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Split Middle Domination in Graphs

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Abstract: The middle graph of a graph G, denoted by $M(G)$ is a graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent if they are adjacent edges of G or one is a vertex and other is an edge incident with it. A dominating set D of $M(G)$ is called split dominating set of $M(G)$ if the induced subgraph $\langle V[M(G)] - D\rangle$ is disconnected. The minimum cardinality of D is called the split middle domination number of G and is denoted by $\gamma_{SM}(G)$.

In this paper many bound on $\gamma_{SM}(G)$ were obtained in terms of the vertices, edges and many other different parameters of G but not in terms of the elements of $M(G)$. Further its relation with other different parameters are also developed.

Key Words: *Middle Graph/ Domination Number/ Independent domination/Edge domination/Connected domination number.*

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1. INTRODUCTION

In this paper, all the graphs consider here are simple and finite. For any undefined terms or notation can be found in Harary [1]. In general, we use $< X >$ to denote the subgraph induced by the set of vertices X and $N(v)$ *and* $N([v])$ denote open (closed) neighborhoods of a vertex v . The notation $\alpha_o(G)(\alpha_1(G))$ is the minimum number of vertices (edges) in vertex (edge) cover of G. The notation $\beta_o(G)(\beta_1(G))$ is the maximum cardinality of a vertex (edge) independent set in G. A set $S \subseteq V(G)$ is said to be a dominating set of G, if every vertex in $V - S$ is adjacent to some vertex in S. The minimum cardinality of vertices in such a set is called the domination number of G and is denoted by $\gamma(G)$. The concept of edge dominating sets were also studied by Mitchell and Hedetniemi in [4,5,6 and 7]. An edge dominating set of G if every edge in $E - F$ is adjacent to at least one edge in F. Equivalently, a set Fedges in G is called an edge dominating set of G if for every edge $e \in E - F$, there exists an edge $e_1 \in F$ such that e and e_1 have a vertex in common. The edge domination number $\gamma'(G)$ of graph G is the minimum cardinality of an edge dominating set of G. The middle graph of a G, denoted by $M(G)$, is a graph whose vertex set is $V(G) \cup E(G)$,and two vertices are adjacent if they are adjacent edges of G or one is a vertex and other is an edge **incident** with it. A set S of vertices of graph $M(G)$ is an independent dominating set of $M(G)$ if S is an independent set and very vertex not in S is adjacent to a vertex in S. The independent middle domination number of G, denoted by $i_M(G)$ is the minimum cardinality of an independent dominating set of $M(G)$. The concept of independent middle dominating sets were also studied by in [2].

The middle graph of a G, denoted by $M(G)$, is a graph whose vertex set is $V(G) \cup E(G)$,and two vertices are adjacent if they are adjacent edges of G or one is a vertex and other is an edge incident with it. A dominating set D of $M(G)$ is called connected dominating set of $M(G)$ if the induced subgraph $\langle D\rangle$ is connected. The minimum **cardinality** of D is called the connected middle domination number of G and is denoted by $\gamma_c[M(G)]$. The concept of connected middle dominating sets were also studied by in [3]. Let $S(G)$ be the subdivision graph of G. The independent graph $i[S(G)]$ of $S(G)$ is a graph whose set of vertices is the union of the set of edges of $S(G)$ in which two vertices are adjacent if and only if the corresponding edges of $S(G)$ are adjacent. A dominating set D of the subdivision graph $S(G)$ is an independent dominating set if $\langle D\rangle$ is independent in $S(G)$ minimum cardinality of the smallest independent dominating set of $i[S(G)]$ is called the independent subdivision dominating set of G and is denoted by $i[S(G)]$.

In this paper, many bounds on $\gamma_{SM}(G)$ were obtained in terms of elements of G but not the elements of $M(G)$. Also its relation with other domination parameters were established.

We need the following theorem for our further results.

Theorem: A[3]: for any non trivial connected (p, q) graph $G, \gamma_c[M(G)] = P - 1$.

Theorem: B[2]: Let G be any connected graph, then $i_M(G) = \alpha_1(G)$.

First we list out the exact values of $\gamma_{SM}(G)$ for some standard graphs.

2. MAIN RESULTS:

Theorem 1:

a. For any path P_p ,

$$\gamma_{SM}(P_p) = \frac{p}{2} \text{ If } p \text{ is even.}$$

$$\gamma_{SM}(P_p) = \left\lceil \frac{p}{2} \right\rceil \text{ If } p \text{ is odd}$$

b. For any path C_p ,

$$\gamma_{SM}(C_p) = \frac{p}{2} \text{ If } p \text{ is even.}$$

$$\gamma_{SM}(C_p) = \left\lceil \frac{p}{2} \right\rceil \text{ If } p \text{ is odd.}$$

c. For any path $K_{1,p}$,

$$\gamma_{SM}(K_{1,p}) = p - 1.$$

d. For any path W_p ,

$$\gamma_{SM}(W_p) = p - 1.$$

Theorem 2: A split middle dominating set $D \subseteq V[M(G)]$ is minimal if and only if for each vertex$x \in D$, one of the following condition holds:

a. There exists a vertex$y \in V[M(G)] - D$ such that$N(y) \cap D = \{x\}$.
b. x is an isolate in$\langle D \rangle$.
c. $\langle V[M(G)] - D \rangle \cup \{x\}$is connected.

Proof: Suppose D is a minimal split middle dominating set of$G$ and there exists a vertex $x \in D$ such that x does not holds any of the above conditions. Then for some vertex$v$, the set $D_1 = D - \{v\}$ forms a split middle dominating set of G by the conditions (a) and (b). Also by (c),$\langle V[M(G)] - D \rangle$ is disconnected. This implies thatD_1 is a split middle dominating set of$G$, a contradiction.

Conversely, suppose for every vertex$x \in D$, one of the above statement hold. Further, if D is not minimal, then there exists a vertex$x \in D$ such that $D - \{x\}$ is a split middle dominating set of G and there exists a vertex $y \in D - \{x\}$ such that $y$dominates x. That is$y \in N(x)$. Therefore,x does not satisfy (a) and (b), hence it must satisfy (c). Then there exists a vertex$y \in V[M(G)] - D$ such that $N(y) \cap D = \{x\}$. Since $D - \{x\}$ is a split middle dominating set of$G$, then there exists a vertex $z \in D - \{x\}$ such that$z \in N(y)$. Therefore$w \in N(y) \cap D$, where $w \neq x$, a contradiction to the fact that $N(y) \cap D = \{x\}$.Clearly,$D$is a minimal split middle dominating set of$G$.

Theorem 3: For any (p, q)graph $G, \gamma(G) + \gamma_{SM}(G) \leq p$.

Proof: Let $C = \{v_1, v_2 \ldots \ldots \ldots \ldots v_n\} \subseteq V(G)$ be the set of all non-end vertices in G.Further let $D \subseteq C$ be the set of vertices with $diam(u_i, v_i) \geq 3, \forall\, u_i, v_i \in D, 1 \leq i \leq k$.Clearly, $N[D] = V(G)$ and D forms a $\gamma - set$ of G, Suppose $diam(u_i, v_i) < 3$,then there exists atleast one vertex $x \in V(G) - D$ such that either $x \in N(v)$ or $x' \in N(v')$,where $v \in D$ and $v' \in D \cup \{x\}$.Then $D \cup \{x\}$ forms a minimal dominating set of G.Now in $M(G)$, $V[M(G)] = V(G) \cup E(G)$.Let $S = \{s_1, s_2, s_3 \ldots \ldots \ldots \ldots \ldots \ldots s_k\}$ be the set of vertices sub dividing each edge in $M(G)$.Again let $S_1 \subseteq S$ be the minimal set of vertices such that $N[S_1] = V[M(G)]$.Then S_1 forms the minimal dominating set in $M(G)$.If $\langle V[M(G)] - S_1 \rangle$ contains atleast two components then $\langle S_1 \rangle$ itself forms the minimal split dominating set of $M(G)$.Otherwise , there exists atleast one vertex $\{u\} \in V[M(G)] - S_1$ such that $\langle V[M(G)] - (S_1 \cup \{u\}) \rangle$ has more than one component. Clearly, $S_1 \cup \{u\}$ forms a minimal split dominating set of $M(G)$.Therefore it follows that $|D \cup$ *xUS1Uu≤VG* and hence *γG+γSMG≤p.*

Theorem 4: For any non-trivial connected graph $G, \gamma_{SM}(G) \geq \alpha_1(G)$.

Proof: Let $E_1 = \{e_1, e_2 \ldots \ldots \ldots \ldots \ldots \ldots e_n\}$ be the minimal set of edges in G such that $|E_1| = \alpha_1(G)$.since $V[M(G)] = V(G) \cup E(G)$, let $S = \{s_1, s_2, s_3 \ldots \ldots \ldots \ldots s_i\}$ be the set of vertices sub dividing each edge in $M(G)$.Now, let $S_1 = \{s_k \setminus 1 \leq k \leq i\} \subseteq S$ be the set of vertices sub dividing each edge $e_k \in E_1$,$1 \leq k \leq i$ in $M(G)$. Clearly $N(S_1) = V(G)$ and also in $M(G)$, $N(S_1) = V(S - S_1)$.Hence $N[S_1] = V(G) \cup V(S - S_1) = V[M(G)]$.Thus $\langle S_1 \rangle$ forms a minimal dominating set in $M(G)$.If the subgraph $\langle V[M(G)] - S_1 \rangle$ contains atleast two components , then S_1 itself forms the minimal split dominating set in $M(G)$.Otherwise there exists atleast one vertex $\{s_j\} \in V[M(G)] - S_1$,$1 \leq j \leq i$ such that the subgraph $\langle V[M(G)] - (S_1 \cup \{s_j\}) \rangle$ is disconnected. Clearly $S_1 \cup \{s_j\}$ forms a minimal split dominating set in $M(G)$.Thus $|E_1| \leq |S_1 \cup \{s_j\}|$ which gives $\gamma_{SM}(G) \geq \alpha_1(G)$.

Theorem 5: For any non-trivial connected (p, q) graph $G, \gamma_{SM}(G) \leq p - 1$.

Proof : We consider the following cases:

Case 1: Let $G = T$ be any non-trivial tree.The vertex set and edge set of T are $V(T) = \{v_1, v_2, \ldots \ldots \ldots v_p\}$ and $E(T) =$

$\{e_1, e_2 \dots \dots e_q\}$.Let $S = \{s_1, s_2 \dots \dots \dots \dots s_q\}$ be the vertices subdividing the edges in $M(T)$,which is also the set of cut vertices in $M(T)$.Suppose $S_1 \subseteq S$ be the minimal set of vertices such that $N[S_1] = V[M(T)]$.Hence S_1 forms the minimal dominating set in $M(T)$.Since S_1 is the set of cut vertices, the subgrap $\langle V[M(T)] - S_1 \rangle$is disconnected. Hence$S_1$ forms the minimal split dominating set in $M(T)$.Therefore$|S_1| \leq |E(T)| = |V(T)| - 1$, which gives $\gamma_{SM}(G) \leq p - 1$.

Case2: Let $G \neq T$,then consider a spanning tree H of G.Let $E_1 = \{e_1, e_2, \dots \dots \dots e_i\}$ be the edges in H.Let $S_1 = \{s_1, s_2, \dots \dots \dots \dots s_i\}$ be the vertices subdividing the edges of E_1 in $M(H)$.Again let $S_2 \subseteq S_1$ be the minimal set of vertices in which covers all the vertices in $M(H)$ and the subgraph $\langle V[M(H)] - D \rangle$ is disconnected. Thus $N[S_2] = V[M(H)]$.By adding the edges $E_2 = E(G) - E_1$ of G to H.Again we consider $S_2' = \{s_1', s_2' \dots \dots \dots \dots s_i'\}$ be the vertices subdividing the edges $E_2 = \{e_1', e_2', \dots \dots \dots \dots e_i'\}$ in $M(G)$.Now since $(s_i) \cap N(s_i') \neq \phi$,$\forall s_i \in S_2$ and $s_i' \in S_2'$in $M(G)$.Clearly $N[S_2] = S_2' \cup V[M(H)] = V[M(G)]$.Thus S_2 forms a minimal split dominating set of $M(G)$ with $|S_2| = \gamma_{SM}(G)$.Since $S_2 \subseteq E_1$,then $|S_2| \leq |E_1| \leq p - 1$.Therefore $\gamma_{SM}(G) \leq p - 1$.

Theorem 6: For any connected graph $G, \gamma_{SM}(G) \leq \gamma_c[M(G)]$.

Proof: By Theorem A $\gamma_c[M(G)] = p - 1$ and also by Theorem 5 $\gamma_{SM}(G) \leq p - 1$.Hence we have $\gamma_{SM}(G) \leq \gamma_c[M(G)]$.

Theorem 7: If G is a connected graph, then $\left\lfloor \frac{diam\,(G)+1}{2} \right\rfloor \leq \gamma_{SM}(G)$.

Proof: Let $S = \{e_1, e_2 \dots \dots \dots \dots \dots \dots e_j\}$ be the set of edges in G which constitute the diametral path in G. Clearly$|S| = diam(G)$. Now without loss of generality, let D_1 be a minimal dominating set in $M(G)$.If $\langle V[M(G)] - D_1 \rangle$ contains atleast two components then $\langle D_1 \rangle$ itself forms the minimal split dominating set of $M(G)$.Otherwise , there exists atleast one vertex $\{u\} \in V[M(G)] - D_1$ such that $\langle V[M(G)] - (D_1 \cup \{u\}) \rangle$ has more than one component. Clearly, $D_1 \cup \{u\}$ forms a minimal split dominating set of $M(G)$.Further since $S \subseteq V[M(G)]$ and $D_1 \cup \{u\}$ is a γ_s setin $M(G)$, the diametral path includes at most $\gamma_{MS}(G) - 1$ edges joining the neighbourhoods of the vertices of $D_1 \cup \{u\}$. Hence $diam(G) \leq \gamma_{SM}(G) + \gamma_{SM}(G) - 1$ which gives$\left\lfloor \frac{diam\,(G)+1}{2} \right\rfloor \leq \gamma_{SM}(G)$.

Theorem 8: For any non-trivial connected graph $G, \gamma_{SM}(G) \geq i_M(G)$.

Proof : The result follows from Theorem B and Theorem 5.

Theorem 9: For any connected graph $G, \gamma_{SM}(G) \leq p - \gamma'(G)$.

Proof: Let $E_1 = \{e_1, e_2, e_3, \dots \dots \dots \dots \dots \dots \dots e_q\} \subseteq E(G)$ be the minimal set of edges, such that for each $e_i \in E_1, i = 1,2,3, \dots \dots \dots \dots \dots \dots \dots \dots . . q$,$N(e_i) \cap E_1 = \phi$.Then $|E_1| = \gamma'(G)$.In $M(G)$,$V[M(G)] = V(G) \cup E(G)$.Let $D = \{v_1, v_2, v_3, \dots \dots \dots \dots \dots \dots v_i\}$ be the set of vertices subdividing the edges of G in $M(G)$.Let $D_1 \subseteq D$,such that each $v_i \in D_1$ subdivides the edges $e_i \in E_1$ in $M(G)$ and $D_2 \subseteq D$ be the set of vertices subdividing the edges in $E(G) - E_1$ in $M(G)$.Suppose $N[D_1] = V[M(G)]$.Then $\langle D_1 \rangle$ forms a minimal dominating set in $M(G)$.Now assume $\langle V[M(G)] - D_1 \rangle$ is disconnected. Then D_1 itself forms the minimal split dominating set in $M(G)$.Otherwise ,let $D_1' \subseteq D_1$ and $D_2' \subseteq D_2$ such that $N[D_1' \cup D_2'] = V[M(G)]$ and the subgraph $\langle V[M(G)] - D_1' \cup D_2' \rangle$ is disconnected. Hence $D_1' \cup D_2'$ forms a minimal split middle dominating set of G.Clearly it follows that $|D_1' \cup D_2'| \leq |V(G)| - |E_1|$ resulting in $\gamma_{SM}(G) \leq p - \gamma'(G)$.

Theorem 10: For any (p, q) graph $G, \gamma_{SM}(G) \leq i[S(G)]$.

Proof: Let $V(G) = \{v_1, v_2, \dots \dots \dots \dots \dots v_i\}$ and $E(G) = \{e_1, e_2, \dots \dots \dots \dots \dots e_j\}$.We consider a set $S = \{u_1, u_2, \dots \dots \dots \dots \dots u_k\}$ be the set of vertices which divides each edge of G in $S(G)$.For $S_1 \subseteq S$,such that $N[S_1] = V[S(G)]$ then S_1 forms a minimal dominating set in $S(G)$.Also since in $S(G)$,$\forall u_i, u_j \in S_1, 1 \leq i, j \leq k$, $N(u_i) \cap (u_j) = \phi$,hence $\langle S_1 \rangle$ itself forms the minimal independent dominating set in $S(G)$.If $N[S_1] \neq V[S(G)]$,we consider a set $I = D_1 \cup D_2$ where $D_1 \subseteq S_1$ and $D_2 \subseteq V[S(G)] - D_1$,such that $N[I] = V[S(G)]$ and for each $u_i \in \langle I \rangle$, $deg(u_i) = 0$,then I forms a minimal independent dominating set of $S(G)$.Further ,without loss of generality $S \subseteq V[M(G)]$.Consider a minimal set $S_2 \subseteq S$ such that $N[S_2] = V[M(G)]$.Then S_2 forms the minimal dominating set in $M(G)$.If the subgraph $\langle V[M(G)] - S_2 \rangle$ is disconnected, then S_2 forms the minimal split dominating set in $M(G)$.Otherwise there exists a vertex $u_i \in S - S_2, 1 \leq i \leq k$ such that $\langle V[M(G)] - S_2 \cup \{u_i\} \rangle$ contains at least two components. Thus $S_2 \cup \{u_i\}$ forms the minimal split dominating set in $M(G)$.Clearly $|I| \geq |S_2 \cup \{u_i\}|$ which gives $\gamma_{SM}(G) \leq i[S(G)]$.

Theorem 11: Let G be graph such that both G and $\overline{G}$ have no isolated edges, then

$$\gamma_{SM}(G) + \gamma_{SM}(\overline{G}) \leq P.$$
$$\gamma_{SM}(G) + \gamma_{SM}(\overline{G}) \leq P^2.$$

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