Lattice Gauge Theory as a Tool to Probe Quark Confinement

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Abstract: Lattice gauge theoryprovides a non-perturbative quantization of gauge fields by a lattice.The results of lattice gauge theory approach to those of continuum gauge theory as the separation of lattice points approaches to zero. Continuum gauge theory provides the linear relationship between quarks potential and the distance between them indicating Quark confinement.

Keywords*: Lattice Gauge Theory, Wilson Loop, Quark Confinement, Asymptotic Scaling*

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I. INTRODUCTION

Wilson loops[1] play a central role in the lattice formulation of gauge theories. These arethe phase factor in Abelian and Non Abelian gauge theories. An analogy of phase factor was first introduced by H.Weyl in 1919[2] to describe gravitational and electromagnetic interactions of an electron. Until 1970s all the predictions of Quantum Chromodynamics (QCD) [3] were restricted to the perturbative regime. In 1974 K.G. Wilson used lattice regularization and introduced a phase factor for the simplest closed contour on the lattice called Wilson loop. Wilson loop contains theholonomy of the gauge connection around the given loop and is used for the study of non perturbative phenomenon of confinement of quark in QCD via the static quark antiquark potential. Lattice gauge theory therefore, is a nonperturbative regularization of gauge theory. It uses an analogy between quantum field theory and statistical mechanics and offers a possibility of applying non-perturbative methods such as the strong-coupling expansion or the numerical Monte Carlo method [4] to QCD. It provides a non-perturbative quantization of gauge fields by a lattice.

II. LINK AND PLAQUETTE OF A LATTICE

A lattice approximates continuous space by a discrete set of points. The lattice is along all four coordinates in

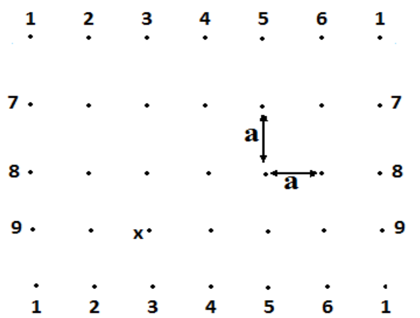

Figure1. Two dimensional lattice with periodic boundary conditions

Euclidian formulation. However, the Hamiltonian approach considers the time coordinate as non-discrete or continuous.The lattice is defined as a set of points, called the lattice sites,in d-dimensional Euclidean space with coordinates

$$x_\mu = n_\mu a \qquad (1)$$

Here ‘a’ is a dimensional constant, called the lattice spacing and is defined as the distance between the neighbouring sites. The spacings are measured in units of a, thereby setting a = 1.Here $n_\mu = (n_1, n_2, \ldots, n_d)$ is a vector whose componentsn_i; i=1,2,…..d are integer numbers, d is the dimension of the space.

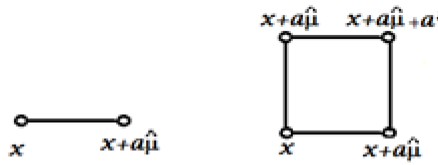

Figure 2. Link (left) and Plaquette (right)

A 2d lattice with periodic boundary conditions is depicted in Figure1. The spatial size of the depicted lattice is $L_1 = 6$ and $L_2 = 4$. An analogous 4d lattice is called hyper cubic.

The link and plaquette of a lattice, which are shown in Figure 2. A link $l = \{x; \mu\}$ connects two neighboring sites xand $x + a\hat{\mu}$, where $\hat{\mu}$ is a unit vector along the μ-direction (μ = 1, . . . , d). A plaquettep $= \{x; \mu, \nu\}$ is an elementary square enclosed byfour links in the directions μ and ν. The set of four linkswhich bound the plaquettepis denoted as ∂p. The number of degrees of freedom for an infinite lattice is infinity (=∞)but enumerable. The lattice is considered to have a finite size $L_1 \times L_2 \times \ldots \ldots \times L_d$ in all directions to limit this number. The periodic boundary conditions are imposed to reduce the finite-size effects.

III. LINK VARIABLES AS GAUGE FIELDS

Matter field, say a quark field, is attributed to the lattice sites; therefore a continuous field $\phi(x)$is approximated by its values at the lattice sites

$$\phi(x) \Rightarrow \phi_x \qquad (2)$$

The lattice field ϕ_x is a good approximation of a continuous field $\phi(x)$when a is much smaller (≪) than the characteristic size of a given configuration. Fine lattice sites in Figure 3(b) with a smaller lattice spacing are closer to continuum field configuration as compared to the "coarse" lattice in Figure 3(a).

The link {x; μ}as we know, connects two neighbouring sites xand $x + a\hat{\mu}$, where $\hat{\mu}$ is a unit vector along the μ-direction (μ = 1, . . . , d).

Thelinkvariable$U_\mu(x)$like $A_\mu(x)$ is characterized by a coordinate and a direction.The gauge field $A_\mu(x)$is therefore, related to the links $U_\mu(x)$

or $A_\mu(x) \Rightarrow U_\mu(x)$ (3)

The link variable$U_\mu(x)$, as we know, is defined as

$$U_\mu(x) = P\, e^{i\int_x^{x+a\hat{\mu}} dz^\mu A_\mu(z)} \quad (4)$$

where the integral is along the link{x; μ}.

Therefore, $U_\mu(x) = e^{iaA_\mu(x)}$as $a \to 0$(5)

This means $U_\mu(x)$ varies as the exponential of the μthcomponent of the vector potential $A_\mu(x)$at the center of the link. Since the path-ordered integral in Eq. (4) depends on the orientation of the link. The same link connecting the points xand $x + a\hat{\mu}$, can be written either as{x; μ} or as { $x + a\hat{\mu}$;−μ}. The orientation of the link {x; μ} is positive that is along the positive of the co-ordinate axis. The orientation of the link{ $x + a\hat{\mu}$;−μ}on the other hand, is negative.The link variable $U_\mu(x)$usually represents the links with positive orientations. The negative orientation link variable is given

$$U_{-\mu}(x + a\hat{\mu}) = U_\mu^\dagger(x) \quad (6)$$

Equation(5) shows the way toevaluate latticephase factors (Wilson loops) by constructing contours from the small links of the lattice.

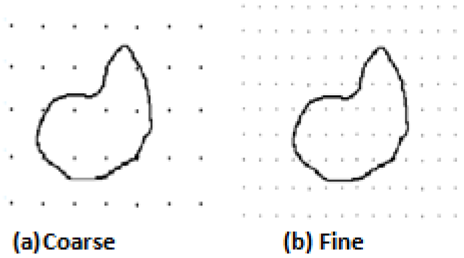

Figure 3.Description of Continuum Configuration by lattices

IV. DERIVATION OF THE WILSON LOOP

Let us choose a pathΓfrom $x \to y$ and divide into infinitesimal small segments$(z_1, z_2 \ldots\ldots z_n)$ as shown in Fig.6.The explicit form of the finite parallel transport or the phase factor as we discussed above is given by

$$U_\Gamma(y,x) = U(y, z_n)U(z_n, z_{n-1}) \ldots\ldots U(z_1, z) \quad (7)$$

Foran infinitesimal segment $z \to z_1$, we can write

$$U(z_1, z) \approx e^{[ieA_\mu(z_1 - z)^\mu]}$$

andtherefore as discussed above

$$U(y,x) = e^{ie\int_\gamma A_\mu(z)dz^\mu} \quad (8)$$

The parallel transport $U_\Gamma(y,x)$is not necessarily path independent$U_{\Gamma 1}(y,x) \neq U_{\Gamma 2}(y,x)$

$$U_\Gamma(y,x) = \lim_{n\to\infty} U(y, z_n)U(z_n, z_{n-1}) \ldots\ldots U(z_1, z)$$

$$= \lim_{\Delta x_j \to 0} \prod_{j=0}^{n}(1 + igA_\mu(z_j)\Delta z_j^\mu)$$

with$\Delta z_j^\mu = z_{j+1}^\mu - z_j^\mu$, $z_0 = x, z_{n+1} = y$.

As $A_\mu(z)$is a matrix and so is non-abelian i.e. $[A_\mu(z_j), A_\upsilon(z_k)] \neq 0$.

$$U_\Gamma(y,x) = 1 + ig\sum_{k=0}^{n} A_\mu(z_k)\Delta z_k^\mu +$$

$$+ (ig)^2 \sum_{k=0}^{n}\sum_{k=0}^{j-1} A_\mu(z_j)\Delta z_k^\mu A_\upsilon(z_l)\Delta z_l^\upsilon \ldots +$$

Now we introduce $z^\mu(r)$ to parameteriseΓ with different segments$r_1, r_2 \ldots\ldots\ldots r_n$.

$$z^\mu(0) = 0, z^\mu(1) = y^\mu, r \in [0,1]$$

$$U_\Gamma(y,x) = 1 + ig\int_0^1 dr_1\, A_\mu(z(r_1))\frac{dx^\mu}{dr_1}$$

$$+(ig)^2\int_0^1 dr_1 \int_0^{r_1} dr_2\, A_\mu(x(r_1))\frac{dx^\mu}{dr_1} A_\upsilon(x(r_2))\frac{dx^\upsilon}{dr_2} + \ldots =$$

$$\sum_{n=0}^{\infty}(ig)^n \int_0^1 dr_1 \int_0^{r_1} dr_2 \ldots \int_0^{r_{n-1}} dr_n\, A_{\mu 1}(x(r_1))\frac{dx^{\mu 1}}{dr_1}$$

$$\ldots\ldots A_{\mu n}(x(r_n))\frac{dx^{\mu n}}{dr_n}$$

$$= \sum_{n=0}^{\infty}(ig)^n \int_{r \geq r_1 \geq r_2 \ldots \geq r_n \geq 0} dr_1 \ldots\ldots dr_n\, A_{\mu 1}(x(r_1))\frac{dx^{\mu 1}}{dr_1}$$

$$\ldots A_{\mu n}(x(r_n))\frac{dx^{\mu n}}{dr_n}$$

$$= Pe^{ig\int_0^1 dr \frac{dx^\mu}{dr} A_\mu(x(r))}$$

$$= Pe^{ig\int_\Gamma A_\mu(x)dx^\mu} \quad (9)$$

For an infinitesimal closed loop $x \to y$,

$$U_\Gamma(x,x) = Pe^{ig\int_\Gamma A_\mu(x)dx^\mu}$$

where the loop Γ approaches a point 'x'

Using Strokes theorem $U_\Gamma(x,x) = Pe^{ig\int_\Gamma F_{\mu\upsilon}\, d\sigma^{\mu\upsilon}}$

where$\sigma^{\mu\upsilon}$ is the area element encircled by the infinitesimal loop and $F_{\mu\upsilon} = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$

Therefore $U_\Gamma(x,x) = e^{igF_{\mu\upsilon}\sigma^{\mu\upsilon}}$

By construction, under a gauge transformation,

$$U_\Gamma(y,x) \to \Omega(y)U_\Gamma(y,x)\Omega^\dagger(x) \quad (10)$$

Therefore for an infinitesimal loop $x \to y$

$$U_\Gamma(x,x) \to \Omega(x)U_\Gamma(x,x)\Omega^\dagger(x)$$

and hence

$$W_\Gamma(x) = \mathrm{Tr}\,(U_\Gamma(x,x) = \mathrm{Tr}(e^{igF_{\mu\upsilon}\sigma^{\mu\upsilon}}) \quad (11)$$

is gauge invariant. This is the non-abelian Wilson loop.

V. PROPERTIES OF WILSON LOOP

Hermiticity –It implies that theHermitian conjugate of a Wilson line gives the same line in opposite direction. Let γ is a Wilson line from a to b along the direction y then-

$$\gamma_x^\dagger[a,b] = \gamma_{-x}[b,a] \quad (12)$$

Causality- If we first have a Wilson line from a to b then a line along the same direction y from b to c, we can glue them together into the Wilson line from a to c-

$$\gamma_y[b,c] = \gamma_y[a,b] = \gamma_y[a,c] \quad (13)$$

Unitarity- If we have a Wilson line from a to b and then a line back from b to a in the opposite direction, they will give 1. $\gamma_y[a,b]\ \gamma_y[a,b] = 1$ (14)

VI. THE WILSON ACTION

The phase factor U(x,y) plays an important role in the lattice formulation of closed contours. Let us consider the simplest closed contour which is the boundary of a plaquette oriented properly (i.e. clockwise or anti-clockwise). Since a plaquette consists of four sides or links forming its sides. The plaguette phase factor is the product of the phase factors of these four links, that is

$$U(\partial p) = U_\nu^\dagger(x)U_\mu^\dagger(x+a\hat{\nu})U_\nu(x+a\hat{\mu})U_\mu(x) \quad (15)$$

The gauge transformation equation for the link variable is given by

$$U_\Gamma(x) \to \Omega(x+a\hat{\mu})U_\mu(x)\Omega^\dagger(x) \quad (16)$$

Lattice gauge transformation is defined as the gauge transformation in which the matrix $\Omega(x)$in above equation is a function of the lattice sites distribution. The plaquette phase factor transforms under the lattice gauge transformation as

$$U(\partial p) \to \Omega(x)U(\partial p)\Omega^\dagger(x) \quad (17)$$

The gauge invariance of the trace of $U(\partial p)$or $trU(\partial p) \to trU(\partial p)$ (18)
is used in constructing the action of the lattice gauge theory called the Wilson action $S_{lat}[U]$which is given as

$$S_{lat}[U] = \sum_p 1 - \left[\frac{1}{N}RetrU(\partial p)\right] \quad (19)$$

The summation in $S_{lat}[U]$is over all the elementary plaquettes p of the lattice (i.e. over all x, μand ν), regardless of their orientations.

The reversal of the orientation means the complex conjugate of the trace, as from Eq.(54), we get

$$trU(\partial p) \xrightarrow{\text{reorientation}} trU^\dagger(\partial p) = [trU(\partial p)]^*(20)$$

The lattice action, in terms of similarly oriented plaquette link variable, is given as

$$S_{lat}[U] = \frac{1}{2}\sum_p\left[1 - \frac{1}{N}trU(\partial p)\right] \quad (21)$$

This lattice action becomes the continuum gauge theory action as $a \to 0$. It can be shown using Strokes theorem that, for abelian fields, link variable at a point x has the form

$$U_\Gamma(x) = e^{igF_{\mu\upsilon}\sigma^{\mu\upsilon}} \quad (22)$$

However, for non-Abelian field strength $F_{\mu\upsilon}(x)$ and for a square lattice, that is $\sigma^{\mu\upsilon} = a^2$, we get

$$U(\partial p) \to e^{[ia^2\mathcal{F}_{\mu\upsilon}(x)+O(a^3)]} \quad (23)$$

The third order term $O(a^3)$ arises because of the non-zero commutator term of $A_\mu(x)$ and $A_\nu(x)$in non-Abelian field strength$F_{\mu\upsilon}(x)$.

Then, as we know $a^4\sum_p \ldots\ldots \xrightarrow{a\to 0} \frac{1}{2}\int d^4x\sum_{\mu,\nu}\ldots\ldots$ (24)

Using expansion of Eq.(23) in Eq. (21), and using result of Eq.(24), we get

$$S_{lat} \xrightarrow{a\to 0} \frac{1}{4N}\int d^4x\sum_{\mu,\nu} tr\mathcal{F}^2{}_{\mu\upsilon}(x) \quad (25)$$

Also the QCD action is written in matrix notation as $S[A,\psi,\bar{\psi}]$ is given by

$$\int d^4x\left[\bar{\psi}\gamma_\mu(\partial_\mu - iA_\mu)\psi + m\bar{\psi}\psi + \frac{1}{4g^2}tr\mathcal{F}_{\mu\nu}^2\right]$$

Where $\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$(26)

VII. WILSON LOOPS ON A LATTICE

Let us represent the lattice contour C by its initial point x and by the directions of the links forming the contour, some of these directions may be negativeas well

$$C = \{x, \mu_1, \ldots\ldots\ldots\ldots, \mu_n\} \quad (27)$$

The lattice phase factor U(C) is given by

$$U(C) = U_{\mu_n}(x + a\hat{\mu}_1 + \cdots + a\hat{\mu}_{n-1})\ldots\ldots\ldots \times U_{\mu_2}(x +$$
aμ1Uμ1x (28)

As discussed earlier the following equation can be used for dealing with the links with negative directions.

$$U_{-\mu}(x + a\hat{\mu}) = U_\mu^\dagger(x) \quad (29)$$

For a closed contour$\mu_1 + \cdots\cdots\cdots + \mu_n = 0$.(30)

The gauge invariant trace of the phase factor for a closed contour is called the Wilson loop.

The Wilson loop average over Contour C is determined as

$$W(C) \equiv \langle\frac{1}{N}trU(C)\rangle$$
$$= Z^{-1}(\beta)\int\prod_{x,\mu} dU_\mu(x)e^{-\beta S(U)}\frac{1}{N}trU(C)$$

where

$$Z(\beta) = \int\prod_{x,\mu} dU_\mu(x)e^{-\beta S(U)} \quad (31)$$

is the partition function for a pure lattice gauge theory at an inverse temperature β which is given by comparing gauge field part $tr\mathcal{F}^2{}_{\mu\upsilon}(x)$results by substituting using QCD action in Eqs.(25) and Eq.(26)

$$\beta = \frac{N}{g^2} \quad (32)$$

This concept of partition function is similar to that in the statistical mechanics. The average of the physical quantities in terms of partition function $Z(\beta)$is given by

$$\langle F[U]\rangle = Z^{-1}(\beta)\int\prod_{x,\mu} dU_\mu(x)e^{-\beta S(U)}F[U](33)$$

$F[U]$is a gauge-invariant function of $U_\mu(x)$ and becomes the expectation value in the continuum theory as $a \to 0$with$\beta = \frac{N}{g^2}$.

It can be shown that the Wilson loop average$W(C)$over a contour C is related to the minimum surface area $A_{min}(c)$(measured in units of a^2) enclosed by the contour $W(C) = [W(\partial p)]^{A_{min}(c)}$ (34)

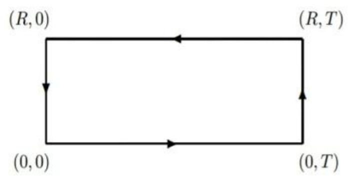

Figure 4. Rectangular Wilson Loop

Where$W(\partial p)$ is the plaquette average and is given by

$$W(\partial p) \begin{cases} = \frac{\beta}{2N^2} \text{ for } SU(N) \text{ with } N \geq 3 \\ = \frac{\beta}{4} \text{ for } SU(2) \end{cases} (35)$$

Wilson loop average over a rectangular contour of dimensions $R \times T$ as shown in Figure 4 is related to the interaction energy of the static quark, separated by the distance R by the formula (24)

$$W(R\times T) \propto e^{-E_0(R)T} \quad (36)$$

Considering the case of axial gauge $A_4 = 0$ and so $U_4(x) = 1$.Only vertical segments contribute to $U(R\times T)$.

Using $\psi_{ij}(t) = \left[Pe^{i\int_0^R dz_1 A_1(z_1,\ldots\ldots,t)}\right]_{ij}$

As we have

$$W(R\times T) = \langle\frac{1}{N}tr\,\psi(0)\psi^\dagger(T)\rangle$$

The sum over a complete set of intermediate states is given by

$$\sum_n|n\rangle\langle n| = 1 \quad (37)$$

Inserting this sum, Wilson loop average $W(R\times T)$is given by

$$W(R\times T) = \sum_n\frac{1}{N}\langle\psi_{ij}(0)|n\rangle\langle n|\psi_{ji}^\dagger(T)\rangle$$

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$$= \sum_n \frac{1}{N} \left|\langle \psi_{ij}(0) | n \rangle\right|^2 e^{-E_n T} \quad (38)$$

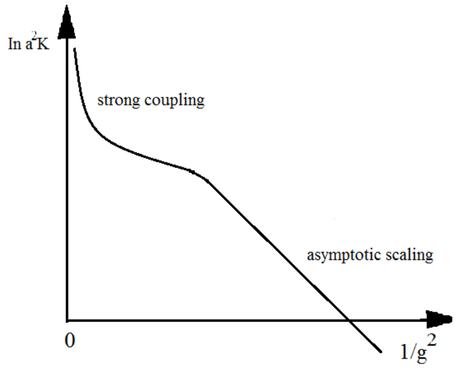

Figure 5. String tension versus ${}^{1}/_{g^2}$.

where E_n is the energy of the state $|n\rangle$. The ground state with the lowest energy is the only state which survives in the sum over the states as $T \to \infty$. Therefore

$$W(R \times T) \xrightarrow{Large\ T} e^{-E_0(R)T} \quad (39)$$

This derivation is independent of the lattice parameters and therefore, the result is equally true for a rectangular loop in the continuum.

VIII. AREA LAW AND CONFINEMENT

The Wilson loop average $W(C)$ over a contour C is related to the minimum surface area $A_{min}(C)$ (measured in units of a^2) enclosed by the contour

$$W(C) = e^{-KA_{min}(C)} \quad (40)$$

The exponential dependence of the Wilson loop average on the area of the minimal surface as in above equation is called the area law. If the area law holds for loops of large area in the pure SU(3) gauge theory then quarks are confined. This means there are no physical $|in\rangle$ or $\langle out|$ quark states and this is the essence of Wilson's confinement criterion. This is because the physical amplitudes such as the polarisation operator do not have quark singularities when the Wilson criterion is satisfied.

We have from Eq.(39) for large T values

$$W(R \times T) \xrightarrow{Large\ T} e^{-E_0(R)T}$$

Comparing with Eq.(40) for contour $C \sim R \times T$, for $A_{min} = R \times T$,

$$E(R) = KR \quad (113)$$

This shows that the potential energy is a linear function of the distance between the quarks.The coefficient K is the string tension given by the energy of the string per unit length. The gluon field between the quarks contracts to a tube or string, with its energy proportional to its length. The string stretches as the distance between quarks increases. It therefore, prevents them from moving apart to macroscopic distances.Equation (34) then gives

$$K = \frac{1}{a^2} \ln \frac{2N^2}{\beta} = \frac{1}{a^2} \ln(2Ng^2) \quad (41)$$

to the leading order of the strong coupling expansion. The next orders of the strong coupling expansion result in corrections in β to this formula. Confinement holds in the lattice gauge theory to any order of the strong coupling expansion.

IX. ASYMPTOTIC SCALING

Eq.(41) establishes the relationship between lattice spacing aand the coupling g^2. Let K equals its experimental value $K = (400\text{MeV})^2 \approx \text{GeV/fm}$

This gives $a \to \infty$ as $g^2 \to \infty$.i.e. the lattice spacing is large in the strong coupling limit, compared to 1fm. This coarse lattice cannot however, describe the continuum,which requires smaller lattice spacings. Also from this equation,a decreases with decreasing g^2.However, this formula ceases to be applicable in the intermediate region of $g^2 \sim 1$or $a \sim 1$fm.For pure SU(3) Yang Mills, Eq.(41) is replaced at small$g^2$by-

$$K = \text{const.} \frac{1}{a^2} \left(\frac{8\pi^2}{11\,g^2}\right)^{\frac{102}{121}} e^{-\frac{8\pi^2}{11 g^2}} \quad (42)$$

where the two loop Gell-Mann Low function is used. Thisexponential dependence of K on $1/g^2$ is called asymptotic scaling and sets in forhigher valuesof ${}^{1}/_{g^2}$. As shown in Figure 5, the strong coupling formula holds for smaller ${}^{1}/_{g^2}$.Both formulas are however, not applicable in the intermediate region ${}^{1}/_{g^2} \sim 1$.For such values ofg^2, where asymptotic scaling holds, the lattice gauge theory is said to have a continuum limit.

CONCLUSION

The QCD studies have been done using the gauge invariance of the Wilson loop. The Wilson loop is a non abelian path ordered phase factor.Continuum gauge theory provides the linear relationship between quarks potential and the distance between them. This linear relationship indicatesthe confined nature of the quarks.The proportionality constant of this relation (string tension) shows strong coupling in large lattice spacing region and the asymptotic scaling in small spacing region. The intermediate region does not show any analytic relationship. Confinement holds in the lattice gauge theory for any order of the strong coupling region.

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