpha_0} \leq \frac{1}{2}.$$

Namely, if μ satisfie the condition (69) then the inverse of the operator

$$I - [H_0 - \lambda - \omega\mu]^{-1} \Pi_c (2\sigma+1) g|\phi_{\lambda,\epsilon}|^{2\sigma}$$

exists. Accordingly, the third equation in (70) has a solution

$$\begin{aligned} u_c &:= u_c(\mu, \lambda) \\ &= Q[\mu, \phi_{\lambda,\epsilon}](a_R\varphi_R + a_L\varphi_L), \end{aligned}$$

where

$$\begin{aligned} Q[\mu, \phi_{\lambda,\epsilon}] = &\left[I - (H_0 - \lambda - \omega\mu)^{-1} \Pi_c (2\sigma+1) g|\phi_{\lambda,\epsilon}|^{2\sigma} \right]^{-1} \\ &\times (H_0 - \lambda - \omega\mu)^{-1} \Pi_c (2\sigma+1) g|\phi_{\lambda,\epsilon}|^{2\sigma} : L^2(\mathbb{R}^d) \to H^2(\mathbb{R}^d), \end{aligned}$$

and

$$\left\| Q[\mu, \phi_{\lambda,\epsilon}] \right\|_{\mathcal{L}(L^2 \to H^2)} \leq C_\sigma |\epsilon| \hbar^{-\alpha_0}.$$

The bound C_σ is uniform in $\hbar$ on D, where D is define in Lemma 3, and for any μ such that $|\mu\omega| \leq C\hbar^2$. In fact, by the same arguments the same estimate holds true also for the derivative of Q with respect to μ:

$$\left\| \frac{\partial Q}{\partial \mu} \right\|_{\mathcal{L}(L^2 \to H^2)} \leq C|\epsilon|\hbar^{-\alpha_0'} \tag{71}$$

for some $\alpha_0' > 0$. We insert this expression of u_c into the system (70), and we have Lyapunov-Schmidt reduction of (70) as follows,

$$\begin{cases} \omega\mu a_R = a_R\Omega - a_L\omega - \lambda a_R - (2\sigma+1)\langle \varphi_R, g|\phi_{\lambda,\epsilon}|^{2\sigma}(I + Q(\mu, \phi_{\lambda,\epsilon}))(a_R\varphi_R + a_L\varphi_L)\rangle, \\ \omega\mu a_L = a_L\Omega - a_R\omega - \lambda a_L - (2\sigma+1)\langle \varphi_L, g|\phi_{\lambda,\epsilon}|^{2\sigma}(I + Q(\mu, \phi_{\lambda,\epsilon}))(a_R\varphi_R + a_L\varphi_L)\rangle. \end{cases}$$

This system can be rewritten under the following form.

$$(N + \mu I - \nu C)\begin{pmatrix} a_R \\ a_L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{72}$$

where we recall that $\lambda = \Omega + \omega E$ and where

$$N = \begin{pmatrix} \alpha, & 1 \\ 1, & \beta \end{pmatrix}, \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} C_1, & C_2 \\ C_3, & C_4 \end{pmatrix},$$

$$\alpha = E + (2\sigma+1)|\eta|\left|a_R^\lambda\right|^{2\sigma}, \qquad \beta = E + (2\sigma+1)|\eta|\left|a_L^\lambda\right|^{2\sigma},$$

and $\nu = e^{-\gamma\rho'/\hbar}$ for any $\rho' \in (0, \rho)$ as in (45). For $\nu \neq 0$, we have put

$$\begin{aligned}
C_1 &= C_1\big(a_R^\lambda, a_L^\lambda, E, \mu; \hbar\big) = -\frac{(2\sigma+1)|\epsilon|}{\nu\omega}\big\{\big\langle\varphi_R, g\big(|\psi_E|^{2\sigma} - |a_R^\lambda \varphi_R|^{2\sigma}\big)\varphi_R\big\rangle \\
&\qquad + \big\langle\varphi_R, g|\psi_E|^{2\sigma} Q(\mu, \phi_{\lambda,\epsilon})\varphi_R\big\rangle\big\}, \\
C_2 &= C_2\big(a_R^\lambda, a_L^\lambda, E, \mu; \hbar\big) = -\frac{(2\sigma+1)|\epsilon|}{\nu\omega}\big\{\big\langle\varphi_R, g|\psi_E|^{2\sigma}\big(I + Q(\mu, \phi_{\lambda,\epsilon})\big)\varphi_L\big\rangle\big\}, \\
C_3 &= \bar{C}_2, \\
C_4 &= C_4\big(a_R^\lambda, a_L^\lambda, E, \mu; \hbar\big) = -\frac{(2\sigma+1)|\epsilon|}{\nu\omega}\big\{\big\langle\varphi_L, g\big(|\psi_E|^{2\sigma} - \big|a_L^\lambda \varphi_L\big|^{2\sigma}\big)\varphi_L\big\rangle \\
&\qquad + \big\langle\varphi_L, g|\psi_E|^{2\sigma} Q(\mu, \phi_{\lambda,\epsilon})\varphi_L\big\rangle\big\}.
\end{aligned}$$

If $\nu = 0$, then μ are the eigenvalues of N and they are the solutions of the equation

$$P(\mu) = \mu^2 + (\alpha+\beta)\mu + \alpha\beta - 1 = 0,$$

which always has only two real different solutions μ_1, μ_2 since $(\alpha+\beta)^2 - 4\alpha\beta + 4 = (\alpha-\beta)^2 + 4 > 0$. In particular, these two real eigenvalues are both negative or both positive if $\alpha\beta > 1$, or only one is negative in $\alpha\beta < 1$.

To investigate the sign of $\alpha\beta - 1$, we consider, at f rst, the case of the symmetric stationary solution corresponding to $z^\lambda = z^s = 0$ (see Theorem 1). Then (hereafter, for the sake of simplicity, we denote by $\sim$ that we have an exponentially small term)

$$a_R^\lambda = a_L^\lambda = \frac{1}{\sqrt{2}}, \qquad E \sim -1 - |\eta|\frac{1}{2^\sigma}$$

and

$$\alpha = \beta = E + |\eta|\frac{2\sigma+1}{2^\sigma} \sim -1 + |\eta|\frac{2\sigma}{2^\sigma}.$$

Hence, condition $\alpha\beta > 1$ is equivalent to the condition $|\eta| > \eta^\star = \frac{2^\sigma}{\sigma}$ (and in such a case both solutions are negative), and condition $\alpha\beta < 1$ is equivalent to the condition $|\eta| < \eta^\star = \frac{2^\sigma}{\sigma}$; provided $\hbar$ is small enough.

We consider next the case of the asymmetrical stationary solution corresponding to $z^\lambda \neq 0$. In such a case we set $a = \frac{1}{2}|\eta|(p^\lambda)^{2\sigma}$ and $b = \frac{1}{2}|\eta|(q^\lambda)^{2\sigma}$, then

$$\alpha \sim E + 2(2\sigma+1)a, \qquad \beta \sim E + 2(2\sigma+1)b$$

and

$$E \sim -\sqrt{1 - \big(z^\lambda\big)^2} - 2\big[a\big(p^\lambda\big)^2 + b\big(q^\lambda\big)^2\big].$$

Hence, condition $\alpha\beta < 1$ is equivalent to the condition

$$\ell\big(z^\lambda, \sigma\big) - 1 < 0,$$

where

$$\ell(z,\sigma) := \left[\sqrt{1-z^2} + 2\left[(p^2-2\sigma-1)a + q^2 b\right]\right]\left[\sqrt{1-z^2} + 2\left[ap^2 + b\left(q^2-2\sigma-1\right)\right]\right]$$
$$\sim \frac{\left[(-1+z+4z\sigma)(1+z)^\sigma + (1-z)^{\sigma+1}\right]}{(1-z^2)\left[(1-z)^\sigma - (1+z)^\sigma\right]^2}$$
$$\times \left[(1+z+4z\sigma)(1-z)^\sigma - (1+z)^{\sigma+1}\right]$$

since

$$p \sim \sqrt{\frac{1+z}{2}} \quad \text{and} \quad q \sim \sqrt{\frac{1-z}{2}}.$$

Then, a straightforward calculation gives that

$$\ell(z,\sigma) - 1 \sim \frac{4z\sigma(1+z)^{2\sigma}}{(1-z^2)\left[(1+z)^\sigma - (1-z)^\sigma\right]^2}$$
$$\times \left[-z-1-(z-1)\frac{(1-z)^{2\sigma}}{(1+z)^{2\sigma}} + 2z(1+2\sigma)\frac{(1-z)^\sigma}{(1+z)^\sigma}\right]$$
$$= \frac{4z\sigma(1+z)^{2\sigma}}{(1-z^2)\left[(1+z)^\sigma - (1-z)^\sigma\right]^2}$$
$$\times \frac{2}{1+y}\left[-1-y^{2\sigma+1} + (1+2\sigma)y^\sigma - (1+2\sigma)y^{\sigma+1}\right]$$

where we have set $y = \frac{1-z}{1+z} \in [0,1]$. We then consider the sign of the following polynomial in the right hand side above,

$$q(y) := -1 + y^{2\sigma+1} + (1+2\sigma)y^\sigma - (1+2\sigma)y^{\sigma+1}.$$

It is in fact easy to conclude that $q(y) \le 0$ for any $y \in [0,1]$. Indeed,

$$q(y) \le -1 + (1+2\sigma)y^\sigma - (1+2\sigma)y^{\sigma+1} \le 0.$$

Now, we wish to investigate the sign of eigenvalues for the case $\nu \neq 0$. Recall that the effective nonlinearity parameter η satisfie $|\eta| \le C$ for some constant $C > 0$. Also there exist $\hbar_0 \in (0, \hbar^\star)$, and a compact interval $K_{\hbar_0}$ such that the two eigenvalues of the matrix N,

$$\mu_1 = \frac{1}{2}\left\{-(\alpha+\beta) - \sqrt{(\alpha-\beta)^2+4}\right\}, \qquad \mu_2 = \frac{1}{2}\left\{-(\alpha+\beta) + \sqrt{(\alpha-\beta)^2+4}\right\}$$

belong to $K_{\hbar_0}$ for any $\hbar \in (0,\hbar_0)$. Then we see that $C_j = C_j(a_R^\lambda, a_L^\lambda, E, \mu, \hbar)$ are bounded, together with their f rst derivatives, on $D \times K_{\hbar_0}$ uniformly for any $\hbar \in (0,\hbar_0)$: indeed, there exists a constant $C > 0$ such that

$$\nu^{-1}\left|\langle \varphi_L, g|\psi_E|^{2\sigma}\left(I + Q(\mu, \phi_{\lambda,\epsilon})\right)\varphi_R\rangle\right|$$
$$\le \nu^{-1}\left[\|g\|_{L^\infty}\|\varphi_R\varphi_L\|_{L^\infty}\|\psi_E\|_{L^{2\sigma}}^{2\sigma} + \|g\|_{L^\infty}\|\phi_L\|\|\phi_R\|_{L^4}^2\|\psi_E\|_{L^{8\sigma}}^{4\sigma}\right],$$

and this right hand side is bounded because of (14), (20) and (36). It also follows that if $1 \le 2\sigma$,

$$\nu^{-1}\left|\langle \varphi_R, g\left(|\psi_E|^{2\sigma} - \left|a_R^\lambda\varphi_R\right|^{2\sigma}\right)\varphi_R\rangle\right| \le \nu^{-1}C\left(\|\varphi_R\varphi_L\| + \|\psi_c\|\right)\left(1 + \|\psi_E\|_{L^{2(2\sigma-1)}}^{2\sigma-1}\right),$$

whose right hand side is bounded, noting (20), (21), (36) and (34). If $0 < 2\sigma < 1$,

$$\left|\langle\varphi_R, g\big(|\psi_E|^{2\sigma} - \left|a_R^\lambda\varphi_R\right|^{2\sigma}\big)\varphi_R\rangle\right| \le C\int \varphi_R^2|g|\left|a_L^\lambda\varphi_L + \psi_c\right|^{2\sigma} dx$$

$$\le C\|g\|_{L^\infty}\int \varphi_R^2\,|\psi_c|^{2\sigma} dx + \|g\|_{L^\infty}\left|a_L^\lambda\right|^{2\sigma}\int \varphi_R^2\varphi_L^{2\sigma}\,dx.$$

The f rst integral is estimated as follows

$$\int \varphi_R^2|\psi_c|^{2\sigma}\,dx \le \left\|\varphi_R^2\right\|_{L^p}\cdot\left\||\psi_c|^{2\sigma}\right\|_{L^q} = \|\psi_c\|^{2\sigma}\|\varphi_R\|^2_{L^{2/(1-\sigma)}} \tag{73}$$

by means of the Hölder inequality, where $q = \frac{1}{\sigma} > 2$ and $p = \frac{1}{1-\sigma}$. Inequalities (34) and (14) yield that this right hand side is exponentially small. Similarly, the estimate of the second integral follows

$$\int \varphi_R^2\varphi_L^{2\sigma}\,dx \le C\hbar^{-\alpha}$$

for some $\alpha > 0$. As for the derivatives of C_j, the analyticity in μ of $(H_0-\lambda-\omega\mu)^{-1}$ ensures their regularity, and the uniform boundedness follows from (71).

We come back to the problem (72). This problem is mapped to the problem to fin the roots of the following characteristic equation,

$$D(a_R, a_L, E, \mu, \nu) = \det(N + \mu I - C) = 0.$$

Concretely,

$$\begin{aligned}\det(N+\mu I - C) &= (\alpha+\mu-\nu C_1)(\beta+\mu-\nu C_4) - (1-\nu C_2)(1-\nu C_3)\\ &= \mu^2 + \big\{(\alpha+\beta) - \nu(C_1+C_4)\big\}\mu + \alpha\beta - 1\\ &\quad - \nu(C_2+C_3+\alpha C_4+\beta C_1) + \nu^2(C_1C_4 - C_2C_3).\end{aligned}$$

Putting $S(\mu,\nu) = -(C_1+C_4)\mu - (C_2+C_3+\alpha C_4+\beta C_1) + \nu(C_1C_4-C_2C_3)$, we have

$$D(\mu,\nu) = P(\mu) - \nu S(\mu,\nu) = 0.$$

We note that by the above arguments, $S(\mu,\nu)$ and $\partial_\mu P(\mu)$ is uniformly bounded on $D\times K_{\hbar_0}$ for any $\hbar\in(0,\hbar_0)$. It is also seen that $D(\mu,\nu)$ is a C^1 function in (μ,ν),

$$D(\mu_1,0) = D(\mu_2,0) = 0,$$

$$\frac{\partial D(\mu_1,0)}{\partial\mu} = 2\mu_1+\alpha+\beta \ne 0, \qquad \frac{\partial D(\mu_2,0)}{\partial\mu} = 2\mu_2+\alpha+\beta\ne 0.$$

By applying Implicit Function Theorem, there exists $\varepsilon_0 > 0$ such that there exist two real solutions $\mu_1(\nu)$ and $\mu_2(\nu)$ of $D(\mu,\nu)=0$ for $|\nu|<\varepsilon_0$ and that

$$\mu_1(\nu) = \mu_1 - \nu\frac{S(\mu_1,0)}{\partial_\mu P(\mu_1)} + O\big(\nu^2\big), \tag{74}$$

$$\mu_2(\nu) = \mu_2 - \nu\frac{S(\mu_2,0)}{\partial_\mu P(\mu_2)} + O\big(\nu^2\big). \tag{75}$$

Therefore, for any $\varepsilon > 0$ there exists $\hbar_1 \in (0, \hbar_0)$ such that $|\mu_1(\nu) - \mu_1| < \varepsilon$, and that $|\mu_2(\nu) - \mu_2| < \varepsilon$ and $\mu_1(\nu), \mu_2(\nu) \in K_{\hbar_0}$ for any $\hbar \in (0, \hbar_1)$. We remark here that $L_+^{\lambda,\epsilon}$ has at least one negative eigenvalue since $\langle L_+^{\lambda,\epsilon}\phi_{\epsilon,\lambda}, \phi_{\epsilon,\lambda}\rangle < 0$. As a consequence, for the symmetric solutions, $L_+^{\lambda,\epsilon}$ has two negative eigenvalues if $|\eta| > \eta^\star$ and has only one negative eigenvalue if $|\eta| < \eta^\star$. For the asymmetric solution, $L_+^{\lambda,\epsilon}$ has only one negative eigenvalue. The proof of Lemma 5 has been completed. □

5.2 $L_-^{\lambda,\epsilon}$ Is a Nonnegative Operator

Next our aim is proving that $L_-^{\lambda,\epsilon}$ has no negative eigenvalues. Since the symmetric solution ψ_E^s, i.e. $\phi_{\epsilon,\lambda}^s$, is positive by means of a suitable choice of the phase, $L_-^{\lambda,\epsilon}[\phi_{\epsilon,\lambda}^s]$ is nonnegative. However, we do not know the sign of the asymmetric solutions and we repeat here the same argument as in Lemma 5 for $L_-^{\lambda,\epsilon}$.

Lemma 6 *Let $\phi_{\lambda,\epsilon}$ be the symmetric and asymmetrical stationary solution associated to the level λ. Then there exists $\hbar_2 \in (0, \hbar_1)$, where $\hbar_1$ has been defined in Lemma 5, such that for any $\hbar \in (0, \hbar_2)$, $L_-^{\lambda,\epsilon}$ has no negative eigenvalues, more precisely, $L_-^{\lambda,\epsilon}$ has a zero eigenvalue and one positive eigenvalue $\omega\mu = O(\hbar^2)$.*

Proof The eigenvalue problem $L_-^{\lambda,\epsilon}u = (\omega\mu)u$ with $u \in H^2(\mathbb{R}^d)$, where $|\omega\mu| \le C\hbar^2$, takes the form

$$\begin{cases} \omega\mu a_R = a_R\Omega - a_L\omega - \lambda a_R - a_R\langle\varphi_R, g|\phi_{\lambda,\epsilon}|^{2\sigma}u\rangle, \\ \omega\mu a_L = a_L\Omega - a_R\omega - \lambda a_L - a_L\langle\varphi_L, g|\phi_{\lambda,\epsilon}|^{2\sigma}u\rangle, \\ \omega\mu u_c = (H_0 - \lambda)u_c - \Pi_c g|\phi_{\lambda,\epsilon}|^{2\sigma}u, \end{cases}$$

where we put $u = a_R\varphi_R + a_L\varphi_L + u_c$, $u_c \in \Pi_c L^2$. We remind that $\mu = 0$ is a solution of the eigenvalue problem since $L_-^{\lambda,\epsilon}\phi_{\lambda,\epsilon} = 0$, we then apply again the same Lyapunov-Schmidt reduction as in Lemma 5 in order to compute the other eigenvalues of $L_-^{\lambda,\epsilon}$ such that $|\mu\omega| \le C\hbar^2$. This eigenvalue problem can be rewritten, assuming $|\omega\mu| \le C\hbar^2$, as follows,

$$(N' + \mu I - \nu C')\begin{pmatrix} a_R \\ a_L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where

$$N' = \begin{pmatrix} \alpha', & 1 \\ 1, & \beta' \end{pmatrix}, \qquad C' = \begin{pmatrix} C_1', & C_2' \\ C_3', & C_4' \end{pmatrix}, \qquad C_3' = \bar{C}_2',$$

$$\alpha' = E + |\eta|\left|a_R^\lambda\right|^{2\sigma}, \qquad \beta' = E + |\eta||a_L^\lambda|^{2\sigma}.$$

Remind that ν is define in Lemma 5. As in the proof of Lemma 5, it suffice to know the sign of $\alpha'\beta' - 1$. We compute the case of the asymmetric solutions corresponding to $z^\lambda \neq 0$ (in the case of the symmetric solution corresponding to $z^\lambda = 0$ we follow the same arguments). In this case,

$$\alpha' \sim -\sqrt{1 - (z^\lambda)^2} - |\eta|\{(p^\lambda)^{2\sigma+2} + (q^\lambda)^{2\sigma+2}\} + |\eta|(p^\lambda)^{2\sigma},$$

$$\beta' \sim -\sqrt{1 - (z^\lambda)^2} - |\eta|\{(p^\lambda)^{2\sigma+2} + (q^\lambda)^{2\sigma+2}\} + |\eta|(q^\lambda)^{2\sigma},$$

$$|\eta| = \left[\left(\frac{1+z^\lambda}{2}\right)^\sigma - \left(\frac{1-z^\lambda}{2}\right)^\sigma\right]^{-1} \times \frac{2z^\lambda}{\sqrt{1-(z^\lambda)^2}}.$$

By direct computations it is not diff cult to obtain that

$$\alpha' \sim \frac{z^\lambda - 1}{\sqrt{1-(z^\lambda)^2}}, \qquad \beta' \sim -\frac{z^\lambda + 1}{\sqrt{1-(z^\lambda)^2}}.$$

Therefore, $\alpha'\beta' \sim 1$ and

$$\alpha' + \beta' \sim -\frac{2}{\sqrt{1-(z^\lambda)^2}},$$

which implies $\mu_1\mu_2 \sim 0$ and $\mu_1 + \mu_2 > 0$ for the eigenvalues of N'. We may assume without generality that $|\mu_1|$ is very small and μ_2 is positive. It follows from the same arguments as in Lemma 5 that the perturbed matrix $N' - \nu C'$ has two different eigenvalues $\mu_1(\nu)$ and $\mu_2(\nu)$ verifying (74) and (75). Since we know that $L_-^{\lambda,\epsilon}$ has always zero eigenvalue, and perturbed eigenvalues are continuous with respect to ν, we conclude that $\mu_1(\nu) = 0$ and $\mu_2(\nu) > 0$. □

5.3 Slope Condition

In order to check the slope condition, we consider the following quantity.

$$F(\lambda) = \|\phi_{\lambda,\epsilon}\|^2 = |\epsilon|^{1/\sigma}$$

and we remark that

$$\frac{dF(\lambda)}{d\lambda} = \left[\frac{d\lambda}{d\epsilon}\right]^{-1} \frac{d}{d\epsilon}(-\epsilon)^{1/\sigma} = -\frac{1}{\omega\sigma}|\epsilon|^{(1-\sigma)/\sigma}\left[\frac{dE}{d\epsilon}\right]^{-1}$$
$$= -\frac{1}{C_R\sigma}|\epsilon|^{(1-\sigma)/\sigma}\left[\frac{dE}{d\eta}\right]^{-1}.$$

Thus, we only have to check the sign of $\frac{dE}{d\eta}$ for the symmetric and asymmetrical stationary solutions.

5.3.1 Estimate of the Stationary Solutions as Function of the Nonlinearity Parameter

The stationary solution

$$\psi = a_R\varphi_R + a_L\varphi_L + \psi_c,$$

of (25) associated to the energy level E depends on the value of the nonlinearity parameter $\eta = \epsilon c/\omega$, where $c = C_R = C_L$ is define in (30).

In particular, in Theorem 1 we have proved that, locally, there is a correspondence one-to-one from η to the solution p, q, α, β and E (up to the gauge choice of the phase, where we set $\theta = \alpha - \beta$) of (47) and ψ_c of (37); provided that $\eta \neq \pm\eta^+$ and $\eta \neq \pm\eta^\star$.

In order to see the sign of $dE/d\eta$, we wish to obtain the estimate of the firs derivative of p, q, α, β, E and ψ_c as function of η. To this end, let

$$D'' = \left\{(p,q,\alpha,\beta,E) \in [0,1]^2 \times [0,2\pi)^2 \times \mathbb{R} : p^2+q^2 \le 1,\ |\omega E| \le C\hbar^2\right\}$$

for some $C > 0$ fixed; and let

$$\begin{array}{rcl}\Phi: \quad \mathbb{R} \times H^2 \times D'' & \to & H^2 \times \mathbb{R}^4, \\ (\eta, \psi_c, p, q, \alpha, \beta, E) & \mapsto & (F(\psi_c), G),\end{array}$$

where $F(\psi_c)$ is defined by (40) and where G is defined by (47) with ϵ replaced by $\omega\eta/c$; $F(\psi_c) = F(\eta, \psi_c, p, q, \alpha, \beta, E)$, and $G = G(\eta, p, q, \alpha, \beta, E) = (G_1, G_2, G_3, G_4)$. For simplicity, we set $y = (\psi_c, p, q, \alpha, \beta, E) \in H^2 \times D''$. Since the mapping $\frac{\partial\Phi}{\partial y}(\eta, \cdot)$: $H^2 \times D'' \to H^2 \times \mathbb{R}^2$ is one-to-one at a point $\eta \neq \pm\eta^\star, \pm\eta^+$, we obtain the unique solution $y = y(\eta)$ of equation $\Phi(\eta, y) = 0$ (up to the gauge choice of the phase). Furthermore, $\Phi(\eta, y)$ is C^1, so the solution $y(\eta)$ is C^1 except for $\eta \neq \pm\eta^\star, \pm\eta^+$, and we have

$$\frac{\partial\Phi}{\partial\eta} + \frac{\partial\Phi}{\partial y} y' = 0. \tag{76}$$

Here, $'$ denotes the derivative with respect to η, and we use this notation hereafter, too. We will in fact see that $\frac{\partial\Phi}{\partial y}(\eta, \cdot)$ is one-to-one for any $\eta \neq \pm\eta^\star, \pm\eta^+$ in the proof of Lemma 8 below. Therefore we do not mention the details about this fact here.

The first equation of (76) takes the form

$$\omega E\psi_c' + \omega E'\psi_c = [H_0 - \Omega]\psi_c' + \frac{\omega}{c}\Pi_c v + \frac{\omega\eta}{c}\Pi_c g W\left(a_R'\varphi_R + a_L'\varphi_L + \psi_c'\right), \tag{77}$$

where $v = g|\psi|^{2\sigma}\psi$, and

$$W = \left[(\sigma+1)|\psi|^{2\sigma} + \sigma\psi^2|\psi|^{2(\sigma-1)}\mathcal{T}\right], \qquad \mathcal{T}u := \bar{u};$$

actually, the stationary solution is a real valued function by means of a gauge choice (see Remark 7).

In order to write the other equations of (76) we make use of the first two equations of (28) and of the normalization condition:

$$\begin{cases} Ea_R = -a_L + \frac{\eta}{c}\langle\varphi_R, v\rangle, \\ Ea_L = -a_R + \frac{\eta}{c}\langle\varphi_L, v\rangle, \\ |a_R|^2 + |a_L|^2 + \langle\psi_c, \psi_c\rangle = 1. \end{cases} \tag{78}$$

Now, we get the estimate of the derivative of ψ_c in Lemma 7 and then the estimate of the derivative of p, q, α, β and E in Lemma 8.

Lemma 7 *Let $(a_R, a_L, E) \in D$ and let η satisfying Hypotesis 3, let ψ_c be the solution of (37). Then*

$$\left\|\frac{\partial\psi_c}{\partial\eta}\right\|_{H^2} = \left[1 + \max\left(\left|\frac{\partial a_R}{\partial\eta}\right|, \left|\frac{\partial a_L}{\partial\eta}\right|, \left|\frac{\partial E}{\partial\eta}\right|\right)\right]\tilde{O}\left(e^{-\rho/\hbar}\right) \quad \text{as } \hbar \to 0. \tag{79}$$

Proof Since (77) can be written as

$$\left[(H_0 - \Omega - \omega E)\Pi_c + \epsilon\Pi_c W\right]\psi_c' = \omega E'\psi_c - \epsilon\Pi_c g W\left(a_R'\varphi_R + a_L'\varphi_L\right) + \frac{\omega}{C_R}\Pi_c v$$

then

$$\psi_c' = \left[I + (H_0 - \Omega - \omega E)^{-1} + \epsilon \Pi_c W\right] [H_0 - \Omega - \omega E]^{-1}$$
$$\times \left[\omega E' \psi_c - \epsilon \Pi_c g W \left(a_R' \varphi_R + a_L' \varphi_L\right) + \frac{\omega}{C_R} \Pi_c v\right]$$

and, by making use of the same ideas applied in the proof of Lemma 5, it turn out that the inverse operator is bounded and (79) follows. □

Lemma 8 *Let* $|\eta| \neq \eta^\star$ *and* $|\eta| \neq \eta^+$. *Then*

$$\max\left[\left|\frac{\partial p}{\partial \eta}\right|, \left|\frac{\partial q}{\partial \eta}\right|, \left|\frac{\partial \alpha}{\partial \eta}\right|, \left|\frac{\partial \beta}{\partial \eta}\right|, \left|\frac{\partial E}{\partial \eta}\right|\right] \leq C \tag{80}$$

for some $C > 0$.

Proof Now, in order to give an estimate of the derivative of p, q, α, β and E we write down the corresponding equations of (76), that is we have to consider the derivate of (78). We assume, for the sake of definiteness that the stationary solution corresponds to $\theta = 0$ (that is ψ is a symmetric or asymmetrical stationary solution). In fact, we rewrite $a_R = pe^{i\alpha}$ and $a_L = qe^{i\beta}$ by means of p, q, α and β (where we set $\theta = \alpha - \beta$); so that (78) takes the form

$$\begin{cases} Ep + q\cos\theta - \frac{\eta}{C_R}\Re[\langle\varphi_R, v\rangle e^{-i\alpha}] = 0, \\ q\sin\theta + \frac{\eta}{C_R}\Im[\langle\varphi_R, v\rangle e^{-i\alpha}] = 0, \\ Eq + p\cos\theta - \frac{\eta}{C_R}\Re[\langle\varphi_L, v\rangle e^{-i\beta}] = 0, \\ -p\sin\theta + \frac{\eta}{C_R}\Im[\langle\varphi_L, v\rangle e^{-i\beta}] = 0, \\ p^2 + q^2 + \|\psi_c\|^2 = 1. \end{cases}$$

We take now the derivative of both sides with respect to η, obtaining that

$$\begin{cases} E'p + Ep' + q'\cos\theta - q\theta'\sin\theta - \frac{\eta}{C_R}\Re[\langle\varphi_R, v'\rangle e^{-i\alpha} - i\alpha'\langle\varphi_R, v\rangle e^{-i\alpha}] \\ \quad = \frac{1}{C_R}\Re[\langle\varphi_R, v\rangle e^{-i\alpha}], \\ q'\sin\theta + q\theta'\cos\theta + \frac{\eta}{C_R}\Im[\langle\varphi_R, v'\rangle e^{-i\alpha} - i\alpha'\langle\varphi_R, v\rangle e^{-i\alpha}] = -\frac{1}{C_R}\Im[\langle\varphi_R, v\rangle e^{-i\alpha}], \\ E'q + Eq' + p'\cos\theta - p\theta'\sin\theta - \frac{\eta}{C_R}\Re[\langle\varphi_L, v'\rangle e^{-i\beta} - i\beta'\langle\varphi_L, v\rangle e^{-i\beta}] \\ \quad = \frac{1}{C_R}\Re[\langle\varphi_L, v\rangle e^{-i\beta}], \\ -p'\sin\theta - p\theta'\cos\theta + \frac{\eta}{C_R}\Im[\langle\varphi_L, v'\rangle e^{-i\beta} - i\beta'\langle\varphi_L, v\rangle e^{-i\beta}] = -\frac{1}{C_R}\Im[\langle\varphi_L, v\rangle e^{-i\beta}], \\ 2pp' + 2qq' = -2\Re\langle\psi_c, \psi_c'\rangle. \end{cases}$$

We remark that

$$\langle\psi_c, \psi_c'\rangle = O\left(\nu^2\right),$$

$$\langle\varphi_R, v\rangle = \langle\varphi_R, g|\varphi_R|^{2\sigma}\varphi_R\rangle|a_R|^{2\sigma}a_R + \tilde{O}(\nu) = C_R p^{2\sigma+1}e^{i\alpha} + \tilde{O}(\nu),$$

$$\langle\varphi_L, v\rangle = \langle\varphi_L, g|\varphi_L|^{2\sigma}\varphi_L\rangle|a_L|^{2\sigma}a_L + \tilde{O}(\nu) = C_L q^{2\sigma+1}e^{i\beta} + \tilde{O}(\nu),$$

$$\langle \varphi_R, v' \rangle = \langle \varphi_R, gW\varphi_R \rangle (p' + pi\alpha') e^{i\alpha} + q' \tilde{O}(\nu) + \beta' \tilde{O}(\nu) + \tilde{O}(\nu),$$

$$\langle \varphi_L, v' \rangle = \langle \varphi_L, gW\varphi_L \rangle (q' + qi\beta') e^{i\beta} + p' \tilde{O}(\nu) + \alpha' \tilde{O}(\nu) + \tilde{O}(\nu),$$

where

$$\nu = e^{-\gamma\rho/\hbar},$$

$$\langle \varphi_R, gW\varphi_R \rangle = C_R \big[(\sigma + 1) + \sigma e^{i2\alpha} \big] p^{2\sigma} + \tilde{O}(\nu),$$

$$\langle \varphi_L, gW\varphi_L \rangle = C_L \big[(\sigma + 1) + \sigma e^{i2\beta} \big] q^{2\sigma} + \tilde{O}(\nu).$$

Therefore, the above system takes the form (where the asymptotics $\sim$ means that the remainder term is of order $\tilde{O}(\nu)$)

$$\begin{cases} E'p + Ep' + q' \cos\theta - q\theta' \sin\theta \\ \quad - \frac{\eta}{C_R} \Re [C_R[(\sigma+1) + \sigma e^{2i\alpha}] p^{2\sigma} (p' + ip\alpha') - i\alpha' C_R p^{2\sigma+1}] \sim p^{2\sigma+1}, \\ q' \sin\theta + q\theta' \cos\theta \\ \quad + \frac{\eta}{C_R} \Im [C_R[(\sigma+1) + \sigma e^{2i\alpha}] p^{2\sigma} (p' + ip\alpha') - i\alpha' C_R p^{2\sigma+1}] \sim 0, \\ E'q + Eq' + p' \cos\theta - p\theta' \sin\theta \\ \quad - \frac{\eta}{C_R} \Re [C_R[(\sigma+1) + \sigma e^{2i\beta}] q^{2\sigma} (q' + iq\beta') - i\beta' C_R q^{2\sigma+1}] \sim q^{2\sigma+1}, \\ -p' \sin\theta - p\theta' \cos\theta \\ \quad + \frac{\eta}{C_R} \Im [C_R[(\sigma+1) + \sigma e^{2i\beta}] q^{2\sigma} (q' + iq\beta') - i\beta' C_R q^{2\sigma+1}] \sim 0, \\ 2pp' + 2qq' \sim 0, \end{cases}$$

that is

$$M\big(1 + \tilde{O}(\nu)\big) \begin{pmatrix} E' \\ p' \\ q' \\ \alpha' \\ \beta' \end{pmatrix} = \begin{pmatrix} p^{2\sigma+1} \\ 0 \\ q^{2\sigma+1} \\ 0 \\ 0 \end{pmatrix}$$

where

$$M = \begin{pmatrix} p & E - \eta[(\sigma+1) + \sigma\cos(2\alpha)]p^{2\sigma} & \cos\theta \\ 0 & \eta\sigma p^{2\sigma} \sin(2\alpha) & \sin\theta \\ q & \cos\theta & E - \eta[(\sigma+1) + \sigma\cos(2\beta)]q^{2\sigma} \\ 0 & -\sin\theta & \eta\sigma q^{2\sigma} \sin(2\beta) \\ 0 & 2p & 2q \end{pmatrix.$$

$$\begin{matrix} -q\sin\theta + \eta\sigma p^{2\sigma+1} \sin(2\alpha) & +q\sin\theta \\ q\cos\theta + \eta p^{2\sigma+1} \sigma[1 + \cos(2\alpha)] & -q\cos\theta \\ -p\sin\theta & +p\sin\theta + \eta\sigma q^{2\sigma+1} \sin(2\beta) \\ -p\cos\theta & p\cos\theta + \eta q^{2\sigma+1} \sigma[1 + \cos(2\beta)] \\ 0 & 0 \end{matrix} \Bigg).$$

We consider now, separately, the symmetric and asymmetrical solutions.

Symmetric Solution In the case of the symmetric solution where $\theta = 0$ we can choose the common phase $\alpha = \beta = 0$, by means of a gauge choice. Since $p = q = \frac{1}{\sqrt{2}}$, then a straightforward calculation gives that the matrix M takes the form

$$\det(M) = -8\sigma\eta 2^{-\sigma}\left(1 + \eta\sigma 2^{-\sigma}\right).$$

Hence, for $\eta < 0$ then $\det(M) \neq 0$ provided that $|\eta| \neq \eta^\star$. Hence, we have that (80) holds true.

Asymmetrical Solution In the case of the asymmetrical solution corresponding to $\eta < 0$ then $\theta = 0$, we can still choose the common phase $\alpha = \beta = 0$ by means of a gauge choice, and $p = \sqrt{\frac{1+z}{2}}$ and $q = \sqrt{\frac{1-z}{2}}$ satisfy equation $f_+(z, \eta) = 0$. Then we can set

$$\eta = -\frac{2z}{\sqrt{1-z^2}}\left[\left(\frac{1+z}{2}\right)^\sigma - \left(\frac{1-z}{2}\right)^\sigma\right]^{-1}$$

and

$$E \sim -\sqrt{1-z^2} + \eta\left[\left(\frac{1+z}{2}\right)^{\sigma+1} + \left(\frac{1-z}{2}\right)^{\sigma+1}\right].$$

By means of a straightforward computation it turns out that

$$\det M = 8\sigma z 2^{\sigma+1}\frac{[-(4\sigma+2)z(1-z^2)^\sigma + (1+z)^{2\sigma+1} - (1-z)^{2\sigma+1}][h(z)]}{(1-z^2)[(1+z)^\sigma - (1-z)^\sigma]^3}$$

where $h(z) = g(z) - g(-z)$ enters in the definitio of η' (see (51)). If we remark that the function

$$Q(z) := \left[-(4\sigma+2)z\left(1-z^2\right)^\sigma + (1+z)^{2\sigma+1} - (1-z)^{2\sigma+1}\right] \tag{81}$$

is such that $Q(0) = 0$ and that

$$\frac{dQ}{dz} = (2\sigma+1)\left[4z^2\sigma\left(1-z^2\right)^{\sigma-1} + \left((1-z)^\sigma - (1+z)^\sigma\right)^2\right] > 0, \quad \forall z \in [-1, +1],\ z \neq 0,$$

then we can conclude that $\det M = 0$ if, and only if, $z = 0$ and z is a zero of the function $h(z)$. Then, as in the case of symmetric solution then (80) holds true. The Lemma is so proved. □

Remark 17 In fact, for symmetric solution a straightforward calculation gives that

$$\begin{pmatrix} E' \\ p' \\ q' \\ \alpha' \\ \beta' \end{pmatrix} \sim M^{-1}\begin{pmatrix} p^{2\sigma+1} \\ 0 \\ q^{2\sigma+1} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2^{-\sigma} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{82}$$

On the other hand, for asymmetrical solution corresponding to $z = z^{as}$ a straightforward calculation gives also that

$$\begin{pmatrix} E' \\ p' \\ q' \\ \alpha' \\ \beta' \end{pmatrix} \sim M^{-1} \begin{pmatrix} p^{2\sigma+1} \\ 0 \\ q^{2\sigma+1} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{Q(z)}{2^{\sigma+1}h(z)} \\ -\frac{\sqrt{2}[(1+z)^\sigma-(1-z)^\sigma]^2(1-z^2)\sqrt{1-z}}{2^{\sigma+3}h(z)} \\ \frac{\sqrt{2}[(1+z)^\sigma-(1-z)^\sigma]^2(1-z^2)\sqrt{1+z}}{2^{\sigma+3}h(z)} \\ 0 \\ 0 \end{pmatrix} \tag{83}$$

where the function $Q(z)$, def ned in (81), is such that $Q(-z) = -Q(z)$, $Q(0) = 0$ and $\frac{dQ}{dz} > 0$ for any $z \in (0, 1]$.

Now, we are ready to go back to the slope condition and to state the following.

Lemma 9 *There exists $\hbar_3 \in (0, \hbar_2)$ such that for any $\hbar \in (0, \hbar_3)$ the following statements are satisfied. Let*

$$F_s(\lambda) = \left\| \phi^s_{\lambda,\epsilon} \right\|^2$$

where $\phi^s_{\lambda,\epsilon}$ is the symmetric stationary solutions. Then

$$\frac{d}{d\lambda} F_s(\lambda) < 0.$$

Moreover,

(i) *Let $\sigma \leq \sigma_{threshold}$ and let*

$$F_{as}(\lambda) = \left\| \phi^{as}_{\lambda,\epsilon} \right\|^2$$

where $\psi^{as}_{\lambda,\epsilon}$ is the asymmetrical stationary solutions. Then

$$\frac{d}{d\lambda} F_{as}(\lambda) < 0.$$

(ii) *Let $\sigma > \sigma_{threshold}$ and let*

$$F_{as1}(\lambda) = \left\| \phi^{as1}_{\lambda,\epsilon} \right\|^2 \quad and \quad F_{as2}(\lambda) = \left\| \phi^{as2}_{\lambda,\epsilon} \right\|^2$$

where $\psi^{as1}_{\lambda,\epsilon}$ and $\psi^{as2}_{\lambda,\epsilon}$ are the asymmetric stationary solutions. Then

$$\frac{d}{d\lambda} F_{as1}(\lambda) < 0 \quad and \quad \frac{d}{d\lambda} F_{as2}(\lambda) > 0.$$

Proof We consider, at f rst, the case of the symmetric stationary solution corresponding to $z^\lambda = z^s = 0$. In such a case from (82) it follows that $\frac{dE}{d\eta} = 2^{-\sigma} > 0$ and thus $\frac{dF_s(\lambda)}{d\lambda} < 0$ proving so the firs statement.

Now, we consider the case of asymmetrical stationary solution corresponding to $z^\lambda \neq 0$. In such a case from (83) it follows that

$$\frac{dE}{d\eta} = \frac{Q(z^\lambda)}{2^{\sigma+1}h(z^\lambda)}$$

is an even function and where $Q(z^\lambda) \cdot z^\lambda > 0$. Hence, the sign of $\frac{dE}{d\eta}$ only depends on the sign of $h(z^\lambda)$. We have then showed all the statements in Lemma 9, recalling that (see the results in Sect. 3)

If $\sigma \le \sigma_{threshold}$, then the asymmetrical stationary solution $\phi^{as}_{\lambda,\epsilon}$ corresponding to $z^\lambda > 0$ satisfie condition $h(z^\lambda) > 0$;
If $\sigma > \sigma_{threshold}$, then the asymmetrical stationary solution $\phi^{as1}_{\lambda,\epsilon}$ corresponding to $z^\lambda > 0$ satisfie condition $h(z^\lambda) > 0$;
If $\sigma > \sigma_{threshold}$, then the asymmetrical stationary solution $\phi^{as2}_{\lambda,\epsilon}$ corresponding to $z^\lambda > 0$ satisfie condition $h(z^\lambda) < 0$. □

Remark 18 In the same way, the monotone decreasing behavior of

$$F_a(\lambda) = \left\| \phi^a_{\lambda,\epsilon} \right\|^2$$

associated to the antisymmetric stationary solution follows.

Finally, collecting the results of Proposition 1 and of Lemmata 5, 6 and 9 then Theorem 3 follows.

Remark 19 In Theorem 3, in case of $\sigma > \sigma_{threshold}$ and $|\eta| = \eta^+$, we did not obtain any conclusion about the orbital stability. Recall that $\eta^+ \in (0,\infty)$ is define by $\eta^+ = |\eta(z^+)|$ with $z^+ \in (0,1)$ such that $\eta'(z^+) = 0$ (see Theorem 1). Let $\phi_{\lambda^+,\epsilon^+}$ be the corresponding asymmetric stationary solution to $\lambda^+ = \Omega + \omega E^+$ where

$$E^+ \sim -\sqrt{1-\left(z^+\right)^2} + \eta^+ \left[\left(\frac{1+z^+}{2} \right)^{\sigma+1} + \left(\frac{1-z^+}{2} \right)^{\sigma+1} \right],$$

and ϵ^+ is given by $\omega\eta^+/c$. According to Remark 17, we see *formally*

$$\left. \frac{dF(\lambda)}{d\lambda} \right|_{\lambda=\lambda^+} = - \frac{1}{C_R \sigma} |\epsilon|^{(1-\sigma)/\sigma} \left. \left(\frac{dE}{d\eta} \right)^{-1} \right|_{|\eta|=\eta^+} = 0, \tag{84}$$

since $\eta'(z^+) = 0$. Thus, we are required to prove the stability/instability for the case $dF/d\lambda = 0$. In fact, this case would be included in (2) of Proposition 1, and we would conclude that, when $\sigma > \sigma_{threshold}$, at the transition point $|\eta| = \eta^+$ from $\phi^{as1}_{\lambda,\epsilon}$ to $\phi^{as2}_{\lambda,\epsilon}$, we should have the instability. To show this fact exactly, it suffice to compute $\frac{d^2F}{d\lambda^2}$ and prove that it is not zero at $\lambda = \lambda^+$, following the argument in Maeda [27] (see also some related conditions in [9, 29]). At least "formally" this may be seen as follows: we note that the use of the argument (57) ensures $d\lambda(z)/dz \sim$ negative for $\hbar$ small. By formal calculations,

$$\frac{d^2F}{d\lambda^2} = \left(\frac{\omega}{C_R} \right)^{1/\sigma} \left\{ \frac{1}{\sigma} \left(\frac{1}{\sigma} - 1 \right) |\eta|^{\frac{1}{\sigma}-2} \left(\frac{d\eta}{d\lambda} \right)^2 - \frac{1}{\sigma} |\eta|^{\frac{1}{\sigma}-1} \frac{d^2\eta}{d\lambda^2} \right\},$$

$$\frac{d\eta}{d\lambda} = \eta'(z) \Big/ \frac{d\lambda}{dz}, \qquad \frac{d^2\eta}{d\lambda^2} = \left\{ \eta''(z) \frac{d\lambda}{dz} - \eta'(z) \frac{d^2\lambda}{dz^2} \right\} \Big/ \left(\frac{d\lambda}{dz} \right)^3.$$

We have seen in Sect. 3 that $\eta''(z^+) \neq 0$, which implies $\frac{d^2F}{d\lambda^2}|_{\lambda=\lambda^+} \neq 0$. However a rigorous justificatio seems more complex and we do not pursue in this direction in the present paper.

Appendix: Stationary States for a Nonlinear Toy Model

Here, we introduce, as a toy model, the semiclassical Schrödinger equation with two attractive symmetric Dirac's δ which is partially investigated in [26].

$$i\hbar\frac{\partial\psi}{\partial t} = H_0\psi + \epsilon g|\psi|^{2\sigma}\psi, \qquad \|\psi(\cdot,t)\| = 1, \quad x \in \mathbb{R},\ t \in \mathbb{R}, \tag{85}$$

where

$$H_0 = -\hbar^2\frac{d^2}{dx^2} + \beta\delta_{-a} + \beta\delta_{+a}$$

for some $a \in \mathbb{R}$ and $\beta < 0$. Hereafter, for the sake of definiteness we assume that $g \equiv 1$.

Even though this operator H_0 with Dirac measures do not satisfy the assumptions for the potential $V(x)$ in the Introduction, the two-level approximation used in the previous sections is directly applicable to this example. In this section, we will give some remarks for the properties of H_0, and the general theory we have used in the previous sections, for example, Cauchy problem and the orbital stability. We remark that a symmetric-breaking phenomenon for the cubic nonlinear Schrödinger equation with double Dirac potential is discussed in [21] too, but not in the semiclassical regime.

6.1 Spectrum of the Linear Operator

The spectral problem

$$\left[-\hbar^2\frac{d^2}{dx^2} + \beta\delta_{-a} + \beta\delta_{+a}\right]\psi = \mathcal{E}\psi$$

for $\beta < 0$ is equivalent to the spectral problem

$$H_\alpha\psi = E\psi$$

where we set $E = \mathcal{E}/\hbar^2$ and where the linear operator

$$H_\alpha = -\frac{d^2}{dx^2}, \quad \text{with } \alpha = \beta/\hbar^2,$$

is self-adjoint on the domain

$$D(H_\alpha) = \left\{\psi \in H^2\big(\mathbb{R}\setminus\{\pm a\}\big) \cap H^1(\mathbb{R}) : \frac{d\psi}{dx}(\pm a + 0) - \frac{d\psi}{dx}(\pm a - 0) = \alpha\psi(\pm a)\right\}.$$

Let us recall some basic properties of the spectrum of H_α (see, e.g., [2, 26] for details).

The essential spectrum of H_α is purely absolutely continuous and coincides with the positive real axis:

$$\sigma_{\text{ess}}(H_\alpha) = \sigma_{\text{ac}}(H_\alpha) = [0, +\infty).$$

The discrete spectrum consists of two eigenvalues, at least, given by means of the Lambert's special function $W(x)$ such that $W(x)e^{W(x)} = x$.

If $\alpha < 0$ the discrete spectrum is not empty, in particular,

– if $a \leq -\frac{1}{\alpha}$, then the discrete spectrum of H_α consists of only one eigenvalue $E_1(a,\alpha)$ define as

$$E_1(a,\alpha) = -\frac{1}{4a^2}\left[W\left(-a\alpha e^{a\alpha}\right) - a\alpha\right]^2;$$

– if $a > -\frac{1}{\alpha}$, then the discrete spectrum of H_α consists of two eigenvalues $E_1(a,\alpha)$ and $E_2(a,\alpha)$ where

$$E_2(a,\alpha) = -\frac{1}{4a^2}\left[W\left(+a\alpha e^{a\alpha}\right) - a\alpha\right]^2.$$

The two associated eigenvectors take the form:

(i) Let

$$k_1 = \sqrt{E_1} = \frac{i}{2a}\left[W\left(-a\alpha e^{a\alpha}\right) - a\alpha\right]$$

then

$$\varphi_1(x) = C_1 \begin{cases} e^{-ik_1x}, & x < -a, \\ \frac{2k_1+i\alpha}{2k_1}(e^{-ik_1x} + e^{ik_1x}), & -a \leq x \leq +a, \\ e^{+ik_1x}, & x > +a, \end{cases}$$

where C_1 is the normalization constant given by

$$C_1 = \frac{|k_1|}{\sqrt{(2|k_1| + \alpha)(2|k_1|a + a\alpha + 1)}}.$$

(ii) Let

$$k_2 = \sqrt{E_2} = \frac{i}{2a}\left[W\left(+a\alpha e^{a\alpha}\right) - a\alpha\right]$$

then

$$\varphi_2(x) = C_2 \begin{cases} e^{-ik_2x}, & x < -a, \\ \frac{2k_2+i\alpha}{2k_2}(e^{-ik_2x} - e^{ik_2x}), & -a \leq x \leq +a, \\ -e^{+ik_2x}, & x > +a, \end{cases}$$

where C_2 is the normalization constant given by

$$C_2 = \frac{|k_2|}{\sqrt{-(2|k_2| + \alpha)(2|k_2|a + a\alpha + 1)}}.$$

Remark 20 Recalling that the Lambert's special function $W(x)$ has the following asymptotic behavior

$$W(x) \sim x - x^2 + \frac{3}{2}x^3 + O\left(x^4\right)$$

then it follows that the splitting is exponentially small, namely

$$|\mathcal{E}_1 - \mathcal{E}_2| \sim \hbar^2\alpha^2 e^{a\alpha} = \frac{\beta^2}{\hbar^2} e^{a\beta/\hbar^2} = \frac{\beta^2}{\hbar^2} e^{-a|\beta|/\hbar^2}.$$

Remark 21 The resolvent formula for H_α is known: let $h \in C_0^\infty(\mathbb{R})$, $k^2 \in \rho(H_\alpha)$, and $\Im k > 0$. The resolvent is expressed as follows,

$$\left(\left[H_\alpha - k^2\right]^{-1} h\right)(x) = \int_{\mathbb{R}} K_\alpha(x, y; k) h(y)\, dy,$$

with the kernel K_α having the following form

$$K_\alpha(x, y; k) = K_0(x, y; k) + \sum_{j=1}^{4} K_\alpha^j(x, y; k)$$

where

$$\begin{aligned}
K_0(x, y; k) &= \frac{i}{2k} e^{ik|x-y|}, \\
K_\alpha^1(x, y; k) &= \frac{\alpha(2k + i\alpha)}{2k((2k + i\alpha)^2 + \alpha^2 e^{i4ka})} e^{ik|x+a|+ik|y+a|}, \\
K_\alpha^2(x, y; k) &= \frac{-i\alpha^2 e^{2ika}}{2k((2k + i\alpha)^2 + \alpha^2 e^{i4ka})} e^{ik|x+a|+ik|y+a|}, \\
K_\alpha^3(x, y; k) &= K_\alpha^2(-x, -y; k), \\
K_\alpha^4(x, y; k) &= K_\alpha^1(-x, -y; k).
\end{aligned}$$

We consider here *the case $a > -1/\alpha$ with $\alpha < 0$*. In such a case we have that the linear problem has two negative nondegenerate eigenvalues:

$$E_1 < E_2 < 0. \tag{86}$$

6.2 Nonlinear Problem

The local existence of solution in $H^1(\mathbb{R})$, and conservation laws of energy and L^2 norm are verifie in a similar way to [13]; the authors in [13] applied Theorem 3.7.1 of [5] to the case of $a = 0$. In our case, we take $-H_\alpha + E_1$ for the operator A of Theorem 3.7.1 of [5]. Then this operator A is a self adjoint operator on $X = L^2(\mathbb{R})$ with the domain $D(A) = D(H_\alpha)$, and also $A \le 0$. We take $X_A = H^1(\mathbb{R})$ whose norm is equivalent to $H^1(\mathbb{R})$ norm

$$\|v\|_{X_A}^2 = \left\|(d/dx)v\right\|^2 + (1 - E_1)\|v\|^2 + \alpha\left(\left|v(a)\right|^2 + \left|v(-a)\right|^2\right).$$

Condition (3.7.2) of Theorem 3.7.1 of [5] is satisfie with $p = 2$, and other conditions hold since we are in one dimensional case.

For the existence of bifurcation of stationary solutions, it suffice to repeat the similar arguments in Sect. 3 (Theorem 1), but in $H^1(\mathbb{R})$ instead of $H^2(\mathbb{R})$.

We can check the assumptions for the orbital stability/instability of stationary states in $H^1(\mathbb{R})$, as in Sect. 5, using the two level approximation. However, due to the singularity of Dirac potentials, we cannot consider the linearized problem with a more smooth domain than $H^1(\mathbb{R})$, as, for ex., was considered in [11]. Remark also that H^2 regularity allows us simply to have the nonlinear instability assuming the existence of an unstable eigenvalue (e.g. [7]). We thus give some explanations here.

We consider as follows the linearized problem around the real valued rescaled stationary state $\phi_{\epsilon,\lambda}$ (ϵ and λ are fixed here to discuss the general theory, so we denote it simply by ϕ from now on).

$$\frac{dv}{dt} = Av + F(v), \quad v = (v_1, v_2) \in D(A) \text{ with } v_1 = \Re v,\ v_2 = \Im v, \tag{87}$$

where $A(v_1, v_2) = (L_-^{\lambda,\epsilon} v_2, -L_+^{\lambda,\epsilon} v_1)$. A is a linear operator in $\mathbb{L}^2(\mathbb{R})$ with domain

$$D(A) = \Big\{ v \in \mathbb{H}^2\big(\mathbb{R} \setminus \{\pm a\}\big) \cap \mathbb{H}^1(\mathbb{R}) : \frac{dv_j}{dx}(\pm a + 0) - \frac{dv_j}{dx}(\pm a - 0) = \alpha v_j(\pm a), \\ j = 1, 2 \Big\},$$

where $\mathbb{H}^m(\mathbb{R}) = H^m(\mathbb{R}) \times H^m(\mathbb{R})$ for $m \in \mathbb{Z}$. The nonlinear term is given by

$$F(v) = i\big\{|\phi + v|^{2\sigma}(\phi + v) - |\phi|^{2\sigma+1} - (\sigma + 1)|\phi|^{2\sigma} v - \sigma|\phi|^{2\sigma}\bar{v}\big\}.$$

This operator A generates its C_0-semigroup on $\mathbb{L}^2$ denoted by e^{tA}. Concerning the spectrum of A, we have the following Lemma. We note that we complexify the space when we consider the spectrum problem of A.

Lemma 10 $\sigma_{\text{ess}}(A) \subset i\mathbb{R}$.

Proof The operator A can be rewritten in the following form (still denoted by A with abuse of notation)

$$Av = -i\big\{H_\alpha - \lambda - (\sigma + 1)|\phi|^{2\sigma} - \sigma|\phi|^{2\sigma}\mathcal{T}\big\}v$$

where $\mathcal{T}v = \bar{v}$ is a nonsymmetric bounded linear operator. We consider the operator iA as the operator A_0 perturbed by the operator C, i.e.

$$iA = A_0 + C,$$

where $A_0 = H_\alpha - \lambda$, and $C = -(\sigma + 1)|\phi|^{2\sigma} - \sigma|\phi|^{2\sigma}\mathcal{T}$. It suffice to prove that $\sigma_{\text{ess}}(iA) \subset \mathbb{R}$. To this end, we remark the following facts.

- Since $\phi \in H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$, C is a bounded operator.
- It is known that $\sigma_{\text{ess}}(A_0) = [-\lambda, +\infty) \subset \mathbb{R}$.
- $C[A_0+\lambda+1]^{-1}$ is a compact operator; indeed, $[A_0+\lambda+1]^{-1}$ is an integral operator with kernel given by $K_\alpha^0(x, y; i) + \sum_{j=1}^4 K_\alpha^j(x, y; i)$. One can see, for e.g., that $K_\alpha^j(x, y; i)$ ($j = 1, 2, 3, 4$) and $|\phi|^{2\sigma} K_\alpha^0(x, y; i)$ are bounded on $L^2(\mathbb{R}^2, dx\,dy)$. This implies that $C[A_0 + \lambda + 1]^{-1}$ is Hilbert-Schmidt.

Then $\sigma_{\text{ess}}(A_0) = \sigma_{\text{ess}}(iA)$ by means of the Weyl criterion. □

As for eigenvalues of A, there are finitely many eigenvalues at the exterior of the essential spectrum for $\hbar$ small. Indeed, $\lambda < 0$ for $\hbar$ small. Our aim is now to conclude the following proposition.

Proposition 2 *Assume that A has an eigenvalue λ_m with $\Re\lambda_m > 0$, and that for any $\varepsilon > 0$, there exists $M > 0$ such that*

$$\left\| e^{tA} v \right\|_{\mathbb{L}^2} \leq M e^{(1+\varepsilon)(\Re\lambda_m)t} \|v\|_{\mathbb{L}^2}, \tag{88}$$

for any $v \in \mathbb{L}^2(\mathbb{R})$ and for any $t \geq 0$. Then, there exists $\varepsilon_0 > 0$, such that for any $\delta > 0$ there exist a time T and an initial data $u_0 \in D(H_\alpha)$ satisfying $\|u_0 - \phi\|_{H^1} < \delta$, and $\inf_{\theta\in\mathbb{R}} \|u(T) - e^{i\theta}\phi\| \geq \varepsilon_0$.

Proposition 2 means that the linearized instability implies the nonlinear instability.

We may prove Proposition 2 as in the proof of Theorem 6.1 of Part II of [18, 19] or in [17]. Note that we have the Dirac measures in the equation and we do not expect that the solution is smooth as we have mentioned before, thus we make use rather of the time derivative, mimicking the proof of [8], than of the way of [18, 19]. Here, for the sake of completeness, we give an outline of proof.

Proof (Sketch of proof) Let z_m be the associated eigenfunction to λ_m. Let $u_\delta(t)$ be the solution of (85) with initial data $u_\delta(0) = \phi + \delta z_m$. Since $\phi, z_m \in D(H_\alpha)$, $u_\delta(\cdot) \in C([0, T], D(H_\alpha)) \cap C^1([0, T], L^2)$ for some $T > 0$ (see Theorem 3.1 of [1]). Remark that $u_\delta(t) = e^{-i\lambda t}(\phi + v_\delta(t))$ with $v_\delta(t)$ satisfying (87) with $v_\delta(0) = \delta z_m$. $v_\delta(t)$ satisfie the following integral equations for any $t \in [0, T]$,

$$v_\delta(t) = \delta e^{\lambda_m} z_m + \int_0^t e^{(\tau-t)A} F\big(v_\delta(\tau)\big)\, d\tau,$$

$$\partial_t v_\delta(t) = \lambda_m \delta e^{\lambda_m} z_m + e^{tA} F(\delta z_m) + \int_0^t e^{(\tau-t)A} \partial_\tau F\big(v_\delta(\tau)\big)\, d\tau.$$

Since we are in the one dimensional case, it is easy to estimate the nonlinear term $F(v)$ as follows,

$$\left\| \partial_t F\big(v_\delta(t)\big) \right\| \leq C\Big(\left\| v_\delta(t) \right\|_{H^1} + \left\| v_\delta(t) \right\|_{H^1}^{2\sigma} \Big) \left\| \partial_t v_\delta(t) \right\|,$$

$$\left\| F\big(v_\delta(t)\big) \right\| \leq C_2\Big(\left\| v_\delta(t) \right\|_{H^1} + \left\| v_\delta(t) \right\|_{H^1}^{2\sigma+1} \Big).$$

Then, for some $C_0 > 0$ and for some T_δ when δ is suff ciently small, we may estimate

$$\left\| v_\delta(t) \right\|_{H^1} + \left\| \partial_t v_\delta(t) \right\| \leq 2C_0 \delta e^{\lambda_m t},$$

for any $t \in [0, T_\delta]$. We apply this quantity $\|v_\delta(t)\|_{H^1} + \|\partial_t v_\delta(t)\|$ as $V_\delta(t)$ in Theorem 2 of [8]. We then repeat their arguments in [8] to get $\|v_\delta(T_\delta)\| \geq (\delta/2)\|z_m\|$. □

We complete our whole arguments with a verificatio of the existence of an eigenvalue λ_m satisfying (88). It follows from [17] or Part I of [18, 19] that there exists a nonzero real eigenvalue of the linearized operator A, if (2) or (3) of Proposition 1 in Sect. 5 hold. Let λ_0 be the maximal positive eigenvalue. Once we have proved the spectral mapping theorem $\sigma(e^{At}) = e^{\sigma(A)t}$, the spectral radius of e^{At} is $e^{\lambda_0 t}$. Thus we have (88) using Lemma 3 of [31]. This implies that we can take λ_0 as λ_m in Proposition 2.

The spectral mapping theorem in fact follows from a resolvent estimate in Lemma 11 below, combined with the arguments in [15].

Lemma 11 *Let $z = a + i\tau$ with $a, \tau \in \mathbb{R}$ and $a \neq 0$. For $|\tau|$ sufficiently large, there exists a constant $C_a > 0$, such that*

$$\left\|(z - A)^{-1}\right\|_{\mathcal{L}(\mathbb{L}^2)} \leq C_a.$$

Proof of Lemma 11 We begin with some preparations. For fi ed $z = a + i\tau$ with $a \in \mathbb{R} \setminus \{0\}$ and $\tau \in \mathbb{R}$, we write the operator $z - A$ as follows,

$$\begin{aligned} z - A = M_z - B_{\lambda,\epsilon} &= \begin{pmatrix} z & -H_\alpha \\ H_\alpha & z \end{pmatrix} + \begin{pmatrix} 0 & \phi^{2\sigma} + \lambda \\ -\lambda - (2\sigma + 1)\phi^{2\sigma} & 0 \end{pmatrix} \\ &= M_z\left[Id - M_z^{-1} B_{\lambda,\epsilon}\right]. \end{aligned}$$

Indeed, we see that $z \notin i\mathbb{R}$, therefore, by Remark 21, the inverse of $H_\alpha^2 + z^2 = (H_\alpha - iz) \times (H_\alpha + iz)$ exists, thus the inverse of M_z exists too. We can express M_z^{-1} as follows,

$$M_z^{-1} = \begin{pmatrix} z\{(H_\alpha)^2 + z^2\}^{-1} & H_\alpha\{(H_\alpha)^2 + z^2\}^{-1} \\ -H_\alpha\{(H_\alpha)^2 + z^2\}^{-1} & z\{(H_\alpha)^2 + z^2\}^{-1} \end{pmatrix}.$$

We estimate now the inverse M_z^{-1} by means of the following lemma.

Lemma 12 *Let $a \neq 0$. There exist $C_a, \tau_0 > 0$ such that for any $z = a + i\tau$ with $|\tau| \geq \tau_0$, we have*

$$\left\|M_z^{-1}\right\|_{\mathcal{L}(\mathbb{L}^2)} \leq \frac{C_a}{1 + |\tau|}.$$

Proof of Lemma 12 We benefi from the explicit resolvent formula of H_α in Remark 21. Let

$$f_\alpha(x) = \left(\left[H_\alpha - k^2\right]^{-1} h\right)(x)$$

and consider $k^2 = iz = -\tau + ia$. First, we remark that

$$f_0(x) = \int_{\mathbb{R}} K_0(x, y; k) h(y)\, dy = \frac{i}{2k}\left(e^{ik|\cdot|} \star h(\cdot)\right)(x)$$

may be estimated, by Young inequality, as follows,

$$\|f_0\| = \frac{1}{2|k|}\left\|e^{ik|\cdot|} \star h(\cdot)\right\| \leq \frac{1}{2|k|}\left\|e^{ik|\cdot|}\right\|_{L^1}\|h(\cdot)\| \leq \frac{C}{|k||\Im k|}\|h\| \leq \frac{C}{\sqrt{\tau}}\|h\|,$$

since

$$\sqrt{\tau + ia} = \sqrt{\tau}\sqrt{1 + \frac{ia}{\tau}} = \sqrt{\tau} + \frac{ia}{2\sqrt{\tau}} + O\tau^{-3/2}, \quad \text{as } |\tau| \to \infty,$$

and $\Im\sqrt{\tau + ia} \sim \frac{1}{2}\frac{a}{\sqrt{\tau}}$ for $\tau \gg 1$.

Next, we set

$$f_\alpha^j(x) = \int_{\mathbb{R}} K_\alpha^j(x, y; k) h(y)\, dy, \quad j = 1, 2, 3, 4.$$

By this definition $f_\alpha = f_0 + \sum_{j=1}^4 f_\alpha^j$. Thus we estimate each term f_α^j. For example,

$$f_\alpha^1(x) = \frac{\alpha(2k+i\alpha)}{2k((2k+i\alpha)^2+\alpha^2 e^{i4ka})} e^{ik|x+a|} \int_{\mathbb{R}} e^{ik|y+a|} h(y)\, dy$$

and then, for sufficientl large $|\tau|$,

$$\begin{aligned}\|f_\alpha^1\| &\le \frac{C}{|k|^2|\Im k|} \left| \int_{\mathbb{R}} e^{ik|y+a|} h(y)\, dy \right| \le \frac{C}{|k|^2|\Im k|} \left\| e^{ik|\cdot+a|} h(\cdot) \right\|_{L^1} \\ &\le \frac{C}{|k|^2|\Im k|} \left\| e^{ik|\cdot+a|} \right\| \|h\| \le \frac{C}{|k|^2|\Im k|^2} \|h\| \le \frac{C}{|\tau|} \|h\|.\end{aligned}$$

Similarly, the other terms f_α^j, $j = 2, 3, 4$, are estimated. Thus, it follows that for $\Im z = a$ fi ed and $\Re z = -\tau$ large enough, then

$$\left\| [H_\alpha - iz]^{-1} h \right\| \le \frac{1}{|\tau|} \|h\|$$

since H_α is a self-adjoint operator. Therefore, decomposing $H_\alpha\{(H_\alpha)^2 + z^2\}^{-1}$ as

$$H_\alpha \left\{ (H_\alpha)^2 + z^2 \right\}^{-1} = (H + iz)^{-1} + iz(H - iz)^{-1}(H + iz)^{-1},$$

we also obtain, for large $|\tau| \gg 1$,

$$\left\| H_\alpha \left\{ (H_\alpha)^2 + z^2 \right\}^{-1} \right\|_{\mathcal{L}(\mathbb{L}^2)} \le \frac{C_a}{1 + |\tau|}.$$

Similarly, for large $|\tau|$,

$$\left\| z \left\{ (H_\alpha)^2 + z^2 \right\}^{-1} \right\|_{\mathcal{L}(\mathbb{L}^2)} \le \frac{C_a}{1 + |\tau|}.$$

□

We go back to the proof of Lemma 11. We put $T_z = M_z^{-1} B_{\varepsilon,\lambda}$, and we write entries of this operator T_z:

$$T_z = \begin{pmatrix} H_\alpha\{(H_\alpha)^2 + z^2\}^{-1}(-\lambda - (2\sigma+1)\phi^{2\sigma}) & z\{(H_\alpha)^2 + z^2\}^{-1}(\phi^{2\sigma} + \lambda) \\ z\{(H_\alpha)^2 + z^2\}^{-1}(-\lambda - (2\sigma+1)\phi^{2\sigma}), & -H_\alpha\{(H_\alpha)^2 + z^2\}^{-1}(\phi^{2\sigma} + \lambda) \end{pmatrix}.$$

Since we are in one dimension, it follows that $\phi \in H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$, thus we can estimate, for example, as

$$\left\| H_\alpha \left\{ (H_\alpha)^2 + z^2 \right\}^{-1} \left(\phi^{2\sigma} + \lambda \right) \right\|_{\mathcal{L}(\mathbb{L}^2)} \le C \left\| H_\alpha \left\{ (H_\alpha)^2 + z^2 \right\}^{-1} \right\|_{\mathcal{L}(\mathbb{L}^2)}.$$

Therefore, combining with the above proof for Lemma 11, we have that for any τ with $|\tau| \ge \tau_0$, $\|T_z\|_{\mathcal{L}(\mathbb{L}^2)} \le 1/2$. This implies immediately for any $u \in \mathbb{L}^2$

$$\left\| (Id - T_z) u \right\|_{\mathbb{L}^2} \ge \|u\|_{\mathbb{L}^2} - \|T_z u\|_{\mathbb{L}^2} \ge (1/2) \|u\|_{\mathbb{L}^2},$$

that is, $Id - T_z$ is invertible for $|\tau| \geq \tau_0$. Then, finall , we get that for any $z = a + i\tau$ with $|\tau| \geq \tau_0$, $a \neq 0$,

$$\left\|(z-A)^{-1}\right\|_{\mathcal{L}(\mathbb{L}^2)} = \left\|(Id - T_z)^{-1} M_z^{-1}\right\|_{\mathcal{L}(\mathbb{L}^2)} \\ \leq \left\|(Id - T_z)^{-1}\right\|_{\mathcal{L}(\mathbb{L}^2)} \left\|M_z^{-1}\right\|_{\mathcal{L}(\mathbb{L}^2)} \leq 2C_a.$$

The proof of Lemma 11 is then completed. □

Lastly, recall that the assumptions (2) or (3) of Proposition 1 in Sect. 5 ensure the existence of a positive real eigenvalue of A. As we checked in Sect. 5, the assumptions (2) or (3) of Proposition 1 in Sect. 5 may be verified for small $\hbar > 0$, depending on σ, η, and the sort of stationary solution. Namely, Theorem 3 in Sect. 5 is valid for (85).

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