Fractal Sequences

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Abstract:- In this paper we shall explore the various ways and properties of fractal sequences. I have given three types of fractal sequences totally in different aspects. The first type is a sequence generated by card shuffling and the second type is the Golden Sequence and the third type is the Signature sequences. The method of their generation is given for each case. Through these types one can explore the beauty and can enjoy with the properties of these amazing sequences.

Keywords*: Fractals, Self-similar, Golden Sequence, Signature Sequences*

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Introduction

First let us look at a formal definition of a fractal sequence given by Benoit Mandelbrot [1].” An unbounded set S is self-similar with respect to the ratio r, when the set r(S) is congruent to S”.

Wewill now consider the following types of fractal sequences.

Type 1:

Make a set of cards numbered 1, 2, 3..., n and hold them face up in your hand. Take the top card and place it face up on the bottom of the deck. Place the next card face up on a table. Continue this process until all n cards are face up on the table. How far down in the pile on the table do you have to look to find the original top card?

The answer relates to a sequence that begins with

1,1,2,1,3,2,4,1,5,3,6,2,7,4,8,1,9,5,10,3,11,6,12,2,13,7,14,4, 15, 8,...

Interestingly the above sequence is fractal containing infinite copies of itself. If you delete the first occurrence of each integer, you’ll see that the remaining sequence is the same as original:**1,1,2,1,3,2,4,1,5,3,6,2,7,4,8,1,9,5,10,3,11,6,12,2,13,7,14,4, 15, 8,...**

Do it again and again and you get the same sequence!

Pertaining to the above definition the above sequence is self-similar with respect to the ratio 2,because $S_2, S_4, S_6, S_8, \ldots$ is identical to the original sequence $S_1, S_2, S_3, S_4, S_5, S_6, \ldots$

We can also consider fractal-like sequences that consist of any string that contains copies of itself, even if the string doesn’t quite conform to the above rules. For example, consider the letter string**:**

a,b,a,c,b,a,d,c,b,e,a,d,c,f,b,e,a,d,g,c,f,b,e, . . .

If you delete the first occurrence of each letter, you’ll see that the remaining string is the same as original: **a,b,a,c,b,a,d,c,b,e,a,d,c,f,b,e,a,d,g,c,f,b,e, . . .**

I refer to this type of sequence as fractal-like because, like most fractals, it has “parts that resemble the whole”.

Type 2:

We will now consider another example of a fractal sequence called "Golden Sequence"

The following sequence is the Golden Sequence:

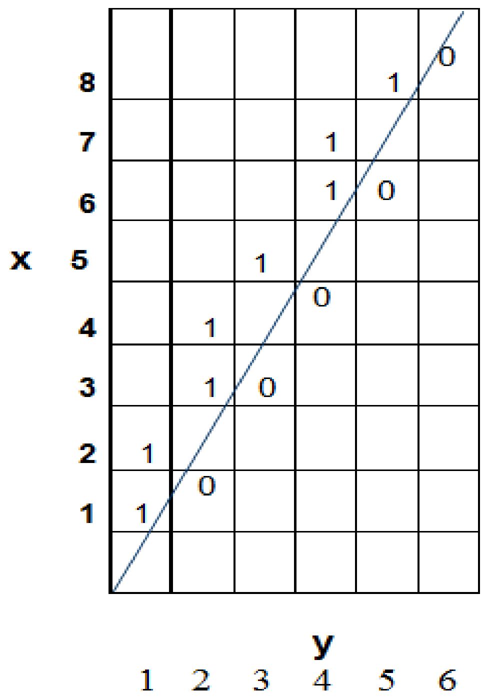
1, 0, 1, 1, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1, 0, 1, 1, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1, 0, 1, 1, 0, 1, . . .

It can be generated by the following algorithm. Start with 1, then replace 1 by 10. From then on, we repeatedly replace 1 by 10 and each 0 by 1.This sequence has many remarkable properties that involve the notorious number called Golden ratio given by 1.6180339... which is provided by the expression $\phi = \dfrac{1+\sqrt{5}}{2}$.

If we draw the line $y = \phi x$ on a graph, then we can see the sequence directly.

Whenever the line crosses the horizontal grid line we write 1 by it on the line and whenever the line crosses the vertical grid line we write a 0.(The line can never cross exactly at an intersection of the vertical and horizontal grid lines, because the slope of the line is ϕ).Now if you run your finger along the line $y = \phi x$ starting at (0,0), you will generate a sequence of 1s and0s – resulting in the Golden Sequence.

Refer the diagram given below:



The diagonal line is $y = \phi x$

Here are some observations about this beautiful sequence:

1. The number of 1s and 0s in this sequence form a Fibonacci sequence and the ratio of 1s to 0s approaches ϕ , [2] as more terms are added.

2. Underline any subsequence of the golden sequence, for example, the subsequence 10: 10 1 1010 110 1 10 . . . You'll find that 10 follows the preceding 10 by the following number of places: 2122121.... If 2 is replaced by

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1 and 1 by 0, the golden sequence is replicated which shows that it is “self-similar” at different scales – that is, it is a fractal sequence.

3. Ron D.Knott of the University of Surrey in the United Kingdom has translated the sequence into an audio file by mapping 1s to A notes (220Hz) and 0s into the A an octave higher (440Hz), played at about 5 notes per second. He notes that the rhythm is hypnotic, having a definite beat that keeps changing but holds one’s attention.

Type 3:

Another example of a fractal sequence is the “Signature Sequence” of a positive irrational number R, such as $\sqrt{2}$. To create this amazing sequence, arrange the set of all numbers i+jR, where i and j are nonnegative integers, in ascending order:

i(1)+j(1)R <i(2)+j(2)R <i(3)+j(3)R <...

Then i(1), i(2), i(3),... defines the signature of R. For example, the signature of the square root of 2 starts with

1,2,1,3,2,1,4,3,2,5,1,4,3,6,2,5,1,4,7,3,6,2,5,8,1,4,7,3,6,9,2,5,8,...

Ifyou delete the first occurrence of each integer, you’ll see that the remaining sequence is the same as original. To compute this sequence, we have to write down the first few possibilities for i+j× $\sqrt{2}$ and arrange them in order from least to greatest:

$$
\begin{aligned}
&1.1+1\times\sqrt{2}=2.414...\\
&2.2+1\times\sqrt{2}=3.414...\\
&3.1+2\times\sqrt{2}=3.828...\\
&4.3+1\times\sqrt{2}=4.414...\\
&5.2+2\times\sqrt{2}=4.828...\\
&6.1+3\times\sqrt{2}=5.243...\\
&7.4+1\times\sqrt{2}=5.414...\\
&8.3+2\times\sqrt{2}=5.828...\\
&9.2+3\times\sqrt{2}=6.243...\\
&10.5+1\times\sqrt{2}=6.414...
\end{aligned}
$$

In this example, i values form the fractal sequence. Of course one could also produce other such Signature Sequences [3] for different irrational numbers.

Some famous Signature Sequences:

$$
\begin{aligned}
&1.73205081\approx\sqrt{3}\to 1,2,1,3,2,4,1,3,5,2,4,6,1,3,5,7,2,4,6,1,8,3,5,7,2,9,4,...\\
&2.718281828\approx e\to 1,2,3,1,4,2,5,3,6,1,4,7,2,5,8,3,6,9,1,4,7,10,2,5,8,1,...\\
&3.141592653\approx \pi\to 1,2,3,4,1,5,2,6,3,7,4,1,8,5,2,9,6,3,10,7,4,1,11,8,5,2,...
\end{aligned}
$$

All of the above sequences are fractal sequences. Thus one can explore as many fractal sequences as possible**.** Fractals are observed as fragments of nature and so Fractal Sequences would be the best possible key to uncover the secrets of nature if we can properly construct such a sequence according to the given circumstances.

References:

[1] Benoit Mandelbrot is the founder of fractal Geometry. He gave the definition in his work ”The Fractal Geometry of Nature”

[2] ϕ is called the Golden Ratio, a very famous irrational number which is even known to Greeks. It has a very traditional background and it is associated with nature in so many ways. There are so many literatures available for exploring the beauty of ϕ. It is also the ratio of the consecutive Fibonacci Numbers.

[3] Kimberling.C.(1995) Numeration systems and fractal sequences. ActaArithmetica 73: 103-117.