Cornering the Queen

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Abstract:- By analyzing a novel game, we can study some fascinating mathematical concepts which leads to surprising conclusions. While doing so, we come across one of the important real number, the Golden Ratio. We introduce Whythoff's Nim game and calculate the safe pairs for our main problem. The paper leads to many further investigations related to connecting mathematics with games.

Keywords: *Games, Cornering of the Queen, Whytoff's Nim Game, Safe Pairs, Golden Ratio (Phi), Lucas Numbers.*

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Introduction

We are going to consider the game probably on a chessboard though it can be played on any square array of board pattern. It has no traditional name though we may now call it "Cornering the Lady" or "Cornering the Queen". It was invented about 1960 by Rufus P. Isaacs, a mathematician at Johns Hopkins University **[1].**

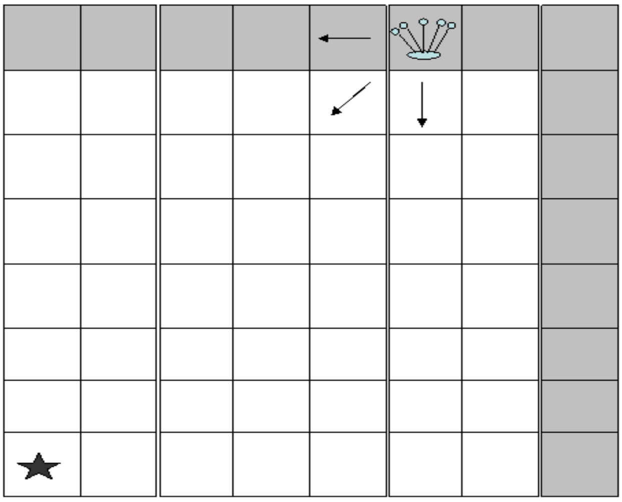

Diagram 1. The Cornering of the Queen

Let us consider that two players say Player A and Player B playing the game. Player A puts the queen on any cell in the top row or in the column farthest to the right of the board; the cells appear in gray in diagram 1. The queen moves in the usual way but only west, south or southwest. Player B moves first, then the players alternate moves. The player who gets the queen to the starred cell at the lower left corner is the winner. No draw is possible, so that A or B is sure to win if both sides play rationally.

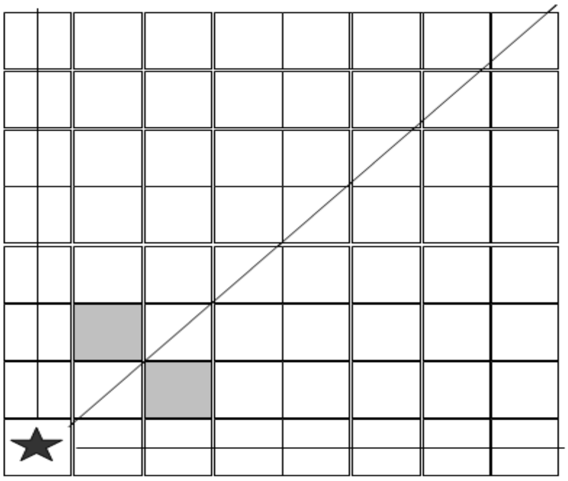

Diagram 2. The Cornering of the Queen

We are interested in finding the winning strategy for the game. If the queen is in row, column or diagonal containing the star (in diagram 1), the person who has the move can win at once. Mark these cells with three straight lines as is shown in diagram2.It is clear that the two shaded cells are "safe", in the sense that if you occupy either one, your opponent is forced to move to a cell that enables you to win on the next move.

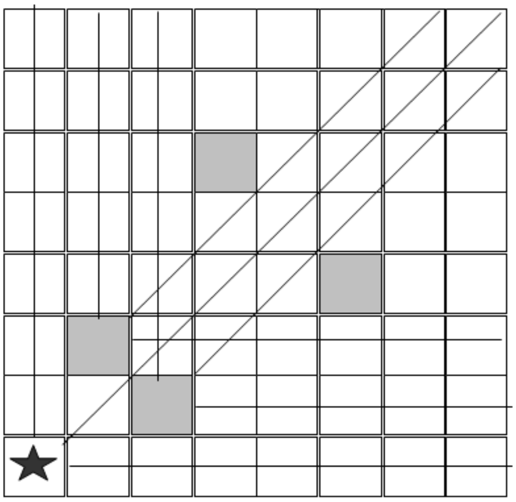

Diagram 3. The Cornering of the Queen

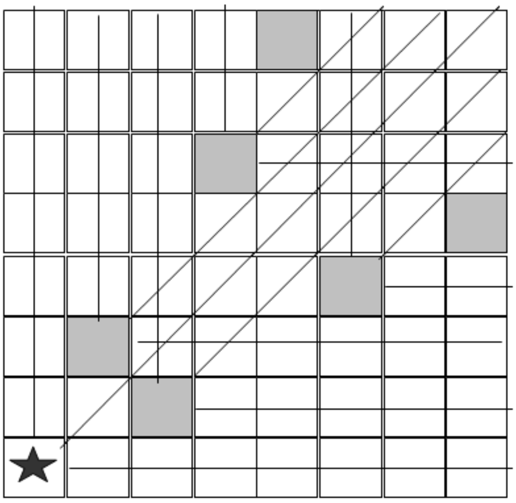

Diagram 4. The Cornering of the Queen

Diagram3. illustrates the next step of our recursive analysis. Add six more lines to mark all the rows, columns and diagonals containing the two previously discovered safe cells. This procedure allows us to shade two more safe cells as shown in diagram3. If you occupy either one, your opponent is forced to move, so that on your next move you can either win at once or move to the pair of safe cells nearer to the star. Repeating this procedure, as shown in diagram4, completes the analysis of the chessboard by finding a third pair of safe cells. It is now clear that Player A can always win by placing the queen on the shaded cell in either the top row or the column farthest to the right. His strategy thereafter is simply to move to a safe cell, B can always win by the same strategy. But we note that the winning moves are not necessarily unique. There are times when the player with the win has two choices; one may delay the win, the other may hasten it.

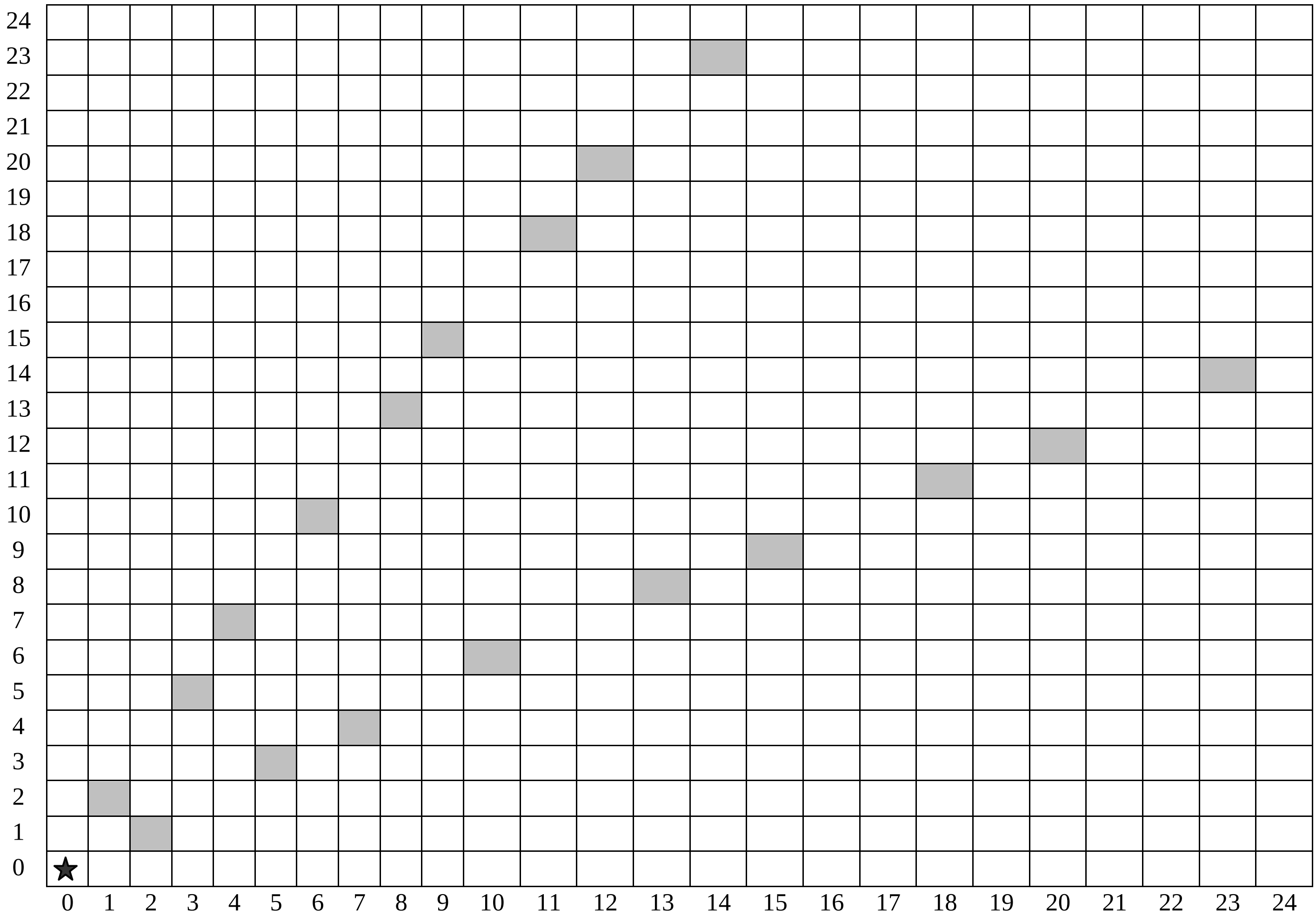

Diagram 5. The First Nine Pairs of Safe Cells

On extending the above recursive analysis to square boards of larger dimensions we can observe some curiosities. For example, in diagram5 we have a square board of dimension 25 by 25 in which all the safe cells are shaded. We note that they are paired symmetrically with respect to the main diagonal and lie almost on two lines that fan outward to infinity. Their locations along these two lines seem to be curiously irregular. Are there formulas by which we can calculate their positions nonrecursively?

Wythoff's Nim Game:

Let us now consider an old counter take-away game said to have been played in china under the name tsyan-shidzi, which means "Choosing stones". The game was reinvented by the Dutch mathematician W.A.Wythoff in 1907. In western mathematics it is called as "Wythoff's Nim."

The game is played with two piles of counters, each pile containing arbitrary number of counters. As in Nim, a move consists in taking any number of counters from either pile. Atleast one counter must be taken. If a player wishes, he may remove an entire pile. A player may take from both piles (which may not be in Nim), provided that he takes the same number of counters from each pile. The player who takes the last counter wins. If both piles have the same number of counters, the next player wins at once by taking both piles. For this reason the game becomes trivial if it starts with equal piles.

We are now ready for the first surprise. Wythoff's Nim is isomorphic with the "Cornering the Queen" game! The isomorphism is easy to see. If we consider diagram5, where we have numbered the 25 columns along the x-coordinate axis, starting with 0; the rows along the y-coordinate axis in the same way, each cell now can be given an x/y number. These numbers correspond to the number of counters in piles x and y. When the queen moves west, pile x is diminished while the queen moves south pile y is diminished. When it moves diagonally southwest, both piles are diminished by the same amount. Moving the cell to 0/0 is equivalent to reducing both piles to 0.

Thus the strategy of winning Wythoff's Nim is to reduce the piles to a number pair that corresponds to the number pair of a safe cell in the Queen game. If the starting pile numbers are safe, the first player loses. He is certain to leave an unsafe pair of piles, which his opponent can always reduce to a safe pair on his next move. If the game begins with unsafe numbers, the first player can always win by reducing the piles to a safe pair and continuing to play to safe pairs. It is also important to note that the order of the two numbers in a safe pair is not important. This is because of the symmetry of any two cells on the chessboard with respect to the main diagonal, they have the same coordinate numbers, one pair being the reverse order of the other. Let us take the safe pairs in sequence, starting with the pair nearest 0/0, and arrange them in a row with each smaller number above its partner, as in diagram6. Above the pairs write their "position numbers." The top numbers of the safe pairs form a sequence we shall call A. The bottom numbers form a sequence we shall call B.

Position (n)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
A. [n φ]	1	3	4	6	8	9	11	12	14	16	17	19	21	22	24
B.[n φ^2]	2	5	7	10	13	15	18	20	23	26	28	31	34	36	39

Diagram 6. The First 15 Safe Pairs in Cornering the Queen

These two sequences, each one strictly increasing, have so many remarkable properties. Note that each B number is the sum of it's A number and its position number. If we add an A number to its B number, the sum is an A number that appears in the A sequence at a position number equal to B. For example 8+13= 21 and the 13^{th} number of the A sequence is 21.). Thus we have constructed two sequences A and B geometrically by drawing lines on the chessboard and shading cells according to a recursive algorithm.

Is there a way to generate the sequences A and B by a recursive algorithm that is purely numerical? Yes! Start with 1 as the top number of the first safe pair. Add this to its position number to obtain 2 as the bottom number. The top number of the next pair is the smallest positive integer not previously used. It is 3. Below it goes 5, the sum of 3 and its position number. For the top of the third pair write again the smallest positive integer not yet used. It is 4. Below it goes 7, the sum of 4 and 3. Continuing in this way we can generate the sequences A and B.

Now we are ready for the second surprise! The numbers in sequence A are simply multiples of the golden ratio(phi), rounded down to nearest integers!. That is, the numbers in the sequence A are given by the formula [n φ], where n is the position number and the brackets signify discarding the fractional part. B numbers can be obtained by adding A numbers to their position numbers, but it turns out that they are rounded -down multiples of the square of phi. The formula for sequence B, therefore, is [n φ^2]. The fact that every positive integer appears once and only once among the safe pairs can be expressed by the following remarkable theorem : "The set of integers that lie between successive multiples of phi and between successive multiples of phi squared is precisely the set of natural numbers."

Two sequences of increasing positive integers that together contain every positive integer just once are called "complementary." Thus our sequences A and B are complementary. The astounding fact that every irrational number generates such complementary sequences was first established by Sam Beatty in 1926[**2**]. Whenever the golden ratio appears, it is a good bet that Fibonacci numbers lurk nearby. The Fibonacci sequence is 1,1,2,3,5,8,13,21,34,55,89,… in which each number after the first two is the sum of the two preceding numbers. If we partition the primary Fibonacci sequence into pairs,1/2,3/5, 8/13,21/34,…, it can be shown that every Fibonacci pair is a safe pair in Wythoff's Nim. The first such pair not in this sequence is 4/7. If we start with another Fibonacci sequence with 4/7 as two initial terms then the successive ratios are 4/7,11/18,29/47,…, all these pairs are also safe in Wythoff's Nim. Indeed, these pairs belong to a Fibonacci sequence of what are called Lucas numbers that begins 2,1,3,4,7,11,18,…

Imagine that we go through the infinite sequence of safe pairs(in the manner of Eratosthenes' sieve for sifting our primes) and cross out the infinite set of all safe pairs that are pairs in the Fibonacci sequence. The smallest pair that is not crossed out is 4/7. We can now cross out a second infinite set of safe pairs, starting with 4/7, that are pairs in the Lucas sequence. An infinite number of safe pairs, of which the lowest is now 6/10, remain. This pair too begins another infinite Fibonacci sequence, all of those pairs are safe. The process continues forever. We call a safe pair "primitive" if it is the first safe pair that generates a Fibonacci sequence. There are an infinite number of primitive safe pairs. Since every positive integer appears exactly once among the safe pairs, we can conclude that there is an infinite sequence of Fibonacci sequences that exactly covers the set of natural numbers.

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14 – 18

Take the primitive pairs 1/2,4/7,6/10,9/15,… in order and write down their position numbers, 1,3,4,6,… Does this sequence look familiar? As we can see that this is none other than sequence A. In otherwords, a safe pair is primitive if and only if its position number is a number in sequence A.

Further Challenges:

I am working out a similar kind of winning strategy for the above mentioned chessboard game by considering a rook or a knight or a bishop piece under the same assumptions. They might provide a clue of whether we can come across any such interesting sequences as we had discussed above.

References:

[1] "The Theory of Graphs and Its Applications", chapter 6. Claude Berge Paris Dunod, 1958; London: Methuen, 1962.
[2] "Problem 3173." Sam Beatty, in The American Mathematical Monthly, 33, 1926, p.159; solutions in 34, 1927, p. 159
[3] "A Fibonacci Property of Wythoff Pairs." Robert Silber; in The Fibonacci Quarterly, 14, November 1976, pp. 380-384.
[4] Challenging Mathematical Problems with elementary Solutions, A.M. Yaglom and I. M. Yaglom. Vol. 2, Holden-Day, 1967, pp.20 and 105.