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NONLINEAR VIBRATIONS OF FUNCTIONALLY GRADED CIRCULAR CYLINDRICAL SHELLS

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In this paper, the effect of the geometry on the nonlinear vibrations of functionally graded cylindrical shells is analyzed. The Sanders-Koiter theory is applied to model nonlinear dynamics of the system in the case of finite amplitude of vibration. Shell deformation is described in terms of longitudinal, circumferential and radial displacement fields. Simply supported boundary conditions are considered. Numerical analyses are carried out in order to characterize the nonlinear response when the shell is subjected to an harmonic external load; different geometries and material distributions are considered. A convergence analysis is carried out in order to determine the correct number of the modes to be used; the role of the axisymmetric and asymmetric modes is carefully analyzed. The analysis is focused on determining the nonlinear character of the response as the geometry (thickness, radius, length) and material properties (power-law exponent N and configurations of the constituent materials) vary. The effect of the constituent volume fractions and the configurations of the constituent materials on the natural frequencies and nonlinear response are studied.

1. Introduction

Functionally graded materials (FGMs) are composite materials obtained by combining and mixing two or more different constituents, which are distributed in the thickness direction in accordance with a volume fraction law. The idea of FGMs was first introduced in 1984 by a group of Japanese material scientists. They studied different physical aspects, such as temperature and thermal stress distributions, static and dynamic responses, in the vibration characteristics of FGM shell.

Loy et al. [**1**] analyzed the vibrations of the cylindrical shells made of a functionally graded material, considering simply supported boundary conditions. They found that the natural frequencies are affected by the constituent volume fractions and the configurations of constituent materials. Touloukian [**2**] studied the thermo-physical properties of the high temperature solid materials, among them FGMs. He developed an equation that allows defining the material property as a function of the environmental temperature. Pellicano [**5**] presented a method for analyzing linear and nonlinear vibrations of cylindrical shells having different boundary conditions. Sanders-Koiter nonlinear theory was applied; the displacement fields were expanded in terms of harmonic functions and Chebyshev polynomials. Pellicano et al. [**6**] studied the effect of the geometry on the nonlinear vibrations of simply supported cylindrical shells. The geometric non-linearities due to finite-amplitude shell motion were considered by using Donnell's non-linear shallow-shell theory.

In the present paper, the effect of the geometry on the nonlinear vibrations of FGM shells is analyzed. The Sanders-Koiter theory [**3-4**] is applied to model the nonlinear dynamics of the system in the case of finite amplitude vibration. The shell deformation is described in terms of longitudinal, circumferential and radial displacement fields. Simply supported boundary conditions are considered. The solution method follows the approach of Refs. [**5**,**6**], it consists of two steps: 1) linear analysis and eigenfunction evaluation; 2) nonlinear analysis using an eigenfunction based expansion. Step 1: the displacement fields are expanded by means of a double mixed series based on harmonic functions for the circumferential variable and Chebyshev polynomials for the longitudinal variable; a Ritz based method allows to obtain approximate natural frequencies and mode shapes. Step 2: the three displacement fields are re-expanded by using the approximate eigenfunctions obtained at step 1; the nonlinear partial differential equations are reduced to a set of ordinary differential equations by using the Lagrange equations. Numerical analyses are carried out to characterize the nonlinear response when the shell is subjected to an harmonic external load; different geometries and material distributions are considered.

2. Fundamental equations of functionally graded materials

The material properties P_{fgm} of FGMs depend on the material properties and the volume fractions of the constituent materials, and they are expressed in the form [**1**]:

$$P_{fgm}(T,z) = \sum_{i=1}^{k} P_i\,(T) V_{fi}(z) \tag{1}$$

where P_i and V_{fi} are the material property and the volume fraction of the constituent material i.

The material properties P of the constituent materials can be described as a function of the environmental temperature $T(\mathrm{K})$ by Touloukian's cubic curve relation [**2**]:

$$P_i(T) = P_0(P_{-1}T^{-1} + 1 + P_1 T + P_2 T^2 + P_3 T^3) \tag{2}$$

where P_0, P_{-1}, P_1, P_2 and P_3 are the coefficients of temperature $T(\mathrm{K})$ of the constituent materials.

In the case of a FGM thin cylindrical shell, the volume fraction V_f can be written as [**1**]:

$$V_f(z) = \left(\frac{z + h/2}{h}\right)^N \tag{3}$$

where the power-law exponent N is a positive real number, with $(0 \le N \le \infty)$.

Young's modulus E, Poisson's ratio ν and mass density ρ are expressed as [**1**]:

$$E_{fgm}(T,z) = \big(E_2(T) - E_1(T)\big)\left(\frac{z\ + h/2}{h}\right)^N + E_1(T), \qquad \nu_{fgm}(T,z) = \big(\nu_2(T) - \nu_1(T)\big)\left(\frac{z\ + h/2}{h}\right)^N + \nu_1(T),$$

$$\rho_{fgm}(T,z) = \big(\rho_2(T) - \rho_1(T)\big)\left(\frac{z + h/2}{h}\right)^N + \rho_1(T) \tag{4}$$

3. Equations of motion

In Figure 1, a FGM circular cylindrical shell having radius R, length L and thickness h is represented; a cylindrical coordinate system $(O; x, \theta, z)$ is considered in order to take advantage from the axial symmetry of structure, having the origin O of reference system located at the centre of one end of the shell. Three displacement fields are represented: longitudinal $u(x,\theta,t)$, circumferential $v(x,\theta,t)$ and radial $w(x,\theta,t)$. The variable t is the time.

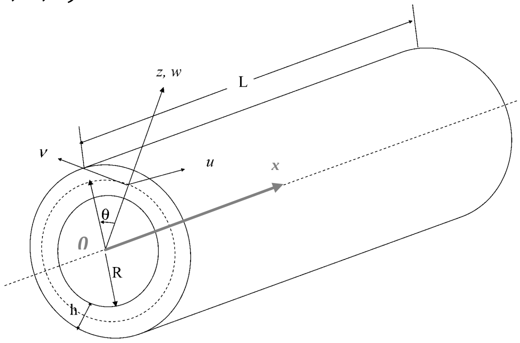

Figure 1. Geometry of a functionally graded circular cylindrical shell.

3.1 Strain energy

The Sanders-Koiter theory of circular cylindrical shells, which is an eight-order shell theory, is based on the first Love approximation [**3**].

The strain components $(\varepsilon_x, \varepsilon_\theta, \gamma_{x\theta})$ at an arbitrary point of the circular cylindrical shell are related to the middle surface strains $(\varepsilon_{x,0}, \varepsilon_{\theta,0}, \gamma_{x\theta,0})$ and to the changes in the curvature and torsion $(k_x, k_\theta, k_{x\theta})$ of the middle surface of the shell by the following relationships [**4**]:

$$\varepsilon_x = \varepsilon_{x,0} + zk_x, \qquad \varepsilon_\theta = \varepsilon_{\theta,0} + zk_\theta, \qquad \gamma_{x\theta} = \gamma_{x\theta,0} + zk_{x\theta} \tag{5}$$

where z is the distance of the arbitrary point of the shell from the middle surface, and (x, θ) are the longitudinal and angular coordinates of the shell, see Figure 1.

According to the Sanders-Koiter nonlinear theory, the middle surface strains and changes in curvature and torsion are given by [**4**]:

$$\varepsilon_{x,0} = \frac{\partial u}{L\partial\eta} + \frac{1}{2}\left(\frac{\partial w}{L\partial\eta}\right)^2 + \frac{1}{8}\left(\frac{\partial v}{L\partial\eta} - \frac{\partial u}{R\partial\theta}\right)^2 + \frac{\partial w}{L\partial\eta}\frac{\partial w_0}{L\partial\eta}, \quad \varepsilon_{\theta,0} = \frac{\partial v}{R\partial\theta} + \frac{w}{R} + \frac{1}{2}\left(\frac{\partial w}{R\partial\theta} - \frac{v}{R}\right)^2 + \frac{1}{8}\left(\frac{\partial u}{R\partial\theta} - \frac{\partial v}{L\partial\eta}\right)^2 +$$
$$\frac{\partial w_0}{R\partial\theta}\left(\frac{\partial w}{R\partial\theta} - \frac{v}{R}\right), \quad \gamma_{x\theta,0} = \frac{\partial u}{R\partial\theta} + \frac{\partial v}{L\partial\eta} + \frac{\partial w}{L\partial\eta}\left(\frac{\partial w}{R\partial\theta} - \frac{v}{R}\right) + \frac{\partial w_0}{L\partial\eta}\left(\frac{\partial w}{R\partial\theta} - \frac{v}{R}\right) + \frac{\partial w}{L\partial\eta}\frac{\partial w_0}{R\partial\theta}, \qquad k_x = -\frac{\partial^2 w}{L^2\partial\eta^2},$$
$$k_\theta = \frac{\partial v}{R^2\partial\theta} - \frac{\partial^2 w}{R^2\partial\theta^2}, \quad k_{x\theta} = -2\frac{\partial^2 w}{LR\partial\eta\partial\theta} + \frac{1}{2R}\left(3\frac{\partial v}{L\partial\eta} - \frac{\partial u}{R\partial\theta}\right) \tag{6}$$

where $(\eta = x/L)$ is the nondimensional longitudinal coordinate.

In the case of FGMs, the stresses $(\sigma_x, \sigma_\theta, \tau_{x\theta})$ are related to the strains $(\varepsilon_x, \varepsilon_\theta, \gamma_{x\theta})$ as follows [**4**]:

$$\sigma_x = \frac{E(z)}{1-\nu^2(z)}(\varepsilon_x + \nu(z)\varepsilon_\theta), \quad \sigma_\theta = \frac{E(z)}{1-\nu^2(z)}(\varepsilon_\theta + \nu(z)\varepsilon_x), \quad \tau_{x\theta} = \frac{E(z)}{2(1+\nu(z))}\gamma_{x\theta} \tag{7}$$

where $E(z)$ is the Young's modulus and $\nu(z)$ is the Poisson's ratio of the shell (plane stress, $\sigma_z = 0$).

The elastic strain energy U_s of a circular cylindrical shell, neglecting the radial stress σ_z (as Love's first approximation), is given by [**4**]:

$$U_s = \frac{1}{2}LR\int_0^1\int_0^{2\pi}\int_{-h/2}^{h/2}(\sigma_x\varepsilon_x + \sigma_\theta\varepsilon_\theta + \tau_{x\theta}\gamma_{x\theta})\,d\eta d\theta dz \tag{8}$$

Using equations (5), (7) and (8), the following expression of U_s can be obtained:

$$U_s = \frac{1}{2}LR\int_0^1\int_0^{2\pi}\int_{-h/2}^{h/2}\frac{E(z)}{1-\nu^2(z)}\left(\varepsilon_{x,0}^2 + \varepsilon_{\theta,0}^2 + 2\nu(z)\varepsilon_{x,0}\varepsilon_{\theta,0} + \frac{1-\nu(z)}{2}\gamma_{x\theta,0}^2\right)d\eta d\theta dz +$$
$$LR\int_0^1\int_0^{2\pi}\int_{-h/2}^{h/2}\frac{E(z)}{1-\nu^2(z)}\left(\varepsilon_{x,0}k_x + \varepsilon_{\theta,0}k_\theta + \nu(z)\left(\varepsilon_{x,0}k_\theta + \varepsilon_{\theta,0}k_x\right) + \frac{1}{2}\gamma_{x\theta,0}k_{x\theta} - \frac{\nu(z)}{2}\gamma_{x\theta,0}k_{x\theta}\right)d\eta d\theta z dz +$$
$$\frac{1}{2}LR\int_0^1\int_0^{2\pi}\int_{-h/2}^{h/2}\frac{E(z)}{1-\nu^2(z)}\left(\frac{1}{2}k_{x\theta}^2 + k_x^2 + k_\theta^2 + 2\nu(z)k_xk_\theta + \frac{1-\nu(z)}{2}k_{x\theta}^2\right)d\eta d\theta z^2 dz + O(h^4) \tag{9}$$

where $O(h^4)$ is a higher-order term in h according to the Sanders-Koiter theory.

The kinetic energy T_s (rotary inertia effect is neglected) is given by [**5**]:

$$T_s = \frac{1}{2}LR\int_0^1\int_0^{2\pi}\int_{-h/2}^{h/2}\rho(z)\,(\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\,d\eta d\theta dz \tag{10}$$

where $\rho(z)$ is the mass density of the functionally graded shell, and $(\dot{\cdot}) = d(\cdot)/dt$.

The virtual work W done by the external forces is written as [**5**]:

$$W = LR\int_0^1\int_0^{2\pi}(q_xu + q_\theta v + q_zw)d\eta d\theta \tag{11}$$

where (q_x, q_θ, q_z) are distributed forces per unit area acting in longitudinal, circumferential and radial direction, respectively.

4. Linear vibration analysis

In order to carry out a linear vibration analysis, in the present section, the Sanders-Koiter linear theory is considered, i.e. in equation (9) only the quadratic terms are retained.

The displacement fields are expanded by means of a double mixed series: axial symmetry of geometry and periodicity of deformation in the circumferential direction lead to use harmonic functions, while Chebyshev orthogonal polynomials are considered in the axial direction.

A modal vibration, i.e. a synchronous motion, is obtained in the form [**5**]:

$$u(\eta,\theta,t) = U(\eta,\theta)f(t), \quad v(\eta,\theta,t) = V(\eta,\theta)f(t), \quad w(\eta,\theta,t) = W(\eta,\theta)f(t) \tag{12}$$

where $u(\eta,\theta,t), v(\eta,\theta,t)$ and $w(\eta,\theta,t)$ are the displacement fields of the system, $U(\eta,\theta)$, $V(\eta,\theta)$ and $W(\eta,\theta)$ represent the modal shape, and $f(t)$ is the time law of the system, which is supposed to be the same for each displacement field (according to the synchronous motion hypothesis).

The modal shape (U,V,W) is then expanded in a double mixed series, in terms of Chebyshev polynomials $T_m^*(\eta)$ and harmonic functions $(\cos n\theta, \sin n\theta)$, in the following form [**5**]:

$$U(\eta,\theta) = \sum_{m=0}^{M_u}\sum_{n=0}^{N} \tilde{U}_{m,n} T_m^*(\eta)\cos n\theta, \qquad V(\eta,\theta) = \sum_{m=0}^{M_v}\sum_{n=0}^{N} \tilde{V}_{m,n} T_m^*(\eta) sin\, n\theta,$$
$$W(\eta,\theta) = \sum_{m=0}^{M_w}\sum_{n=0}^{N} \tilde{W}_{m,n} T_m^*(\eta)\cos n\theta \tag{13}$$

where $T_m^*(\eta) = T_m(2\eta - 1)$ and $T_m(\cdot)$ is the mth-order Chebyshev polynomial; m is the number of the longitudinal half-waves, n is the number of nodal diameters and $\left(\tilde{U}_{m,n}, \tilde{V}_{m,n}, \tilde{W}_{m,n}\right)$ are the generalized coordinates.

4.1 Boundary conditions

In the present work, simply supported circular cylindrical shells are considered; the boundary conditions are imposed by applying constraints to the free coefficients $\left(\tilde{U}_{m,n}, \tilde{V}_{m,n}, \tilde{W}_{m,n}\right)$ of the expansions (13). Simply supported boundary conditions are given by [**5**]:

$$w = 0, \quad v = 0, \quad M_x = 0, \quad N_x = 0 \quad \text{for} \quad \eta = (0,1) \tag{14}$$

The previous conditions imply the following equations [**5**]:

$$W(\eta,\theta) = \sum_{m=0}^{M_w}\sum_{n=0}^{N} \tilde{W}_{m,n}\, T_m^*(\eta)\cos n\theta = 0, \qquad V(\eta,\theta) = \sum_{m=0}^{M_v}\sum_{n=0}^{N} \tilde{V}_{m,n} T_m^*(\eta) sin\, n\theta = 0$$
$$W_{,\eta\eta}(\eta,\theta) = \sum_{m=0}^{M_w}\sum_{n=0}^{N} \tilde{W}_{m,n}\, T_{m,\eta\eta}^*(\eta)\cos n\theta = 0, \qquad U_{,\eta}(\eta,\theta) = \sum_{m=0}^{M_u}\sum_{n=0}^{N} \tilde{U}_{m,n} T_{m,\eta}^*(\eta)\cos n\theta = 0 \tag{15}$$

The linear algebraic system given by equations (15) can be solved analytically in terms of the coefficients $\left(\tilde{U}_{1,n}, \tilde{U}_{2,n}, \tilde{V}_{0,n}, \tilde{V}_{1,n}, \tilde{W}_{0,n}, \tilde{W}_{1,n}, \tilde{W}_{2,n}, \tilde{W}_{3,n}\right)$, for $n\epsilon[0,N]$.

4.2 Lagrange equations

The equations (12) and (13) are inserted in the expressions of the kinetic energy T_s and the linearized potential energy U_s (equations $(9-10)$); a set of ordinary differential equations (ODE) is then obtained by using Lagrange equations.

An intermediate step is the reordering of the variables in a vector [**5**]:

$$q = \left[\tilde{U}_{0,0}, \tilde{U}_{3,0}, \tilde{U}_{4,0}, \dots\dots, \tilde{U}_{M_u,0}, \tilde{U}_{0,1}, \tilde{U}_{3,1}, \tilde{U}_{4,1}, \dots\dots, \tilde{U}_{M_u,1}, \tilde{V}_{2,0}, \tilde{V}_{3,0}, \dots\dots, \tilde{V}_{M_v,0}, \tilde{V}_{2,1}, \tilde{V}_{3,1}, \dots\right.$$
$$\left.\dots, \tilde{V}_{M_v,1}, \tilde{W}_{4,0}, \tilde{W}_{5,0}, \dots\dots, \tilde{W}_{M_w,0}, \tilde{W}_{4,1}, \tilde{W}_{5,1}, \dots\dots, \tilde{W}_{M_w,1}\right] f(t) \tag{16}$$

The maximum number of variables needed for describing a generic vibration mode, with m longitudinal half-waves and n circumferential waves, can be calculated by the following relation $\left(N_p = M_u + M_v + M_w - 5\right)$, with $(M_u = M_v = M_w)$ as maximum degree of Chebyshev orthogonal polynomials, by considering the previous equations (14) for the boundary conditions of the shell.

The number of degrees of freedom of the system can be computed by the following relation $\left(N_{max} = N_p \times N\right)$, where N describes the maximum number of nodal diameters considered.

The Lagrange equations for free vibrations read [**5**]:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \quad \text{for } i\epsilon[1, N_{max}] \quad (L = T_s - U_s) \quad \left(N_{max} = N_p \times N\right) \tag{17}$$

Considering an harmonic motion $(f(t) = e^{j\omega t})$, we obtain the secular equation [**5**]:

$$(-\omega^2 \boldsymbol{M} + \boldsymbol{K})\boldsymbol{q} = \boldsymbol{0} \tag{18}$$

which is the classical nonstandard eigenvalue problem, that furnishes frequencies (eigenvalues) and modes of vibration (eigenvectors).

The modal shape, corresponding to the jth mode, is given by the equations (13), where the coefficients $(\tilde{U}_{m,n}, \tilde{V}_{m,n}, \tilde{W}_{m,n})$ are substituted with $(\tilde{U}_{m,n}^{(j)}, \tilde{V}_{m,n}^{(j)}, \tilde{W}_{m,n}^{(j)})$, i.e. the components of the jth eigenvector of equation (18);

$$\boldsymbol{U}^{(j)}(\eta,\theta) = \left[U^{(j)}(\eta,\theta), V^{(j)}(\eta,\theta), W^{(j)}(\eta,\theta)\right]^T \tag{19}$$

represents the jth eigenfunction vector of the original problem. The linear model is validated with respect to natural frequencies, details are omitted for the sake of brevity.

5. Nonlinear vibration analysis

In the nonlinear vibration analysis, the full expression of the potential energy (9), containing terms up to the fourth order (cubic nonlinearity), is considered. The displacement fields $u(\eta,\theta,t)$, $v(\eta,\theta,t)$ and $w(\eta,\theta,t)$ are expanded by using the linear mode shapes $U(\eta,\theta)$,$V(\eta,\theta)$, $W(\eta,\theta)$ obtained in the previous section [**6**]:

$$u(\eta,\theta,t) = \sum_{j=1}^{N_{max}} U^{(j)}(\eta,\theta)\, f_{u,j}(t),\;\; v(\eta,\theta,t) = \sum_{j=1}^{N_{max}} V^{(j)}(\eta,\theta)\, f_{v,j}(t),\;\; w(\eta,\theta,t) = \sum_{j=1}^{N_{max}} W^{(j)}(\eta,\theta)\, f_{w,j}(t) \tag{20}$$

These expansions respect the simply supported boundary conditions (14); the mode shapes $U^{(j)}(\eta,\theta), V^{(j)}(\eta,\theta), W^{(j)}(\eta,\theta)$ are known functions expressed in terms of the polynomials and harmonic functions.

The Lagrange equations for forced vibrations are expressed in the following form [**6**]:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i, \quad \text{for } i\epsilon[1, N_{max}] \qquad (L = T_s - U_s) \qquad \left(N_{max} = N_p \times N\right) \tag{21}$$

The generalized forces Q_i are then obtained by differentiation of the Rayleigh's dissipation function F and the virtual work done by external forces W, in the form [**6**]:

$$Q_i = -\frac{\partial F}{\partial \dot{q}_i} + \frac{\partial W}{\partial q_i} \tag{22}$$

Expansions (20) are inserted into strain energy (9), kinetic energy (10) and virtual work of external forces (11), in the case of external excitation; using Lagrange equations (21), a system of ordinary differential equations is then obtained.

6. Numerical results

In this section, the nonlinear vibrations of simply supported functionally graded circular cylindrical shells with different geometries and material distributions are analyzed.

Chebyshev polynomials used in the approximate method have degree equal to $m = 11$. The functionally graded material is composed by stainless steel and nickel, its properties are graded in the thickness direction according to a volume fraction distribution, where N is the power-law exponent. The material properties are reported in Table 1 [**1**].

Table 1. Properties of stainless steel and nickel against coefficients of temperature evaluated at $(\mathrm{T} = 300\mathrm{K})$.

	stainless steel			*nickel*		
	$E(\mathrm{Nm}^{-2})$	ν	$\rho(\mathrm{kgm}^{-3})$	$E(\mathrm{Nm}^{-2})$	ν	$\rho(\mathrm{kgm}^{-3})$
P_0	$+2.01 \times 10^{11}$	0.326	8166	$+2.24 \times 10^{11}$	0.3100	8900
P_{-1}	0	0	0	0	0	0
P_1	$+3.08 \times 10^{-4}$	-2.002×10^{-4}	0	-2.79×10^{-4}	0	0
P_2	-6.53×10^{-7}	$+3.797 \times 10^{-7}$	0	-3.99×10^{-9}	0	0
P_3	0	0	0	0	0	0
P	$+2.08 \times 10^{11}$	0.318	8166	$+2.05 \times 10^{11}$	0.3100	8900

In the nonlinear model (expansions (20)), the following modes having m longitudinal half-waves and n nodal diameters (m, n) are selected: $(1,0), (3,0), (1,6)$ in the longitudinal displacement field; $(1,6), (1,12), (3,12)$ in the circumferential displacement field; $(1,0), (3,0), (1,6)$ in the radial displacement field. After selecting such modes, each expansion present in equation (20) is reduced to a three-terms modal expansion; the resulting nonlinear system has 9 degrees of freedom.

The circular cylindrical shell is excited by means of an external modally distributed radial force $q_r = f_{1,6} \sin \eta \cos 6\theta \cos \omega t$, having amplitude of excitation equal to $f_{1,6} = 0.0012 h^2 \varrho \omega_{1,6}^2$ and frequency of excitation ω close to the $(1,6)$ frequency, $\omega \cong \omega_{1,6}$. The driven mode is $(1,6)$, and the external forcing $f_{1,6}$ is normalized with respect to mass, acceleration and thickness; the damping ratio is $\xi_{1,6} = 0.0005$. In Figure 2, the amplitude-frequency response is shown (modal coordinate $(1,6)$ of w); a FGM shell having stainless steel on the outer surface and nickel on the inner surface is considered ($h/R = 0.002, L/R = 20$ and the power-law exponent $N = 1$). A hardening nonlinear behaviour is observed.

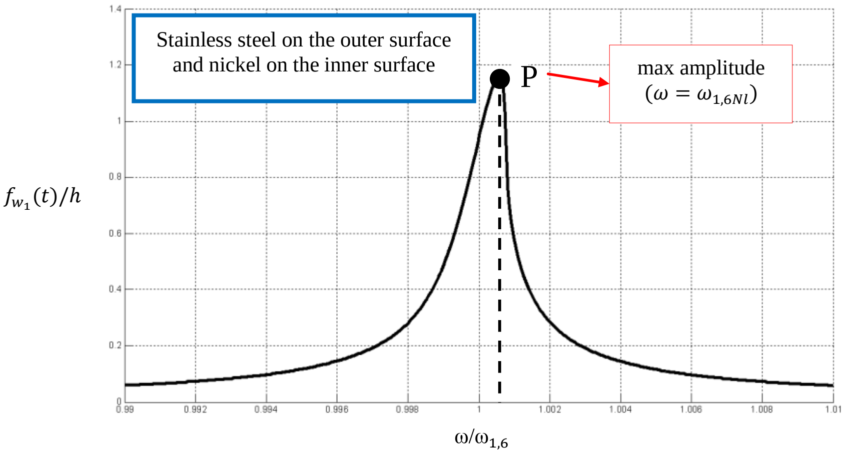

Figure 2. Nonlinear amplitude-frequency curve of the FGM cylindrical shell $(h/R = 0.002, L/R = 20, N = 1)$.

6.1 Nonlinear response convergence

The convergence analysis is developed by introducing in longitudinal, circumferential and radial displacement fields a different number of asymmetric and axisymmetric modes: a 6 dof model with modes $(1,0), (1,6), (1,12)$, a 9 dof model with modes $(1,0), (3,0), (1,6), (1,12), (3,12)$, a 12 dof model with modes $(1,0), (3,0), (1,6), (3,6), (1,12), (3,12)$, a 15 dof model with modes $(1,0)$, $(3,0), (5,0), (1,6), (3,6), (1,12), (3,12), (1,18)$ and a 18 dof model with modes $(1,0), (3,0), (5,0)$, $(7,0), (1,6), (3,6), (1,12), (3,12), (1,18), (3,18)$.

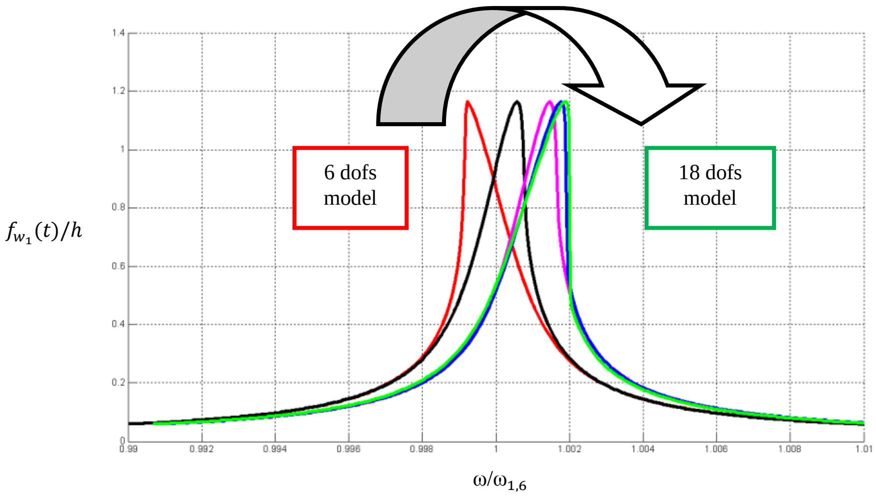

Figure 3. Comparison of nonlinear amplitude-frequency curves of the FGM shell $(h/R = 0.002, L/R = 20, N = 1)$.
–, 6 dofs model; –, 9 dofs model; –, 12 dofs model; –, 15 dofs model; –, 18 dofs model.

In Figure 3, a comparison of nonlinear amplitude-frequency curves of the FGM shell ($h/R = 0.002, L/R=20, N=1$ is shown: the nonlinear 6 dofs model describes a wrong softening nonlinear behaviour, the higher-order nonlinear expansions converge to a strongly hardening nonlinear behaviour, that is the correct character of the shell response.

6.2 Effect of the geometry on the nonlinear response

In Figure 4, the transitions from hardening to softening nonlinear behaviours (ratio $h/R = 0.003 \div 0.004$) and from softening to hardening ($h/R = 0.046 \div 0.047$) can be clearly observed. Let us define a nonlinearity indicator $NL_b = \frac{\omega_{1,6Nl} - \omega_{1,6}}{\omega_{1,6}}$, where $NL_b > 0$ means a hardening and $NL_b < 0$ means a softening nonlinear behaviour.

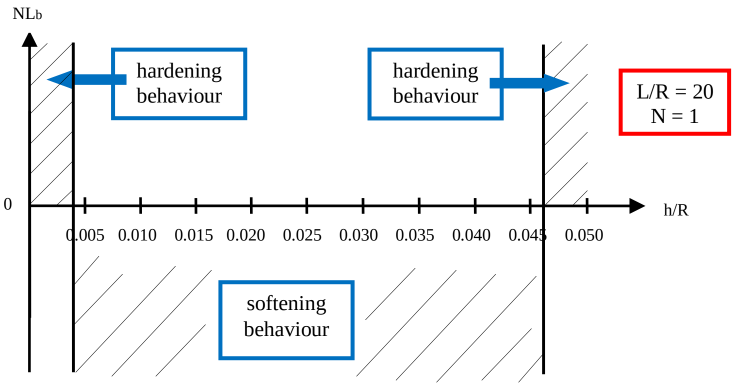

Figure 4. Nonlinear behaviour NL_b transitions of the FGM shells for different values of the ratio (h/R).

Very thin or thick shells show a hardening nonlinearity, conversely, a softening nonlinearity is found in a wide range of the FGM circular cylindrical shell geometries. Note that beyond $h/R > 0.05$ the thin walled approx looses accuracy [**3**].

6.3 Effect of the material distribution on the nonlinear response

The effect of the material distribution on the nonlinear response is analyzed by considering two different FGM shells: Type I FGM shell has nickel on its inner surface and stainless steel on its outer surface, and Type II FGM shell has stainless steel on its inner surface and nickel on its outer surface. In Figure 5(a), the nonlinear behaviour NL_b of Type I FGM shell is shown: as the value of the exponent N increases, the value of the natural frequency $\omega_{1,6}$ decreases and the character of the nonlinear behaviour decreases from strongly hardening ($N < 1$) to weakly hardening ($N > 5$). In Figure 5(b), the nonlinear behaviour NL_b of Type II FGM shell is shown: as the value of the exponent N increases, the value of the natural frequency $\omega_{1,6}$ increases and the character of the nonlinear behaviour increases from weakly hardening ($N < 1$) to strongly hardening ($N > 5$) behaviour.

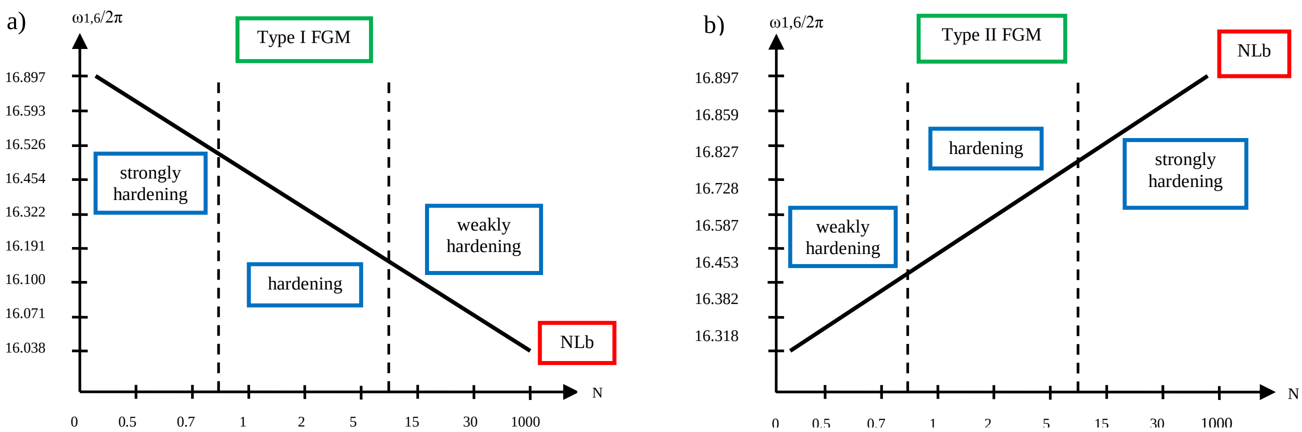

Figure 5. Nonlinear behaviour NLb of the shell $(h/R = 0.002, L/R = 20)$. (a) Type I FGM; (b) Type II FGM.

7. Conclusions

In this paper, the effect of the geometry on the nonlinear vibrations of FGM cylindrical shells is analyzed; different configurations and constituent volume fractions are considered. The Sanders-Koiter theory is applied to model nonlinear dynamics of the system in the case of finite amplitude of vibration. The shell deformation is described in terms of longitudinal, circumferential and radial displacement fields. Simply supported boundary conditions are considered. Displacement fields are expanded by means of a double mixed series based on harmonic functions for the circumferential variable and Chebyshev polynomials for the longitudinal variable.

Numerical analyses are carried out in order to characterize the nonlinear response of the shells; the effect of the geometry on the nonlinear vibrations of the shells is analyzed, and a comparison of nonlinear amplitude-frequency curves of functionally graded cylindrical shells with different geometries is carried out.

Different nonlinear behaviours, depending on the geometric characteristics of the FGM shells, are obtained: very thin or thick shells show a hardening nonlinearity, conversely, a softening nonlinearity is found in a wide range of the FGM shell geometries.

A convergence analysis is developed by introducing in longitudinal, circumferential and radial displacement fields a different number of asymmetric and axisymmetric modes; the correct number of modes to describe the actual nonlinear behaviour of the cylindrical shells is determined.

The influence of the constituent volume fractions and the effect of the configurations of the constituent materials on the natural frequencies and nonlinear responses of the shells are analyzed.

Natural frequencies computed for the different values of the power-law exponent are observed to lie between the natural frequencies of the homogeneous stainless steel and homogeneous nickel circular cylindrical shell.

In Type I FGM shell, as the value of the power-law exponent increases, the value of the corresponding natural frequency decreases and the hardening character of the nonlinear behaviour decreases from a strongly hardening to a weakly hardening behaviour.

In Type II FGM shell, as the value of the power-law exponent increases, the value of the corresponding natural frequency increases and the hardening character of the nonlinear behaviour increases from a weakly hardening to a strongly hardening behaviour.

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