

Crack Growth under Random Loading

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Abstract—The basic fatigue crack growth is important consideration in structural design and constrained operations of safety in critical structural component. The basic law used in this study has been modified for analyzing the fracture mechanics of structures under random loading. In any mechanical system the behavior is modeled by differential equations deterministic and stochastic in nature. We give Paris law in random loading and differential equation governing such real time situations.

Keywords— *Fatigue crack growth, Paris Law, Random loading, Stochastic differential equation, Paris Erdogan model.*

I. INTRODUCTION

We give the various models for random fatigue crack growth. We give an elaborate account of the work by Sobczyk. Crack growth model under random loading is dealt at length. Crack growth is given as a differential equation model and as a Markov chain model. Crack propagation is illustrated as a diffusion model. We have found that stochastic modeling given by Keith Ortiz is best suited for fatigue crack growth. Fatigue crack growth has always been much greater than predicted and with the development of new materials whose properties cannot be obtained using empirical methods, the need to predict the remaining lives of pavements and the design of new pavements to withstand heavier traffic loading with new axle and suspension configurations require the use of an analytical method as the traditional empirical ones cannot cope.

It is desirable to formulate probabilistic models for fatigue phenomena that deal with all physical and chemical processes within a considered material which we observe are likely to be responsible for the generation of fatigue. While the existing physical theories are helpful in explaining qualitatively, the nature of fatigue, they do not give a base for treating fatigue problems quantitatively to yield results that are valid at the macroscopic level. In modeling fatigue processes, it seems to be important to relate the random factors and processes provoking fatigue, the mechanism of fatigue crack growth. It deals with the randomization methodology of investigation of fatigue crack growth through various models. It considers the Paris law with modified versions and discusses the fatigue failure or equivalently the propagation of cracks under random loading. The standard Paris Erdogan model is considered and some of the parameters in the governing equation are

randomized for studying crack growth mechanism of materials.

Probabilistic damage models based on Markov chain theory is studied. A basic concept in the model is a duty cycle which is repetitive period of operation on the life of a component during which damage accumulates in a probabilistic manner only on the duty cycle itself and on the value of the damage accumulated at the start of the duty cycle.

Oh considered the growth of fatigue crack as a continuous stochastic process and obtained it as a diffusion process. Diffusion model of crack propagation is studied. It deals with the integrated of the two probabilistic approaches involving randomization of fatigue growth law coefficient.

II. CRACK GROWTH MODEL UNDER RANDOM LOADING

It has been recognized that fatigue failure in materials result from the nucleation and propagation of cracks. However during their course of propagation, the cracks encounter various types of metallurgical structural imperfections so that the rate of propagation is in general varying in time. Randomness of a fatigue process is completely evident if a structure is subjected to time varying random loading.

In modeling fatigue processes it seems to be important to relate the random factors and processes provoking fatigue to the mechanism of fatigue crack growth. This part deals with the randomization methodology of investigation of fatigue crack growth as dealt with by Sobczyk.

This first part considers the Paris law along with its subsequent modifications. The next part discusses Sobczyk's modified crack growth equation for random loading and its implications. The last part considers the variation of a crack growth equation taking into account the randomness of few other factors.

A. Paris Law

In general, the fatigue crack growth can be expressed as

$$dL = F(L, S, C, T, s; t)dN \dots\dots (1)$$

Where S, C and T are the qualities indicated above and s denotes all other parameters important in fatigue problems (chemical properties, internal stresses etc.). L denotes the length of a dominant crack and N denotes the number of cycles corresponding to the crack length L. But the information about the influence of each of these parameter are insufficient.

Recent studies of fatigue crack growth in elastic materials have shown that the stress intensity factor K is the primary quantity for characterizing the fatigue growth rate.

From Paris Law, we have

$$\frac{dL}{dN} = C (\Delta K)^n \dots\dots\dots (2)$$

Where $\Delta K = K_{max} - K_{min}$ and $K = S\sqrt{\pi L}$. Also C and n are assumed constants for a given material. The above Paris law (2) has essential deficiencies as the constants n and c depends on many factors and their numerical values deviate greatly from experiment to experiment.

As the stress ratio $R = S_{min} / S_{max}$ was recognised to have sufficient influence on fatigue crack growth (2) can be replaced by

$$\frac{dL}{dN} = F(R, \Delta K) \dots\dots\dots (3)$$

The influence of R on fatigue crack length depends on the structure and mechanical properties of the material considered. The experiments show that the effect of mean stress under the tensile stresses results on increasing of fatigue crack growth rate. Thus (3), can be expressed as

$$\frac{dL}{dN} = Cg(R)(\Delta K)^n \dots\dots\dots (4)$$

Brock and Schifne (1963) generalized (4) in the form

$$\frac{dL}{dN} = C \left[\frac{1}{1-R} \right]^2 (\Delta K)^3 \dots\dots\dots (5)$$

Newmen et al.(1972) suggested the modification

$$\frac{dL}{dN} = C(1+\alpha R)^q (\Delta K)^q \dots\dots\dots (6)$$

These modifications were mainly based on experimental data pertaining to specific materials.

These laws are based on fixed stress level fatigue experiments (constant amplitude cyclic loadings). Bell and Wolfmann discussed the Eller's crack closure model based on effective stress range concept $K_{eff} = [S_{max} - S_c]\sqrt{\pi L}$ where S_c is the crack closure stress. Since $S_{max} - S_c = S_{max}(1 - C_f)$ where C_f is the closure factor, the proposed fatigue crack growth equation is

$$\begin{aligned} \frac{dL}{dN} &= A \left[\frac{\Delta S}{1-R} (1 - C_f)\sqrt{\pi L} \right]^n \\ &= \left(\frac{1-C_f}{1-R} \Delta k \right)^n \dots\dots\dots (7) \end{aligned}$$

The analysis of experimental data, revealed that C_f can be expressed as

$$C_f = a + b(1 + R)^q \dots\dots\dots (8)$$

$$a = C_f$$

$$b = C_{fo} - C_{f-1}$$

Where C_{fo} and C_{f-1} are the values of C_f at $R = 0$ and $R = -1$ respectively and q is a positive constant.

Since (7) is of the form (4), we can use (4) as a base for further analysis.

Consider a situation when the structural element is subjected to time varying random loading.

We assumed that, the material considered is homogeneous, linearly elastic for which the stress intensity factor is expressible in terms of the basic parameters of the material and applied stress. While there may be a number of cracks in a specimen it is assumed that the final damage is due to the growth of the dominant crack. The configuration of a dominant crack depends on one quantity – the length L (t). Randomness in crack evolution is caused by random applied stress and some other uncertainties relevant to fatigue.

Also,

Let $S(t)$ be a stochastic process characterizing the random applied stress and $M_s(t)r_s(t_1, t_2)$ denote respectively the mean and autocorrelation function for the

process. If $S(t)$ is a stationary process then $M_s(t) = M_s$, a constant which is independent of time.

$$r_s(t_1, t_2) = r_s(t_2 - t_1) \text{ and}$$

$$S_{rms}(t) = S_o = \text{Constant}$$

Recent studies strongly suggest that ΔK should be replaced by the root mean square K_{rms} of the stress intensity factor.

$$\text{i.e. } K_{rms} = S_{rms} \sqrt{\pi L} \dots\dots\dots (9)$$

The mean stress effect is modified by the new ratio.

$$Q = \frac{\langle S \rangle}{S_{rms}} \dots\dots\dots (10)$$

The further modification is concerned with the cycle which plays a conventional role in fatigue analysis. In the case of random we replace N by $\eta(t)$ where the relation between number of cycles N and time 't' is random and $\eta(t)$ is a point stochastic process. Assuming that $\eta(t)$ characterizes a number of local maxima of a stress process in the interval $[t_0, \tau]$ the quantity $n(t)$ is defined as

$$n(t) = \int_{t_0}^T n(s) ds$$

$$\text{Also } \langle n(t) \rangle = \int_{t_0}^T \mu(s) ds,$$

$$\mu(t) = \langle n(t) \rangle \dots\dots\dots (11)$$

So $\mu(t)$ describes the average number of cycles of a stress process per unit time.

Thus the relation between the cyclic and temporal description can be put forth as

$$\frac{dL}{dN} = \mu(t) \frac{dL}{dN} \dots\dots\dots (12)$$

If the process $S(t)$ is sufficiently regular (if atleast twice mean square differentiable) then the expected number of maxima per unit time of $S(t)$ above a certain levels S_o is,

$$\mu(t; s_o) = - \int_{-\infty}^0 ds'' \int_{s_o}^{\infty} S'' P(s, 0, s'', t) ds \dots\dots\dots (13)$$

Where $P(s, s', s'', t)$ is the joint density function of $s(t), s'(t), s''(t)$ at time t.

The average crack growth rate under random loading takes the modified form.

$$\frac{dL}{dN} = \mu(t) F(Q, K_{rms}) \dots\dots\dots (14)$$

In which K_{rms}, Q and $\mu(t)$ characterize the effect of stress intensity range, stress ratio and frequency content respectively. The correlation properties of random applied stress are introduced through $\mu(t)$. Then,

$$\frac{dL}{dN} = C \mu(t) g(Q) (K_{rms})^n; \quad L(t_0) = L_o \dots\dots (15)$$

Using (9). The above equation (15) can be written as

$$\frac{dL}{dt} = f(t) L^p(t),$$

$$f(t) = A \mu(t) g[Q(t)] S_{rms}^n(t)$$

$$L(t_0) = L_o, \quad P = n/2, \quad A = C \sqrt{\pi}.$$

$$\int_{t_0}^{\tau} \frac{L}{L^p(t)} = \int_{t_0}^{\tau} f(t) dt$$

$$\text{i.e. } \left(\frac{L^{-p+1}(t)}{1-p} \right)_{t_0}^{\tau} = \phi(\tau) \dots\dots\dots (16)$$

$$\text{Where } \phi(\tau) = \int_{t_0}^{\tau} f(s) ds$$

$$\text{i.e. } \frac{1}{(1-p)} [L^{1-p}(\tau) - L_o^{1-p}] = \phi(\tau)$$

$$L^{1-p}(t) = L_o^{1-p} - (p-1)\phi(\tau)$$

$$L(t) = \frac{1}{\left[\frac{1}{L_o^{(p-1)} - (p-1)\phi(\tau)} \right]^{\frac{1}{p-1}}} \dots\dots\dots (17)$$

The explosion time τ is obtained from the equation

$$(p-1)\phi(\tau) = \frac{1}{L_o^{p-1}} \dots\dots\dots (18)$$

If $S(t)$ is stationary then $Q(t) = Q_o$; $S_{rms}(t) = S_o$; $\mu(t) = \mu_o$; $f(t) = f_o$ and we have

$$\tau = t_o + \frac{1}{f_o(p-1)L_o^{p-1}} \dots\dots\dots (19)$$

For the non-stationary case, assume that the time t^* to reach the crack size L^* as

$$\phi(t^*) = \frac{1}{(1-p)} (L^*(1-p) - L_o^{1-p}) \dots\dots\dots (20)$$

And for the stationary case

$$t^* = t_o + \frac{L_o^{1-p} - L^{*1-p}}{f(p-1)} \dots\dots\dots (21)$$

The modified growth equation gives a satisfactory picture for the case of stationary loading when both the stress range (characterized by S_{rms}) and expected frequency μ are constant. For the non-stationary case, the

determination of $\mu(t)$ in analytical form possess certain difficulties.

B. Illustration 01

Let the random loading be the stochastic process characterized by the wave form

$S(t) = a \sin(\omega t + \psi)$ Where a and ω are the constants and ψ is a random variable uniformly distributed on $(0, 2\pi)$. Then

$$\langle S \rangle = \int_0^{2\pi} a \sin(\omega t + X) \frac{1}{2\pi} dX = 0$$

$$r_s(t_1, t_2) = r_s(t, t + \tau)$$

$$= \int_0^{2\pi} a \sin(\omega t + X) a \sin(\omega t + \omega \tau + X) \frac{1}{2\pi} dX$$

$$= \frac{a^2}{2} \cos \omega \tau$$

$$S_{rms} = \frac{a}{\sqrt{2}}, \quad Q = 0, \quad \mu(t) = \mu_o = \frac{\omega}{2\pi}$$

The function $f(t)$ takes the form

$$f(t) = A \frac{\omega}{2\pi} g(0) \left(\frac{a}{\sqrt{2}} \right)^n$$

This can be applied to the crack induced by the vibrations of random loading by vehicles in under simple harmonic motion.

C. Illustration 02

Fatigue cracks in a pavement layer are caused by the combination of repetitive strains and apparent reduction of tensile strength caused by fatigue of the layer material. This type of failure occurs when the pavement has been a stress to the limit of its fatigue life by repetitive axle load applications. It is often called alligator cracking. The fatigue cracking is often caused by high moisture content, poor sub grade or some other local problems that can be repaired without major reconstruction. In these instances the poor material is removed and replaced with good material and drainage improved. If the failed area is extensive one typical repair strategy is to place a HMA overlay over the entire surface. Investigation should involve determination of the thickness of layers and the material quality so that a suitable solution is selected.

Crack in pavement under varied stress with the help of illustration 01, we can illustrate the fatigue crack growth in the pavement is illustrated given in the figure.



III. STOCHASTIC CRACK GROWTH EQUATION

We improve the model for random fatigue by accounting for randomness of other factors provoking fatigue.

The model proposed is

$$\frac{dL}{dt} = C\mu(t)g(Q)(K_{rms})^n X(t, \gamma) \dots \dots \dots (22)$$

$$L(t_o) = L_o$$

Here $X(t, \gamma)$ represents the combined effect of unknown random factors of external origin such as environment, creep etc.,

If $X(t, \gamma) = m_x + \bar{X}(t, \gamma)$ where M_x is the average value of $X(t, \gamma)$

$$\frac{dL}{dt} = C\mu(t)g(Q)(K_{rms})^n [m_x + \bar{X}(t, \gamma)] \dots \dots \dots (23)$$

The above equation can be rewritten as

$$\frac{dL}{dt} = C\mu(t)m_x g(Q)(K_{rms})^n + C\mu(t)g(Q)(K_{rms})^n \bar{X}(t, \gamma) \dots \dots \dots (24)$$

It is convenient to write the stochastic growth model (23),(24) in the form

$$\frac{dL}{dt} = a(L, t) + \sigma(L, t)\bar{X}(t, \gamma) \dots \dots \dots (25)$$

$$L(t_o) = L_o$$

Or

$$\frac{dL}{dt} = f_1(t)L^p(t) + f_2(t)L^p(t) \bar{X}(t, \gamma) \dots\dots\dots (26)$$

$$L(t_o) = L_o$$

Where $p = n/2, C = c\pi^p$

$$a(L, t) = f_1(t)L^p(t)$$

$$\sigma(L, t) = f_2(t)L^p(t)$$

$$f_1(t) = m_x C\mu(t)g(Q)S_{rms}^{2p}(t) \dots\dots\dots (27)$$

$$f_2(t) = 1/m_x f_1(t)$$

There are three basic factors which determine properties and admissibility of the model proposed. They are

- i. Random applied stress $S(t, \gamma)$. If it is stationary

$$S_{rms}(t) = S_o = Constant$$

$$Q(t) = Q_o = Constant$$

$$\mu(t) = \mu_o$$

$$f_1(t) = f_1^0 = Constant$$

$$f_2(t) = f_2^0 = Constant$$
- ii. Random multiplicative noise $X(t, \gamma)$
- iii. Constant experimental parameters especially realistic values of 'p'.

A. Properties of Stochastic growth model

Since stochastic differential equation (22) and (23) has been introduced in a somewhat artificial way (via randomization of experimental laws one should check carefully its properties and usefulness.

In order to recognize the basic properties of the model introduced we consider a special but important case when random disturbance $X(t, \gamma)$ is a white Gaussian noise, It is worth nothing that because of normality of the distribution it can theoretically take negative values with positive probability. This deficiency is not serious since the deterministic term in (25) which is positive dominates the tendency of the motion.

Let us assume that in equations (24) and (25)

$$\bar{X}(t, \gamma) = \xi(t, \gamma) \dots\dots\dots (28)$$

$$\langle \xi(t_1, \gamma)\xi(t_2, \gamma) \rangle = 2D\delta(t_2 - t_1)$$

Where D is a constant intensity of noise and $\langle . \rangle$ denotes averaging. Although (25) together with $X(t, \gamma)$ given by

(28) looks like a differential equation, it is really G formal record of symbols since $\xi(t, \gamma)$ is an abstraction and not a physical process. Equation (25) with (28) does not define a stochastic process $L(t, \gamma)$ yet, it is a pre-equation.

There are two well known interpretations of our pre-equation turning it into a meaningful stochastic differential equation defining process $L(t, \gamma)$. These are the Ito and Stratonovich interpretation. Here we adopt Stratonovich interpretation. This means that (20) can be understood as the following equivalent Ito equation,

$$dL(t) = a^*(L, t)dt + \sigma(L, t)dw(t) \dots\dots\dots (29)$$

$$L(t_o) = L_o$$

Where $w(t)$ is the Brownian motion starting from $t = t_o$ and $w(t_o) = 0$ almost surely white noise $\xi(t)$ is a generalized derivative of $w(t)$ and

$$a^*(L, t) = a(L, t) + D\sigma(L, t) \partial\sigma/\partial L = f_1(t)L^p(t) + pDf_2^2(t)L^{2p-1}(t) \dots\dots\dots (30)$$

It should be noted that in the Stratonovich interpretation the deterministic (drift) term $a^*(L, t)$ differs from the macroscopic deterministic law equations (14) and (15).

B. Case 1

Adopting the results of general theory of stochastic systems, analysis of equation (29) yields the following conclusions.

1. In the case when $p = n/2$ occurring in experimental laws is not greater than one ($p \leq 1$) the crack growth is stable in a sense that on each finite time interval $L(t)$ takes finite values with probability one.

When $p=1$, what is often met in experimental predictions, the crack size $L(t)$ is expressed explicitly by qualities occurring in the problem i.e. by characteristics of random applied stress [functions: $f_1(t), f_2(t)$], the Brownian motion process $W(t)$ characterizing other uncertainties and initial crack length L_o . When random applied stress is stationary the appropriate formulae are very simple. The probability distribution of the crack size is for each fixed t and deterministic L_o log-normal.

2. In the case when $p = n/2 > 1$ which concern large variety of experimental laws, situation predicted by the model differs from this when ≤ 1 .

- The parameters occurring in the model proposed which need to be estimated from experiments are the following:

Two material constants:

C and $p = \frac{n}{2}$ occurring in crack growth law.

One Constant: D characterizing the intensity of white noise $\xi(t)$.

Three functions: $m_s(t) = \langle S(t, \gamma) \cdot S_{rms}(t) \text{ and } \mu(t) \rangle$; if random applied stress is stationary then: $m_s(t) = m_s = \text{constant}$.
 $S_{rms}(t) = f_o = \text{constant}$ and $\mu(t) = \mu_o = \text{constant}$.

IV. A DIFFERENTIAL EQUATION MODEL FOR RANDOM FATIGUE GROWTH

Stochastic models for cumulative damage describe the probabilistic mechanism of fatigue accumulation from which fatigue life can be predicted. These models can be broadly classified into two categories. (a) Time to failure models where the system is characterized through a distribution function for the time to failure. (b) 'Phenomenological' models which characterize explicitly the underlying physical mechanism which causes the failure. Cumulative damage models are phenomenological models in which shock causes a certain amount of damage is additive. System failure occurs when total damage exceeds a critical level.

Most of our basic knowledge on fatigue behavior comes from experiments. The experimental data from engineering laboratories constitute a basic source of information about the fatigue behavior of materials subjected to various loading conditions. An important problem is to represent the information contained in the fatigue data in the language of Mathematics.

In the study of crack propagation in materials, most of the researchers take into consideration the Paris-Erdogan equation.

$$\frac{da}{dN} = C(\Delta K)^m$$

For the rate of fatigue growth under homogeneous cyclic stressing which was attained from experimental results as the linear regression of $\log \left[\frac{da}{dN} \right]$ on $\log(\Delta K)$ where 'a' is the crack length. 'N' is the number of cycles and k is the range of stress intensity factor at the crack tip.

The Paris-Erdogan equation

We consider the Paris-Erdogan model from two points of view namely.

Crack length 'a' is the dependent variable.

Cycle number 'n' is the dependent variable.

It is generally accepted that fatigue crack under constant amplitude cyclic loading can be related to the stress intensity factor Δk through the first order differential equation.

$$\frac{da}{dN} = C(\Delta K)^m \dots \dots \dots (31)$$

Where $a = a(n)$ the crack length at time n. In general 'f' is an experimentally determined function and 'c' is an experimentally determined constant. Clearly $\frac{da}{dn} \geq 0$ implies that c and f are non -ve. From fracture mechanics, one can relate Δk to crack length 'a' via

$$\Delta k = \alpha \Delta S a^{1/2} \dots \dots \dots (32)$$

Where 'a' is a geometrically related parameter and ΔS is the applied stress amplitude. (31) can be written as

$$\frac{da}{dn} = c f(\alpha \Delta S a^{1/2}) \dots \dots \dots (33)$$

On the basis of number of experimental investigations 'f' can be approximated as a power function so that

$$\frac{da}{dn} = c(a \Delta S)^\beta a^\beta = K a^{\beta/2} \dots \dots \dots (34)$$

Where k contains all geometric, stress and material factors or parameters. The above equation (34) contains three parameters k, β and a. Thus any randomization of the P.E. equation must be based upon a randomization of these three parameters.

$$\text{Now } \int \frac{da}{a^{\beta/2}} = \int k dn$$

$$\text{i.e. } \frac{a^{-\beta/2+1}}{-\beta/2+1} = kn + C$$

We have the initial condition $a(0) = a_o$.

$$\frac{2}{2-\beta} a_o^{2-\beta/2} = C$$

$$\frac{2-\beta}{2} a^{2-\beta/2} = kn + \frac{2}{2-\beta} a_o^{2-\beta/2}$$

$$a(n) = a_o^{2-\beta/2} + \frac{2-\beta}{2} kn^{2/(2-\beta)} = k(a_o) \dots \dots \dots (35)$$

V. CONCLUSION

We have given the behavior of crack growth under random loading using continuous stochastic models of stress

behavior and stochastic differential equation model of fatigue behavior. Illustration is given to explain these modifications in the basic law under random loading.

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