Regular number of Middle graph of a graph

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***Abstract*:-** For any $(p\ ,q)$ graph G, the middle graph of a graph G, is denoted by M(G), is a graph whose vertex set is $V(G) \cup E(G)$, and two vertices are adjacent if they are adjacent edges of G or one is a vertex and other is an edge incident with it. The regular number of the M(G) is the minimum number of subsets into which the edge set of M(G) should be partitioned so that the subgraph induced by each subset is regular and is denoted by $r_M(G)$. In this paper some results on regular number of $r_M(G)$ were obtained and expressed in terms of elements of G.

Keywords : Regular number / middle graph / domination number / total domination number / binary tree.

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I. INTRODUCTION :

All graphs considered here are simple, finite, and non-trivial. As usual p and q denote the number of vertices and edges of a graph G and the maximum degree of a vertex in G is denoted by $\Delta(G)$. A vertex v is called a cutvertex if removing it from G increases the number of components of G. A graph G is called trivial if it has no edges. The maximum distance between any two vertices in G is called a diameter and is denoted by diam(G). The path and tree numbers were introduced by Stanton James and Cown in[15]. Any undefined term in this paper may be found in[3].A tree is called a binary tree if it has one vertex of degree 2, and each of the remaining vertices is of degree 1 or 3. The middle graph of G, is defined with the vertex set $V(G) \cup E(G)$ where two vertices are adjacent if and only if they are either adjacent edges of G or one is a vertex and the other is an edge incident with it and it is denoted by M(G). The edge set independence number $\beta_1^*(G)$ is the minimum order of partition of $E(G)$ into subsets so that the subgraph induced by each set must be independent. The independence number $\beta_1(G)$ is the maximum cardinality of an edge independent set in G. Let $G = (\ V\ ,E\ )$ be a graph. A set $D' \subseteq V$ is said to be a dominating set of G, if every vertex in $(\ V - D'\ )$ is adjacent to some vertex in D'. The minimum cardinality of vertices in such a set is called the domination number of G and is denoted by $\gamma(G)$. A dominating set is said to be total dominating set of G, if N$(D') = V$ or equivalently, if for every $v \in V$, there exists a vertex $u \in D'$, $u \neq v$, such that u is adjacent to v. The total domination number of G, denoted by $\gamma_t(G)$ is the minimum cardinality of total dominating set of G. Domination related parameters are now well studied in graph theory. Total domination in graphs was studied by E.J.Cockayne, R.M.Dawes, and S.T.Hedetniemi in[1]. This concept was studied by M.A.Henning in[4] and was studied, for example in[5, 6, 8, 9]. A dominating set D of L(G) is a regular total dominating set (RTDS) if the induced subgraph $< D >$ has no isolated vertices and deg$(v) = 1$, $\forall\ v \in D$. The regular total domination number $\gamma_{rt}(L(G))$ is the minimum cardinality of a regular total dominating set. The regular total domination in line graphs was studied by M.H.Muddebihal, U.A.Panfarosh and Anil.R.Sedamkar in[10]. Total domination and total domination subdivision numbers of graphs were studied by O. Favaron, H. Karami and S. M. Sheikholeslami in[2]. On complementary graphs was studied by E. A. Nordhaus and J. W. Gaddum in[14]. The regular number of graph valued function were studied by M.H.Muddebihal, Abdul Gaffar, and Shabbir Ahmed in[11] and also developed in[7, 12, 13].

II. RESULTS :

The following results are obvious, hence we omit the proof .

Theorem 1 : For any graph G, M(G) is not regular.

Proof : We discuss the regularity of middle graph M(G) of a graph G in the following two cases.

Case 1 : Suppose G is $r(> 1)$ regular. Then every edge of G is adjacent with $r + k$ number of edges, $r = 2, 3, 4, \ldots$; $k = 0, 1, 2, \ldots$.. Since S$(G) \subset$ M(G) and $\forall\ v_i \in$ S(G), $deg_{S(G)}(v_i) > deg_{M(G)}(v_j)$, then M$(G)$ is not regular.

Case 2 : Suppose G is not a regular graph and consider a vertex v in G such that deg$(v) \neq \Delta(G)$. Then in M(G), $deg_G(v) = deg_{M(G)}(v)$. Since G is not a regular graph, then M(G) is also not regular. Hence for any graph G, M(G) is not regular.

Now, we give the exact value of the regular number of a middle graph of a path with $p \geq 4$ vertices.

Theorem 2 : For any path P_p , with $p \geq 4$, then $r_M(P_p) = 3$.

Proof : Let $P_p : e_1 = v_1v_2 , e_2 = v_2v_3 , e_3 = v_3v_4 , \ldots , e_{p-2} = v_{p-2}v_{p-1} , e_{p-1} = v_{p-1}v_p$ be a path. Now, in $M(P_p)$, $V[M(P_p)] = \{ v_1 , v_2 , v_3 , \ldots , v_{p-1} , v_p \} \cup \{ v_1' , v_2' , v_3' , \ldots , v_{p-1}' \}$ and $E[M(P_p)] = \{ e_1' = v_1'v_2' , e_2' = v_2'v_3' , e_3' = v_3'v_4' , \ldots , e_{p-3}' = v_{p-3}'v_{p-2}' , e_{p-2}' = v_{p-2}'v_{p-1}' \}$. Let $F_1 = \{ v_1'v_2v_2' , v_3'v_4v_4' , v_5'v_6v_6' , \ldots , v_{p-3}'v_{p-2}v_{p-2}' \}$ $F_2 = \{ v_2'v_3v_3' , v_4'v_5v_5' , v_6'v_7v_7' , \ldots , v_{p-2}'v_{p-1}v_{p-1}' \}$ and $F_3 = \{ v_1v_1'$ and $v_{p-1}'v_p \}$ be the minimum regular partition of $M(P_p)$.

Hence,

$$r_M(P_p) = |\{ F_1 , F_2 , F_3 \}|$$

$$r_M(P_p) = 3.$$

In the following theorem we establish the regular number of a middle graph of a complete graph.

Theorem 3 **:** For any complete graph K_p with $p \geq 3$, then $r_M(K_p) = \frac{p+1}{2}$; if p is odd.

$$= \left\lceil \frac{p+1}{2} \right\rceil + 1 \quad ; \text{if } p \text{ is even.}$$

Proof **:** Suppose $G = K_p$,with $V[K_p] = \{ v_1 , v_2 , v_3 , \ldots , v_{p-1} , v_p \}$; $E[K_p] = \{e_1 , e_2 , e_3 , \ldots , e_{\frac{p(p-1)}{2}} \}$. Now we consider $e_1 = v_1' , e_2 = v_2' , e_3 = v_3' , \ldots , e_{\frac{p(p-1)}{2}} = v_{\frac{p(p-1)}{2}}'$ be the set of vertices and $V[M(K_p)] = \{ v_1 , v_2 , v_3 , \ldots , v_{p-1} , v_p \} \cup \{ v_1' , v_2' , v_3' , \ldots , v_{\frac{p(p-1)}{2}}' \}$. In K_p , every edge is adjacent to $2(p-2)$ edges. Then clearly in $M(G)$, the vertices $\{ v_1' , v_2' , v_3' , \ldots , v_{\frac{p(p-1)}{2}}' \}$ are adjacent to $2(p-2)$ vertices. Then we consider the following two cases.

Case 1 : If p is odd, then

$F_1 = \{ v_1' , v_2' , v_3' , \ldots , v_{\left[\frac{p(p-1)}{2}-1\right]}' , v_{\frac{p(p-1)}{2}}' \}$ is a single partition such that $\forall\, v_i \in F_1$, $deg_{F_1}(v_i) = 2(p-2)$.

$F_2 = \{ v_1v_1'v_2v_2'v_3v_3'v_4 \ldots v_{p-1}v_{p-1}'v_pv_p'v_1 \}$ is a 2- regular partition. , F_3 , F_4 , $\ldots$,

$F_n = \{ v_1v_{\frac{p(p-1)}{2}}'v_3v_{\left[\frac{p(p-1)}{2}-1\right]}'v_5v_{\left[\frac{p(p-1)}{2}-2\right]}'v_7 \ldots v_{p-3}v_{\left[\frac{p(p-1)}{4}+2\right]}'v_{p-1}v_{\left[\frac{p(p-1)}{4}+1\right]}'v_1\}$.

Thus,

$$r_M(K_p) = |\{ F_1 , F_2 , F_3 , \ldots , F_n \}|$$

$$r_M(K_p) = \frac{p+1}{2}.$$

Case 2 : If p is even, then

$F_1 = \{ v_1' , v_2' , v_3' , \ldots , v_{\left[\frac{p(p-1)}{2}-1\right]}' , v_{\frac{p(p-1)}{2}}' \}$ is a single partition with $2(p-2)$ regular.

$F_2 = \{ v_1v_1'v_2v_2'v_3v_3'v_4 \ldots v_{p-1}v_{p-1}'v_pv_p'v_1 \}$ is a 2-regular partition.

$F_3 = \{ v_1v_{p+1}'v_3v_{p+3}'v_5 \ldots v_{p-1}v_{2p-1}'v_1 , v_2v_{p+2}'v_4v_{p+4}'v_6 \ldots v_pv_{2p}'v_2 \}$

$F_4 = \{v_1v_{2p+1}'v_4v_{2p+2}'v_7 \ldots v_{p-2}v_{3p}'v_1\} , \ldots ,$

$F_{n-1} = \{ v_1 v'_{\frac{p(p-1)}{2}} , v_2 v'_{\left[\frac{p(p-1)}{2}-1\right]} , v_3 v'_{\left[\frac{p(p-1)}{2}-2\right]} , \ldots , v_p v'_{\left[\frac{p(p-1)}{2}-(p-1)\right]} \}$

$F_n = \{ v'_{\frac{p(p-1)}{2}} v_{\left[\frac{p}{2}+1\right]} , v'_{\left[\frac{p(p-1)}{2}-1\right]} v_{\left[\frac{p}{2}+2\right]} , v'_{\left[\frac{p(p-1)}{2}-2\right]} v_{\left[\frac{p}{2}+3\right]} , \ldots , v'_{\left[\frac{p(p-1)}{2}-(p-1)\right]} v_{\left[\frac{p}{2}\right]} \}$.

Thus,

$r_M(K_p) = |\{ F_1 , F_2 , F_3 , \ldots , F_{n-1} , F_n \}|$

$r_M(K_p) = |\{ F_1 , F_2 , F_3 , \ldots , F_{n-1} \}| + 1.$

$r_M(K_p) = \left\lceil \frac{p+1}{2} \right\rceil + 1$.

Next we establish the sharp value for $r_M(G)$ of a cubic graph.

Theorem 4 : For any cubic graph G, with $p \geq 4$, then $r_M(G) = 4$.

Proof : Let $v_1 , v_2 , v_3 , \ldots , v_{p-1} , v_p$ be the vertices of a cubic graph such that $\deg v_i = 3$ for $1 \leq i \leq p$. Let $e_1 , e_2 , e_3 , \ldots , e_{\left(\frac{3p}{2}-1\right)} , e_{\left(\frac{3p}{2}\right)}$ be the edges of a cubic graph such that $e_1 = v_1 v_2 = v'_1$, $e_2 = v_2 v_3 = v'_2$, $e_3 = v_3 v_4 = v'_3$, $\ldots$, $e_{\left(\frac{p}{2}-1\right)} = v_{\left(\frac{p}{2}-1\right)} v_{\left(\frac{p}{2}\right)} = v'_{\left(\frac{p}{2}-1\right)}$, $e_{\left(\frac{p}{2}\right)} = v_{\left(\frac{p}{2}\right)} v_1 = v'_{\left(\frac{p}{2}\right)}$, $e_{\left(\frac{p}{2}+1\right)} = v_{\left(\frac{p}{2}+1\right)} v_{\left(\frac{p}{2}+2\right)} = v'_{\left(\frac{p}{2}+1\right)}$, $e_{\left(\frac{p}{2}+2\right)} = v_{\left(\frac{p}{2}+2\right)} v_{\left(\frac{p}{2}+3\right)} = v'_{\left(\frac{p}{2}+2\right)}$, $e_{\left(\frac{p}{2}+3\right)} = v_{\left(\frac{p}{2}+3\right)} v_{\left(\frac{p}{2}+4\right)} =$

$v'_{\left(\frac{p}{2}+3\right)}$, $\ldots$, $e_{p-1} = v_{p-1} v_p = v'_{p-1}$, $e_p = v_p v_{\left(\frac{p}{2}+1\right)} = v'_p$ and $e_{(p+1)} = v_1 v_{\left(\frac{p}{2}+1\right)} = v'_{p+1}$, $e_{(p+2)} = v_2 v_{\left(\frac{p}{2}+2\right)} = v'_{p+2}$, $e_{(p+3)} = v_3 v_{\left(\frac{p}{2}+3\right)} = v'_{p+3}$, $\ldots$, $e_{\left(\frac{3p}{2}-1\right)} = v_{\left(\frac{p}{2}-1\right)} v_{p-1} = v'_{\left(\frac{3p}{2}-1\right)}$, $e_{\left(\frac{3p}{2}\right)} = v_{\left(\frac{p}{2}\right)} v_p = v'_{\left(\frac{3p}{2}\right)}$.

Now, in M(G), V[M(G)] = $\{v_1 , v_2 , v_3 , \ldots , v_{p-1} , v_p\} \cup \{v'_1 , v'_2 , v'_3 , \ldots , v'_{\left(\frac{3p}{2}-1\right)} , v'_{\left(\frac{3p}{2}\right)}\}$. In any cubic graph G, every edge is adjacent to 4 edges. Then, clearly the degree of each vertex of the vertex set $\{v'_1 , v'_2 , v'_3 , \ldots , v'_{\left(\frac{3p}{2}-1\right)} , v'_{\left(\frac{3p}{2}\right)}\}$ is 4. Further, $F_1 = \{ v'_1 v'_2 v'_3 \ldots v'_{\left(\frac{p}{2}-1\right)} v'_{\left(\frac{p}{2}\right)} v'_{\left(\frac{p}{2}+1\right)} v'_{\left(\frac{p}{2}+2\right)} \ldots v'_{p-1} v'_p v'_{p+1} v'_{p+2} \ldots v'_{\left(\frac{3p}{2}-1\right)} v'_{\left(\frac{3p}{2}\right)} \}$ is a single partition of 4 regular. Now A = $\{ v_1 v'_1 v_2 v'_2 v_3 v'_3 v_4 \ldots v_{\left(\frac{p}{2}-1\right)} v'_{\left(\frac{p}{2}-1\right)} v_{\left(\frac{p}{2}\right)} v'_{\left(\frac{p}{2}\right)} v_1 \}$ and B = $\{ v_{\left(\frac{p}{2}+1\right)} v'_{\left(\frac{p}{2}+1\right)} v_{\left(\frac{p}{2}+2\right)} v'_{\left(\frac{p}{2}+2\right)} v_{\left(\frac{p}{2}+3\right)} \ldots v_{p-1} v'_{p-1} v_p v'_p v_{\left(\frac{p}{2}+1\right)} \}$ be the two sets such that each $<$ A $>$ and $<$ B $>$ are edge disjoint cycles. Thus $F_2 = \{A \cup B\}$. Further $F_3 = \{ v_1 v'_{p+1} , v_2 v'_{p+2} , v_3 v'_{p+3} , \ldots , v_{\left(\frac{p}{2}-1\right)} v'_{\left(\frac{3p}{2}-1\right)} v_{\left(\frac{p}{2}\right)} v'_{\left(\frac{3p}{2}\right)} \}$ and $F_4 = \{ v'_{p+1} v_{\left(\frac{p}{2}+1\right)} , v'_{p+2} v_{\left(\frac{p}{2}+2\right)} , v'_{p+3} v_{\left(\frac{p}{2}+3\right)} , \ldots , v'_{\left(\frac{3p}{2}-1\right)} v_{p-1} , v'_{\left(\frac{3p}{2}\right)} v_p \}$. Let F be the minimum regular partition of the cubic graph, then

$r_M(G) = |F|$

$r_M(G) = 4.$

In the next result we obtain the regular number of a middle graph of a complete bipartite graph.

Theorem 5: For any complete bipartite graph $K_{m,n}$ for $1 \leq m \leq n$ then

$r_M(K_{m,n}) = \frac{n}{m} + 1$; if $n \equiv 0 \ (mod\ m)$

$= \left\lfloor \frac{n}{m} \right\rfloor + m + 1$; if $n \equiv 1 \ (mod\ m)$.

Proof : Let $K_{m,n}$ be a complete bipartite graph with $1 \leq m \leq n$. In M($K_{m,n}$), V[M($K_{m,n}$)]= $V(G) \cup q$, since $V_1 \cup V_2 = V(G)$ then V[M($K_{m,n}$)]= $V_1 \cup V_2 \cup V_3$ where $\forall\, q \in V_3$. Every edge of $K_{m,n}$ is divided by a new vertex. In $K_{m,n}$ there are mn number of edges then, clearly there are mn number of vertices $mn = q \in V_3$. In M($K_{m,n}$) $\forall\, v_i \in V_3$ are adjacent to each other and form $(m + n$ - 2) regular. Now, we consider the following two cases.

Case 1 : If $n \equiv 0\ (mod\ m)$, then clearly $F = \{ F_1 , F_2 , F_3 , \ldots , F_t \}$ is the minimum regular partition of $K_{m,n}$ such that the subgraph induced by each F_i i.e , $< F_i >$ is $K_{m,m}$ for $1 \le i \le t$. Thus,

$$r_M(K_{m,n}) = |F| + 1.$$

$$= t + 1.$$

$$= \frac{nm}{m^2} + 1.$$

$$= \frac{n}{m} + 1.$$

Case 2 : If $n \equiv 1\ (mod\ m)$, then $n - 1 \equiv 0\ (mod\ m)$ and hence,

$$r_M(K_{m,n}) = r_M(K_{m,n-1}) + r_M(K_{m,1}) + 1.$$

$$= \frac{n-1}{m} + m + 1.$$

$$= \left\lfloor \frac{n}{m} \right\rfloor + \text{m} + 1.$$

In the following theorem we determine the exact value of a regular number of a middle graph of a star.

Theorem 6 : For any star $K_{1,n}$, then $r_M(K_{1,n}) = 2$.

Proof : Let $G = K_{1,n}$, E(G) = $\{ e_1 = vv_1 , e_2 = vv_2 , e_3 = vv_3 , \ldots , e_n = vv_n \}$. Further $v_1^{'} , v_2^{'} , v_3^{'} , \ldots , v_n^{'}$ are the vertices divide each edge of E(G) such that $\{ vv_1^{'} , vv_2^{'} , vv_3^{'} , \ldots , vv_n^{'} \}$ gives an induced subgraph $< K_{1+n} >$ in M(G). Now we consider a partition such as $F_1 = \{ vv_1^{'} , v_2^{'} , vv_3^{'} , \ldots , vv_n^{'} \}$ and $F_2 = \{ v_1^{'}v_1 , v_2^{'}v_2 , v_3^{'}v_3 , \ldots , v_n^{'}v_n \}$. Hence $r_{M(G)} = |F_1 , F_2| = 2$.

Next, we develop the result which establishes the relationship between $r_M(G)$ and diam(G).

Theorem 7 : For any non trivial graph G, $r_M(G) \le p -$diam$(G) + 2$.

Proof :Let $P_p : v_1v_2 , v_2v_3 , v_3v_4 , \ldots , v_{p-1}v_p$ be a path on diam(G) + 1. : Let $e_1 = v_1v_2 , e_2 = v_2v_3 , e_3 = v_3v_4 , \ldots , e_{p-2} = v_{p-2}v_{p-1} , e_{p-1} = v_{p-1}v_p$ be the edges of P_p . In M(P_p) , the edges

$\{ e_1 , e_2 , e_3 , \ldots , e_{p-1} \}$ are divided by the new vertex set $\{ v_1^{'} , v_2^{'} , v_3^{'} , \ldots , v_{p-1}^{'} \}$ with $e_i = v_i^{'}$ for $1 \le i \le p-1$ and join these vertices by the new edge set $\{ e_1^{'} , e_2^{'} , e_3^{'} , \ldots , e_{p-3}^{'} , e_{p-2}^{'} \}$ such that $e_1^{'} = v_1^{'}v_2^{'}$, $e_2^{'} = v_2^{'}v_3^{'}$, $e_3^{'} = v_3^{'}v_4^{'} , \ldots , e_{p-3}^{'} = v_{p-3}^{'}v_{p-2}^{'}$, $e_{p-2}^{'} = v_{p-2}^{'}v_{p-1}^{'}$. Then clearly $F_1 = \{ v_1^{'}v_2v_2^{'} , v_3^{'}v_4v_4^{'} , v_5^{'}v_6v_6^{'} , \ldots , v_{p-3}^{'}v_{p-2}v_{p-2}^{'} \}$, $F_2 = \{ v_2^{'}v_3v_3^{'} , v_4^{'}v_5v_5^{'} , v_6^{'}v_7v_7^{'} , \ldots , v_{p-2}^{'}v_{p-1}v_{p-1}^{'} \}$ and $F_3 = \{ v_1v_1^{'} , v_{p-1}^{'}v_p \}$ is the minimum regular partition of M(P_p).

Hence,

$$r_M(G) \le |F|$$

$$r_M(G) \le p - \text{diam}(G) + 2.$$

Now, we prove the following result to prove our next result.

Theorem 8 : For any graph G, $r_M(G) \le q - \beta_1(G) + 2$.

Proof : Let S be a maximum edge independent set in G. Then $E - S$ has at most $|E - S|$ edge independent sets. Thus,

$$r_M(G) \le |E - S| + 2.$$

$$r_M(G) \le q - \beta_1(G) + 2.$$

Now, the following result determines the upper bound on $r_M(G)$.

Theorem 9 : For any graph G, $r_M(G) \leq 2q - p + 2$.

Proof : By Theorem 8, we have

$r_M(G) \leq q - \beta_1(G) + 2$.

Since, $\beta_1(G) \geq \gamma'(G)$.

This implies,

$r_M(G) \leq q - \gamma'(G) + 2$.

Where $\gamma'(G)$ is the edge domination number of G.

Also, $p - q \leq \gamma'(G)$.

Thus,

$r_M(G) \leq q - (p - q) + 2$.

$r_M(G) \leq q - p + q + 2$.

$r_M(G) \leq 2q - p + 2$.

In the next result we obtain Nordhaus-Gaddum type result on $r_M(G)$.

Theorem 10 : For any graph G, $G \neq K_p$, $r_M(G) + r_M(\bar{G}) \leq p(p-3) + 4$.

Proof : By Theorem 9, we have

$r_M(G) \leq 2q - p + 2$.

$r_M(\bar{G}) \leq 2\bar{q} - p + 2$.

$r_M(G) + r_M(\bar{G}) \leq 2(q + \bar{q}) - 2p + 4$.

$r_M(G) + r_M(\bar{G}) \leq 2\binom{p}{2} - 2p + 4$.

$r_M(G) + r_M(\bar{G}) \leq 2\frac{p(p-1)}{2} - 2p + 4$.

$r_M(G) + r_M(\bar{G}) \leq p(p-1) - 2p + 4$.

$r_M(G) + r_M(\bar{G}) \leq p(p-3) + 4$.

Next, we develop the regular number of a middle graph of a wheel.

Theorem 11 : For any wheel W_p , with $p \geq 4$ vertices then

$r_M(W_p) = 3$; if p is odd.

$= 4$; if p is even.

Proof : Let $V_1 = \{ v_1 , v_2 , v_3 , \ldots , v_p \}$ be the vertices of W_p such that $\deg v_i = 3$ for $1 \leq i \leq p - 1$ and $\deg v_p = p - 1$. Let $\{ e_1 , e_2 , e_3 , \ldots , e_{p-1} \}$, $\{ e_1' , e_2' , e_3' , \ldots , e_{p-1}' \}$ be the edges of W_p such that $e_i = v_i v_{i+1}$ for $1 \leq i \leq p - 2$, $e_{p-1} = v_1 v_{p-1}$ and $e_i' = v_i v_p$ for $1 \leq i \leq p - 1$. In M(W_p), the edge set $\{ e_1 , e_2 , e_3 , \ldots , e_{p-1} \}$ and $\{ e_1' , e_2' , e_3' , \ldots , e_{p-1}' \}$ which divides by the new set $V_2 = \{ v_1^{'} , v_2^{'} , v_3^{'} , \ldots , v_{p-1}^{'} \}$ and $V_3 = \{ v_1^{''} , v_2^{''} , v_3^{''} , \ldots , v_{p-1}^{''} \}$ respectively. Then, clearly $V_1 \cup V_2 \cup V_3 \cup v_p \in V_M(W_p)$. Now, we consider the following two cases.

Case 1 : If p is odd, then $p-1$ is even and hence,

$F_1 = \{ v_1^{'}v_2^{''}v_2^{'}v_2 , v_3^{'}v_4^{''}v_4^{'}v_4 , v_5^{'}v_6^{''}v_6^{'}v_6 , \ldots , v_{p-2}^{'}v_{p-1}^{''}v_{p-1}^{'}v_{p-1} \}$ be a 3-regular partitions.

$F_2 = \{ v_2^{'}v_3^{''}v_3^{'}v_3 , v_4^{'}v_5^{''}v_5^{'}v_5 , v_6^{'}v_7^{''}v_7^{'}v_7 , \ldots , v_1^{'}v_1^{''}v_{p-1}^{'}v_1 \}$ be a 3-regular partitions.

$F_3 = \{ v_1^{''}v_2^{''}v_3^{''}v_4^{''} , \ldots , v_{p-2}^{''}v_{p-1}^{''}v_p \}$ is a complete graph and it is $(p-1)$ regular graph.

Let F be the minimum regular partition of M(W_p).

Thus,

$$r_M(W_p) = |F|$$

$$r_M(W_p) = 3.$$

Case 2 : If p is even, then $p-1$ is odd and thus,

$F_1 = \{ v_1^{'}v_2^{''}v_2^{'}v_2 , v_3^{'}v_4^{''}v_4^{'}v_4 , v_5^{'}v_6^{''}v_6^{'}v_6 , \ldots , v_{p-3}^{'}v_{p-2}^{''}v_{p-2}^{'}v_{p-2} \}$ be a 3-regular partitions.

$F_2 = \{ v_2^{'}v_3^{''}v_3^{'}v_3 , v_4^{'}v_5^{''}v_5^{'}v_5 , v_6^{'}v_7^{''}v_7^{'}v_7 , \ldots , v_{p-2}^{'}v_{p-1}^{''}v_{p-1}^{'}v_{p-1} \}$ be a 3-regular partitions.

$F_3 = \{ v_{p-1}^{'}v_1^{'}v_1^{''}v_1 \}$ be a 3-regular partition and

$F_4 = \{ v_1^{''}v_2^{''}v_3^{''}v_4^{''} , \ldots , v_{p-2}^{''}v_{p-1}^{''}v_p \}$ be a complete graph and it is $(p-1)$ regular graph.

Let F be the minimum regular partition of M(W_p).

Hence,

$$r_M(W_p) = |F|$$

$$r_M(W_p) = 4.$$

Next, we establish the result which gives the relationship between $r_M(W_p)$ and $\Delta(W_p)$.

Where $\Delta(W_p)$ is the maximum degree of W_p .

Corollary 12 : For any wheel W_p , with $p \geq 5$ vertices then

$$r_M(W_p) \leq \Delta(W_p) - 1. \text{ Where } \Delta(W_p) \text{ is the maximum degree of } W_p .$$

Proof : By Theorem 11, if p is odd, then $r_M(W_p) = 3$ and if p is even, then $r_M(W_p) = 4$. Clearly, if $p = 5$, then $\Delta(W_p) = 4$ and By Theorem 11, we have $r_M(W_p) = 3$.

Hence,

$$r_M(W_p) = \Delta(W_p) - 1.$$

Similarly, if $p = 6$, then $\Delta(W_p) = 5$ and By Theorem 11, we have

$$r_M(W_p) = 4.$$

Thus,

$$r_M(W_p) = \Delta(W_p) - 1.$$

In succession if $p = 7$, then $\Delta(W_p) = 6$ and By Theorem 11,

$$r_M(W_p) = 3.$$

Hence,

$$r_M(W_p) \leq \Delta(W_p) - 1.$$

In the following theorem we establish the regular number of a middle graph of a binary tree.

Theorem 13 : For any non trivial binary tree T,

$$r_M(T) = 2 \;\; ; \; if \; p = 3.$$

$$= 3 \;\; ; \; if \; p = 5 \text{ and } p = 7.$$

$$= 4 \;\; ; \; if \; p \geq 9.$$

Proof : Let T be a non trivial binary tree. Then, we consider the following three cases.

Case 1: If $p = 3$, let v_1 , v_2 , v_3 be the vertices of T, such that $\deg(v_1) = \deg(v_3) = 1$ and $\deg(v_2) = 2$. Let $e_1 = v_1v_2 \; and \; e_2 = v_2v_3$ be the edges of T. In M(T), the edges e_1 and e_2 which divides by the new vertex set $\{ v_1' , v_2' \}$ and join these vertices by the new edge. Let $F_1 = \{ v_1'v_2v_2' \}$ and $F_2 = \{v_1v_1' \; and \; v_2'v_3 \}$ be the minimum regular partition.

Hence,

$$r_M(T) = |\{ F_1 , F_2 \}|$$

$$= 2.$$

Case 2 : If $p = 5$, let v_1 , v_2 , v_3 , v_4 , v_5 be the vertices of T, such that $\deg(v_1) = 2$, $\deg(v_2) = 3$ and $\deg(v_3) = \deg(v_4) = \deg(v_5) = 1$. Let $e_1 = v_1v_2$, $e_2 = v_1v_3$, $e_3 = v_2v_4$ and $e_4 = v_2v_5$ be the edges of T. In M(T), the edge set $\{ e_1 , e_2 , e_3 , e_4 \}$ which divides by the new vertex set $\{ v_1' , v_2' , v_3' , v_4' \}$and join these vertices by the new edges. Let $F_1 = \{ v_1v_1'v_2' \}$, $F_2 = \{ v_1'v_3'v_4'v_2 \}$ and $F_3 = \{ v_2'v_3 , v_4'v_5$ and $v_3'v_4 \}$ be the minimum regular partition.

Thus,

$$r_M(T) = |\{ F_1 , F_2 , F_3 \}|$$

$$r_M(T) = 3.$$

If $p = 7$, let v_1 , v_2 , v_3 , v_4 , v_5 , v_6 , v_7 be the vertices of T, such that $\deg(v_1) = 2$, $\deg(v_2) = \deg(v_3) = 3$ and $\deg(v_4) = \deg(v_5) = \deg(v_6) = \deg(v_7) = 1$. Let $e_1 = v_1v_2$, $e_2 = v_1v_3$, $e_3 = v_2v_4$, $e_4 = v_2v_5$, $e_5 = v_1v_2$, and $e_6 = v_3v_7$ be the edges of T. In M(T), the edge set $\{ e_1 , e_2 , e_3 , e_4 , e_5 , e_6 \}$ which divides by the new vertex set $\{ v_1' , v_2' , v_3' , v_4' , v_5' , v_6' \}$ and join these vertices by the new edges. Let $F_1 = \{ v_1v_1'v_2' \}$, $F_2 = \{ v_1'v_3'v_4'v_2 \; and \; v_2'v_5'v_6'v_3 \}$ and $F_3 = \{ v_3'v_4 , v_4'v_5 , v_5'v_6$ and $v_6'v_7 \}$ be the minimum regular partition.

Hence,

$$r_M(T) = |\{ F_1 , F_2 , F_3 \}|$$

$$r_M(T) = 3.$$

Case 3 : If $p = 9$, let $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9$ be the vertices of T, such that $\deg(v_1) = 2$, $\deg(v_2) = \deg(v_3) = \deg(v_4) = 3$ and $\deg(v_5) = \deg(v_6) = \deg(v_7) = \deg(v_8) = \deg(v_9) = 1$. Let $e_1 = v_1v_2$, $e_2 = v_1v_3$, $e_3 = v_2v_4$, $e_4 = v_2v_5$, $e_5 = v_3v_6$, $e_6 = v_3v_7$, $e_7 = v_4v_8$ and $e_8 = v_4v_9$ be the edges of T. In M(T), the edge set $\{ e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8 \}$ divided by the new vertex set $\{ v_1', v_2', v_3', v_4', v_5', v_6', v_7', v_8' \}$ respectively and join these vertices by the new edges to the adjacent vertices. Let $F_1 = \{ v_1v_1'v_2' \}$, $F_2 = \{ v_1'v_3'v_4'v_2$ and $v_2'v_5'v_6'v_3 \}$ $F_3 = \{ v_3'v_7'v_8'v_4 \}$ and $F_4 = \{ v_7'v_8, v_8'v_9, v_4'v_5, v_5'v_6$ and $v_6'v_7 \}$ be the minimum regular partition.

Thus,

$$r_M(T) = |\{ F_1, F_2, F_3, F_4 \}|$$

$$r_M(T) = 4.$$

If $p > 9$. In T, $\Delta(T) = 3$, and there exists only one vertex of degree 2 let v be a vertex with $\deg(v) = 2$. $N(v) = v_1', v_2'$ and in M(T) which forms a block of 2-regular. Clearly, those vertices of degree 3 which forms the blocks of 3-regular which are adjacent to each other. Hence these adjacent blocks belongs to either F_2 or F_3 and the remaining edges which are of 1-regular can be partitioned in F_4 only. Hence, in general for $p \geq 9$, we have $r_M(T) = 4$.

Now, we give the exact value of $r_M(T)$.

Where T is a non-trivial tree with n-cut vertices with same degree and $n \geq 2$.

Theorem 14 : For any non-trivial tree T, with n-cut vertices with same degree and $n \geq 2$, then $r_M(T) = 3$.

Proof : For any tree T, $V_1 = \{ v_1, v_2, v_3, \ldots, v_n \}$ be the subset of V(T) be the set of all non-end vertices and the degree of each vertex is same. Suppose, $\deg(v_1) = \deg(v_2) = \deg(v_3) = , \ldots, = \deg(v_n) = m$ (say). Then in M(T), $\forall\, v_i \in V_1$ such that $1 \leq i \leq n$, gives n-number of m-regular blocks. Let $v_i \in V_1$, $1 \leq i \leq n$ such that $\{ N(v_i) \cup v_i \} \in F_1$ and $v_j \in V_1$ such that $N(v_j) = v_i$ in G. Further $\{ N(v_j) \cup v_j \} \in F_2$. Hence $V[M(G)] - \{ N(v_i) \cup v_i \} \cup \{ N(v_j) \cup v_j \} \in F_3$. Since, $V[M(G)] = F_1 \cup F_2 \cup F_3$ and each $< N(v_i) \cup v_i >$ and $< N(v_j) \cup v_j >$ is edge disjoint regular subgraph of M(G), then $r_M(T) = |\{ F_1, F_2, F_3 \}| = 3$.

In the following theorem we establishes the relationship between $r_M(T)$ and $\gamma(T)$.

Theorem 15 : For any non-trivial tree T, with n-cut vertices with same degree and $n \geq 3$ and $T \neq P_p$ for $p \leq 6$, then $r_M(T) \leq \gamma(T)$.

Proof : For any tree T, $V_1 = \{ v_1, v_2, v_3, \ldots, v_n \}$ be the subset of V(T) be the set of all non-end vertices and the degree of each vertex is same. Suppose, $\deg(v_1) = \deg(v_2) = \deg(v_3) = , \ldots, = \deg(v_n) = m$ (say). Then in M(T), $\forall\, v_i \in V_1$ such that $1 \leq i \leq n$, gives n-number of m-regular blocks. Suppose $r_M(T) \leq \gamma(T)$. Now we consider $T = P_p$ with $p \leq 6$ vertices. Then for $\gamma(P_p) = 2$. Since $r_M(P_p)$ for $p \leq 6$ is 3. Clearly, $\gamma(P_p) < r_M(P_p)$, a contradiction. Further for any nontrivial tree if $p \leq 6$ and $T \neq P_p$, $\gamma(T) \geq r_M(T)$. Since by Theorem 14, $r_M(T) = 3$, then $r_M(T) \leq \gamma(T)$.

In the next result we developed a relationship between $r_M(T)$ and $\gamma_t(T)$.

Theorem 16 : For any non-trivial tree T , with n-cut vertices with same degree and $n \geq 3$, then $r_M(T) \leq \gamma_t(T)$.

Proof : By Theorem 15, we have

$$r_M(T) \leq \gamma(T).$$

Since, $\gamma(T) \leq \gamma_t(T)$.

This follows,

$$r_M(T) \leq \gamma_t(T).$$

III. CONCLUSION:

We studied the property of our concept by applying to some standard graphs. We also established the regular number of middle graph of some standard graphs by dividing the each edge by a new vertex and joining the new adjacent vertices by the new edges. Also many results established are sharp.

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