ET-Hollow Module and ET-Lifting Module

Firas sh. Fandi[1],
[1] Mathematics Department College of Education for Pure Sciences,
University Of Anbar, Ramadi, Iraq
Frisshker1978@gmail.com

Sahira M. Yaseen[2]
[2] Mathematics Department College of Science
University of Baghdad, Iraq
Sahira.mahmood@gmail.com

Abstract: Let M be a R-module , where R be a commutative ring with identity,
In this paper, we defined a new types of module namely "ET-hollow(ET-holl.) and ET-lifting(ET-lift.) modules". An R-module M is called ET-holl. module, if for all sub-module H of M then H $\ll_{ET}$M. An R-module M is called An R-module M is called ET-lifting module, if for all H ≤ M , there exists X $\leq_{\oplus}$M and L $\ll_{ET}$M, such that H=X+L. We give many characterizations of ET-holl. and ET-lifting modules, Also we give the relation between T-hollow and ET-holl. and relation between T-lifting modules and ET-lift. modules.

Key Words: T-small sub-module, ET-small sub-module, e-hollow module, T-hollow module,ET-hollow module, T-lifting module, ET-lifting module

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1. Introduction

Throughout this paper, M be an R-module; where R be any a commutative ring and R with unity, a proper sub-module H of M is called small(H≪ M), if for all X≤ M, such that M=H+X then M=X[1]. A sub-module H of M is called essential of M , if H∩B=0 , ∀ B ≤ M, then B=0[2]. A sub-module H of M is e-small (H$\ll_e$M), if for every sub-module B of M such that H+B=M, then B=M[3]. Let T ≤ M, A sub-module H of M is said to be "T-small sub-module of M", if for all K ≤ M such that T ⊆ H+K , then T ⊆ K [4]. Recalled a module M is e-hollow module, if for all proper sub-module H of M is e-small of M[5]. M is e-lifting module if for all sub-module H of M, there exists K ≤ M such that $\frac{H}{K} \ll_e \frac{M}{K}$[6]. In[7] we defined ET-small sub-module of M, Let T ≤ M and A sub-module H of M is "ET-small sub-module of M", if for all K $\leq_e$ M such that T ⊆ H+K , then T ⊆ K, and we give some properties of ET-small sub-module of M, Clearly. Every T-small sub-module of M is ET-small sub-module of M but the converse is not true. Let M be an R-module and Let T, N_1 and N_2 ≤ M, Then $N_1 \ll_{ET}$ M and $N_2 \ll_{ET}$ M if and only if $N_1+N_2 \ll_{ET}$ M. assume that T,H and L are sub-modules of any R-module M such that T ≤ N and H ≤ L ≤ M and L$\ll_e$ M, if H$\ll_{ET}$ M then H$\ll_{ET}$ L, and there are many of other properties of ET-small [7].

In this work, we give the concept of ET-holl. module and ET-lifting module. An module M is called ET-holl. $module$, if ∀ H ≤ M then H$\ll_{ET}$M. An R-module M is called ET-lifting module, if for all H ≤ M , there exists X $\leq_{\oplus}$M and L $\ll_{ET}$M such that H=X+L. Also We give many properties of ET-holl. module and ET-lifting module.

In section 1, we defined ET-holl. module with we give some properties of ET-holl. module. Every T-hollow module is ET-holl. module, but the converse is not true, but if M be a non-zero module and T =M. thus M is ET-holl. module iff M is e-holl. $module$. Let T ≤ M .If M is ET-holl. module, then every essential sub-module K of M such that T⊆K, then K is a T-hollow module. Let (T≠ 0) be an essential sub-module of a module M, if M is ET-holl. module, then T is indecomposable. there are some of other properties of ET-holl. module.

In section 2, we defined ET-lifting module with we give some properties of ET-lifting module. Also, we list many of their important properties. Also we give the relation between T-hollow and ET-holl. and relation between T-lifting modules and ET-lifting modules.

Section 1: ET-hollow module

In section this, we defined ET-holl. module and we give many properties of ET-holl. module.

Definition 1.1: Let T be a sub-module of ($M \neq 0$) module. We say that M is a ET-holl. module if for every sub-module H of M is a ET-small sub-module of M.

Remark and Example 1.2:

1. Consider Z_6 as Z-module .Let T={0,3}, then Z_6 is ET-holl. since the only essential sub-module of Z_6. Then the sub-modules H={0} , {0,3}, {0,2,4} and Z_6,then T ⊆ H+ Z_6. Implies that T ⊆Z_6 , then for every sub-module H of Z_6 is a ET-small sub-module of Z_6.

2. Every ($M \neq 0$) T-hollow module is ET-holl. module, but the converse is not true, for example Consider Z_4 as Z-module. let T={0,2}and B={0,2}, then B is not T- small sub-module of Z_4 , since T ⊆ B+ {0}, but T⊈ {0}. So Z_4 is not T-hollow module . but Z_4 is ET-holl. module. Since for every sub-module H of M is a ET-small sub-module of M, since H ={0},{0,2}and Z_4 are sub-modules of Z_4 then T ⊆ H+ X, ∀ X $\leq_e$ Z_4, but the only essential sub-modules of Z_4 are {0,2}and Z_4 since T ⊆ {0,2} and T ⊆ Z_4. So every sub-module H of Z_4 is ET-small sub-module of Z_4. Then Z_4 is ET-hollow module.

3. Let ($M \neq 0$) be an uniform module and let T =M , so M is holl. module iff M is ET-holl. $module$.

4. Let ($M \neq 0$) be an module and T =M. so M is ET-holl. module iff M is e-holl. $module$.

5. Let ($M \neq 0$) be an module and T =o, hence M is ET-holl. $module$. by remark (1.2-3) [7].

Proof 3**:** Assume that ($M \neq 0$) be an uniform module such that T =M and M is hollow module to prove M is ET-holl. module . suppose that H ≤ M to prove $H \ll_{ET}$ M, let T ⊆ H+ X, for every essential sub-module X of M , to prove T ⊆ X, since T =M so M ⊆ H+ X and H+ X ⊆ M therefor M = H+ X , but M is hollow module then H ≪ M, so M=X implies that T=X therefor T ⊆ X then $H \ll_{ET}$ M . thus M is ET-holl. module. Conversely let M is ET-holl. module, to show M is ET-holl. module . let H ≤ M , to show H ≪ M , so M=H+X then T ⊆ H+X , ∀X ≤ M, but M is uniform module then X $\leq_e$ M, since M is ET-holl. module, then $H \ll_{ET}$ M, therefor T ⊆ X then M ⊆ X and X ⊆ M , thus M=X, implies that H ≪ M, then M is hollow module.

4. Let M is hollow module and T =M and M is hollow module to prove M is e-hollow module . let H ≤ M to show $H \ll_e M$, let M = H+ B, for every essential sub-module B of M , to prove B = M, since T =M so T= H+ B therefor T ⊆ H+B, but M is ET-holl. module, then $H \ll_{ET} M$, so T⊆ B thus M ⊆ B and B⊆ M then B = M , hence $H \ll_e M$. then M is e-hollow module. Conversely let M is e-hollow module, to show M is ET-hollow module . let H ≤ M , to prove $H \ll_{ET}$ M , let T ⊆ H+A , ∀ A $\leq_e$M, since T =M so M ⊆ H+A and H+A ⊆ M thus M= H+ A, since M is e-hollow module then $H \ll_e M$, therefor A=M, then T=A therefor T ⊆ A, thus $H \ll_{ET}$ M then M is ET-hollow module.

Proposition 1.3: Let T be sub-module of a non -zero module M .If M is ET-holl. module, then every essential sub-module H of M such that T⊆H, then H is a T-hollow module.

Proof:- Let H be an essential sub-module of M such that T⊆H. to prove H is a ET-holl. module. so Let L be a sub-module of H, show to $L \ll_{ET} H$, Since H ≤ M and M be an ET-holl. module, then $L \ll_{ET} M$ .Then, by (1. 4) [7]. then $L \ll_{ET} H$. Thus H is a T-hollow module .

Proposition 1.4: *Let* T be a non-zero essential sub-module of a module M .If M is ET-holl. module .Then T is indecomposable.

Proof:- Let M be an ET-holl. module and T is decomposable, then T = H⨁X, for some H and X are non-trivial sub-modules of T. Therefore T⊈H. since X ≤ T = (H⨁X) $\leq_e$M, then X $\leq_e$M, since M is ET-holl. module and T⊆ ⊆ H⨁X, then $H \ll_{ET} M$. therefore T⊆X and X⊆T, hence T = X which is a contradiction .Then T is indecomposable .

Proposition 1.5: let f : M → N be an epimorphism , where M and N are a non -zero modules, if M be an ET-holl. module, Then N is Ef(T)-hollow module .

Proof:**-** Let M is an ET-holl. module and let f : M → N be an epimorphism .To prove N is Ef (T)-hollow module. Let H be a sub-module of N, to prove $H \ll_{Ef(T)} N$, since f be an epimorphism , there exist K ≤ M such that f(K) = H , But M is ET-holl. module, then K$\ll_{ET}$M, and by lemma (1.8)[7], we get f(k) $\ll_{Ef(T)}$N, implies that H$\ll_{Ef(T)}$N. then N is Ef(T)-hollow module.

Proposition 1.6: Let T and H be sub-modules of a module M such that H ⊆ T. H $\leq_c$ M. If H$\ll_{ET}$M , $\frac{M}{H}$ be any E($\frac{T}{H}$) - holl. module, hence M is ET-holl.

Proof:- Let H$\ll_{ET}$M and $\frac{M}{H}$ is E($\frac{T}{H}$) –holl. module. To prove M is ET-holl. let B be a sub-module of M, To prove B$\ll_{ET}$M, let T ⊆B+L, ∀ L $\leq_e$M. Then $\frac{T}{H} \subseteq \frac{B+L}{H}$ and hence $\frac{T}{H} \subseteq \frac{B+H}{H} + \frac{L+H}{H}$. since (L ≤ L+H ≤ M and L $\leq_e$M then L+H $\leq_e$M and since H $\leq_c$ M then $\frac{L+H}{H} \leq_e \frac{M}{H}$), but $\frac{M}{H}$ is E($\frac{T}{H}$) -hollow module, therefore $\frac{B+H}{H} \ll_{E(\frac{T}{H})} \frac{M}{H}$, hence $\frac{T}{H} \subseteq \frac{L+H}{H}$, Thus T⊆L+H . Since H$\ll_{ET}$M , then T⊆L. hence M is ET-holl. module.

Proposition 1.7: Let M be a module and H and T are sub-modules of M, such that H⪇T and H $\leq_e$M. If M is ET-holl. Module, and $\frac{T}{H}$ is non-zero $finitely\ generated$; hence T be a finitely generated.

Proof: Let $\frac{T}{H}$ = $R(x_1$+H)+$R(x_2$+H)+…+$R(x_n$+H)= (Rx_1+H)+(Rx_2+H)+…+(Rx_n+H) where x_i ∈T, for all i =$1,…,n$. then Rx_1+ Rx_2+…+ Rx_n ⊆T .To prove T ⊆ Rx_1+Rx_2+…+ Rx_n , let a ∈T .Then a +H∈ $\frac{T}{H}$, then there exist $r_1,…,r_n$ ∈R, such that a +H = r_1 (x_1+H) + r_2 (x_2+H) +…+ r_n (x_n+H) = (r_1x_1 + r_2x_2 +…+ r_nx_n) +H. Hence a = (r_1x_1 + r_2x_2 +…+ r_nx_n) + h ,for some h∈H .Therefore T = (Rx_1 + Rx_2 +…+ Rx_n) + H .Since H⪇ T, since M is ET-holl. module, so H$\ll_{ET}$M. So T ⊆ Rx_1 + Rx_2 + …+ Rx_n . hence T = (Rx_1 + Rx_2 + …+ Rx_n).

Section 2: *ET-lifting module*

In section this, we defined ET-lift. module with we give many properties of ET-lifting module.

Definition 2.1:- suppose T is sub-module of a module M. M is said to be a ET-lifting module, if every sub-module H of M, there exists a direct summand K of M and L$\ll_{ET}$M such that H=K+L .

Remark and Example 2.2:

1. Z_4 as Z-module is E(Z_4)-lifting module.

2.Let M be a module and T=M .Then M is ET-lifting module if and only if M is e-lifting module.

Proof:- Let T=M and M is E(T)-lifting module, to prove M is e-lifting module. Let H be sub-module of M, hence there exist $B \leq_{\oplus} M$ and A$\ll_{ET}$M such that H=B+A. By the modular law H=H⋂M=H⋂(B⨁B`)=B⨁(H⋂B`) .Take A=H⋂B` and hence A$\ll_e$M, by [(1.2-4)[7].Thus M is e-lift. module. For the converse, suppose that M is ET-lifting module and let H be sub-module of M, then H=B⨁L, where B $\leq_{\oplus}$M and A$\ll_e$M. by [(1.2-4))[7]. hence H$\ll_{ET}$M. then M is E(T)-lift. module.

3. every T-lifting module is ET-lift. module. But the converse is not true, for the example consider Z_8 as Z-module and let T={$\underline{0}$,$\underline{4}$}then Z_8 is not T–lifting module, To prove that , suppose that is not and let H={$\underline{0}$,$\underline{4}$}, then there exists K $\leq_{\oplus}$ Z_8 and L$\ll_T$ Z_8 such that H=K+L .Since Z_8 indecomposable, then K=0 and H={$\underline{0}$,$\underline{4}$}. But H=L= {$\underline{0}$,$\underline{4}$} is not T-small sub-module in Z_8 .implies that Z_8 is not T-lifting module. But H=L is ET-small sub-module in Z_8 [7], then Z_8 is an ET-lifting module.

Remark 2.3: assume that M be a semisimple module. So M is ET-lifting module, for all sub-module $T\ of\ M$.

Proof:- Let M is an semisimple module and H be sub-module of M .Thus H=H+0, where H $\leq_{\oplus}$M and 0 $\ll_{ET}$M .Then M is ET-lifting module.

Remark 2.4: Let M be an ET-lifting module .Then for all sub-module B of M such that T⊆ B is also ET-lifting.

Proof:- let M is ET-lifting and B be a sub-module of M such that T⊆B. To prove B is ET-lifting, let H is a sub-module of B .thus $H \subseteq M$ and since M is ET-lifting, then H=K+L, where K $\leq_{\oplus}$M and L$\ll_{ET}$M and since K ≤ B ≤ M and K $\leq_{\oplus}$M then K $\leq_{\oplus}$ B [2]. Since T⊆B, then L$\ll_{ET}$B, by (1.4)[7].Thus B is ET-lifting .

Recall that a sub-module H of a module M is called ((projective invariant)), if for every $P=P^2 \in End$ (M), P(H) ≤ H, see [8].

Proposition 2.5: Let T be a submodule of a module M .Consider the following statements:

1. For each submodule X of M, there exists a decomposition M=K⊕K` such that K⊆ H and (H∩K`) $\ll_{ET}$ M.

2. M is ET-lifting module.

Then **1⇒2**

2⇒1 If for all ET-small sub-module of M is projective invariant.

Proof: **(1⇒2)** Let H be a sub-module of M, there exists a decomposition M=K⊕K` such that K⊆H and (H∩K`) $\ll_{ET}$ M. Now H=H∩M=H∩(K⊕K`)= K⊕(H∩K`), by the Modular Law and let L=(H∩K`), then H= K+L .Thus M is ET-lifting module.

(2⇒1) Let M is ET-lifting module and for every ET-small sub-module of M is (projective invariant) . Let H be sub-module of M, Then H=K+L, where K$\leq_{\oplus}$M and L$\ll_{ET}$M. to prove H∩K` $\ll_{ET}$ M. H=H∩M= H∩(K⊕K`) =K⊕(H∩K`), by the Modular Law . Let P: M→K` be the projection map . P(L)=P(K+L)=P(H)=P(K⊕(H∩K`))=P(H∩K`)=H∩K` .Since L is ET-small in M, then by our assumption L is projective invariant . So P(L)=H∩K`≤ L . Thus H∩K`$\ll_{ET}$ M ,by (1.8) [7].

Theorem 2.6: Let T be sub-module of a module M .Then the following statements are equivalent:

1. For all sub-module H of M, there exists a decomposition M=K⊕K`, K⊆H and H∩K`$\ll_{ET}$M.

2. For all sub-module H of M, there exists $\beta \in End$ (M) such that $\beta^2=\beta$, β(M)⊆H and (1- β)(H) $\ll_{ET}$M.

Proof: **(1⇒2)** Let H be sub-module of M .Then there exists a decomposition M=K⊕K` such that K⊆H and H∩K`$\ll_{ET}$M. Let β : M→K be the projection map .Clearly that $\beta^2=\beta$ and M=K⊕K`= $\beta$(M)⊕(1-$\beta$)(M), since $\alpha$ is onto then $\beta$(M)=K⊆H. Now (1-$\beta$)(H)=H∩(1-$\beta$)(M)=H∩K` but H∩K`$\ll_{ET}$M. then (1-β)(H) $\ll_{ET}$M.

(2⇒1) Let H be sub-module of M .Then there exists $\beta \in End$ (M) such that $\beta^2=\beta$, β(M)⊆H and (1-β)(H) $\ll_{ET}$M. Clearly that M= β(M)⊕(1-β)(M), then M= (K⊕K`). Let D= $\beta$(M) and D`=(1-β)(M) .Then H∩K`=H∩(1- β)(M). To prove H∩(1-β)(M)=(1- β)(H), let a=(1-β)(b) ∈ (H∩(1- β)(M)). Since $(1-\beta)^2=(1-\beta)$,then a=$(1-\beta)^2$(b)=(1-β)((1-β)(b))= (1-β)(a)∈ (1-β)(H). Now let a=(1-β)(b) ∈(1- β)(H) , b ∈H ,then a ∈(1-β)(M). a=(1-β)(b) =b- β(b) ∈ H .Thus a ∈H∩(1-β)(M) .

Now (1-β)(H)= H∩(1-β)(M) = H∩K`, But (1-$\beta$)(X) $\ll_{ET}$M, then H∩K` $\ll_{ET}$M .

Proposition 2.7: let M be ET-lifting module and let B be a sub-module of M such that for every direct summand K of M, $\frac{K+B}{B}$ is direct summand of $\frac{M}{B}$, then $\frac{M}{B}$ is E($\frac{T+B}{B}$)-lifting module.

Proof:- Let $\frac{H}{B}$ is a sub-module of $\frac{M}{B}$. but M is ET-lifting , so H=K+L , ∀ K$\leq_{\oplus}$M and L$\ll_{ET}$ M .Hence $\frac{H}{B}=\frac{K+L}{B}=\frac{K+B}{B}+\frac{L+B}{B}$ .By our assumption, $\frac{K+B}{B} \leq_{\oplus} \frac{M}{B}$ .To prove $\frac{L+B}{B} \ll_{E(\frac{T+B}{B})} \frac{M}{B}$. Let $\frac{Z}{B} \leq_e \frac{M}{B}$ such that $\frac{T+B}{B} \subseteq \frac{H+B}{B}+\frac{Z}{B}=\frac{H+B+Z}{B}$, then T⊆T+B⊆H+(B+Z). (Since $\frac{Z}{B} \leq_e \frac{M}{B}$ then Z $\leq_e$M and Z ≤ B+Z≤ M, then B+Z $\leq_e$M). Since L$\ll_{ET}$ M ,then T⊆B+Z and hence $\frac{T+B}{B} \subseteq \frac{Z+B}{B}=\frac{Z}{B}$. Thus $\frac{M}{B}$ is E($\frac{T+B}{B}$) -lifting module.

Recall that M is called a (distributive module) if for every sub-modules A, B and C of M, A+(B∩C)=(A+B)∩(A+C) and A∩(B+C)=(A∩B)+(A∩C), see[9]. Corollary2.8: suppose that M is ET-lifting , M is distributive module and A be a sub-module of M .Then $\frac{M}{B}$ is $\frac{T+B}{B}$ -lifting module.

Proof: Let K be a direct summand of M .Then M=K⊕K`, for some sub-module K` of M .Hence $\frac{M}{B}=\frac{K+K`}{B}=\frac{K+B}{B}+\frac{K`+B}{B}$ .Since M is distributive, then (K+B)∩(K`+B)=((K+B)∩K`)+((K+B)∩B) = (K∩K`) + (B∩K`)+(K∩B)+B = B .Hence $\frac{M}{B}=\frac{K+B}{B} \oplus \frac{K`+B}{B}$ .Thus, by (2.8), then $\frac{M}{B}$ is E($\frac{T+B}{B}$)-lifting module.

Proposition2.9*:* Let M=$M_1 \oplus M_2$ be a module such that R=A_{nn} (M_1) + A_{nn} (M_2) . If M_1 is ET_1-lifting and M_2 is ET_2-lifting, then M=$M_1 \oplus M_2$ is E($T_1 \oplus T_2$)-lifting module.

Proof:- Let H be sub-module of a module M. Since R=A_{nn} (M_1)+A_{nn} (M_2), then then H=$H_1 \oplus H_2$,where $H_1 \leq M_1$ and $H_2 \leq M_2$ [10] . But M_1 is ET_1-lifting and M_2 is ET_2-ifting ,then $H_1=K_1+L_1$ and $H_2=K_2+L_2$,where $K_1 \leq_{\oplus} M_1$ and $L_1 \ll_{ET} M_1$, K_2

$\leq_{\oplus}M_2$ and $L_2 \ll_{ET} M_2$ .Then H= $H_1 \oplus H_2$ = $(K_1+L_1) \oplus (K_2+L_2)$ = $(K_1 \oplus K_2)$ + $(L_1 \oplus L_2)$ .to prove $(K_1 \oplus K_2)$ is a direct summand of M . since $K_1 \leq_{\oplus} M_1$ then there exist $A_1 \leq M_1$ and $A_2 \leq M_2$ such that $M_1 = K_1+A_1$ and $M_2 = K_2+A_2$, then M_1+M_2=(K_1+A_1)+ (K_2+A_2) =(K_1+K_2)+ (A_1+A_2) and (K_1+K_2)∩ (A_1+A_2)=0, then (K_1+K_2) $\leq_{\oplus}(M_1+M2)$=M By Proposition (2.12) in [7], then($L_1 \oplus L_2) \ll_{E(T1 \oplus T2)}$ M. Thus M=$(M_1 \oplus M_2)$ is E$(T_1 \oplus T_2)$-lifting module.

Proposition 2.10:- Let M = $\oplus_{i\in I} M_i$ be a" *fully stable module*" ; and T= $\oplus_{i\in I} T_i$, where $T_i \leq M_i$. If M_i is ET_i-lifting, for each i∈I , then M is a ET-lifting module .

Proof: Let B be sub-module of M .For each i∈I, let f_i : M → M_i be the projection map .To prove B=$\oplus_{i\in I}(B \cap M_i)$ .Let y∈B ,then y∈$\oplus_{i\in I} M_i$ and hence y=$\sum_{i\in I} y_i$,for $y_i \in M_i$ and $y_i \neq 0$ for at most a finite number of i∈I .Since M is fully stable, then π_i(y) ∈ B and hence f_i (y) ∈ B∩Mi . Now f_i(y) = π_i ($\sum_{i\in I} y_i$) = y_i . So y_i ∈ B∩Mi and hence y=$\sum_{i\in I} y_i \in \oplus_{i\in I}(B \cap M_i)$. Thus B⊆$\oplus_{i\in I}(B \cap M_i)$. Thus B=$\oplus_{i\in I}(B \cap M_i)$. Since $B \cap M_i \subseteq M_i$ and M_i is ET_i-lifting ,then $B \cap M_i = K_i + L_i$,where $K_i \leq_{\oplus} M_i$ and $L_i \ll_{ETi} M_i$ .Then $\oplus_{i\in I}(B \cap M_i)$=$\oplus_{i\in I}(K_i+L_i)$= $\oplus_{i\in I}K_i$+$\oplus_{i\in I}L_i$. One can easily show that $\oplus_{i\in I}K_i \leq \oplus_{i\in I}M_i$ .By Proposition (2.13) in [7], then $\oplus_{i\in I}Li \ll_{E\oplus_{i\in I}Ti}$ M .ThenM is ET-lifting module.

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