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Finite-Generalized-Laplace-Hankel-Clifford-Transformation

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Abstract— In this paper, finite-generalized-Laplace-Hankel-Clifford-transformation is established. The relation of the conventional Laplace transformation is applied over finite-generalized-Hankel-Clifford and derived further results. Analyticity and boundedness condition is established. The Operation transform and an inversion formula for a distributional finite-generalized-Laplace-Hankel-Clifford-transformation is developed. A problem is solved in the text to support development of the finite-generalized-Laplace-Hankel-Clifford-transformation.

Keywords*- Finite generalized Hankel-Clifford transformation, Laplace transforms, boundedness, analyticity,applications.*

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I. INTRODUCTION

The conventional Laplace transformation is defined by

$$\bar{f}(s)=\int_0^\infty f(t)e^{-st}dt; s=\sigma+iR;\ \sigma_1<\operatorname{Re}s<\sigma_2 \tag{1.1}$$

and its inversion formula for every value of $'s'$ in complex plane $\sigma_1<\operatorname{Re}s<\sigma_2$ where σ_1 and σ_2 are some real numbers, is given by

$$f(t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\bar{f}(s)e^{st}ds \tag{1.2}$$

The conventional generalized Hankel-Clifford transform is defined by

$$F(m)=\int_0^\infty y^{-\alpha-\beta}\mathrm{J}_{\alpha,\beta}(\lambda_m y)f(y)dy, \tag{1.3}$$

where $\mathrm{J}_{\alpha,\beta}(y)=y^{(\alpha+\beta)/2}J_{\alpha-\beta}(2\sqrt{y}),\ J_{\alpha-\beta}(y)$ being the Bessel function of the first kind of order $(\alpha\text{-}\beta)$ [5], y and m are positive real variables. The corresponding inversion, under certain conditions Malgonde [6], is given by

$$f(x)=\int_0^\infty\frac{2\mathrm{J}_{\alpha,\beta}(\lambda_m y)}{\lambda_m\,\mathrm{J}^2_{\alpha,\beta-1}(\lambda_m)}F(m)dm \tag{1.4}$$

R. V. Churchill [1], Colombo [2] have published for Fourier and Mellin transformation. Dorta Diaz and Méndez-Pérez developed in [3] Dini's series expansions and the Finite Hankel-Clifford transformations. B. R. Bhonsle and R. A. Prabhu, [10] developed an inversion formula for a distributional finite- Hankel- Laplace transformation. In their work, an extension of the classical Finite-Hankel- Laplace transform to generalized functions was done. Malgonde and Gorty [7] introduced the corresponding finite transformation through the equation

$$(\hbar_{1,\alpha,\beta}f)(n)=F_{1,\alpha,\beta}(n)=\int_0^a x^{-\alpha-\beta}\mathrm{J}_{\alpha,\beta}(j_n x)f(x)dx\,, \tag{1.5}$$

which is called finite generalized Hankel-Clifford transformation of first kind of order $(\alpha-\beta)$. Its inversion theorem is stated as

Theorem 1.1. Let $f(t)$ be a function defined in $(0,1)$ and assumed to be absolutely summable over the same interval. Let $(\alpha-\beta)\ge-\frac{1}{2}$ and

$$a_n=\frac{1}{J^2_{\alpha,\beta-1}(j_n)}\int_0^1 t^{-\alpha-\beta}J_{\alpha,\beta}(j_n t)\ f(t)dt,\quad n=1,2,\dots .$$

If $f(t)$ is of bounded variation in $(a,b),\ (0<a<b<1)$ and if $x\in(a,b)$, then the series (1.2) converges to $\frac{1}{2}[f(x+0)+f(x-0)]$.

It is quite well known that there are several problems which can be solved by the repeated applications of the transformations and in particular the transformations in (1.1) and (1.3). If constructed an integral transform for which the kernel is the product of the kernels of the Laplace and Hankel-Clifford transformation of the first kind, integral (special) transform as Laplace Hankel-Clifford transform which has been successfully applied to deal with the problems occurring in mathematical physics. This motivated to study the transform in the distributional setting also.

The finite-generalized-Laplace-Hankel-Clifford transform is defined by

$$L\hbar_{\alpha,\beta}(f)(s,\lambda_m)=F(s,\lambda_m)$$
$$=\int_{-\infty}^{\infty}\int_{0}^{1}e^{-sx}y^{-\alpha-\beta}J_{\alpha,\beta}(\lambda_m y)f(x,y)dxdy, \quad (1.6)$$

where $f(x,y)$ belongs to an appropriate function space $(-\infty<x<\infty, 0<y<1)$ and where $J_{\alpha,\beta}(y)=y^{(\alpha+\beta)/2}J_{\alpha-\beta}(2\sqrt{y})$, $J_{\alpha-\beta}(y)$ being the Bessel function of the first kind of order $(\alpha-\beta)$, with $(\alpha-\beta)\geq-\frac{1}{2}$ and $s=\sigma+iR$ is a restricted complex variable. The finite-generalized-Laplace-Hankel-Clifford transform will be represented as FGLHCT.

The aim of the present paper is to outline $L\hbar_{\alpha,\beta}(f)$ transform and to establish analytic and boundedness condition. Also motivates the authors to initiate use of finite-generalized-Laplace-Hankel-Clifford transform in Engineering applications in the field of Electronics & Communication, Electrical Engineering. In recent study 2011 [11], studied Control System Design Using Finite Laplace Transform Theory. All microprocessor based electronic systems are designed as repetition of finite time activities. The classical infinite Laplace transform (ILT) theory violates this fundamental requirement of engineering systems. In [12] presents a modeling, analysis, and design approach for linear time invariant systems using the theory of the finite Laplace transform (FLT). The study developed in this paper will help in modeling, analyzing, and designing approach for linear time invariant systems in finite-generalized-Laplace-Hankel-Clifford transform which not only helps in solving problems in finite range as in single integral, but in finite area with two parameters at a time.

Throughout the work the notation and terminology of [8] and [9] will be used.

II. PRELIMINARY RESULTS

Let $c,\alpha-\beta$ satisfy $\alpha-\beta\geq-1/2, c\geq1/2$. Call a function '$f$' for FGLHCT, as if it belongs to $LH'(w,z,c,\alpha,\beta)$ for some real number w,z. Let σ_f and ρ_f defined as follows:

$$\sigma_f=\inf\{w/f\in LH'(w,z,c,\alpha,\beta)\}$$
$$\rho_f=\sup\{z/f\in LH'(w,z,c,\alpha,\beta)\}.$$

Now define the FGLHCT by $L\hbar_{\alpha,\beta}$. For given FGLHCT function f, let Δ_f, denote the strip $\{s/\sigma_f<\mathrm{Re}(s)<\rho_f\}$ and let $\{\lambda_m\}$ be the positive zeros of $J_{\alpha,\beta}(z)$ arranged in ascending order. Then, the FGLHCT $F(s,\lambda_m)$ of f is defined as the application of f to the kernel $e^{-sx}y^{-\alpha-\beta}J_{\alpha,\beta}(\lambda_m y)$ i.e.,

$$L\hbar_{\alpha,\beta}(f)(s,\lambda_m)=F(s,\lambda_m)=\langle f(x,y),e^{-sx}y^{-\alpha-\beta}J_{\alpha,\beta}(\lambda_m y)\rangle \quad (2.1)$$

where $s\in\Omega_f$ and $\{\lambda_m\}$ are the positive zeros of $J_{\alpha,\beta}(z)$.

For any $s\in\Omega_f$, and all λ_m, the right-hand-side of (2.1) has meaning as the application of $f\in LH'(\sigma_f,\rho_f,c,\alpha,\beta)$ to $e^{-sx}y^{-\alpha-\beta}J_{\alpha,\beta}(\lambda_m y)\in LH(\sigma_f,\rho_f,c,\alpha,\beta)$ (or equivalently) the application of $f\in LH'_{a,b,c,\alpha,\beta}$ to $e^{-sx}y^{-\alpha-\beta}J_{\alpha,\beta}(\lambda_m y)\in LH_{a,b,c,\alpha,\beta}$ for any tip, $\sigma_f<a\leq\mathrm{Re}(s)\leq b<\rho_f$.

If $f(x,y)$ is a locally integrable function such that $e^{-sx}y^{-\alpha-\beta}J_{\alpha,\beta}(\lambda_m y)$ is absolutely integrable $\sigma_f<a<b<\rho_f$, then its FGLHCT $\int_0^1\int_{-\infty}^{\infty}y^{-\alpha-\beta}e^{-sx}J_{\alpha,\beta}(\lambda_m y)f(x,y)dxdy$ exists for at least one $s\in\Delta_f$ and for all λ_m, where $\{\lambda_m\}$ are positive zeros of $J_{\alpha,\beta}(z)$ and can be identified with FGLHCT (2.1). The analyticity theorem for the FGLHCT is established.

Theorem 2.1 (The Analyticity theorem)

If $L\hbar_{\alpha,\beta}(f)(s,\lambda_m)=F(s,\lambda_m)$ for $s\in\Delta_f$ such that $\{s/\sigma_f<\mathrm{Re}(s)<\rho_f\}$ and $\{\lambda_m\}$ are the positive zeros of $J_{\alpha,\beta}(z)$, $F(s,\lambda_m)$ is analytic in S for some fixed λ_m and

$$D_s[F(s,\lambda_m)]=\langle f(x,y),-xe^{-sx}y^{-\alpha-\beta}J_{\alpha,\beta}(\lambda_m y)\rangle \quad (2.2)$$

Proof. Let s be an arbitrary fixed point in $\Delta_f=\{s/\sigma_f<\mathrm{Re}(s)<\rho_f\}$ and $\{\lambda_m\}$ are the positive zeros of $J_{\alpha,\beta}(z)$ in [4]. Choose the positive real numbers a and r such that

$\sigma_f < a \le \operatorname{Re}(s) - r \le \operatorname{Re}(s) + r \le b < \rho_f$. Also Δs be the complex increment such that $|\Delta s| < r$. For $\Delta s \ne 0$, invoke the definition of $F(s, \lambda_m)$ to write

$$\frac{F(s+\Delta s, \lambda_m) - F(s, \lambda_m)}{\Delta s} - \left\langle f(x,y), \frac{\partial}{\partial s}\left(e^{-sx} y^{-\alpha-\beta} \mathrm{J}_{\alpha,\beta}(\lambda_m y)\right)\right\rangle = \left\langle f(x,y), \Psi_{\Delta s}(x,y)\right\rangle \quad (2.3)$$

where

$$\Psi_{\Delta s}(x,y) = \left\{\frac{e^{-(s+\Delta s)x} - e^{-sx}}{\Delta s} - \frac{\partial}{\partial s}\left(e^{-sx}\right)\right\} y^{-\alpha-\beta} \mathrm{J}_{\alpha,\beta}(\lambda_m y),$$

since $\Psi_{\Delta s}(x,y) \in LH_{a,b,c,\alpha,\beta}(I)$, so that (2.3) has sense. As $|\Delta s| \to 0$, then $\Psi_{\Delta s}(x,y)$ converges in $LH_{a,b,c,\alpha,\beta}(I)$ to zero. Because $f(x,y) \in LH'_{a,b,c,\alpha,\beta}(I)$, it implies that $\left\langle f(x,y), \Psi_{\Delta s}(x,y)\right\rangle \to 0$ as $|\Delta s| \to 0.$

To proceed, let $\mathbb{C}$ be the circle in the $s-$plane with centre at s and radius r, where $0 < r < r_1 < \min(\operatorname{Re}(s) - a, b - \operatorname{Re}(s))$. In the view of analyticity of e^{-sx} and invoking Cauchy's integral formula, $\Psi_{\Delta s}(x,y)$ is as follows:

$$\Psi_{\Delta s}(x,y) = y^{-\alpha-\beta} \mathrm{J}_{\alpha,\beta}(\lambda_m y) \frac{\Delta s}{2\pi i} \int_{\mathbb{C}} \frac{e^{-zx}}{(z-s)^2 (z-s-\Delta s)} dz.$$

Hence,

$$k_{a,b}(x) y^c D_x^k \Delta_{\alpha,y}^{k'}\left[y^{\alpha+\beta} \Psi_{\Delta s}(x,y)\right] = (-1)^{k+k'} \lambda_m^{2k'} y^c \mathrm{J}_{\alpha,\beta}(\lambda_m y) \frac{\Delta s}{2\pi i} \int_{\mathbb{C}} \frac{z^k k_{a,b}(x) e^{-zx}}{(z-s)^2 (z-s-\Delta s)} dz$$

Implies

$$\left|k_{a,b}(x) y^c D_x^k \Delta_{\alpha,y}^{k'}\left[y^{\alpha+\beta} \Psi_{\Delta s}(x,y)\right]\right| = \left|(-1)^{k+k'} \lambda_m^{2k'} y^c \mathrm{J}_{\alpha,\beta}(\lambda_m y) \frac{\Delta s}{2\pi i} \int_{\mathbb{C}} \frac{z^k k_{a,b}(x) e^{-zx}}{(z-s)^2 (z-s-\Delta s)} dz\right|$$
$$\le \lambda_m^{2k'} A_{\alpha,\beta} \left|\frac{\Delta s}{2\pi}\right| \left|\int_{\mathbb{C}} \frac{k}{(r_1)^2 (r_1 - r)}\right| |dz| \le \frac{\lambda_m^{2k'+\alpha+\beta} A_{\alpha,\beta} |\Delta s| k}{(r_1)^2 (r_1 - r)}.$$

For all $z \in \mathbb{C}$ and $-\infty < x < \infty$, $\left|z^k k_{a,b}(x) e^{-zx}\right| < k$, where k is a constant independent of z and x and for all $0 < y < 1$, $\alpha - \beta \ge -\frac{1}{2}$, $\left|(\lambda_m y)^{\alpha+\beta} \mathrm{J}_{\alpha,\beta}(\lambda_m y)\right| < A_{\alpha,\beta}$, $A_{\alpha,\beta}$ is also independent of λ_m and y. Therefore as $|\Delta s| \to 0$, $\sup_{(x,y)\in I} \left|k_{a,b}(x) y^c D_x^k \Delta_{\alpha,y}^{k'}\left[y^{\alpha+\beta} \Psi_{\Delta s}(x,y)\right]\right| \to 0$ which means that as $|\Delta s| \to 0$, $P_{a,b,k,k'}^{c,\alpha,\beta}\left[\Psi_{\Delta s}(x,y)\right] \to 0$. This shows that $\Psi_{\Delta s}(x,y)$ converges to zeros in $LH_{a,b,c,\alpha,\beta}(I)$ as $\Delta s \to 0$.

Hence if $f \in LH'_{a,b,c,\alpha,\beta}(I)$, then $\left\langle f(x,y), \Psi_{\Delta s}(x,y)\right\rangle \to 0$ as $\Delta s \to 0$, which is the aim.

The boundedness property of the FGLHCT has been established by proving the theorem.

Theorem 2.2 *Let f be a member of $LH'_{a,b,c,\alpha,\beta}(I)$, $c \ge \frac{1}{2}$, $\alpha - \beta \ge -\frac{1}{2}$, $a \le \operatorname{Re}(s) \le b$, $m = 1,2,3,\ldots$ and $F(s, \lambda_m)$ be defined by*

$$F(s, \lambda_m) = \left\langle f(x,y), e^{-sx} y^{-\alpha-\beta} \mathrm{J}_{\alpha,\beta}(\lambda_m y)\right\rangle \quad (2.4)$$

For $s \in \Delta_f = \left\{s / \sigma_f < \operatorname{Re}(s) < \rho_f\right\}$ and $\{\lambda_m\}$ are the positive zeros of $J_{\alpha,\beta}(z)$. Then $F(s, \lambda_m)$ satisfies the inequality: $\left|F(s, \lambda_m)\right| \le (\lambda_m)^{\alpha+\beta} P(|s| \lambda_m)$, where the polynomial $P(|s| \lambda_m)$ will depend upon the choice of a, b, α and β..

Proof. Since $f \in LH'_{a,b,c,\alpha,\beta}(I)$, then there exists a non-negative integer r' and a positive constant c' such that

$$|F(s,\lambda_m)|$$
$$=\left|\left\langle f(x,y), e^{-sx}y^{-\alpha-\beta}J_{\alpha,\beta}(\lambda_m y)\right\rangle\right|$$
$$\le c' \max_{\substack{0\le k\le r'\\0\le k'\le r'}} \left[P^{c,\alpha,\beta}_{a,b,k,k'}\left[e^{-sx}y^{-\alpha-\beta}J_{\alpha,\beta}(\lambda_m y)\right]\right]$$
$$=c' \max_{\substack{0\le k\le r'\\0\le k'\le r'}} \sup_{(x,y)\in I}\left|k_{a,b}(x)y^c D_x^k\Delta^{k'}_{a,y}\left[e^{-sx}J_{\alpha,\beta}(\lambda_m y)\right]\right|$$
$$=c' \max_{\substack{0\le k\le r'\\0\le k'\le r'}} |s|^k(\lambda_m)^{2k'} \sup_{(x,y)\in I}\left|k_{a,b}(x)y^c e^{-sx}J_{\alpha,\beta}(\lambda_m y)\right|$$
$$=c' \max_{\substack{0\le k\le r'\\0\le k'\le r'}} |s|^k(\lambda_m)^{2k'+\alpha+\beta}\times$$
$$\sup_{(x,y)\in I}\left|k_{a,b}(x)y^{c+\alpha+\beta}(\lambda_m y)^{-\alpha-\beta}e^{-sx}J_{\alpha,\beta}(\lambda_m y)\right|$$

Since $c\ge\frac{1}{2},\alpha-\beta\ge-\frac{1}{2}$, $a\le\operatorname{Re}(s)\le b$, then $\left|k_{a,b}(x)y^{c+\alpha+\beta}(\lambda_m y)^{-\alpha-\beta}e^{-sx}J_{\alpha,\beta}(\lambda_m y)\right|\le A_{\alpha,\beta}$ for all $(x,y)\in I$.

Hence

$$|F(s,\lambda_m)|\le c'A_{\alpha,\beta}\max_{\substack{0\le k\le r'\\0\le k'\le r'}}|s|^k(\lambda_m)^{2k'+\alpha+\beta}$$
$$\le c'A_{\alpha,\beta}(\lambda_m)^{\alpha+\beta}\max_{\substack{0\le k\le r'\\0\le k'\le r'}}|s|^k(\lambda_m)^{2k'}$$
$$\le c'A_{\alpha,\beta}(\lambda_m)^{\alpha+\beta}P(|s|\lambda_m)$$
$$\le(\lambda_m)^{\alpha+\beta}P(|s|\lambda_m)$$

This completes the proof of the theorem 2.2.

For $c\ge\frac{1}{2},\alpha-\beta\ge-\frac{1}{2}$, define a operator $\left[D_x\Delta_{\alpha,\beta,y}\right]^*$ on $LH'_{a,b,c,\alpha,\beta}(I)$ as the adjoint of the operator $\left[y^{-\alpha-\beta}(-D_x)\Delta_{\alpha,\beta,y}y^{\alpha+\beta}\right]$ on $LH_{a,b,c,\alpha,\beta}(I)$. More specifically, for arbitrary $\phi(x,y)\in LH_{a,b,c,\alpha,\beta}(I)$ and $f\in LH'_{a,b,c,\alpha,\beta}(I)$

$$\left\langle\left[D_x\Delta_{\alpha,\beta,y}\right]^* f(x,y),\phi(x,y)\right\rangle = \left\langle f(x,y), y^{-\alpha-\beta}(-D_x)\Delta_{\alpha,\beta,y}\left[y^{\alpha+\beta}\phi(x,y)\right]\right\rangle \tag{3.1}$$

The right hand side of (3.1) has a sense, because $y^{-\alpha-\beta}(-D_x)\Delta_{\alpha,\beta,y}\left[y^{\alpha+\beta}\phi(x,y)\right]$ is a member of $LH_{a,b,c,\alpha,\beta}(I)$ whenever $\phi(x,y)\in LH_{a,b,c,\alpha,\beta}(I)$. Since the mapping $\phi(x,y)\to y^{-\alpha-\beta}(-D_x)\Delta_{\alpha,\beta,y}\left[y^{\alpha+\beta}\phi(x,y)\right]$ is continuous linear on $LH_{a,b,c,\alpha,\beta}(I)$ into itself, then $\left[D_x\Delta_{\alpha,\beta,y}\right]^*$ is also continuous linear mapping on $LH'_{a,b,c,\alpha,\beta}(I)$ into itself. Consequently since mapping is linear continuous on $LH(w,z,c,\alpha,\beta)$ into itself, then $\left[D_x\Delta_{\alpha,\beta,y}\right]^*$ is also continuous linear mapping on $LH'(w,z,c,\alpha,\beta)$ into itself. By induction

$$\left\langle\left[D_x^k\Delta^{k'}_{\alpha,\beta,y}\right]^* f(x,y),\phi(x,y)\right\rangle = \left\langle f(x,y), y^{-\alpha-\beta}(-D_x)^k\Delta^{k'}_{\alpha,\beta,y}\left[y^{\alpha+\beta}\phi(x,y)\right]\right\rangle \tag{3.2}$$

which leads to the following operation transform formula:

$$\left(L\hbar_{\alpha,\beta}\left[D_x^k\Delta^{k'}_{\alpha,\beta,y}\right]^* f\right)=(-1)^{k'}\lambda_m^{2k'}s^k\left(L\hbar_{\alpha,\beta}f\right)(s,\lambda_m)$$

for $s\in\Delta_f=\{s/\sigma_f<\operatorname{Re}(s)<\rho_f\}$ and $\{\lambda_m\}$ are the positive zeros of $J_{\alpha,\beta}(z)$. Indeed, $f\in LH'(w,z,c,\alpha,\beta)$ and $e^{-sx}y^{-\alpha-\beta}J_{\alpha,\beta}(\lambda_m y)\in LH(w,z,c,\alpha,\beta)$.

$$\begin{aligned}&\left\langle\left[D_x^k\Delta^{k'}_{\alpha,\beta,y}\right]^* f(x,y),e^{-sx}y^{-\alpha-\beta}J_{\alpha,\beta}(\lambda_m y)\right\rangle\\&=\left\langle f(x,y),y^{-\alpha-\beta}(-D_x)^k\Delta^{k'}_{\alpha,\beta,y}\left[e^{-sx}J_{\alpha,\beta}(\lambda_m y)\right]\right\rangle\\&=(-1)^{k'}\lambda_m^{2k'}s^k\left\langle f(x,y),e^{-sx}y^{-\alpha-\beta}J_{\alpha,\beta}(\lambda_m y)\right\rangle\\&=(-1)^{k'}\lambda_m^{2k'}s^k\left(L\hbar_{\alpha,\beta}f\right)(s,\lambda_m)\end{aligned} \tag{3.3}$$

when $\left(L\hbar_{\alpha,\beta}f\right)(s,\lambda_m)=\left\langle f(x,y),e^{-sx}y^{-\alpha-\beta}J_{\alpha,\beta}(\lambda_m y)\right\rangle$.

The formula (3.3) represents the property of FGLHCT which makes it useful as an operational tool for solving differential equations.

Theorem 3.1 *If f is regular distribution in $LH'_{a,b,c,\alpha,\beta}(I)$ generated by elements of $D(I)$, then*

$$\left[D_x\Delta_{\alpha,\beta,y}\right]^* f(x,y)=\left[D_x\Delta_{\alpha,\beta,y}\right]f(x,y) \tag{3.4}$$

Proof. For any $f \in LH'_{a,b,c,\alpha,\beta}(I)$ and $\phi \in D(I)$,

$$\left\langle \left[D_x\Delta_{\alpha,\beta,y}\right]^* f(x,y),\phi(x,y)\right\rangle$$
$$=\left\langle f(x,y), y^{1-\alpha-\beta}(-D_x)\Delta_{\alpha,\beta,y}y^{-1+\alpha+\beta}\phi(x,y)\right\rangle$$
$$=\int_0^1\int_{-\infty}^{\infty} f(x,y)y^{1-\alpha-\beta}(-D_x)\Delta_{\alpha,\beta,y}y^{-1+\alpha+\beta}\phi(x,y)\,dxdy$$
$$-\left[\begin{array}{l}\int_0^1\left[\left[y^{1-\alpha-\beta}f(x,y)\right]\Delta_{\alpha,\beta,y}\left[y^{-1+\alpha+\beta}\phi(x,y)\right]\right]_{-\infty}^{\infty}dy\\ -\int_0^1\int_{-\infty}^{\infty}D_x\left[y^{1-\alpha-\beta}f(x,y)\right]\Delta_{\alpha,\beta,y}\left[y^{-1+\alpha+\beta}\phi(x,y)\right]dxdy\end{array}\right].$$

Since $f(x,y)$ has a compact support then first term vanishes and hence
$$\left\langle f(x,y), y^{1-\alpha-\beta}(-D_x)\Delta_{\alpha,\beta,y}y^{-1+\alpha+\beta}\phi(x,y)\right\rangle$$
$$=\int_0^1\int_{-\infty}^{\infty}D_x\left[y^{1-\alpha-\beta}f(x,y)\right]\Delta_{\alpha,\beta,y}\left[y^{-1+\alpha+\beta}\phi(x,y)\right]dxdy$$

Now,

$$\int_0^1\int_{-\infty}^{\infty}D_x\left[y^{1-\alpha-\beta}f(x,y)\right]\Delta_{\alpha,\beta,y}\left[y^{-1+\alpha+\beta}\phi(x,y)\right]dxdy$$
$$=\int_0^1\int_{-\infty}^{\infty}yD_x\left[y^{-\alpha-\beta}f(x,y)\right]\times$$
$$\left[D_y^{\,2}+\frac{(1-\alpha-\beta)}{y}D_y+\frac{\alpha\beta}{y^2}\right]\left[y^{-1+\alpha+\beta}\phi(x,y)\right]dxdy$$

First on evaluating the integral,
$$\int_0^1\int_{-\infty}^{\infty}yD_x\left[y^{-\alpha-\beta}f(x,y)\right]D_y^{\,2}\left[y^{-1+\alpha+\beta}\phi(x,y)\right]dxdy.$$

Integrating the integral,

$$\int_0^1\int_{-\infty}^{\infty}yD_x\left[y^{-\alpha-\beta}f(x,y)\right]D_y^{\,2}\left[y^{-1+\alpha+\beta}\phi(x,y)\right]dxdy$$
$$=\left[\begin{array}{l}\left[\int_{-\infty}^{\infty}yD_x\left[y^{-\alpha-\beta}f(x,y)\right]D_y\left[y^{-1+\alpha+\beta}\phi(x,y)\right]dx\right]_0^1\\ -\int_0^1\int_{-\infty}^{\infty}D_y\left[yD_xy^{-\alpha-\beta}f(x,y)\right]D_y\left[y^{-1+\alpha+\beta}\phi(x,y)\right]dxdy\end{array}\right]$$

Since $f(x,y)$ has a compact support, the first term vanishes:

$$\int_0^1\int_{-\infty}^{\infty}yD_x\left[y^{-\alpha-\beta}f(x,y)\right]D_y^{\,2}\left[y^{-1+\alpha+\beta}\phi(x,y)\right]dxdy$$
$$=-\int_0^1\int_{-\infty}^{\infty}D_x\left[y^{-\alpha-\beta}f(x,y)\right]D_y\left[y^{-1+\alpha+\beta}\phi(x,y)\right]dxdy \quad (3.5)$$

Now

$$-\int_0^1\int_{-\infty}^{\infty}yD_yD_x\left[y^{-\alpha-\beta}f(x,y)\right]D_y\left[y^{-1+\alpha+\beta}\phi(x,y)\right]dxdy$$
$$=-\left[\int_{-\infty}^{\infty}yD_y\left[D_xy^{-\alpha-\beta}f(x,y)\right]\left[y^{-1+\alpha+\beta}\phi(x,y)\right]dx\right]_0^1$$
$$+\int_0^1\int_{-\infty}^{\infty}D_y\left[yD_yD_xy^{-\alpha-\beta}f(x,y)\right]\left[y^{-1+\alpha+\beta}\phi(x,y)\right]dxdy.$$

Since $f(x,y)$ has a compact support, the first term vanishes at limit points hence,

$$-\int_0^1\int_{-\infty}^{\infty}yD_yD_x\left[y^{-\alpha-\beta}f(x,y)\right]D_y\left[y^{-1+\alpha+\beta}\phi(x,y)\right]dxdy$$
$$=\int_0^1\int_{-\infty}^{\infty}yD_y^{\,2}D_x\left[y^{-\alpha-\beta}f(x,y)\right]y^{-1+\alpha+\beta}\phi(x,y)\,dxdy$$
$$+\int_0^1\int_{-\infty}^{\infty}(1-\alpha-\beta)D_y\left[D_xy^{-\alpha-\beta}f(x,y)\right]\left[y^{-1+\alpha+\beta}\phi(x,y)\right]dxdy \quad (3.6)$$

From (3.5) and (3.6).

$$\int_0^1\int_{-\infty}^{\infty}yD_x\left[y^{-\alpha-\beta}f(x,y)\right]D_y^{\,2}\left[y^{-1+\alpha+\beta}\phi(x,y)\right]dxdy$$
$$=\int_0^1\int_{-\infty}^{\infty}D_x\left[f(x,y)\right]\left[D_y^{\,2}+\frac{(1-\alpha-\beta)}{y}D_y\right]\phi(x,y)\,dxdy$$
$$-\int_0^1\int_{-\infty}^{\infty}D_x\left[y^{-\alpha-\beta}f(x,y)\right]\frac{(1-\alpha-\beta)}{y}D_y\left[y^{-1+\alpha+\beta}\phi(x,y)\right]dxdy$$

Hence

$$\left\langle \left[D_x\Delta_{\alpha,\beta,y}\right]^* f(x,y),\phi(x,y)\right\rangle$$
$$=\int_0^1\int_{-\infty}^{\infty}yD_x\left[y^{-\alpha-\beta}f(x,y)\right]\times$$
$$\left[D_y^{\,2}+\frac{(1-\alpha-\beta)}{y}D_y+\frac{\alpha\beta}{y^2}\right]\left[y^{-1+\alpha+\beta}\phi(x,y)\right]dxdy \quad (3.7)$$

$$=\int_0^1\int_{-\infty}^{\infty} D_x\left[y^{-\alpha-\beta}f(x,y)\right]$$
$$\times\left[D_y^{\,2}+\frac{(1-\alpha-\beta)}{y}D_y+\frac{\alpha\beta}{y^2}\right]\left[y^{\alpha+\beta}\phi(x,y)\right]dxdy$$
$$=\int_0^1\int_{-\infty}^{\infty} D_x\left[y^{-\alpha-\beta}f(x,y)\right]\Delta_{\alpha,\beta,y}y^{\alpha+\beta}\phi(x,y)dxdy$$
$$=\int_0^1\int_{-\infty}^{\infty} D_x\Delta_{\alpha,\beta,y}\left[y^{-\alpha-\beta}f(x,y)\right]y^{\alpha+\beta}\phi(x,y)dxdy.$$

Since $f\in LH'_{a,b,c,\alpha,\beta}(I)$, then $D_x\Delta_{\alpha,\beta,y}\left[y^{-\alpha-\beta}f(x,y)\right]\in LH'_{a,b,c,\alpha,\beta}(I)$. Hence $D_x\Delta_{\alpha,\beta,y}\left[y^{-\alpha-\beta}f(x,y)\right]$ also generate a regular distribution in $LH_{a,b,c,\alpha,\beta}(I)$. Hence (3.7) can be written as

$$\left\langle\left[D_x\Delta_{\alpha,\beta,y}\right]^* f(x,y),\phi(x,y)\right\rangle=\left\langle D_x\Delta_{\alpha,\beta,y}f(x,y),\phi(x,y)\right\rangle.$$

Hence $\left[D_x\Delta_{\alpha,\beta,y}\right]^* f(x,y)=D_x\Delta_{\alpha,\beta,y}f(x,y)$. The equality (3.4) also holds, if f as regular distribution function and if put some suitable restriction on f so that the limit term should vanish.

Multiplier in $LH_{a,b,c,\alpha,\beta}(I)$:

Let $\boxed{H}$ be a linear space of smooth functions $\theta(x,y)$ defined on I such that for each pair of non-negative integer k,k', there exists a pair of non-negative integers N_k and N'_k for which $\dfrac{D_x^k\Delta_{\alpha,\beta,y}^{k'}\theta(x,y)}{(1+x^2)^{N_k}(1+y^{N'_k})}$ is bounded on I.

If $\alpha-\beta\ge-\frac{1}{2},c\ge\frac{1}{2}$ and $a<b,e<b$, then for $\theta\in\boxed{H}$, the operation $\phi\to\theta$; ϕ is a continuous linear mapping of $LH_{d,e,c,\alpha,\beta}(I)$ into $LH_{a,b,c,\alpha,\beta}(I)$.

Thus,

$$k_{a,b}(x)y^cD_x^k\Delta_{\alpha,\beta,y}^{k'}\left[y^{\alpha+\beta}\theta\phi\right]$$
$$=\sum_{i=0}^{k}\sum_{j=0}^{k'}\binom{k}{i}\binom{k'}{j}$$
$$\times\frac{\left[\begin{array}{l}k_{a,b}(x)\left[D_x^{k-i}\Delta_{\alpha,\beta,y}^{k'-j}(\theta)\right](1+x^2)^{N_k}(1+y^{N'_k})\\ \times y^ck_{d,e}(x)y^cD_x^i\Delta_{\alpha,\beta,y}^j\left[y^{\alpha+\beta}\phi\right]\end{array}\right]}{(1+x^2)^{N_k}(1+y^{N'_k})k_{d,e}(x)}$$

Since $\phi\in LH_{d,e,c,\alpha,\beta}(I)$ and $\theta\in\boxed{H}$, then (3.8) can be written as

$$\sup_{(x,y)\in I}\left|k_{a,b}(x)y^cD_x^k\Delta_{\alpha,\beta,y}^{k'}\left(y^{\alpha+\beta}\theta\phi\right)\right|$$
$$\le\sum_{i=0}^{k}\sum_{j=0}^{k'}\binom{k}{i}\binom{k'}{j}B_{k,k'}\left[P_{d,e,i,j}^{c,\alpha,\beta}(\phi)+P_{d,e,i,j}^{c+N_{k'},\alpha,\beta}(\phi)\right],$$

where $B_{k,k'}$ are constants. This proves that $\phi\to\theta\phi$ is a continuous linear mapping from $LH_{d,e,c,\alpha,\beta}(I)$ into $LH_{a,b,c,\alpha,\beta}(I)$.

It is clear that every choice of w and z and any $\theta\in\boxed{H}$, $\phi\to\theta\phi$ is a linear mapping from $LH(w,z,c,\alpha,\beta)$ itself. To show the continuity, let $\{\phi_m\}_{m=1}^{\infty}$ be any sequence that converge to ϕ in $LH(w,z,c,\alpha,\beta)$. Choose real numbers a and b such that $w<a<d$, $e<b<z$ with $\phi_m\to\phi$ in $LH_{d,e,c,\alpha,\beta}(I)$. Our previous results, show that $\{\theta\phi_m\}$ converges in $LH_{a,b,c,\alpha,\beta}(I)$ and hence in $LH(w,z,c,\alpha,\beta)$. Thus the mapping $\phi\to\theta\phi$ is a continuous linear mapping from $LH(w,z,c,\alpha,\beta)$ into itself.

Problem 1: Let
$L\hbar_{\alpha,\beta}(f)(s,\lambda_m)=F(s,\lambda_m)=\left\langle f(x,y),e^{-sx}y^{-\alpha-\beta}\mathrm{J}_{\alpha,\beta}(\lambda_m y)\right\rangle$

for $s\in\Delta_f=\{s/\sigma_f<\mathrm{Re}(s)<\rho_f\}$ and $\{\lambda_m\}$ are the positive zeros of $\mathrm{J}_{\alpha,\beta}(z)$. Show that $\phi(x,y)\to\phi(-x,y)$ is isomorphism from $LH_{-b,-a,c,\alpha,\beta}(I)$ into $LH_{a,b,c,\alpha,\beta}(I)$. Then for $f\in LH'_{a,b,c,\alpha,\beta}(I)$ define the mapping $f(x,y)\to f(-x,y)$ by $\langle f(x,y),\phi(x,y)\rangle=\langle f(x,y),\phi(-x,y)\rangle$. Finally show that $L\hbar_{\alpha,\beta}f(-x,y)=F(-s,\lambda_m)$.

Solution: First to show that $\phi(x,y)\to\phi(-x,y)$ is a continuous linear mapping from $LH_{-b,-a,c,\alpha,\beta}(I)$ onto $LH_{a,b,c,\alpha,\beta}(I)$.

$$P_{-b,-a,k,k'}^{c,\alpha,\beta}(\phi(x,y))=\sup_{(x,y)\in I}\left|k_{-b,-a}(x)y^cD_x^k\Delta_{\alpha,\beta,y}^{k'}\left(y^{\alpha+\beta}\phi(x,y)\right)\right|$$

where

$$k_{-b,-a}(x)=\begin{cases} e^{-bx};\ 0\le x<\infty \\ e^{-ax};-\infty<x<0 \end{cases}.$$

Now,

$$k_{-b,-a}(x)=\begin{cases} e^{-bx};\ 0\le x<\infty \\ e^{-ax};-\infty<x<0 \end{cases}$$

$$=\begin{cases} e^{-ax};0\le -x<\infty \\ e^{-bx};-\infty\le -x<0 \end{cases}=k_{a,b}(-x)$$

Now,

$$P^{c,\alpha,\beta}_{a,b,k,k'}\left(\phi(-x,y)\right)$$
$$=\sup_{(-x,y)\in I}\left|k_{a,b}(-x)\,y^{c}D_x^k\Delta^{k'}_{\alpha,\beta,y}\left(y^{\alpha+\beta}\phi(x,y)\right)\right|$$
$$=\sup_{(x,y)\in I}\left|k_{-b,-a}(x)\,y^{c}D_x^k\Delta^{k'}_{\alpha,\beta,y}\left(y^{\alpha+\beta}\phi(x,y)\right)\right|$$
$$=P^{c,\alpha,\beta}_{-b,-a,k,k'}\left(\phi(x,y)\right)$$

Hence, the mapping $\phi(x,y)\to\phi(-x,y)$ is a continuous linear mapping from $LH_{-b,-a,c,\alpha,\beta}(I)$ onto $LH_{a,b,c,\alpha,\beta}(I)$. The linearity of mapping is obvious. There exists a unique inverse mapping $\phi(x,y)\to\phi(-x,y)$ from $LH_{-b,-a,c,\alpha,\beta}(I)$ onto $LH_{a,b,c,\alpha,\beta}(I)$ which is also a continuous linear mapping. Hence $\phi(x,y)\to\phi(-x,y)$ isomorphism from $LH_{-b,-a,c,\alpha,\beta}(I)$ onto $LH_{a,b,c,\alpha,\beta}(I)$.

Now for $f\in LH'_{a,b,c,\alpha,\beta}(I)$, define the mapping $f(x,y)\to f(-x,y)$ by

$\left\langle f(x,y),\phi(x,y)\right\rangle=\left\langle f(x,y),\phi(-x,y)\right\rangle$. For $f\in LH'_{a,b,c,\alpha,\beta}(I)$ and
$\phi(x,y)=e^{-sx}y^{-\alpha-\beta}\mathrm{J}_{\alpha,\beta}(\lambda_m y)\in LH_{a,b,c,\alpha,\beta}(I)$.

$$L\hbar_{\alpha,\beta}(f)(s,\lambda_m)=\left\langle f(-x,y),e^{-sx}y^{-\alpha-\beta}\mathrm{J}_{\alpha,\beta}(\lambda_m y)\right\rangle\mathbf{R}$$
$$=\left\langle f(x,y),e^{-(-sx)}y^{-\alpha-\beta}\mathrm{J}_{\alpha,\beta}(\lambda_m y)\right\rangle$$
$$=F(-s,\lambda_m)$$

By applying as in [12], relations of finite-generalized-Laplace-Hankel-Clifford transform with other transforms and can be presented the application in electronics engineering field. Finite-generalized-Laplace-Hankel-Clifford transform is a very effective mathematical tool to simplify very complex problems in the area of stability and control.

IV. An Inversion Formula for a Distributional FGLHCT: The Notation and Terminology.

In notation and terminology it is followed from [2] and [5]. The open set $(-\infty,\infty)\times(0,1)$ will be denoted by Ω. The operators are:

$$D_x^k\Delta^{k'}_{\alpha-\beta,y}=D_x^k\left(D_y^2+\frac{(1-\alpha-\beta)}{y}D_y+\frac{\alpha\beta}{y^2}\right)^{k'},k,k'=0,1,2,3,\ldots \quad (4.1)$$

where it $(\alpha-\beta)\ge-\frac{1}{2}$ and the expression,

$$T_N(y,\tau)=\sum_{m=1}^{n}\frac{\mathrm{J}_{\alpha,\beta}(\lambda_m y)\mathrm{J}_{\alpha,\beta}(\lambda_m\tau)}{\lambda_m\mathrm{J}^2_{\alpha,\beta-1}(\lambda_m)}.$$

V. Conclusion.

The relation of the conventional Laplace transformation is applied over finite-generalized-Laplace-Hankel-Clifford is established. Analyticity and boundedness condition are proved. The Operation transform and an inversion formula for a distributional finite-generalized-Laplace-Hankel-Clifford-transformation is also developed. The study developed in this paper will be useful to extend recently developed applications in Electronics & Communication, Electrical Engineering innovative tools to improve outcomes.

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