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Strong Non Split Block Domination in Graphs

[1]M. H. Muddebihal and [2]Nawazoddin U. Patel

Department of Mathematics Gulbarga University, Gulbarga-585106

Karnataka, India.

E-mail: mhmuddebihal@yahoo.co.in and nawazpatel.88 @gmail.com

Abstract: For any graph $G = (V, E)$, the block graph $B(G)$ is a graph whose set of vertices is the union of the set of blocks of G in which two vertices are adjacent if and only if the corresponding blocks of G are adjacent. A dominating set D of a graph $B(G)$ is a strong non split block dominating set if the induced sub graph $\langle V[B(G)] - D\rangle$ is complete. The strong non split block domination number $\gamma_{snsb}(G)$ of G is the minimum cardinality of strong non split block dominating set of G. In this paper, we study graph theoretic properties of $\gamma_{snsb}(G)$ and many bounds were obtain in terms of elements of G and its relationship with other domination parameters were found.

Keywords: *Dominating set/ Independent domination/Block graph /Strong split block domination/ Edge domination/Roman domination/ strong non split block domination.*

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1. Introduction:

In this paper, all the graphs consider here are simple and finite. For any undefined terms or notation can be found in Harary [3]. In general, we use $< X >$ to denote the subgraph induced by the set of vertices X and $N(v)$ and $N([v])$ denote open (closed) neighborhoods of a vertex v .

The concept of Roman domination function (RDF) was introduced by E.J. Cockayne, P.A.Dreyer, S.M.Hedetiniemi and S.T.Hedetiniemi in [2]. A Roman dominating function on a graph $G = (V, E)$ is a function $f: V \to \{0,1,2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex of v of G for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V) = \sum_{v\in V} f(v)$. The Roman domination number of a graph G, denoted by $\gamma_R(G)$,equals the minimum weight of a Roman dominating function on G.

The notation $\alpha_o(G)(\alpha_1(G))$ is the minimum number of vertices (edges) in vertex (edge) cover of G. The notation $\beta_o(G)(\beta_1(G))$ is the maximum cardinality of a vertex (edge) independent set in G.A block graph $B(G)$ is the graph whose vertices correspond to the blocks of G and two vertices in $B(G)$ are adjacent if and only if the corresponding blocks in G are adjacent.

We begin by recalling some standard definitions from domination theory (see [6]). A dominating set D of a graph $G = (V, E)$ is an independent dominating set if the induced subgraph $< D >$ has no edges. The independent domination number $i(G)$ of a graph G is the minimum cardinality of an independent dominating set(see[5]). The concept of domination in graphs, with its many variations, is now well studied in graph theory (see [1,5, and 6]). In this article, we study a variation on the domination theme, which is called Strong split Block domination in Graphs, recently introduce by M.H.Muddebihal et.al. [4].

A dominating set D of a graph $B(G)$ is a strong non split block dominating set if the induced subgraph $\langle V[B(G)] - D\rangle$ is complete. The strong non split block domination number $\gamma_{snsb}(G)$ of G minimum cardinality of strong non split block dominating set of G. In this paper, many bounds on $\gamma_{snsb}(G)$ were obtained in terms of elements of G but not the elements of $B(G)$. Also its relation with other domination parameters were established.

2. Main Results

We establish the lower bounds for $\gamma_{snsb}(G)$.

1. LOWER BOUNDS PROBLEMS FOR $\gamma_{snsb}(G)$:

Many lower bounds for $\gamma_{snsb}(G)$ are established in the following theorems.

Theorem 1: For any connected (p,q)graph G and $(G) \neq K_P$, then $\gamma_{snsb}(G) \geq \left\lceil \frac{p}{\Delta(G)+1} \right\rceil$.

Proof: Suppose $B(G) = K_P$. Then by definition of $\gamma_{snsb}(G)-set$ does not exists. Now we consider a set $C = \{v_1, v_2, v_3, \dots\dots\dots, v_n\} \subseteq V(G)$ a set of all non end vertices in G. Assume there exists at least one vertex $v \in C$ such that $\deg(v) = \Delta(G)$. Then consider $S \subseteq C$ such that $N[v_i] = V(G), \forall v_i \in S, 1 \leq i \leq k$. Now without loss of generality, let $C' = \{b_1, b_2, b_3, \dots\dots\dots, b_i\}$ be the set of cut vertices in $B(G)$. Since each block in $B(G)$ is complete and each cut vertex is incident with at least two blocks. Let $C'' = V[B(G)] - C'$ and consider a set $C'' \subseteq C'$ such that $V[B(G)] - \{C'' \cup C_1''\} = S', \forall b_i \in S'$ gives $< S' >$ which is complete. Suppose there exists vertex $v \in G$ such that , $\forall v_i \in C$ are adjacent to v, if $\deg(v) = \Delta(G)$ and $N(v_i) \geq \Delta(G) + 1$. Then $C_1'' \cup C'' \geq \frac{V[G]}{\Delta(G)+1}$, which gives$|C_1'' \cup C''| \geq \left\lceil \frac{V[G]}{\Delta(G)+1} \right\rceil$. So that $\gamma_{snsb}(G) \geq \left\lceil \frac{p}{\Delta(G)+1} \right\rceil$.

Theorem 2: For any connected (p,q) tree $T$and $B(T) \neq K_P$, then $\gamma_{snsb}(T) \geq \alpha_o(T) - 1$. Where $\alpha_o(T)$ is the vertex covering number of T.

Proof: Suppose $B(T) = K_P$. Then $\gamma_{snsb} - set$ does not exists. We consider a tree T with $V(T) = \{v_1, v_2, v_3, \dots\dots\dots, v_p\}$. Let $V_1 = \{v_1, v_2, v_3, \dots\dots\dots, v_i\}, 1 \leq i \leq p$ be the set of cut vertices which are adjacent to end vertices and $V_2 = \{v_1, v_2, v_3, \dots\dots\dots, v_l\}, 1 \leq l \leq p$ be the set of cut vertices such that $\forall v_k \in N(v_l)$ are non-end vertices $1 \leq l \leq p$. Suppose a set $V_j \subseteq V_1$ *or* V_2. Then we consider another subset $V_2' = \{v_1, v_2, v_3, \dots\dots\dots, v_n\}, 1 \leq n \leq l$ which are at a odd distance from the vertices of T with $\deg(v_p) \geq 3$. Then every vertex belongs to $V_1 \cup V_2 \cup V_j \cup V_2'$ which covers all the edges of T. Hence $|V_1| \cup |V_2| \cup |V_j| \cup |V_2'| = \alpha_o(T)$. To get $\gamma_{snsb}(T)$, since each block in $B(T)$ is complete.

Now suppose each block of $B(T)$ is an edge. Then $V[B(T)] = \{v_1, v_2, v_3, \dots\dots\dots, v_n\}$ and there exists a set $H = \{v_1, v_2, v_3, \dots\dots\dots, v_i\}, 1 \leq i \leq n, H \subseteq V[B(T)]$ such that $v_j v_k \in V[B(T)]$ and $v_j v_k \in E[B(T)]$. Hence $V[B(T)] - \{H\} = v_j v_k$ is complete. Clearly H is $\gamma_{snsb} - set$. Now $|V_1| \cup |V_2| \cup |V_j| \cup |V_2'| \leq |H| + 1$ which gives $\gamma_{snsb}(T) \geq \alpha_o(T) - 1$.

Theorem 3: For any (p,q)acyclic graph G with $B(G) \neq K_P$, then $\gamma_{snsb}(G) \geq \gamma(G)$. Equality holds for a path p_p with $p \geq 3$.

Proof: Suppose $V = \{v_1, v_2, v_3, \dots\dots\dots, v_p\}$ be the set of vertices of G. Let $D = \{v_1, v_2, v_3, \dots\dots\dots, v_k\}$, $1 \leq k \leq p$ be a minimal dominating set of G such that $|D| = \gamma(G)$. Further $B = \{B_1, B_2, B_3, \dots\dots\dots, B_n\}$ be the number of blocks in G. In $B(G)$, $V[B(G)] = \{b_1, b_2, b_3, \dots\dots\dots, b_n\}$ be the set of vertices corresponding to the blocks $\{B_1, B_2, B_3, \dots\dots\dots, B_n\}$ of G. Suppose there exists a block H in $B(G)$ with maximum number of vertices and each block in $B(G)$ is complete. Now we consider the following cases.

Case 1: Assume H is an end block in $B(G)$ with m vertices. Then $H' \subset H$ with $(m-1)$ vertices where $\{H'\} \in V[B(G)] - D'$ and is complete. Hence D' is a $\gamma_{snsb} - set$ and $|D'| \geq |D|$ which gives $\gamma_{snsb}(G) \geq \gamma(G)$.

Case 2: Assume H is not an end block in $B(G)$ with m vertices. Then $V[H] - \{v_1, v_2, v_3, \dots\dots\dots, v_m\}$. If at least two vertices of H are cut vertices, then every vertex of H is adjacent to at least one vertex of $V[B(G)] - H$. Since H is complete. Hence $|H|$ is a $\gamma_{snsb} - set$. Suppose for some $v \in H$ such that $\{V[B(G)] - H\} \cup \{v\}$ is not minimal then H itself is a minimal $\gamma_{snsb} - set$. Hence $|H| \geq |D|$ which gives $\gamma_{snsb}(G) \geq \gamma(G)$.

For equality, suppose $G = P_p$ with$P \leq 2$, $\gamma_{snsb}(G)$ does not exists. Hence we consider $G = P_p$ with $P \geq 3$. Suppose $G = P_p$ with $P \geq 3$. Let $G = P_p: \{v_1, v_2, v_3, \dots\dots\dots, v_p\}$ be a path with $P \geq 3$ then we consider a set $D = \{v_2, v_5, v_8, \dots\dots\dots, v_{p-n}\}$ such that $N(v_{p-n}) \cap N(v_{p-n-1}) = \emptyset$. Hence D be a $\gamma - set$ of P_p. In $B(P_p), V[B(P_p)] = P - 1$, then we consider a set $K \subset V[B(P_p)]$ such that $V[B(P_p)] - K = M$ where each element in M is complete. Clearly $|M| = |D|$ which gives $\gamma_{snsb}(P_p) = \gamma(P_p)$.

Theorem 4: For any (p,q)acyclic graph G and $B(G) \neq K_P$, then $\gamma_{snsb}(G) \geq i(G)$. Where $i(G)$ is an independent domination number.

Proof: Let $m = \gamma_{snsb}(G)$ and let $D_{-1} = \{w_0, w_1 \dots\dots\dots, w_{u-1}\} \subseteq V$ be a dominating set. Also for any non empty $V' \subseteq V$ let $a(V')$ denote the number of edges in the subgraph induced by V'. Clearly $0 \leq a(D_{-1}) \leq (2^m)$. If $a(D_{-1}) = 0$ then D_{-1} is an independent set and $i(G) \leq m = \gamma_{snsb}(G)$. Therefore without loss of generality we may assume that $w_0 w_1 \in E(G)$.

Now the set $N_0 = \{u \in V - D_{-1} | N(u) \cap D_{-1}| = \{w_0\}\}$ is not empty. Let u and w be any two distinct elements of N_0 and consider $\{w_0, w_1 \dots\dots\dots, u, w\} \subseteq V$. The subgraph induced by this set certainly contains $\{w_0 w_1, w_0 u, w_0 w\}$. By hypothesis$\{w_1 u, w_1 w, uw\} \cap E(G) \neq \emptyset$. But since $N(u) \cap D_{-1} = \{w_0\} = N(w) \cap D_{-1}$ it must be that $\in E(G)$. Now we see that any two distinct elements of $N_0 \cup \{w_0\}$ are adjacent. Take $u_0 \in N_0$ and consider $D_0 = \{u_0, w_1, \dots\dots\dots, w_{m-1}\}$. Let $Z \in V - D_0 = M \cup K$, where $M = (N_0 - \{u_0\}) \cup \{w_0\}$ and $K = V - (N_0 \cup D_{-1})$. If $Z \in M$ then $Z u_0 \in E(G)$ and if $Z \in K$ then $N(Z) \cap D_{-1} \supseteq \{w_i\}$, where $1 \leq i \leq m-1$, which says $Zw_i \in E(G)$. Suppose D_0 is a block dominating set such that $|D_0| = m$. Now $N(u_0) \cap D_0 = \emptyset$ and hence $0 \leq a(D_0) \leq (2^m)$. Let $D_k = \{u_0, u_1, w_2, \dots\dots\dots, w_m\}$,$1 \leq k \leq m$ such that $|D_k| \leq m = \gamma_{snsb}(G)$. Hence D_k is an independent dominating set which is implies that D_k is a maximal independent set. Hence $i(G) \leq |D_k| \leq m = \gamma_{snsb}(G)$ and which gives $\gamma_{snsb}(G) \geq i(G)$.

Theorem 5: For any non trivial tree T with $C, C \geq 2$ cut vertices, if every non end vertex of a tree T is adjacent to at least one end vertex, then $\gamma_{snsb}(T) \geq C - 1$.

Proof: Let $F = \{v_1, v_2, v_3, \dots\dots\dots, v_m\} \subseteq V(T)$ be the set of all cut vertices in T with $|F| = C$. Further, $A = \{e_1, e_2, e_3, \dots\dots\dots, e_k\}$ be the set of edges which are incident with the vertices of F. Now by block graph, suppose $D = \{b_1, b_2, b_3, \dots\dots\dots, b_i\} \subseteq A$ be the set of vertices which covers all the vertices in $B(T)$. Let $D' = \{b_1, b_2, b_3, \dots\dots\dots, b_m\}$ where $m < i$ is a minimal dominating set of $B(T)$ such that $V[B(T)] - D' = N$ is a complete, then $|D'| = \gamma_{snsb}(T)$. Hence $|D'| \geq |F| - 1$ which gives $\gamma_{snsb}(T) \geq C - 1$.

Now we obtain an upper bounds of $\gamma_{snsb}(G)$.

2. UPPER BOUNDS PROBLEMS FOR $\gamma_{snsb}(G)$:

Many upper bounds for $\gamma_{snsb}(G)$ are established in the following theorems.

Theorem 6: For any (p, q) graph G and $B(G) \neq K_P$, then $\gamma_{snsb}(G) \leq P - \Delta(G)$.

Proof: Suppose $B(G) = K_P$. Then by definition $\gamma_{snsb} - set$ does not exist. Hence $B(G) \neq K_P$. Assume every block of G is an edge, let $A = \{B_1, B_2, B_3, \dots\dots\dots, B_n\}$ be the blocks of G and $M = \{b_1, b_2, b_3, \dots\dots\dots, b_n\}$ be the block vertices in $B(G)$. Let $\{B_i\} \subset A$ such that each B_i is an non end block of G. Then $\{b_i\} \subseteq V[B(G)]$ which are vertices corresponding to the set $\{B_i\}$ since each block is complete in $B(G)$.

Again we consider a subset $\{b_i'\}$ such that $\{b_i'\} \subset V[B(G)] - \{b_i\}$. Then$V[B(G)] - \{b_i'\} = \{b_i\}$. If $i = 1$, then $\{b_i\}$ is a $\gamma_{snsb} - set$ of G. Otherwise if there exists $i > 1$ for $\{b_i\}$, we choose $\forall v_i \in N[b_i]$ such that $V[B(G)] - \{b_i'\} \cup \{v_i\} = b_i$ gives for $i > 1$. Hence $< b_i >$ is complete. Then $|V[B(G)] - \{b_i'\} \cup \{v_i\}| = \gamma_{snsb}(G)$. Which gives $\gamma_{snsb}(G) \leq P - \Delta(G)$.

Theorem 7: If every non end vertex of a tree T is adjacent to at least one end vertex and $B(T) \neq K_P$, then $\gamma_{snsb}(T) \leq 2P - 2M(T) + 1$. Where $M(T)$ is the number of end vertices in T.

Proof: Let $F = \{v_1, v_2, v_3, \dots\dots\dots, v_m\} \subseteq V(T)$ be the set of all end vertices in T with $|F| = M$. Further, $A = \{e_1, e_2, e_3, \dots\dots\dots, e_k\}$ be the set of edges which are incident with the vertices of $C \in N(F)$, where $C \subseteq V(T)$. In $B(T)$,$\{A\} \subseteq V[B(T)]$ and each block of $B(T)$ is complete. Suppose there exists a block B in $B(T)$ with maximum vertices. Then $D' = V[B(T)] - \{B\}$ and $\forall v_i \in B$ is adjacent to at least one vertex of D'. Clearly D' is a $\gamma_{snsb} - set$ of T. Hence $|D'| \leq 2|V(T)| - 2|F| + 1$ gives $\gamma_{snsb}(T) \leq 2P - 2M(T) + 1$.

Theorem 8: For any (p, q) graph G with $B(G) \neq K_P$, then $\gamma_{snsb}(G) \leq \gamma_R(G) + \gamma_t(G) - 3$.

Proof: Let $f = (V_0, V_1, V_2)$ be any γ_R –function of G. Then V_2 is a $\gamma - set$ of $H = G[V_0 \cup V_2]$ such that $|H| = \gamma_R(G)$. Let $S_1 = \{v_1, v_2, v_3, \dots\dots\dots, v_k\} \subseteq V(G)$ be the set of all non end vertices in G. Suppose $S_2 \subseteq S_1$ be the minimum set of vertices in G and if $\deg(v_i) \geq 1, \forall v_i \in S_2$, $1 \leq i \leq n$ in the sub graph $< S_2 >$. Then S_2 forms a total dominating set of G. Otherwise , if $\deg(v_i) < 1$, then attach the vertices $w_i \in N(v_i)$ to make $\deg(v_i) \geq 1$ such that $< S_2 \cup \{w_i\} >$ does not contain any isolated vertex. Clearly, $S_2 \cup \{w_i\}$ forms a minimal total dominating set of G such that $|S_2 \cup \{w_i\}| = \gamma_t(G)$.

Now we consider $\{b_1, b_2, b_3, \dots\dots\dots, b_n\}$ be the set of vertices of $B(G)$ corresponding to the blocks $\{B_1, B_2, B_3, \dots\dots\dots, B_n\}$ of G. Let $D' = \{b_1, b_2, b_3, \dots\dots\dots, b_m\}$ where $m < n$ is a minimal dominating set of $B(G)$ such that $V[B(G)] - D' = N$ is complete then

$| D^{'} | = \gamma_{snsb}(G)$. Hence $\gamma_{snsb}(G) = | D^{'} | \leq |H| \cup |S_2 \cup \{w_i\}| = \gamma_R(G) + \gamma_t(G) - 3$ which gives $\gamma_{snsb}(G) \leq \gamma_R(G) + \gamma_t(G) - 3$.

Theorem 9: For any (p, q) graph G with C number of cut vertices, then $\gamma_{snsb}(G) \leq \gamma(G) + \gamma^{'}(G) + \left\lfloor \frac{C}{2} \right\rfloor$.

Proof: Suppose $B = \{B_1, B_2, B_3, \dots\dots\dots, B_n\}$ is the set of blocks in G. Then $\{B\} = V[B(G)]$. Let $A = \{B_1, B_2, B_3, \dots\dots\dots, B_i\}, 1 \leq i \leq n$ such that $A \subseteq B$ and $\forall b_i \in A$ are the non- end blocks in G which gives cut vertex in $B(G)$. Also $C^{'} = \{b_1, b_2, b_3, \dots\dots\dots, b_j\}, 1 \leq j \leq n$ be the set of end blocks in G and $C^{'} \subseteq B$. Let $\{v_1, v_2, v_3, \dots\dots\dots, v_p\}$ be the set of vertices of G and D= $\{v_1, v_2, v_3, \dots\dots\dots, v_m\}$ where $m \leq p$ be a dominating set of G such that $\gamma(G) = |D|$. Let F be minimal edge dominating set of G. Suppose $E - F$ is not an edge dominating set. Then there exists an edge f such that $f \epsilon F$ is adjacent to any edge in $E - F$. Since G has no isolated edges then f is dominated by at least one edge in $F - \{f\}$. Thus $F - \{f\}$ is edge dominating set , a contradiction to the minimality of F. Therefore F is edge dominating set. Such that $|F| = \gamma^{'}(G)$.

Suppose $M = \{b_1, b_2, b_3 \dots\dots\dots, b_n\}$ be the set of vertices in $B(G)$ corresponding to blocks in G. Let $M_1 = \{b_1, b_2, b_3 \dots\dots\dots, b_j\} \subseteq M$ where$1 \leq j \leq n$ be the set of all end vertices in $B(G)$. Also $M_2 = \{b_1, b_2, b_3 \dots\dots\dots, b_i\} \subseteq M$, $1 \leq i \leq n$ be the set of all cut vertices in $B(G)$. Further we consider a set $M_3 = \{b_1, b_2, b_3 \dots\dots\dots, b_s\}$, $1 \leq s \leq i$ such that $M_3 \subset M_2$. Now $\{V[B(G)] - (M_1 \cup M_3)\}$ which is complete which gives a strong non split block domination in $B(G)$. Hence $|M_1 \cup M_3| = \gamma_{snsb}(G)$. Suppose in G every non end block has at least two blocks which are adjacent with different cut vertices and is denoted these cut vertices by a set C. Then by the definition of $B(G)$ which gives $|D| + |F| + \left\lfloor \frac{C}{2} \right\rfloor \geq |M_1 \cup M_3|$. Hence $\gamma_{snsb}(G) \leq \gamma(G) + \gamma^{'}(G) + \left\lfloor \frac{C}{2} \right\rfloor$.

Theorem 10: For any (p, q) graph G with $B(G) \neq K_P$, then $\gamma_{ssb}(G) \leq \gamma_{snsb}(G)$.

Proof: Let $D = \{b_1, b_2, b_3, \dots\dots\dots, b_i\}$ be the set of cut vertices in $B(G)$. Since each block in $B(G)$ is complete and each cut vertex is incident with at least two blocks. Let $D^{'} = V[B(G)] - D$ and consider a set $D_1^{'} \subseteq D^{'}$ such that $V[B(G)] - \{D^{'} \cup D_1^{'}\} = S$ where $\forall b_i \in S$ is an isolates. Hence $|D^{'} \cup D_1^{'}| = \gamma_{ssb}(G)$. Now we consider a subset $\{b_i^{'}\}$ such that $\{b_i^{'}\} \subset V[B(G)] - \{b_i\}$. Then $V[B(G)] - \{b_i^{'}\} = \{b_i\}$. If $i = 1$, then $\{b_i\}$ is a $\gamma_{snsb} - set$ of G. Otherwise if there exists $i > 1$ for $\{b_i\}$, we choose $\forall v_i \in N[b_i]$ such that $V[B(G)] - \{b_i^{'}\} \cup \{v_i\} = b_i$ gives for $i > 1$. Hence $< b_i >$ is complete. Then $|V[B(G)] - \{b_i^{'}\} \cup \{v_i\}| = \gamma_{snsb}(G)$. Which gives $\gamma_{ssb}(G) \leq \gamma_{snsb}(G)$.

Further we developed our concept by comparing other different domination parameters.

Theorem 11: For any connected (p, q) graph G and $B(G) \neq K_P$, then $\gamma_{snsb}(G) + \gamma_c(G) \geq P + \gamma(G) - \Delta(G)$.

Proof: Let $V(G) = \{v_1, v_2, v_3, \dots\dots\dots, v_n\}$ be the set of vertices in G. Suppose there exists a minimal set of vertices $S = \{v_1, v_2, v_3, \dots\dots\dots, v_k\} \subseteq V(G)$ such that $N[v_i] = V(G)$, $\forall v_i \in S$, $1 \leq i \leq k$. Then S forms a minimal dominating set of G. Further, if the sub graph $< S >$ has exactly one component, then S is itself is a connected dominating set of G. Suppose S has more than one component, then attach the minimal set of vertices $S^{'}$ of $V(G) - S$, which are every in $u - w$ path, $\forall u, w \in S$ gives a single component $S_1 = S \cup S^{'}$. Clearly, S_1 forms a minimal $\gamma_c - set$ of G.

Let $F = \{b_1, b_2, b_3, \dots\dots\dots, b_m\}$ be the set of vertices corresponding to the blocks of G. Suppose there exists a set of vertices $D' = \{b_1, b_2, b_3, \dots\dots\dots, b_j\} \subseteq V[B(G)] - D$, where $D \subseteq F$ such that $< D' >$ is complete. Since for any graph G, there exists at least one vertex $v \in V(G)$ with $\deg(v) = \Delta(G)$, it follows that $|D'| \cup |S_1| \geq |V(G)| \cup |S| - \Delta(G)$. Hence $\gamma_{snsb}(G) + \gamma_c(G) \geq P + \gamma(G) - \Delta(G)$.

Theorem 12: For any connected (p, q) graph G and $B(G) \neq K_P$, then $\gamma_{snsb}(G) + 2\,\gamma_c(G) \geq \alpha_o(G) + \beta_o(G) + 1$.

Proof: Let $A = \{v_1, v_2, v_3, \dots\dots\dots, v_n\} \subseteq V(G)$ be the set of all end vertices in G and $V_1 = V(G) - A$. Suppose there exists set of vertices $C \subseteq V_1$ such that $N(u) \cap N(w) \neq \emptyset, \forall u, w \in C$. Further $N(x) \cap N(y) \neq \emptyset, \forall x \in A, y \in C$. Then $C \cup A$ forms a maximal independent set of vertices. If $A = \emptyset$, then C itself forms a maximal independent set of vertices in G. Let $B = \{v_1, v_2, v_3, \dots\dots\dots, v_k\} \subseteq V_1$ be the set of vertices with $\operatorname{dist}(v_i, v_j) \geq 2, 1 \leq i \leq j \leq k, \forall v_i, v_j \in B$, covers all edges in G. Clearly, B forms a vertex covering set. Suppose the set $C \cup A$ covers all the vertices in G and if the sub graph $< V(G) - C \cup A >$ does not contain any isolated vertex. Then $C \cup A$ itself connected dominating set of G. Otherwise, if there exists a vertex $v \in V(G) - \{C \cup A\}$ with $\deg(v) = 0$ in the sub graph $< V(G) - \{C \cup A\} >$. Then $C \cup A \cup \{v\}$ forms a minimal $\gamma_c - set$ of G. Further if $A = \emptyset$, then $< V(G) - C >$ gives a connected dominating set of G. Now in $B(G)$, let $F = \{B_1, B_2, B_3, \dots\dots\dots, B_m\} \subseteq V[B(G)]$ be the set of vertices corresponding to the blocks of G. Suppose $D = \{v_1, v_2, v_3, \dots\dots\dots, v_k\} \subseteq F$ be the set of vertices such that $N(u_i) = V[B(G)], \forall u_i \in D, 1 \leq i \leq k$ and $< V[B(G)] - D >$ is complete. Now $|D| \cup 2|C \cup A \cup \{v\}| \geq |B| \cup |C| + 1$ and hence $\gamma_{snsb}(G) + 2\,\gamma_c(G) \geq \alpha_o(G) + \beta_o(G) + 1$. If $A = \emptyset$, then $|D| \cup 2|C| \geq |B| \cup |C| + 1$, gives $\gamma_{snsb}(G) + 2\,\gamma_c(G) \geq \alpha_o(G) + \beta_o(G) + 1$.

Theorem 13: For any connected (p, q) graph G and $B(G) \neq K_P$, then $\gamma_{snsb}(G) + \gamma(G) \leq P + \gamma_c(G) - 2$.

Proof: Let $V'_1 = \{v_1, v_2, v_3, \dots\dots\dots, v_i\} \subseteq V_1(G)$ be the set of all non end vertices in G. Suppose there exists a minimal set of vertices $V'' = \{v_1, v_2, v_3, \dots\dots\dots, v_j\} \subseteq V'_1$ such that $N[v_k] = V_1(G), \forall v_k \in V'', 1 \leq k \leq n$. Then V'' forms a minimal dominating set of G. Further, if the sub graph $< V'' >$ has exactly one component, then V'' is itself is a connected dominating set of G. Suppose V'' has more than one component, then attach the minimal set of vertices V_1'' of $V'_1 - V''$, which are every in $u - w$ path, $\forall u, w \in V''$ gives a single component $V_2 = V'' \cup V_1''$. Clearly, V_2 forms a minimal $\gamma_c - set$ of G. Let $D = \{b_1, b_2, b_3, \dots\dots\dots, b_m\}$ be the set of vertices corresponding to the block which are incident with the vertices of V'' in G. Suppose there exists a set of vertices $D' = \{b_1, b_2, b_3, \dots\dots\dots, b_j\} \subseteq V[B(G)] - D''$, where $D'' \subseteq D$ such that $< D' >$ is complete, which gives a $\gamma_{snsb} - set$ in $B(G)$. it follows that $|D'| \cup |V''| \leq |V_1(G)| \cup |V_2| - 2$. Hence $\gamma_{snsb}(G) + \gamma(G) \leq P + \gamma_c(G) - 2$.

Theorem 14: For any non trivial tree T with n-blocks and $B(T) \neq K_P$, then $\gamma_{snsb}(T) + \gamma(T) \leq n(T) + \Delta(T)$.

Proof: Suppose $B = \{b_1, b_2, b_3, \dots\dots\dots, b_n\}$ is the set of blocks in T. Then $\{B\} = V[B(T)]$. Let $A = \{b_1, b_2, b_3, \dots\dots\dots, b_i\}, 1 \leq i \leq n$ such that $A \subseteq B$ and $\forall b_i \in A$ are the non- end blocks in T which gives cut vertex in $B(T)$. Also $C = \{b_1, b_2, b_3, \dots\dots\dots, b_j\}, 1 \leq j \leq n$ be the set of end blocks in T and $C \subseteq B$. Let $\{v_1, v_2, v_3, \dots\dots\dots, v_p\}$ be the set of vertices of T and D= $\{v_1, v_2, v_3, \dots\dots\dots, v_m\}$ where $1 \leq m \leq p$ be a dominating set of T such that $\gamma(T) = |D|$. Now we consider $A' \subset A$ and $C' \subset C$. Since $\{A\} \cup \{C\} = V[B(T)]$ then $V[B(T)] - \{A'\} \cup \{C'\} = \{K\}$ is complete. Hence $|A'| \cup |C'| = \gamma_{snsb}(T)$. For any graph T, there exists at least one vertex $v \in V(T)$ with $\deg(v) = \Delta(T)$. Clearly $|A'| \cup |C'| + |D| \leq |A| \cup |C| + \Delta(T)$ which gives $\gamma_{snsb}(T) + \gamma(T) \leq n(T) + \Delta(T)$.

Theorem 15: If $\{v_i\}$ be the set of all end vertices in $B(G)$, then $\{v_i\}$ is in every $\gamma_{snsb} - set$.

Proof: Suppose D be a block dominating set in $B(G)$. Let $D^{'} = \{v_1, v_2, v_3, \dots \dots \dots, v_n\} \subseteq V[B(G)] - D$ be the set of vertices which covers all the vertices in $B(G)$. Clearly, $D^{'}$ forms an $\gamma_{snsb} - set$. Now assume $\{v_i\} \notin D^{'}$, and consider the vertices $\{u_i, w_i\} \notin D^{'}$. Then clearly, v_i is in every $u_i - w_i$ path in $B(G)$. Since $B(G) - \{v_i\}$ has exactly one component, it follows that the set $D_1^{'} = (D^{'} - \{u_i, w_i\}) \cup \{v_i\}$ is also a block dominating set. Clearly, $|D_1^{'}| = |D^{'}| = 1$, a contradiction to the fact that $D^{'}$ is a $\gamma_{snsb} - set$. Hence $\{v_i\} \notin D^{'}$.

Theorem 16: If v be an end vertex of $B(G)$, then v is in every strong non split block dominating set of $B(G)$.

Proof: Let $D = \{v_1, v_2, v_3, \dots \dots \dots, v_n\} \subseteq V[B(G)]$ be the minimal block dominating set of $B(G)$. Suppose there exists a vertex set $D^{'} \subseteq V[B(G)] - D$ be the $\gamma_{snsb} - set$ of $B(G)$, assume there exists an end vertex $v \in V[B(G)]$, $v \notin D^{'}$. Now consider any two vertex u and w such that $u, w \notin D^{'}$. Since $v \notin D^{'}, v$ is in every $u - w$ path in $B(G)$ further, since $\deg(v) = 1$, where $v \in V[B(G)]$ it follows that the set $D^{''} = \{D^{'} - \{u, w\}\} \cup \{v\}$ is also a minimal block dominating set of $B(G)$. Clearly, $|D^{''}| = |D^{'}| = 1$, a contradiction to the fact that $D^{'}$ is also a $\gamma_{snsb} - set$ of $B(G)$. Hence, $v \in D^{'}$ and v is in every $\gamma_{snsb} - set$ of $B(G)$.

3. References

[1] Anwar Alwardi and N.D.Soner, Equitable edge Domination in graphs, BIMVI, 3(2013) 7-13.

[2] E.J.Cockayne, P.A.Dreyer. Jr,S.M.Hedetiniemi and S.T.Hedetiniemi, Roman domination in graphs, Discrete maths,278(2004),11-22.

[3] F.Harary, graph Theory, Adison Wesley,Reading mass,(1972).

[4] M.H.Muddebihal et.al. strong split lict domination of a graph, IJESR, 2(2014) 187-198.

[5] Robert B.ALLAN and Renu Laskar, on domination and independent domination number of a graph, Discrete Mathematics, 23(1978) 73-76.

[6] T.W.Haynes,S.T.Hedetiniemi and P.J.Slater, Fundamentals of domination in graphs. Marcel-Dekker,Inc.(1997).