Generalized Laplace-Fractional Mellin Transform and its Analytical structure

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Abstract:- Fractional MellinTransform is integral part of mathematical modeling method because of its scale invariance property. Since its success in description of anomalous diffusion, non-integer calculus, both in one and multidimensional space, it has become an important tool in many areas of physics mechanics, chemistry finance and bioengineering.

In this paper Generalization of Laplace-Fractional Mellin transform is presented. Analyticity theorem for the Laplace-Fractional Mellin Transform is also proved.

Keywords:- *Laplace transform, Mellin transform, Fractional Mellin transform, Laplace-Fractional Mellin Transform, Generalized function.*

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1. INTRODUCTION

The investigation of transform is the most effective procedure for solving many problems concerning PDE and functional equations. Fractional Meliln transform play an important role in technology. It is veryuseful in different areas like Signalprocessing, 2D Matching [1], Digital watering [2], Pattern recognition etc. Main advantage of fractional derivative provided an excellent instrument for the description of memory and hereditary properties various materials and processes.

Mellin transformation is one kind of nonlinear transformation , it is a form of a signal representation similar to Fourier transform which had been widely used in signal processing owing to its distinct properties like scale invariance. Mellin transform is implemented as a Fast Mellin transforms [3]. Fractional Mellin transform is the extension of Mellin transform. In visual navigation, algorithms are usually known to be computationally heavy and time consuming [4]. The Fractional Mellin transform can expedite the computing speed and reduce the cost devices, which indicate it will have an amazing vista. Fractional Mellin transform was firstly introduced into the field of image encryption [5].

Mellin transform is closely connected to Laplacetransform. Owing to great efforts of researchers there have been rapid developments on theory of fractional transform and its application, of which Laplace transform is frequently applied. Laplace–Fractional Mellin transform is a relatively new algorithm, it has a great potential in the future in different area of mathematical science.

This paper is organized in the following manner. In section 2 we define Laplace-fractional Mellin transform and testing function space. Section 3 deals with the Definition of distributional generalized Laplace-Fractional Mellin transform. In section 4 discuss Analyticity theorem of Laplace-Fractional Mellin transform. Section 7 concludes the result. Notation and terminology as per zemanion [10].

2. LAPLACE-FRACTIONAL MELLIN TRANSFORM:

2.1 Definition

The Laplace-Fractional Mellin transform with parameter θ of $f(x,y)$ denoted by $LFrMT\{f(t,x)\}$ performs a linear operation, given by the integral transform

$$LFrMT\{f(t,x)\} = F_\theta\{f(t,x)\}(s,u) = F_\theta(s,u) = \int_0^\infty \int_0^\infty f(t,x) K_\theta(t,s,x,u)\, dt\, dx \quad \ldots\ldots\ldots (2.1)$$

where the kernel,

$$K_\theta(t,s,x,u) = e^{-st} x^{\frac{2\pi iu}{\sin\theta}-1} e^{\frac{\pi i}{\tan\theta}(u^2+log^2 x)}$$

$$= x^{\frac{2\pi iu}{\sin\theta}} e^{\frac{\pi i}{\tan\theta}(u^2+log^2 x)-st}, \qquad 0<\theta\le\frac{\pi}{2} \qquad \dots\dots\dots(2.2)$$

2.2 Test function space $LM_{r,b}^{\alpha,\beta}$

An infinitely differentiable complex valued smooth function φ on R^n belongs to $E(R^n)$, if for each compact set $I \subset S_a$, where $S_a = \{x: x \in R^n, |x| \le a, a > 0\}, I \in R^n$. And k be the open sets in $R_+ \times R_+$ such that

$$\gamma_{E,b,l,q}(\varphi) = \sup_{\substack{t\in k\\ x\in I}} \left|e^{bt} D_t^l D_x^q \varphi(t,x)\right| \qquad \dots\dots\dots(2.3)$$

$$< \infty, \qquad l,q = 0,1,2 \dots.$$

The space $LM_{r,b}^{\alpha,\beta}$ are equipped with their natural Hausdoff locally convex topology $\mathcal{T}_{r,b}^{\alpha,\beta}$. This topology is respectively generates by the total families of seminorms $\{\gamma_{E,b,l,q}\}$ given by (2.3).

3. DISTRIBUTIONAL GENERALIZED LAPLACE-FRACTIONAL MELLIN TRANSFORM (LMRT)

Let $LM_{r,b}^{\alpha,\beta*}$ is the dual space of $LM_{r,b}^{\alpha,\beta}$. This space $LM_{r,b}^{\alpha,\beta*}$ consist of continuous linear functionalon $LM_{r,b}^{\alpha,\beta}$. The distributional Laplace-Fractional Mellin transform of $f(t,x) \in E^*(R^n)$ is defined as

$$LFrMT\{f(t,x)\} = F_\theta\{f(t,x)\}(s,u)$$

$$= \langle f(t,x), K_\theta(t,s,x,u)\rangle \qquad \cdots\cdots\cdots(3.1)$$

where the kernel

$$K_\theta(t,s,x,u) = x^{\frac{2\pi iu}{\sin\theta}} e^{\frac{\pi i}{\tan\theta}(u^2+log^2 x)-st} \quad \dots\dots\dots(3.2)$$

For each fixed $t(0<t<\infty)$, $s>0$ and $0<\theta\le\frac{\pi}{2}$, the right hand side of (3.1) has sense as the application of $f(t,x) \in LM_{r,b}^{\alpha,\beta*}$ to $K_\theta(t,s,x,u) \in LM_{r,b}^{\alpha,\beta}$.

4. ANALYTICITY THEOREM

Let $f(t,x) \in LM_{r,b}^{\alpha,\beta}$ and Let its Laplace Fractional Mellin Transform is defined (3.1) then $F_\theta(s,v)$ is analytic on C^n if the $suppf \subset S_a$ where $S_a = \{x: x\in R^n, |x| \le a, a>0\}$. Moreover $F_\theta(s,v)$ is differentiable and $D_s F_\theta(s,v) = < f(t,x), D_s K_\theta(t,s,x,v) >$.

Proof:

Let S be an arbitrary but fixed point in Ω_f. Choose the real +ve number a,b,r such that $\sigma_1 < a < s-r < s+r < b < \sigma_2$.

Also, let Δs be complex increment such that $\theta < |\Delta s| < r$.

Consider

$$\frac{F_\theta(s+\Delta\theta,v) - F_\theta(s,v)}{\Delta s} - \langle f(t,x), \frac{\partial}{\partial s} K_\theta(t,s,x,v)\rangle$$

$$= \langle\ f(t,x), \frac{\partial}{\partial s} K_\theta(t,s,x,v)\rangle$$

$$= \langle f(t,x), \quad K_\theta(t,s+\Delta\theta,x,v) - K_\theta(t,s,x,v)\rangle - \langle f(t,x), \frac{\partial}{\partial s} K_\theta(t,s,x,v)\rangle$$

$$= \langle f(t,x), \quad \left[K_\theta(t,s+\Delta\theta,x,v) - K_\theta(t,s,x,v) - \frac{\partial}{\partial s} K_\theta(t,s,x,v)\right]\rangle$$

$$= \langle f(t,x), \quad \psi_{\Delta s}(t,x)\rangle$$

Where $\psi_{\Delta s} = \left\{K_\theta(t,s+\Delta\theta,xv) - K_\theta(t,s,xv) - \frac{\partial}{\partial s} K_\theta(t,s,x,v)\right\}$

To Prove $\psi_{\Delta s}(t,x) \epsilon LM_{r,b}^{\alpha,\beta}$

We shall show that as $|\Delta s| \to 0$. $\psi_{\Delta s}(t,x)$Converges in $LM_{r,b}^{\alpha,\beta}$ to zero.

Let c be the circle with center at S and radius r_1 where $0 < r < r_1 < \min(s-a, b-s)$

We may interchange differentiation on S with differentiation on t and by using Cauchy integral formula.

$$(-D_t^l)\psi_{\Delta s}(t,x) = \frac{1}{\Delta s}\left(x^{\frac{2\pi i\mu}{\sin\theta}}.e^{\frac{\pi i}{\tan\theta}(v^2+\log^2 x)}\right).\left[(-1)^l(s+\Delta s)^l.e^{-(s+\Delta s)t} - (-1)^l s^l e^{-st}\right] - \frac{\partial}{\partial s}(-s)^l e^{-st}.\left(x^{\frac{2\pi i\mu}{\sin\theta}}.e^{\frac{\pi i}{\tan\theta}(v^2+\log^2 x)}\right)$$

$$= \frac{1}{2\pi i\Delta s}\left(x^{\frac{2\pi i\mu}{\sin\theta}}.e^{\frac{\pi i}{\tan\theta}(v^2+\log^2 x)}\right)\int_c \left[\frac{1}{z-s-\Delta s} - \frac{1}{z-s}\right](-1)_z^l l e^{-zt}\,dz - \frac{1}{2\pi i}\int_c \frac{(-z)^l e^{-zt}\left(x^{\frac{2\pi i\mu}{\sin\theta}}.e^{\frac{\pi i}{\tan\theta}(v^2+\log^2 x)}\right)}{(z-s)^2}dz$$

$$= \frac{1}{2\pi i}\left(x^{\frac{2\pi i\mu}{\sin\theta}}.e^{\frac{\pi i}{\tan\theta}(v^2+\log^2 x)}\right)\int_c \left[\frac{1}{(z-s-\Delta s)(z-s)} - \frac{1}{(z-s)^2}\right](-z)^l e^{-zt}\,dz$$

$$= \frac{1}{2\pi i}\left(x^{\frac{2\pi i\mu}{\sin\theta}}.e^{\frac{\pi i}{\tan\theta}(v^2+\log^2 x)}\right)\int_c \frac{(-z)^l e^{-zt}}{(z-s-\Delta s)(z-s)^2}dz$$

$$\therefore D_t^l D_x^q \psi_{\Delta s}(t,x) = \frac{\Delta s}{2\pi i}\left[\sum_{r=0}^{q}\sum_{\sigma=0}^{r}\binom{q}{r}.p(v).r!\,x^{-q+r}\left(\frac{2\pi i}{\tan\theta}\right)^{r-q}(\log x)^{r-q}C_r(x)\,K_\theta(x,v)\right].\int_c \frac{(-z)^l e^{-zt}}{(z-s-\Delta s)(z-s)^2}dz$$

Where, $P(v)$ is polynomials in v and $c_2(x) = \frac{1}{(r-2b)!}\left(\frac{1-\log x}{2.xlog\ x}\right)^r$

Now for all z $\in$ c and0< t <∞ ,

$$\sup_I \left|K_\theta(t,s,x,v) D_x^q\left(x^{\frac{2\pi i\mu}{\sin\theta}}.e^{\frac{\pi i}{\tan\theta}(v^2+\log^2 x)}\right)\right| \le N$$

for some constant N
Where, N is a constant independent of z and t.
Moreover $|z - s - \Delta s| > r_1 - r > 0$ and $|z - s| = r_1$
$C_1 = \text{Max}\{|z|^{\ell}, z \in c\}$

Consequently

$$\left|K_\theta(t,s,x,v)D_t^l D_x^q \psi_{\Delta s}(t,x)\right|$$

$$= \sup_I \left|K_\theta(t,s,x,v)\frac{\Delta s}{2\pi i}\left[D_x^q\left(x^{\frac{2\pi i\mu}{\sin\theta}}.e^{\frac{\pi i}{\tan\theta}(v^2+\log^2 x)}\right)\right]\int_c \frac{(-z)^l e^{-zt}}{(z-s-\Delta s)(z-s)^2}dz\right|$$

$$\le \sup_I \left|K_\theta(t,s,x,v)\left[D_x^q\left(x^{\frac{2\pi i\mu}{\sin\theta}}.e^{\frac{\pi i}{\tan\theta}(v^2+\log^2 x)}\right)\right]\right|\frac{|\Delta s|}{|2\pi|}\int_c \frac{|z|^l|e^{-zt}|}{|z-s-\Delta s||z-s|^2}|dz|$$

$$\le \frac{|\Delta s|}{2\pi}\int_c \frac{N.C_1}{(r_1-r)r_1^2}|dz|$$

$$\le \frac{|\Delta s|C_2}{2\pi(r_1-r)r_1^2}\int_c dz \qquad ,\{C_2 = NC_1\}$$

$$\le \frac{|\Delta s|C_2}{2\pi(r_1-r)r_1^2}2\pi r_1$$

$$\le \frac{|\Delta s|C_2}{(r_1-r)r_1}$$

∴ The R.H.S. is independent of t and Converges to zero of $|\Delta s| \to 0$

This shows that $\psi_{\Delta s}$ (t, x) converges to zero in $LM_{r,b}^{\alpha,\beta}$ as $|\Delta s| \to 0$

Hence Proved

5. CONCLUSION

This paper presents extension of generalized Laplace-Fractional MellinTransform. Analyticity theorem for the generalized Laplace-Fractional Mellin Transform is proved .

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