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Adjoint Operators of Two Dimensional Fractional Fourier-Mellin Transform

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Abstract- Methods based on the Fourier transform and Mellin transform are used in virtually all areas of engineering and science and by virtually all engineers and scientists. These transform play a important role in signal processing, algorithm, watermarking, pattern recognition, correlators, navigation, vowel recognition, cryptographic scheme, quantum calculus, radar system and have applications in agriculture, medical stream, detection of watermark in images regardless of the scaling and rotation.

In this paper we present an adjiont shifting operator, adjoint scaling operator and adjoint shifting-scaling operator of two dimensional fractional Fourier-Mellin transform. Also we discuss some transform formulae using adjoint differential operator.

Keywords: *Two-Dimensional Fractional Fourier-Mellin Transform, Testing Function Space, Generalized function*

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I. INTRODUCTION:

Integral transforms play wide and important role in mathematical physics, theoretical physics. The Fourier transform is no longer the appropriate transform to change the representation space of these signals. It has to be replaced by a new transform, the Mellin transform, which is invariant in modulus to dilations and decomposes the signal on a basis of hyperbolic signals.

Namias introduced Fractional Fourier Transform the field of quantum mechanics for solving some classes of differential equations efficiently. Later, Ozaktas etal came up with the discrete implementation of FrFT. Since then, a number of applications of FrFT have been developed, mostly in the field of optics.

Mellin's transformation has been applied in many different areas of physics and engineering research area. May be the most famous application is the computation of the solution of a potential problem in a wedge-shaped region where the unknown function (e.g., temperature or electrostatic potential) is supposed to satisfy Laplace's equation with given boundary conditions on the edges [9].

Milanese, R.,Cherbuliez, M.,Pun, T. was looking for the invariant content of images in multimedia archives using Fourier-Mellin transformation[1]. In signal processing terms, the MI of a sound is the Mellin transform of a stabilized wavelet transform of the sound. Toshio Irino a, Roy D. Patterson discussed in their article that the MI provides a good model of auditory vowel normalization, and that this provides a good framework for auditory processing from cochlea to cortex [2]. P.Ghosh, , E.D.Gelasca, K. R.Ramakrishnan, B. S. Manjunath detected duplicate images in large databases[3]. The Mellin transformation is a basic tool for analyzing the behavior of many important functions in mathematics and mathematical physics, such as the zeta functions occurring in number theory and in connection with various spectral problems [4]. Its also use for the registration of medical images. It have done the registrations of images which have been shifted, rotated and have modified scale. Pratt, J. G. used the Fourier-Mellin transformation for the comparison of plant leaves[5]. Watermarking method for the protection of multimedia signals (image, sound) in which Fourier-Mellin transform is used as a tool was described by Kin etal [6]. The Fourier-Mellin transformation can also be used for the registration of images, watermarks, invariant pattern recognition, preprocessing of images. Fourier-Mellin transformation and registration of images take up the work of multiple authors. Turski, J. suggested some Fourier's transformation for the subgroups, including tapering transforms[7]. S.Derrode, F.Ghorbe used the approximation of Fourier-Mellin transformation for the reconstruction of the grayscale images [8].

In this paper we discussed adjiont shifting operator, adjoint scaling operator and adjoint shifting-scaling operator and some transform formulae using differential operator of two dimensional fractional Fourier-Mellin transform.

In our previous work we define some terminology [4,10,11] is as follows.

II. TWO-DIMENSIONAL FRACTIONAL FOURIER-MELLIN TRANSFORM

A. Definition of two-dimensional fractional Fourier-Mellin transform

The two-dimensional fractional Fourier-Mellin transform with parameters α and θ of $f(x,y,t,q)$ denoted by 2DFRFMT$\{f(x,y,t,q)\}$ performs a linear operation, given by the integral transform. 2DFRFMT

$$\{f(x,y,t,q)\} = F_{\alpha,\theta}(\xi,\eta,\lambda,\chi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} f(x,y,t,q)K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)dxdydtdq$$ ----(1)

where $K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi) =$

$$\sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{1}{2sin\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha-2(x\xi+y\eta)}\, t^{\frac{2\pi i\lambda}{sin\theta}-1}\, q^{\frac{2\pi i\chi}{sin\theta}-1}\, e^{\frac{\pi i}{tan\theta}[\lambda^2+\chi^2+log^2 t+log^2 q]}$$

$$= C_{1\alpha} e^{iC_{2\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha-2(x\xi+y\eta)}\, t^{C_{1\theta}i\lambda-1}$$

$q^{C_{1\theta} i\chi-1} e^{C_{2\theta} i[\lambda^2+\chi^2+\log^2 t+\log^2 q]}$

where $C_{1\alpha} = \sqrt{\frac{1-icot\alpha}{2\pi}}$, $C_{2\alpha} = \frac{1}{2sin\alpha} C_{1\theta} = \frac{2\pi}{sin\theta}$,

$C_{2\theta} = \frac{\pi}{tan\theta} \quad 0 < \alpha < \frac{\pi}{2},\ 0 < \theta < \frac{\pi}{2}.$ ---(2)

B. The Test Function

An infinitely differentiable complex valued smooth function $\emptyset(x,y,t,q)$ on R^n belongs to $E(R^n)$, if for each compact set $I \subset S_{a,b}$, $J \subset S_{c,d}$ where

$S_{a,b} = \{x,y: x,y\epsilon R^n, |x| \le a, |y| \le b, a > 0, b > 0\}$

$S_{c,d} = \{t,q: t,q\epsilon R^n, |t| \le c, |q| \le d, c > 0, d > 0\}$

$\gamma_{E,m,n,k,l}[\emptyset(x,y,t,q)] = \sup_{x,y\epsilon I, t,q\epsilon J} \left| D^{m,n,k,l}_{x,y,t,q} \emptyset(x,y,t,q)\right| < \infty$ ---(3)

Thus $E(R^n)$ will denote the space of all $\emptyset(x,y,t,q)\epsilon E(R^n)$ with compact support contained in $S_{a,b} \cap S_{c,d}$.

Note that the space E is complete and therefore a Frechet space. Moreover, we say that $f(x,y,t,q)$ is a fractional Fourier-Mellin transformable if it is a member of E.

III. DISTRIBUTIONAL TWO DIMENSIONAL FRACTIONAL FOURIER-MELLIN TRANSFORM (2DFRFMT)

The two dimensional distributional Fractional Fourier -Mellin transform of $f(x,y,t,q)\epsilon E^*(R^n)$ can be defined by

$2DFRFMT\{f(x,y,t,q)\} = F_{\alpha,\theta}(\xi,\eta,\lambda,\chi) = \langle f(x,y,t,q), K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\rangle$---(4)

where,

$$K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi) = \sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{1}{2sin\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha - 2(x\xi+y\eta)}\, t^{\frac{2\pi i\lambda}{sin\theta}-1}\, q^{\frac{2\pi i\chi}{sin\theta}-1}\, e^{\frac{\pi i}{tan\theta}[\lambda^2+\chi^2+\log^2 t+\log^2 q]}$$

$$= C_{1\alpha} e^{iC_{2\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha - 2(x\xi+y\eta)}\, t^{C_{1\theta} i\lambda-1}\, q^{C_{1\theta} i\chi-1} e^{C_{2\theta} i[\lambda^2+\chi^2+\log^2 t+\log^2 q]}$$

where $C_{1\alpha} = \sqrt{\frac{1-icot\alpha}{2\pi}}$, $C_{2\alpha} = \frac{1}{2sin\alpha}, C_{1\theta} = \frac{2\pi}{sin\theta}$,

$C_{2\theta} = \frac{\pi}{tan\theta} \quad 0 < \alpha < \frac{\pi}{2},\ 0 < \theta < \frac{\pi}{2}.$ ---(5)

Right hand side of equation (4) has a meaning as the application of $f(x,y,t,q)\epsilon E^*(R^n)$ to $K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\epsilon E$.

It can be extended to the complex space as an entire function given by

$2DFRFMT\{f(x,y,t,q)\} = F_{\alpha,\theta}(\xi',\eta',\lambda',\chi')$

$= \langle f(x,y,t,q), K_{\alpha,\theta}(x,y,t,q,\xi',\eta',\lambda',\chi')\rangle$---(6)

The right hand side is meaningful because for each $\xi',\eta',\lambda',\chi'\epsilon C^n$, $K_{\alpha,\theta}(x,y,t,q,\xi',\eta',\lambda',\chi')\epsilon E$ as a function of x,y,t,q.

IV. ADJOINT OPERATORS-

1. Adjoint shifting operator-

The adjoint shifting operator is a continuous function from E^* to E^*. The adjoint operator $f(x,y,t,q) \to f(x-\omega, y-\delta, t, q)$ leads to the operator transform formula

$$2DFRFMT\{f(x-\omega,y-\delta,t,q)\} = e^{iC_{2\alpha}[(\omega^2+\delta^2)cos\alpha - 2(\omega\xi+\delta\eta)}\, 2DFRFMT\{e^{2iC_{2\alpha}(x\omega+y\delta)cos\alpha} f(x,y,t,q)\}$$

Proof-

Consider,

$2DFRFMT\{f(x-\omega,y-\delta,t,q)\} = \langle f(x-\omega, y-\delta,t,q, K\alpha,\theta x,y,t,q,\xi,\eta,\lambda,\chi$

$$= \langle f(x-\omega,y-\delta,t,q), C_{1\alpha} e^{iC_{2\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha - 2(x\xi+y\eta)}\, t^{C_{1\theta} i\lambda-1} q^{C_{1\theta} i\chi-1} e^{C_{2\theta} i[\lambda^2+\chi^2+\log^2 t+\log^2 q]}\rangle$$

where $C_{1\alpha} = \sqrt{\frac{1-icot\alpha}{2\pi}}$, $C_{2\alpha} = \frac{1}{2sin\alpha}, C_{1\theta} = \frac{2\pi}{sin\theta}$,

$C_{2\theta} = \frac{\pi}{tan\theta} \quad 0 < \alpha < \frac{\pi}{2},\ 0 < \theta < \frac{\pi}{2}.$

$$= \langle f(x,y,t,q), C_{1\alpha} e^{iC_{2\alpha}[[(x+\omega)^2+(y+\delta)^2+\xi^2+\eta^2]cos\alpha - 2[(x+\omega)\xi+(y+\delta)\eta]}\, t^{C_{1\theta} i\lambda-1} q^{C_{1\theta} i\chi-1} e^{C_{1\theta} i[\lambda^2+\chi^2+\log^2 t+\log^2 q]}\rangle$$

$$= \langle f(x,y,t,q), C_{1\alpha} e^{iC_{2\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha - 2(x\xi+y\eta)}\, e^{iC_{2\alpha}[(2x\omega+\omega^2+2y\delta+\delta^2)cos\alpha - 2(\omega\xi+\delta\eta)}\, t^{C_{1\theta} i\lambda-1} q^{C_{1\theta} i\chi-1} e^{C_{2\theta} i[\lambda^2+\chi^2+\log^2 t+\log^2 q]}\rangle$$

$$= e^{iC_{2\alpha}[(\omega^2+\delta^2)cos\alpha - 2(\omega\xi+\delta\eta)}\, \langle f(x,y,t,q), e^{2iC_{2\alpha}[(x\omega+y\delta)cos\alpha]} K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\rangle$$

$$= e^{iC_{2\alpha}[(\omega^2+\delta^2)cos\alpha - 2(\omega\xi+\delta\eta)}\, 2DFRFMT\{e^{2iC_{2\alpha}(x\omega+y\delta)cos\alpha} f(x,y,t,q)\}$$

2- Adjoint scaling operator

The adjoint scaling operator is a continuous function from E^* to E^*. The adjoint operator $f(x,y,t,q) \to \frac{1}{\mu\tau} f\left(x,y,\frac{t}{\mu},\frac{q}{\tau}\right)$ leads to the operator transform formula

$$2DFRFMT\left\{f\left(x,y,\frac{t}{\mu},\frac{q}{\tau}\right)\right\} = A\,2DFRFMT\{e^{i2C_{2\theta}(\log\mu\ \log t+\log\tau\ \log q)} f(x,y,t,q)\}$$

where $A = \mu^{C_{1\theta} i\lambda-1} \tau^{C_{1\theta} i\chi-1} e^{iC_{2\theta}[\log^2\mu+\log^2\tau]}$

Proof-

Consider,

$2DFRFMT\left\{\frac{1}{\mu\tau}f\left(x,y,\frac{t}{\mu},\frac{q}{\tau}\right)\right\}$

$=\langle\frac{1}{\mu\tau}f\left(x,y,\frac{t}{\mu},\frac{q}{\tau}\right),K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\rangle$

$=\langle\frac{1}{\mu\tau}f\left(x,y,\frac{t}{\mu},\frac{q}{\tau}\right),C_{1\alpha}e^{iC_{2\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha-2(x\xi+y\eta)}$
$t^{C_{1\theta}i\lambda-1}q^{C_{1\theta}i\chi-1}e^{C_{2\theta}i[\lambda^2+\chi^2+log^2t+log^2q]}\rangle$

where $C_{1\alpha}=\sqrt{\frac{1-icot\alpha}{2\pi}}$, $C_{2\alpha}=\frac{1}{2sin\alpha}$, $C_{1\theta}=\frac{2\pi}{sin\theta}$, $C_{2\theta}=\frac{\pi}{tan\theta}$ $\quad 0<\alpha<\frac{\pi}{2},\ 0<\theta<\frac{\pi}{2}$.

$=\langle f(x,y,t,q),C_{1\alpha}e^{iC_{2\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha-2(x\xi+y\eta)}$
$(\mu t)^{C_{1\theta}i\lambda-1}(\tau q)^{C_{1\theta}i\chi-1}e^{C_{2\theta}i[\lambda^2+\chi^2+log^2(\mu t)+log^2(\tau q)]}\rangle$

$=\langle f(x,y,t,q),C_{1\alpha}e^{iC_{2\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha-2(x\xi+y\eta)}$
$(\mu)^{C_{1\theta}i\lambda-1}(t)^{C_{1\theta}i\lambda-1}(\tau)^{C_{1\theta}i\chi-1}(q)^{C_{1\theta}i\chi-1}$
$e^{C_{2\theta}i[\lambda^2+\chi^2+log^2t+log^2q]}\ e^{e^{C_{2\theta}i[log^2\mu+log^2\tau+2(log\mu logt\ +log\tau logq\ )]}}\rangle$

$=(\mu)^{C_{1\theta}i\lambda-1}(\tau)^{C_{1\theta}i\chi-1}e^{C_{2\theta}i[log^2\mu+log^2\tau]}$
$\langle f(x,y,t,q),e^{2C_{2\theta}i(log\mu logt\ +log\tau logq\ )]}K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\rangle$

$=A$
$\langle f(x,y,t,q),e^{2C_{2\theta}i(log\mu logt\ +log\tau logq\ )]}K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\rangle$

where, $A=\mu^{C_{1\theta}i\lambda-1}\tau^{C_{1\theta}i\chi-1}e^{iC_{2\theta}[log^2\mu+log^2\tau]}$

$=\ A\ 2DFRFMT\{e^{2iC_{2\theta}(log\mu\ logt+log\tau\ logq)}f(x,y,t,q)\}$

3- Adjoint shifting scaling operator

The adjoint shifting scaling operator is a continuous function from E* to E*. The adjoint operator $f(x,y,t,q)\rightarrow\frac{1}{\mu\tau}f\left(x-\omega,y-\delta,\frac{t}{\mu},\frac{q}{\tau}\right)$ leads to the operator transform formula

$2DFRFMT\left\{f\left(x-\omega,y-\delta,\frac{t}{\mu},\frac{q}{\tau}\right)\right\}=$
$B2DFRFMT\{e^{2iC_{2\alpha}(x\omega+y\delta)cos\alpha}\ e^{2iC_{2\theta}[log\mu logt\ +log\tau logq\ ]}f(x,y,t,q)\}$

where, $B=$
$e^{iC_{2\alpha}[(\omega^2+\delta^2)cos\alpha-2(\omega\xi+\delta\eta)]}e^{C_{2\theta}i[log^2\mu+log^2\tau]}(\mu)^{C_{1\theta}i\lambda-1}(\tau)^{C_{1\theta}i\chi-1}$

Proof-

Consider,

$2DFRFMT\left\{\frac{1}{\mu\tau}f\left(x-\omega,y-\delta,\frac{t}{\mu},\frac{q}{\tau}\right)\right\}$

$=\langle\frac{1}{\mu\tau}f\left(x-\omega,y-\delta,\frac{t}{\mu},\frac{q}{\tau}\right),K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\rangle$

$=\langle\frac{1}{\mu\tau}f\left(x-\omega,y-\delta,\frac{t}{\mu},\frac{q}{\tau}\right),C_{1\alpha}e^{iC_{2\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha-2(x\xi+y\eta)}$
$t^{C_{1\theta}i\lambda-1}q^{C_{1\theta}i\chi-1}e^{C_{2\theta}i[\lambda^2+\chi^2+log^2t+log^2q]}\rangle$

where $C_{1\alpha}=\sqrt{\frac{1-icot\alpha}{2\pi}}$, $C_{2\alpha}=\frac{1}{2sin\alpha}$, $C_{1\theta}=\frac{2\pi}{sin\theta}$, $C_{2\theta}=\frac{\pi}{tan\theta}$ $\quad 0<\alpha<\frac{\pi}{2},\ 0<\theta<\frac{\pi}{2}$.

$=\langle f(x,y,t,q),C_{1\alpha}e^{iC_{2\alpha}[[(x+\omega)^2+(y+\delta)^2+\xi^2+\eta^2]cos\alpha-2[(x+\omega)\xi+(y+\delta)\eta]}$
$(\mu t)^{C_{1\theta}i\lambda-1}(\tau q)^{C_{1\theta}i\chi-1}e^{C_{2\theta}i[\lambda^2+\chi^2+log^2(\mu t)+log^2(\tau q)]}\rangle$

$=\langle f(x,y,t,q),$
$C_{1\alpha}e^{iC_{2\alpha}[[x^2+\omega^2+2x\omega+y^2+\delta^2+2y\delta+\xi^2+\eta^2]cos\alpha-2x\xi-2\omega\xi-2y\eta-2\delta\eta]}$
$(\mu)^{C_{1\theta}i\lambda-1}(t)^{C_{1\theta}i\lambda-1}(\tau)^{C_{1\theta}i\chi-1}(q)^{C_{1\theta}i\chi-1}$
$e^{C_{2\theta}i[\lambda^2+\chi^2+(log\mu+logt)^2+(log\tau+logq)^2]}\rangle$

$=e^{iC_{2\alpha}[(\omega^2+\delta^2)cos\alpha-2(\omega\xi+\delta\eta)]}e^{C_{2\theta}i[log^2\mu+log^2\tau]}(\mu)^{C_{1\theta}i\lambda-1}(\tau)^{C_{1\theta}i\chi-1}$
$\langle f(x,y,t,q),$
$e^{2iC_{2\alpha}(x\omega+y\delta)cos\alpha}e^{2iC_{2\theta}[log\mu logt\ +log\tau logq\ ]}K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\rangle$

$=B2DFRFMT\left\{\begin{matrix}e^{2iC_{2\alpha}(x\omega+y\delta)cos\alpha}\ e^{2iC_{2\theta}[log\mu logt\ +log\tau logq\ ]}\\ f(x,y,t,q)\end{matrix}\right\}$

where, $B=\ e^{iC_{2\alpha}[(\omega^2+\delta^2)cos\alpha-2(\omega\xi+\delta\eta)]}e^{C_{2\theta}i[log^2\mu+log^2\tau]}$
$(\mu)^{C_{1\theta}i\lambda-1}(\tau)^{C_{1\theta}i\chi-1}$

4- Theorem

The adjoint differential operator $f(x,y,t,q)\rightarrow D_{x,y}f(x,y,t,q)$ is continuous linear mapping from the dual space E* into itself the corresponding transform formula is

$2DFRFMT\ \{D_{x,y}f(x,y,t,q)\}$
$=C2DFRFMT\{[xcos\alpha-\xi][ycos\alpha-\eta]f(x,y,t,q)\}$

Proof- consider,

$2DFRFMT\ \{D_{x,y}f(x,y,t,q)\}$
$=\langle D_{x,y}f(x,y,t,q),K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\rangle$

$=\langle f(x,y,t,q),-D_{x,y}\{K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\}\rangle$

$=$
$\langle f(x,y,t,q),-D_{x,y}\{\ C_{1\alpha}e^{iC_{2\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha-2(x\xi+y\eta)}\}$

$t^{C_{1\theta}i\lambda-1}q^{C_{1\theta}i\chi-1}e^{C_{2\theta}i[\lambda^2+\chi^2+log^2t+log^2q]}\rangle$

$=\langle f(x,y,t,q),-\{\ C_{1\alpha}e^{iC_{2\alpha}[(\xi^2+\eta^2)cos\alpha\ ]}\}$
$D_x\{e^{iC_{2\alpha}[x^2cos\alpha-2x\xi]}\}D_y\{e^{iC_{2\alpha}[y^2cos\alpha-2y\eta]}\}$
$t^{C_{1\theta}i\lambda-1}q^{C_{1\theta}i\chi-1}e^{C_{2\theta}i[\lambda^2+\chi^2+log^2t+log^2q]}\rangle$

$=\langle f(x,y,t,q),4{C_{2\alpha}}^2[xcos\alpha-\xi][ycos\alpha-\eta]\ K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\rangle$

$=C\langle[xcos\alpha-\xi][ycos\alpha-\eta]f(x,y,t,q),K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\rangle$

where, $C=4{C_{2\alpha}}^2$

$=C2DFRFMT\{[xcos\alpha-\xi][ycos\alpha-\eta]f(x,y,t,q)\}$

5- Theorem

The adjoint differential operator $f(x,y,t,q)\rightarrow D_{t,q}f(x,y,t,q)$ is continuous linear mapping from the dual space E* into itself' the corresponding transform formula is

$2DFRFMT\ \{D_{t,q}f(x,y,t,q)\}$
$=2DFRFMT\left\{\left[-\frac{1}{tq}\left(\frac{2\pi ilogt}{tan\theta}+\right.\right.\right.$
2πiλsinθ−12πiloggtanθ +2πiχsinθ−1fx,y,t,q

Proof- consider,

$2DFRFMT\ \{D_{t,q}f(x,y,t,q)\}$
$=\langle D_{t,q}f(x,y,t,q),K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\rangle$

$= \langle f(x,y,t,q), -D_{t,q}\{K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\}\rangle$

$= \langle f(x,y,t,q), -D_{t,q}\{ C_{1\alpha} e^{iC_{2\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha -2(x\xi+y\eta)}\}$

$t^{C_{1\theta} i\lambda -1} q^{C_{1\theta} i\chi -1} e^{C_{2\theta} i[\lambda^2+\chi^2+log^2 t+log^2 q]}\rangle$

$= \langle f(x,y,t,q), -\{ C_{1\alpha} e^{iC_{2\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha -2(x\xi+y\eta)}\}$

$D_{t,q}\{t^{C_{1\theta} i\lambda -1} q^{C_{1\theta} i\chi -1} e^{C_{2\theta} i[\lambda^2+\chi^2+log^2 t+log^2 q]}\}$

$=$

$\langle f(x,y,t,q), -\{ \sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{i}{2sin\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha -2(x\xi+y\eta)}\}$

$e^{\frac{2\pi i}{sin\theta}[\lambda^2+\chi^2]} D_t\left\{t^{\frac{2\pi i\lambda}{sin\theta}-1} e^{\frac{\pi i}{tan\theta}[log^2 t]}\right\} D_q\left\{q^{\frac{2\pi i\chi}{sin\theta}-1} e^{\frac{\pi i}{tan\theta}[log^2 q]}\right\}\rangle$

$=$

$\langle f(x,y,t,q), -\{ \sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{i}{2sin\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha -2(x\xi+y\eta)}\}$

$e^{\frac{\pi i}{sin\theta}[\lambda^2+\chi^2]} t^{\frac{2\pi i\lambda}{sin\theta}-1} q^{\frac{2\pi i\lambda}{sin\theta}-1}$

$\left\{\frac{\pi i}{tan\theta}\frac{2logt}{t} + \left(\frac{2\pi i\lambda}{sin\theta}-1\right)t^{-1}\right\}$

$\left\{\frac{\pi i}{tan\theta}\frac{2logq}{q} + \left(\frac{2\pi i\chi}{sin\theta}-1\right)q^{-1}\right\} e^{\frac{\pi i}{tan\theta}[log^2 t+log^2 q]}\rangle$

$= \langle f(x,y,t,q),$

$-\left\{\frac{1}{tq}\left[\frac{2\pi ilogt}{tan\theta} + \left(\frac{2\pi i\lambda}{sin\theta}-1\right)\right]\left[\frac{2\pi ilogq}{tan\theta} + \left(\frac{2\pi i\chi}{sin\theta}-1\right)\right]\right\}$

$K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\rangle$

$= \langle -\left\{\frac{1}{tq}\left[\frac{2\pi ilogt}{tan\theta} + \left(\frac{2\pi i\lambda}{sin\theta}-1\right)\right]\left[\frac{2\pi ilogq}{tan\theta} +\right.\right.$

2πiχsinθ−1fx,y,t,q,

$K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\rangle$ $2DFRFMT\,\{D_{t,q} f(x,y,t,q)\}$

$= 2DFRFMT\left\{\left[-\frac{1}{tq}\left(\frac{2\pi ilogt}{tan\theta} +\right.\right.\right.$

2πiλsinθ−12πilogqtanθ +2πiχsinθ−1fx,y,t,q

Conclusion-

S.N.	Operators	Mapping
1	Adjoint shifting operator $f(x,y,t,q) \to f(x-\omega, y-\delta, t, q)$	$2DFRFMT\{f(x-\omega,y-\delta,t,q)\} = e^{iC_{2\alpha}[(\omega^2+\delta^2)cos\alpha -2(\omega\xi+\delta\eta)}$ $2DFRFM$
2	Adjoint scaling operator $f(x,y,t,q) \to \frac{1}{\mu\tau} f\left(x,y,\frac{t}{\mu},\frac{q}{\tau}\right)$	$2DFRFMT\left\{f\left(x,y,\frac{t}{\mu},\frac{q}{\tau}\right)\right\} =$ $A2DFRFMT$ $\{e^{i2C_{2\theta}(log\mu\ logt+log\tau\ logq)} f(x,y,t,q)\}$
3	Adjoint shifting-scaling operator $f(x,y,t,q) \to \frac{1}{\mu\tau} f\left(x-\omega, y-\right.$ $\delta,t\mu,q\tau$	$2DFRFMT\left\{f\left(x-\omega,y-\delta,\frac{t}{\mu},\frac{q}{\tau}\right)\right\}$ $= B2DFRFMT\{e^{2iC_{2\alpha}(x\omega+y\delta)cos\alpha}$ $e^{2iC_{2\theta}[log\mu logt+log\tau logq]} f(x,y,t,q)\}$
4	Adjoint differential operator- 1 $f(x,y,t,q) \to D_{x,y} f(x,y,t,q)$	$2DFRFMT\,\{D_{x,y} f(x,y,t,q)\}$ $= C2DFRFMT\{[xcos\alpha-\xi][ycos\alpha-\eta] f(x,y,t,q)\}$
5	$f(x,y,t,q) \to D_{t,q} f(x,y,t,q)$	$2DFRFMT\,\{D_{t,q} f(x,y,t,q)\}$ $= 2DFRFMT\left\{\left[-\frac{1}{tq}\left(\frac{2\pi ilogt}{tan\theta} + \frac{2\pi i\lambda}{sin\theta}-1\right)\left(\frac{2\pi ilogq}{tan\theta}+\frac{2\pi i\chi}{sin\theta}-1\right)\right] f(x,y,t,q)\right\}$

V. CONCLUSION

In the present work discussed adjoint shifting operator, adjoint scaling operator and adjoint shifting-scaling operator and some transform formulae using adjoint differential operator of two dimensional fractional Fourier-Mellin transform.

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