ISSN: 2321-8169
3926 - 3931

Algorithms to Find Linear Geodetic Numbers and Linear Edge Geodetic Numbers in Graphs

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Abstract:-Given two vertices u and v of a connected graph G=(V, E), the closed interval I[u, v] is that set of all vertices lying in some u-v geodesic in G. A subset of V(G) $S=\{v_1,v_2,v_3,\ldots,v_k\}$ is a linear geodetic set or sequential geodetic set if each vertex x of G lies on a $v_i - v_{i+1}$ geodesic where $1 \le i < k$. A linear geodetic set of minimum cardinality in G is called as linear geodetic number *lgn(G)* or sequential geodetic number *sgn(G).* Similarly, an ordered set $S=\{v_1,v_2,v_3,\ldots,v_k\}$ is a linear edge geodetic set if for each edge e = xy in G, there exists an index i, $1 \le i < k$ such that e lies on a $v_i - v_{i+1}$ geodesic in G. The cardinality of the minimum linear edge geodetic set is the linear edge geodetic number of G denoted by *legn(G).* The purpose of this paper is to introduce algorithms using dynamic programming concept to find minimum linear geodetic set and thereby linear geodetic number and linear edge geodetic set and number in connected graphs.

Keywords*:* *distance, linear geodetic, linear geodetic number, linear edge geodetic, linear edge geodetic number, sequential geodetic, sequential geodetic cover, sequential geodetic number.*

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1. INTRODUCTION

Given two vertices u and v of an undirected connected simple graph G=(V, E), the distance d(u,v) from u to v is the length of a shortest u-v path in G. A u-v path of length d(u,v) is called a u-v geodesic. A vertex w is said to lie on an x–y geodesic P if w is a vertex of P including the vertices x and y. For vertex v in a connected graph G, the eccentricity e(v) is the distance between v and a vertex farthest from v in G. The minimum eccentricity among vertices of G is its radius and the maximum eccentricity is its diameter, which are denoted by rad(G) and diam(G), respectively. A vertex v in G is a central vertex if e(v)=rad(G). Graph notation and terminology is referred from the books [1, 2]. For two vertices u and v in a connected graph G, the closed interval I[u, v] of two vertices u and v in G is the set of those vertices of G belonging to at least one u-v geodesic that is the shortest path between u and v. Note that a vertex w belongs to I[u,v] if and only if there is a shortest path between **(u,w)** and **(w,v)**. For a set S of vertices, let the closed interval I[S] of S be the union of the closed intervals I[u, v] over all pairs of vertices u and v in S. A set of vertices S is called geodetic if I[S] contains all vertices V of G. Harary et al. [3] define the geodetic number ***gn(G)*** of a graph G as the minimum cardinality of a geodetic set. The geodetic number of a graph was introduced in [3, 5, 6] and its different dimensions are further studied in [7-17]. An algorithm to find geodetic numbers and edge geodetic numbers in graphs has been introduced in [18].

There are interesting applications of Geodetic number concepts to the problem of designing the route for a shuttle and communication network design. The edge geodetic set is more advantageous to the real life application of routing problem. In particular, the edge geodetic sets are more useful than geodetic sets in the case of regulating and routing the goods vehicles to transport the commodities to important places. The different other areas that apply geodetic number concepts are telephone switching centres, facility location, distributed computing, Image and Video Editing, Neural networks and Data mining.

2. LINEAR / SEQUENTIAL GEODETIC NUMBER

Let G be a connected graph. An ordered set S= $\{v_1, v_2, v_3, \ldots, v_k\}$ of vertices in G is a linear geodetic set of G if for each vertex x in G, there exists an index i, $1 \le i < k$ such that x lies on a v_i - v_{i+1} geodesic in G. A linear geodetic set of minimum cardinality is the linear geodetic number **lgn(G).** The concept of linear geodetic number and related results has been studied in [15]. Let $S_1, S_2, \ldots\ldots, S_k$ are the sequences of subsets of V (G) where $S_1 = \{v_1\}$, $S_2 = S_1 \cup \{v_2\}$, $S_3 = S_2 \cup \{v_3\}$ and $S_k = S_{k-1} \cup \{v_k\} = \{v_1, v_2, v_3, \ldots, v_k\}$. A subset $S_k = \{v_1, v_2, v_3, \ldots, v_k\}$ is called as sequential geodetic cover of G if $\{v_1, v_2, v_3, \ldots, v_k\}$ is a geodetic sequence in G and S_k=V(G). A sequential geodetic cover of minimum cardinality is called sequential geodetic basis of G. The minimum cardinality of a sequential basis is called as Sequential Geodetic Number **sgn(G)**. The concept of Sequential geodetic number and related results has been studied in [16]. When it is compared, linear geodetic number and sequential geodetic number are found to be more or less equivalent.

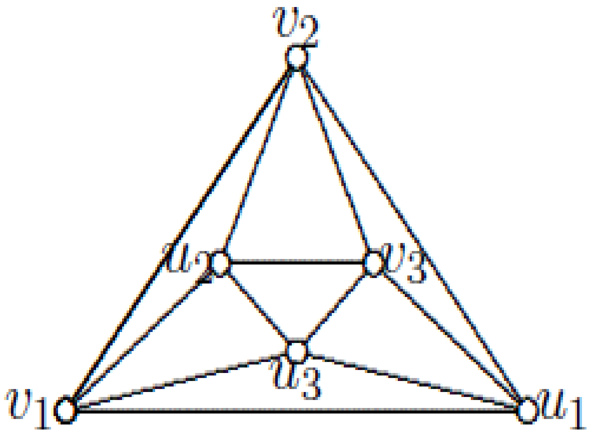

Figure (a)

Let us take for example the graph given in Figure (a) to identify the linear / sequential geodetic set. Let $S_1 = \{v_1\}$ and $S_2 = \{v_1, v_2\}$. While finding closed interval of S_2, I $[S_2] = \{v_1, v2\}$ and found that S_2 is not a linear / sequential geodetic set since $I[S_2] \neq V(G)$. The subset $S_3 = S_2 \cup \{v_3\} = \{v_1,v_2,v_3\}$ and the closed interval set of S_3 can be evaluated as,

$I[S_3] = \{v_1,v_2,v_3\}$ **U** $I[v_1,v_2]$ U $I[v_2,v_3]$ that is $S_3=\{v_1,v_2,v_3\}$ **U** $\{v_3,u_1,u_2,u_3, v_2,v_1\}$ **U** $\{v_3,v_2\}$ Hence $I[S_3]= \{v_1,v_2,v_3,u_1,u_2,u_3\}$ and so $I[S_3]$=V(G). S_3 is found to be the linear/sequential geodetic basis. The cardinality of S_3 is 3 and hence linear/sequential geodetic number of the graph G is ***lgn(G)*** or ***sgn(G)***=3.

3. LINEAR EDGE GEODETIC NUMBER

As edge geodetic number was introduced and studied in [10], an edge geodetic set S of G is a set of vertices of G such that every edge of G is contained in a geodesic joining some pair of vertices u and v in S. The edge geodetic number **egn(G)** of G is the minimum cardinality of its edge geodetic sets. Let G be a connected graph. An ordered set $S=\{u_1, u_2, \ldots, u_k\}$ of vertices in G is a linear edge geodetic set of G if for each edge **e = xy** in G, there exists an index i, $1 \le i < k$ such that e lies on a $u_i - u_{i+1}$ geodesic in G, and a linear edge geodetic set of minimum cardinality is a minimum linear edge geodetic set, and the cardinality of a minimum linear edge geodetic set is the linear edge geodetic number **legn(G)** of G. A graph G is called a linear edge geodetic graph if it has a linear edge geodetic set. Every linear edge geodetic set of a graph is an edge geodetic set of G.

For a non-trivial connected graph G, a linear set $S \subseteq V(G)$ is called an linear edge geodetic cover of G if every edge of G is contained in a geodesic joining some pair of vertices in S. The set $I_e[u, v]$ consists of all edges of G lying in any u - v geodesic in G. If $S \subseteq V(G)$, then the set $I_e[S]$ denotes the union of all $I_e[u, v]$, where $u, v \in S$. A subset S of V (G) is a linear edge geodetic set or a linear edge geodetic cover of G if every edge of G is contained in $I_e[S]$, that is, $I_e[S] = E(G)$. The linear edge geodetic number ***legn*** (G) of G is the minimum order of its linear edge geodetic covers and any linear edge geodetic cover of order ***legn (G)*** is a linear edge geodetic basis. Concept of linear edge geodetic number of a graph and related results has been studied from [17].

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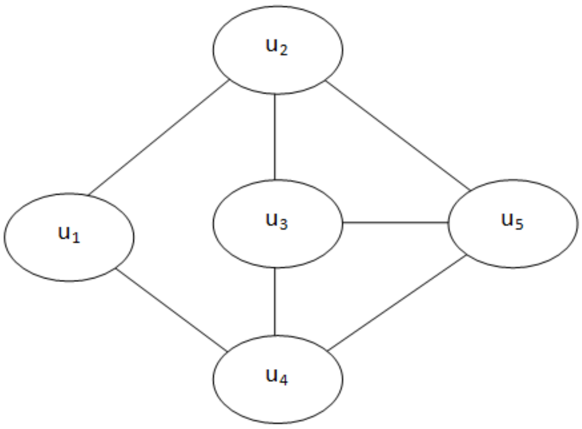

Figure (b)

Let us consider the graph given in Figure (b) to identify the linear edge geodetic set. Let $S_1 = \{u_1\}$ and $S_2 = \{u_1, u_2\}$. While finding closed interval of S_2, $\mathbf{I_e[S_2]} = \{(\boldsymbol{u_1}, \boldsymbol{u_2})\}$ and found that S_2 is not a linear edge geodetic set since $I_e[S_2] \neq E(G)$.

The subset $S_3 = S_2 \mathbf{U} \{u_3\} = \{u_1, u_2, u_3\}$ and the closed interval set of S_3 can be evaluated as $\mathbf{I_e[S_3]} = \{(\boldsymbol{u_1}, \boldsymbol{u_2}), (\boldsymbol{u_1}, \boldsymbol{u_4}), (\boldsymbol{u_2}, \boldsymbol{u_3}), (\boldsymbol{u_3}, \boldsymbol{u_4})\}$. $I_e[S_2] \neq E(G)$ and hence S_3 is not a linear edge geodetic set. As per the process, the subset S_4 is formed as $S_3 \, U \, \{u_4\}$ and hence $S_4 = \{u_1, u_2, u_3, u_4\}$. The closed interval of S_4 is evaluated as $\mathbf{I_e[S_4]} = \{(\boldsymbol{u_1}, \boldsymbol{u_2}), (\boldsymbol{u_1}, \boldsymbol{u_4}), (\boldsymbol{u_2}, \boldsymbol{u_3}), (\boldsymbol{u_3}, \boldsymbol{u_4}), (\boldsymbol{u_2}, \boldsymbol{u_5}), (\boldsymbol{u_5}, \boldsymbol{u_4})\}$. The closed interval of S_4 contains all the edges except $(\boldsymbol{u_3}, \boldsymbol{u_5})$ and hence S_4 is not a linear edge geodetic set. Finally, $S_5 = \{u_1, u_2, u_3, u_4, u_5\}$ and its closed interval $\mathbf{I_e[S_5]} = \{(\boldsymbol{u_1}, \boldsymbol{u_2}), (\boldsymbol{u_1}, \boldsymbol{u_4}), (\boldsymbol{u_2}, \boldsymbol{u_3}), (\boldsymbol{u_3}, \boldsymbol{u_4}), (\boldsymbol{u_2}, \boldsymbol{u_5}), (\boldsymbol{u_3}, \boldsymbol{u_5}), (\boldsymbol{u_4}, \boldsymbol{u_5})\}$. $I_e[S_5] = E(G)$ and hence S_5 is called as the linear edge geodetic set. The cardinality of S_5 is 5 and hence linear edge geodetic number of the above graph in Figure (b) is ***legn(G)*** = 5.

4. ALGORITHM FOR COMPUTING LINEAR GEODETIC NUMBER

In this section we introduce an algorithm using dynamic programming approach for evaluating the linear geodetic number of a connected graph G. This process is implemented in the proposed algorithm in two stages. The first stage of the algorithm records the shortest path between each vertex to every other vertices for further process using all source shortest path algorithm introduced by Floyd. The second stage of the algorithm uses the recorded shortest path and finds the closed interval I[S] which is the union of the intervals I[u, v] over all pairs of vertices u and v and thereby determines the linear geodetic number of a given graph G.

4.1 Algorithm for Linear Geodetic Number

Algorithm LinearGeodeticNumber(G)
// Input: An undirected graph $G=(V,E)$ $|V| \geq 2$
// Output: Linear Geodetic Number of G

1. Using Floyd's All source shortest path algorithm, find the shortest path between each vertex and every other vertices and vertices in the shortest path are recorded in a set named Path[u,v].
2. k = 1
3. Found = false
4. $S=\{v_1\}$
5. While ($k \leq |V|$ and Found$\neq$ true)
6.   k = k+1
7. $S= S \, U \, \{v_k\}$ // $\forall\, v_k \in V$
8.   While Found$\neq$ true
9.   For every vertex u and v in S
10.   Find the closed interval I[S] union of the intervals I[u, v] over all pairs of vertices u and v in S from set **Path[u, v]** computed above in Step 1.
11.   Loop
12.   If I[S]=V Then Found=true
13.   Loop
14. Loop
15. Return k

Theorem 4.1: For any undirected connected graph G such that $|V| \geq 2$, the algorithm 4.1 finds the linear geodetic number **lgn(G)** and $\mathbf{2 \leq lgn(G) \leq |V|}$.

Proof: Let G be a connected graph with $V= \{v_1, v_2, v_3, \ldots\ldots, v_n\}$. Step 1 of the Algorithm 4.1 evaluates the shortest path between each vertex and every other vertex. The algorithm begins with the set S having v_1. Step 7 of the Algorithm 4.1 reconstructs the set S by adding the vertex next in the list of vertices of V such that $S_k=S_{k-1}Uv_k$.

ISSN: 2321-8169
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Cardinality of generated subset S starts with minimum value 2 and it subset increases by one during each iteration until linear geodetic set is found. Step 10 of the Algorithm 4.1 executes the main the process of finding the linear geodetic number by evaluating closed interval I[S] which has the set of vertices of G that forms shortest path between vertices in S from the Path set that contains the shortest path between each vertex and every other vertex. Step 12 of the Algorithm 4.1 checks if I[S]=V(G), , if it is true then S is considered as the minimal linear geodetic set and returns **|S|**$\geq$2 to be the linear geodetic number. Steps 5 to step 14 of the Algorithm 4.1 are repeated until a minimum linear geodetic set is identified.

If I[S] = V(G) in the first iteration itself, the linear geodetic number is 2 otherwise it linearly increases by one during each iteration. For the worst case S contains all the vertices of G and so the linear geodetic number **lgn(G)** is **|V|.** Hence it is proved that **2$\leq$ lgn(G) $\leq$|V|**.

Theorem 4.2: Algorithm 4.1 computes Linear geodetic number of an undirected graph G in efficiency **$\Omega(|V|^3)$.**

Proof: Let G be a connected undirected graph with V= {v_1, v_2, $v_{3, \ldots}$ v_n}. Algorithm 4.1 initiates the task of finding linear geodetic number by evaluating all source shortest path and records in a set named Path using the Floyd's algorithm which is proved having the efficiency **$O(|V|^3)$** and records the path in a Set data structure and it is referred later. The process continues by constructing the subset of V(G) that is S (linearly as per Step 7 of Algorithm 4.1) and evaluating the closed interval I[S] of S be the union of the intervals I[u, v] over all pairs of u-v geodesic of vertices u and v in S . The computing of I[S] which is the linear geodetic closure of a set of vertices is done by referring the Path set constructed in Step 1 of the Algorithm 4.1. For each subset S, computing of I[S] is to be performed repeatedly until the linear geodetic set is finalized. The closed interval evaluation is performed a minimum of 1 time and maximum of **|V|** number of times and therefore efficiency of this process is **O(|V|).** Therefore total computing time can be represented as **$O(|V|^3+|V|)$**. It is clearly known that the computing time will be just more than **$|V|^3$** and so represented with notation-**Ω.** Hence the computing of linear geodetic number **lgn(G)** using Algorithm 4.1 can be completed in **$\Omega(|V|^3)$.**

5. ALGORITHM FOR COMPUTING LINEAR EDGE GEODETIC NUMBER

We introduce an algorithm using dynamic programming approach in this section for evaluating the edge geodetic number of a connected graph G. The proposed process has two divisions as in the Algorithm 4.1. The first division of the algorithm records the edges that fall in the shortest path between each vertex to every other vertex for further process using Floyd's all source shortest path algorithm. The second part of the algorithm uses the recorded shortest path and finds the closed interval $I_e[S]$ which is the union of the intervals $I_e[u, v]$ over all pairs of vertices u and v and thereby determine the edge geodetic number of the given graph G.

5.1 Algorithm for LinearEdge Geodetic Number

Algorithm LinearEdgeGeodeticNumber(G)
// Input: An undirected graph G=(V,E) |V| $\geq$2
// Output: Linear Edge Geodetic Number of G

1. Using Floyd's All source shortest path algorithm, find the shortest path between each vertex and every other vertices and edges in the shortest path are recorded in a set named Path[u,v].
2. k = 1
3. Found = false
4. S={v_1}
5. While (k $\leq$ |V| and Found$\neq$ true)
6. k = k+1
7. S= S U {v_k} // $\forall$ $v_k \in$V
8. While Found$\neq$ true
9. For every vertex u and v in S
10. Find the closed interval I_e[S] union of the intervals I_e [u, v] over all pairs

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of vertices u and v in S from set **Path[u, v]** computed above in Step 1.

11. Loop
12. If $I_e[S]$ = E Then Found=true
13. Loop
14. Loop
15. Return k

Theorem 5.1: Algorithm 5.1 finds the linear edge geodetic number **legn(G)** and **2≤ legn(G)** ≤|**V**| for any undirected connected graph G such that |V|≥2.

Proof: Let G be a connected graph with $V=\{v_1,v_2,v_3,\ldots\ldots,v_n\}$. Step 1 of the Algorithm 5.1 evaluates the shortest path between each vertex and every other vertex. Step 7 of the Algorithm 4.1 reconstructs the set S by adding the vertex next in the list of vertices of V such that $S_k=S_{k-1}Uv_k$. Cardinality of generated subset S starts with minimum value 2 and it subset increases by one during each iteration until linear geodetic set is found. Step 10 of the Algorithm 5.1 executes the main the process of finding the linear edge geodetic number by evaluating closed interval $I_e[S]$ which has the set of linear edges of G that forms shortest path between vertices in S from the Path set that contains the linear edges fall in the shortest path between each vertex and every other vertex in S. Step 12 of the Algorithm 5.1 checks if $I_e[S]$=E, if it is true then S is considered as the minimal linear edge geodetic set and returns |**S**|≥2 to be the linear edge geodetic number. Steps 5 to step 14 of the Algorithm 5.1 are repeated until a minimum linear edge geodetic set is identified.

If $I_e[S]$=E in the first iteration itself then the linear edge geodetic number is 2 otherwise it linearly increases by one during each iteration. For the worst case S contains all the vertices of G and so the linear edge geodetic number **legn(G)** is |**V**|. Hence it is proved that **2≤ legn(G)** ≤|**V**|.

Theorem 5.2: Linear edge Geodetic number of an undirected graph G can be computed using Algorithm 5.1 in **Ω(|V|^3).**

Proof: Let G be a connected undirected graph with V= $\{v_1, v_2, v_{3,} \ldots v_n\}$ and the edges E=$\{e_1, e_2, e_{3,} \ldots e_n\}$. Algorithm 5.1 initiates the task of finding linear edge geodetic number by evaluating all source shortest path using the Floyd's algorithm which is proved having the efficiency **O(|V|^3)** and records the path in a Set data structure and it is referred later. The process continues by constructing the subset of V(G) that is S (linearly as per Step 7 of Algorithm 5.1) and evaluating the closed interval $I_e[S]$ of S be the union of the intervals $I_e[u, v]$ over all pairs of u-v geodesic of vertices u and v in S . Computing of $I_e[S]$, the linear edge geodetic closure of a set of edges is done by referring the Path set constructed in Step 1 of the Algorithm 5.1. For each subset S, computing of $I_e[S]$ is to be performed repeatedly until finalizing the linear edge geodetic set. The closed interval evaluation is performed a minimum of 1 time and maximum of |**V**| number of times and therefore efficiency of this process is **O(|V|).** Therefore total computing time can be represented as **O(|V|^3+|V|).** It is clearly known that the computing time will be just more than |**V**|^3 and so represented with notation-**Ω.** Hence the computing of linear edge geodetic number **legn(G)** using Algorithm 5.1 can be completed in **Ω(|V|^3).**

6. CONCLUSION

The main contribution of this paper is in having presented algorithm using dynamic programming concept for computing the linear geodetic number and linear edge geodetic number of undirected graphs and analyzing its efficiency. We believe that our algorithms can be extended to address further tasks related to Distributed computing, Image processing, Telephone switching centres, Facility location, Neural networks and Data mining.

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