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Semi- Ideal and Semi-Filter in Distributive Q-lattices

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Abstract: In this paper, We define concepts of prime semi ideal, prime semi filter in distributive q-lattice A and we prove A non empty subset F of A (F≠ A) is prime semi filter if and only if (A-F) is prime semi ideal. We define concepts of semi a-ideal, semi a-filter and prove if F be semi a-filter in A then F is maximal semi a-filter if for x ∉ F there exists y ∈ F such that a = y ∧ x , and some equivalent conditions. We define a semi-ideal $P^{\perp}$ = { x ∈ A / x ∧ a = s for all a ∈ P } and some properties. If f: A → A′ be an onto homomorphism then we prove for any semi ideal $I^{\perp}$ of A , f($I^{\perp}$) is semi ideal of A′

KEYWORDS: Distributive q-lattice, Ideal, Filter, Semi a-ideal ,Semi a-filter, Prime semi-ideal, Prime semi-filter.

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1. INTRODUCTION

Ivan Chajda [2] introduced the concept of a q-lattice and defined distributive q-lattice. After that G. C. Rao, P. Sundarayya, S. Kalesha vali, and Ravi Kumar Bandaru [1] defined ideals of distributive q-lattice , A. D. Lokhande, Ashok S Kulkarni [4] in paper **'**Filter and Annihilator in Distributive q-lattices' defined Filter in a distributive q-lattice and proved if A be a distributive q-lattice then F(A), the set of all filters of A is a lattice under set inclusion. G. C. Rao and M. Sambasiva Rao [5] defined ' annihilator ' ideal in Almost Distributive Lattice (ADL_s) and derived some properties, In paper [4] A. D. Lokhande, Ashok S Kulkarni defined annihilator in distributive q-lattice A and proved for any ideal I of distributive q-lattice A and a∈ A , the annihilator (a:I) is an ideal of A and derived some properties. Ashok S Kulkarni and A. D. Lokhande [5] defined J(P) and prove if J be an ideal of distributive q-lattice A then for any prime ideal P containing J, J(P) is an ideal of A such that J⊆J(P) ⊆P also if P be a prime ideal containing an ideal J of distributive q-lattice A and Q be a prime ideal such that J ⊆ Q ⊆ P then J(P) ⊆ Q. Also shown (a] = { a ∧ x / a ∈ {a} , x ∈ A } is an ideal of A. In this paper, We define concepts of prime semi ideal, prime semi filter and semi a-ideal, semi a-filter in distributive q-lattice A. we prove A non empty subset F of A (F≠ A) is prime semi filter if and only if (A-F) is prime semi ideal. We prove if F be semi a-filter in A then F is maximal semi a-filter if for x ∉ F there exists y ∈ F such that a = y ∧ x , and some equivalent conditions. We define a semi-ideal $P^{\perp}$ = { x ∈ A / x ∧ a = s for all a ∈ P } and some properties.

2. PRELIMINARIES

Some of the following definitions and results are taken from [1] and [4]

Definition 2.1:[1]. An algebra (A, ∨, ∧) whose binary operations ∨, ∧ satisfy the following is called a q-lattice.

(i) a ∨ b = b ∨ a ; a ∧ b = b ∧ a (commutativity)
(ii) a ∨ (b ∨ c) = (a ∨ b) ∨ c ; a ∧ (b ∧ c) = (a ∧ b) ∧ c (associatativity)
(iii) a ∨ (a ∧ b) = a ∨ a ; a ∧ (a ∨ b) = a ∧ a (weak- absorption)
(iv) a ∨ b = a ∨ (b ∨ b) ; a ∧ b = a ∧ (b ∧ b) (weak- idempotence)
(v) a ∨ a = a ∧ a (equalization)

Definition 2.2:[1]. A q-lattice (A, ∨, ∧) is distributive if it satisfies the identity

x ∨ (y ∧ z) = (x ∨ y) ∧ (x ∨ z) for all x ,y, z ∈ A

Lemma 2.1 :[1]. Let A be a distributive q-lattice then the following identity hold

a ∧ (b ∨ c) = (a ∧ b) ∨ (a ∧ c) for all a, b, c ∈ A.

Definition 2.3:[1]. Ideal of a distributive q-lattice:

A nonempty subset I of a distributive q-lattice A is called an ideal of A if

i) $x, y \in I \Rightarrow x \vee y \in I$

ii) $x \in I$ and $a \in A \Rightarrow a \wedge x \in I$

Definition 2.4:[4]. Filter of a distributive q- lattice :

A nonempty subset F of a distributive q-lattice A is called a filter of A, if.

i) $x, y \in F \Rightarrow x \wedge y \in F$

ii) $x \in F$ and $a \in A \Rightarrow a \vee x \in F$

3. Semi- ideal and semi-filter

Definition: 3.1.

Semi ideal of a distributive q-lattice:

A nonempty subset I of a distributive q-lattice A is called semi ideal of A if

$x, y \in I \Rightarrow x \vee y \in I$

Definition: 3.2.

Semi-filter of a distributive q- lattice :

A nonempty subset F of a distributive q-lattice A is called semi filter of A, if.

$x, y \in F \Rightarrow x \wedge y \in F$

Definition: 3.3.

Prime semi ideal of a distributive q-lattice:

A proper semi ideal I of a distributive q-lattice A is called prime semi ideal of A if for all x and y in A,

$x \wedge y \in I \Rightarrow x \in I$ or $y \in I$

Definition: 3.4.

Prime semi-filter of a distributive q- lattice , Maximal semi filter of a distributive q-lattice:

A proper semi filter F of a distributive q-lattice A is called prime semi filter of A, if for all x and y in A, $x \vee y \in F \Rightarrow x \in F$ or $y \in F$.

A proper semi filter F of A is said to be maximal if it is not properly contained in any proper semi filter of A

Theorem: 3.1.

A non empty subset F of A ($F \neq A$) is prime semi filter if and only if (A-F) is prime semi ideal

Proof: only if part:

Let F be prime semi filter. As $F \neq A$ implies A-F is nonempty

Let $x , y \in A-F$

Implies $x \notin F$, $y \notin F$ and since F is prime semi filter

Implies $x \vee y \notin F$

Implies $x \vee y \in (A-F)$

Therefore (A-F) is semi ideal.

Now for $x , y \in (A-F)$

If $x \wedge y \in (A-F)$

Implies $x \wedge y \notin F$ and since F is semi filter

Implies either $x \notin F$ or $y \notin F$

Implies $x \in (A-F)$ or $y \in (A-F)$

This shows that (A-F) is prime semi ideal.

If part:

Let (A-F) be prime semi ideal in A

Already we have taken F is non empty

Let $x , y \in F$

Implies $x \notin (A-F)$, $y \notin (A-F)$ and since (A-F) is prime semi ideal

Implies $x \wedge y \notin (A-F)$

Implies $x \wedge y \in F$, Hence F is semi filter.

To prove that F is prime:
Let x , y ∈ A if x ∨ y ∈ F
Implies x ∨ y ∉ (A-F) and since (A-F) is semi ideal in A
Implies x ∉ (A-F) or y ∉ (A-F)
Implies x ∈ F or y ∈ F
Therefore F is prime semi filter.
Similarly we can prove,
A non empty subset P (P≠ A) of A is prime semi ideal if and only if (A-P) is prime semi filter in A

Definition :3.5. Semi a-filter in distributive q-lattice:
Let a be any fixed element in A then we define semi a-filter is a semi filter in A not containing a

Theorem:3.2.
Let F be semi a-filter in A then F is maximal semi a-filter if for x ∉ F there exists y ∈ F such that a = y ∧ x
Proof: Let F be semi a-filter in A satisfying the given condition
Now we prove F is maximal semi a-filter in A :
Let if possible there exists semi a-filter J in A such that F ⊂ J ⊂ A
As F ⊂ J there exists x ∈ J such that x ∉ F
Hence by assumption there exists y ∈ F such that a = y ∧ x
Now F ⊂ J and y ∈ F implies y ∈ J
As x ∈ J , y ∈ J and J is an filter so y ∧ x ∈ J
That is y ∧ x = a ∈ J which is a contradiction
Hence F is maximal semi a-filter in A.

Theorem: 3.3.
The following statements are equivalent in A.
1) Every maximal semi a-filter is prime.
2) Every semi a-filter which is contained in some maximal semi a-filter is disjoint from prime semi a-ideal.
3) Every semi a-filter which is contained in some maximal semi a-filter , is contained in prime semi a-filter.
Proof: (1) ⇒ (2)
Let F be any semi a-filter in A
Suppose F be contained in some maximal semi a-filter say M
By assumption (1) , M be prime
Hence (A-M) is prime semi a-ideal , further as F ∩ (A-M) = ϕ
So every semi a-filter is disjoint from prime semi a-ideal
(2) ⇒ (3)
Let F be semi a-filter in A which is contained in some maximal semi a-filter
By assumption (2) There exists prime semi a-ideal P such that P ∩ F = ϕ
This implies F ⊆ (A-P) and by theorem 3.1, (A-P) is prime semi a-filter
So the implication follows.
(3) ⇒ (1)
Let M be any maximal semi a-filter , clearly M is semi a-filter and as M ⊆ M
By assumption (3) M is contained in prime semi a-filter say Q in A
But then ,maximality of M will imply that M = Q
Hence M is prime.
Thus (1) ⇒ (2) ⇒ (3) ⇒ (1) so all the statements are equivalent.

Definition: 3.6.
For any non empty set P and special element s ∈ A satisfying properties s ∨ s = s. we define , $P^{\perp}$ = { x ∈ A / x ∧ a = s for all a ∈ P }

Theorem: 3.4.
For distributive q-lattice A with element s ∈ A satisfying property s ∨ s = s and for any non empty set I and J of A we have the following
(1) $I^{\perp}$ is a semi ideal.

(2) If $I \subseteq J$ then (a) $J^{\perp} \subseteq I^{\perp}$ and (b) $(I^{\perp})^{\perp} = I^{\perp\perp} \subseteq (J^{\perp})^{\perp} = J^{\perp\perp}$
(3) $(I \vee J)^{\perp} = I^{\perp} \cap J^{\perp}$
(4) $(I \cap J)^{\perp} = I^{\perp} \cap J^{\perp}$
(5)) If $J \subseteq I$ then prove that $(I^{\perp} \vee J^{\perp})^{\perp} \subseteq I^{\perp\perp}$
(6) $I^{\perp\perp} \cap J^{\perp\perp} \subseteq (I^{\perp} \vee J^{\perp})^{\perp}$
(7) $I \subseteq I^{\perp\perp}$

Proof:

(1) To prove $I^{\perp}$ is a semi ideal

Let $x, y \in I$

Implies $x \wedge a = s = y \wedge a$, for all $a \in I$

Now $(x \vee y) \wedge a = a \wedge (x \vee y)$
$= (a \wedge x) \vee (a \wedge y)$
$= (x \wedge a) \vee (y \wedge a)$
$= s \vee s$
$= s$ for all $a \in I$

Hence $x \vee y \in I^{\perp}$

Therefore $I^{\perp}$ is a semi ideal.

2) If $I \subseteq J$ then to prove that (a) $J^{\perp} \subseteq I^{\perp}$ and (b) $(I^{\perp})^{\perp} = I^{\perp\perp} \subseteq (J^{\perp})^{\perp} = J^{\perp\perp}$

(a) Let $x \in J^{\perp}$

Implies $x \wedge a = s$ for all $a \in J$

Implies $x \wedge a = s$, for all $a \in I$ since $I \subseteq J$

Implies $x \in I^{\perp}$

So $J^{\perp} \subseteq I^{\perp}$

(b) Let $x \in I^{\perp\perp} = (I^{\perp})^{\perp}$

Implies $x \wedge a = s$ for all $a \in I^{\perp}$

Implies $x \wedge a = s$ for all $a \in J^{\perp}$ since $J^{\perp} \subseteq I^{\perp}$

Implies $x \in (J^{\perp})^{\perp} = J^{\perp\perp}$

Therefore $(I^{\perp})^{\perp} = I^{\perp\perp} \subseteq (J^{\perp})^{\perp} = J^{\perp\perp}$

3) To prove

$(I \vee J)^{\perp} = I^{\perp} \cap J^{\perp}$

Let $x \in I^{\perp} \cap J^{\perp}$

Implies $x \in I^{\perp}$ and $x \in J^{\perp}$

Implies $x \wedge a = s$ for all $a \in I$ and $x \wedge b = s$ for all $b \in J$

Let $t = a \vee b \in I \vee J$ where $a \in I$ and $b \in J$

Now $x \wedge t = x \wedge (a \vee b)$
$= (x \wedge a) \vee (x \wedge b)$
$= s \vee s = s$ for all $t \in I \vee J$

Implies $x \in (I \vee J)^{\perp}$ implies $I^{\perp} \cap J^{\perp} \subseteq (I \vee J)^{\perp}$ ------(1)

Now let $x \in (I \vee J)^{\perp}$

Implies $x \wedge t = s$ for all $t \in I \vee J$ Let $t = (a \vee b)$

Implies $x \wedge t = x \wedge (a \vee b) = s$

Implies $(x \wedge a) \vee (x \wedge b) = s$ but s is an element satisfying property $s \vee s = s$

Implies $(x \wedge a) = s$ and $(x \wedge b) = s$ for all $a \in I$ and for all $b \in J$

Implies $x \in I^{\perp}$ and $x \in J^{\perp}$

Implies $x \in I^{\perp} \cap J^{\perp}$

Therefore $(I \vee J)^{\perp} \subseteq I^{\perp} \cap J^{\perp}$ ---------(2)

From (1) and (2) we get

$(I \vee J)^{\perp} = I^{\perp} \cap J^{\perp}$

(4) To prove $(I \cap J)^{\perp} = I^{\perp} \cap J^{\perp}$
Let $x \in (I \cap J)^{\perp}$
Implies $x \wedge a = s$ for all $a \in I \cap J$
Implies $x \wedge a = s$ for all $a \in I$ and $a \in J$
Implies $x \wedge a = s$ for all $a \in I$ and $x \wedge a = s$ for all $a \in J$
Implies $x \in I^{\perp}$ and $x \in J^{\perp}$
Implies $x \in I^{\perp} \cap J^{\perp}$
Therefore $(I \cap J)^{\perp} \subseteq I^{\perp} \cap J^{\perp}$ ------- (1)
Now let $x \in I^{\perp} \cap J^{\perp}$
Implies $x \in I^{\perp}$ and $x \in J^{\perp}$
Implies $x \wedge a = s$ for all $a \in I$ and $x \wedge b = s$ for all $b \in J$
Implies $x \wedge a = s$ for all $a \in I$ and $a \in J$
Implies $x \wedge a = s$ for all $a \in I \cap J$
Implies $x \in (I \cap J)^{\perp}$
Therefore $I^{\perp} \cap J^{\perp} \subseteq (I \cap J)^{\perp}$ ------- (2)
From (1) and (2) we get
$(I \cap J)^{\perp} = I^{\perp} \cap J^{\perp}$
(5) If $J \subseteq I$ then to prove that $(I^{\perp} \vee J^{\perp})^{\perp} \subseteq I^{\perp\perp}$
Let $x \in I^{\perp}$
Then $x \wedge a = s$ for all $a \in I$
Implies $x \wedge a = s$ for all $a \in J$ since $J \subseteq I$
Let $t = a \vee b \in I \vee J$ where $a \in I$ and $b \in J$
Now $x \wedge t = x \wedge (a \vee b) = (x \wedge a) \vee (x \wedge b)$
$= s \vee s = s$
Implies $x \in I^{\perp} \vee J^{\perp}$
Therefore $I^{\perp} \subseteq I^{\perp} \vee J^{\perp}$ and using property (2)
$(I^{\perp} \vee J^{\perp})^{\perp} \subseteq I^{\perp\perp}$
(6) To prove that $I^{\perp\perp} \cap J^{\perp\perp} \subseteq (I^{\perp} \vee J^{\perp})^{\perp}$
Let $x \in I^{\perp\perp} \cap J^{\perp\perp}$
Implies $x \in I^{\perp\perp}$ and $x \in J^{\perp\perp}$ and let $y \in I^{\perp} \vee J^{\perp}$
Then $y = i \vee j$ for some $i \in I^{\perp}$ and $j \in J^{\perp}$
Now $x \wedge y = x \wedge (i \vee j) = (x \wedge i) \vee (x \wedge j)$
$= s \vee s = s$
Implies $x \in (I^{\perp} \vee J^{\perp})^{\perp}$
Therefore $I^{\perp\perp} \cap J^{\perp\perp} \subseteq (I^{\perp} \vee J^{\perp})^{\perp}$
(7) prove that $I \subseteq I^{\perp\perp}$
Proof: Let $p \in I$, $y \in I^{\perp}$, $x \in I^{\perp\perp}$
As $y \in I^{\perp}$ implies $y \wedge p = s$ for all $p \in I$
Also as $x \in I^{\perp\perp}$ implies $x \wedge y = s$ for all $y \in I^{\perp}$
Implies $p \wedge y = s$ for all $y \in I^{\perp}$
Implies $p \in I^{\perp\perp}$
Therefore $I \subseteq I^{\perp\perp}$

Definition: 3.7.
Let A and A′ be two distributive q-lattices with special elements s and s′ respectively. Then a mapping $f: A \to A'$ is called a homomorphism if it satisfies $f(a \vee b) = f(a) \vee f(b)$ and $f(a \wedge b) = f(a) \wedge f(b)$

Theorem: 3.5.
Let $f: A \to A'$ be an onto homomorphism then for any semi ideal $I^{\perp}$ of A then
$f(I^{\perp})$ is semi ideal of A′

Proof: Let f: A → A′ be an onto homomorphism. Let $I^{\perp}$ be semi ideal of A we have to show that f($I^{\perp}$) is semi ideal of A′

Let f(x), f(y) ∈ f($I^{\perp}$)

Now f(x) ∨ f(y) = f(x ∨ y)

As x, y ∈ $I^{\perp}$ and $I^{\perp}$ is semi ideal hence x ∨ y =x′ (say) ∈ $I^{\perp}$

Implies f(x) ∨ f(y) = f(x ∨ y)

= f(x′) ∈ f($I^{\perp}$)

Therefore f($I^{\perp}$) is semi ideal of A′

Theorem: 3.6.

Let f: A → A′ be a homomorphism with property f(s) = s′ where s′ is special element of A′ satisfying s′ ∨ s′ = s′ then for any non empty subset I of A we have f($I^{\perp}$) ⊆ $(f(I))^{\perp}$

Proof: Let a ∈ f($I^{\perp}$) and y ∈ f(I)

Then there exists b ∈ $I^{\perp}$ and x ∈ I such that a = f(b) and y = f(x)

Now a ∧ y = f(b) ∧ f(x) = f(b ∧x) = f(s) for all y ∈ f(I)

Hence a ∈ $(f(I))^{\perp}$

Therefore f($I^{\perp}$) ⊆ $(f(I))^{\perp}$.

Theorem:3.7.

Let A and A′ be two distributive q-lattice with special element s and s′ respectively and f: A → A′ be one one, onto homomorphism then prove that $\{f^{-1}(B)\}^{\perp} \subseteq \{ f^{-1}(B^{\perp})\}$

Proof: Let x ∈ $\{f^{-1}(B)\}^{\perp}$

Implies x ∧ b = s for all b ∈ f^{-1} (B)

Implies x ∧ b = s for all f(b) ∈ B

Implies f(x ∧ b) = f(s) for all f(b) ∈ B

Implies f(x) ∧ f(b) = f(s) = s′ for all f(b) ∈ B

Implies f(x) ∈ $B^{\perp}$

Implies x ∈ $f^{-1}(B^{\perp})$

Hence $\{f^{-1}(B)\}^{\perp} \subseteq \{ f^{-1}(B^{\perp})\}$

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