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Homometric Number of Graphs

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Abstract: Given a graph $G = (V, E)$, two subsets S_1 and S_2 of the vertex set V are homometric, if their distance multi sets are equal. The homometric number $h(G)$ of a graph G is the largest integer k such that there exist two disjoint homometric subsets of cardinality k. We find lower bounds for the homometric number of the Mycielskian of a graph and the join and the lexicographic product of two graphs. We also obtain the homometric number of the double graph of a graph, the cartesian product of any graph with K_2 and the complete bipartite graph. We also introduce a new concept called weak homometric number and find weak homometric number of some graphs.

Keywords: *Homometric sets, Homometric number, Weak Homometric number*
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I. Introduction

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. If there is no ambiguity in the choice of G, then we write $V(G)$ and $E(G)$ as V and E respectively. For any set $S \subseteq V$ the cardinality of S is denoted by $|S|$. The distance multi set of S, denoted by $DM(S)$, is the multi set of all pair-wise distances between any two vertices of S. Two subsets S_1 and S_2 of the vertex set V are said to be homometric, if their distance multi sets are equal. The homometric number $h(G)$ of a graph G is the largest integer k such that there exist two disjoint homometric subsets, S_1 and S_2 of the vertex set V, each of cardinality k. Clearly, $h(G) \leq \lfloor \frac{n}{2} \rfloor$, where $\lfloor x \rfloor$ denotes the greatest whole number less than or equal to x. Even though there is a concept of infinite distance in the case of disconnected graphs, to avoid ambiguity we consider only connected graphs. For a family of graphs $\mathcal{G}$, $h(\mathcal{G})$ is the largest integer h such that $h(G) \geq h$, for every $G \in \mathcal{G}$. For any positive integer n, $h(n) = h(\mathcal{G}_n)$, where $\mathcal{G}_n$ denotes the class of all graphs on n vertices.

In 2010, Albertson, Pach and Young [1] initiated the study of homometric sets in graphs. They proved that every graph on n vertices, $n > 3$, contains homometric sets of size at least $\frac{c \log n}{\log \log n}$, for a constant c. On the other hand, they constructed a class of graphs where the size of homometric sets cannot exceed $\frac{n}{4}$, where $n > 3$. The lower bound was apparently improved by Alon in [11] as $h(n) \geq \frac{c\,(\log n)^2}{(\log \log n)^2}$. Axenovich and Özkahya [3] gave a better lower bound on the maximal size of homometric sets in trees. They showed that every tree on n vertices contain homometric sets of size at least $\sqrt[3]{n}$. A haircomb tree on n vertices contains homometric sets of size at least $\frac{\sqrt{n}}{2}$. They also proved that, for any graph G of diameter d, $h(G) \geq cn^{\frac{1}{2d-2}}$. R. Fulek and S. Mitrović [6] improved the result on trees by proving that there exist disjoint homometric sets of size at least $\sqrt{\frac{n}{2}} - \frac{1}{2}$. A better lower bound for haircomb trees is also given in [6]. Lemke, Skiena and Smith [8] showed that if G is a cycle of length $2n$ then every subset of $V(G)$ with n vertices and its complement are homometric sets. In [2], it is proved that the above result works not only for cycles but for all vertex transitive graphs.

1.1 Basic Definitions and Preliminaries

For any graph G the number of vertices in G is denoted by $n(G)$. For any vertex $v \in V$ the degree of v, denoted by $d_G(v)$, is the number of edges incident to v. The distance between any two vertices u and v in V is the length of the shortest path joining u and v in G and is denoted by $d_G(u, v)$. The maximum distance between any pair of vertices in G is the diameter of the graph G and is denoted by $diam(G)$. Any induced path $P = u_1, u_2, \dots, u_l$ in G where $d_G(u_1, u_l) = diam(G)$ is called a diametral path with end vertices u_1 and u_l. Since $\{u_1, u_2, \dots, u_{\lfloor \frac{l}{2} \rfloor}\}$ and $\{u_{\lfloor \frac{l}{2} \rfloor + 1}, \dots, u_{2\lfloor \frac{l}{2} \rfloor}\}$ are disjoint homometric subsets,

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$\lceil \frac{diam\,(G)}{2} \rceil \leq h(G)$, where $\lceil x \rceil$ denotes the least whole number greater than or equal to x.

A subset $S \subseteq V$ of vertices is said to be independent if no two vertices of S are adjacent to each other in G. The maximum cardinality of an independent set of vertices in G is the independence number, denoted by $\alpha(G)$. The girth of a graph G is the length of the shortest cycle in G and is denoted by $g(G)$.

The Mycielskian $M(G)$ of a graph G is the graph with vertex set $V(G) \cup V'(G) \cup \{w\}$ where $V'(G) = \{v_i : u_i \in V(G)\}$ and edge set $E(G) \cup \{u_i v_j : u_i u_j \in E(G)\} \cup \{w v_i : v_i \in V'(G)\}$. In [5], it has been proved that for a connected noncomplete graph G, $diam(M(G)) = \min\{diam(G), 4\}$.

The join of two graphs G and H, denoted by $G \vee H$, is defined as the graph with $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{uv, where\ u \in V(G)\ and\ v \in V(H)\}$. The cartesian product of two graphs G and H, denoted by $G \square H$, is the graph with vertex set $V(G) \times V(H)$ and any two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \square H$ if (i) $u_1 = u_2$ and $v_1 v_2 \in E(H)$, or (ii) $u_1 u_2 \in E(G)$ and $v_1 = v_2$. It is known that [7], if (u_1, v_1) and (u_2, v_2) are two vertices in $G \square H$, then $d_{G\square H}((u_1, v_1)(u_2, v_2)) = d_G(u_1, u_2) + d_H(v_1, v_2)$. The lexicographic product of two graphs G and H is the graph $G[H]$ with vertex set $V(G) \times V(H)$ and any two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G[H]$ if and only if (i) $u_1 u_2 \in E(G)$, or (ii) $u_1 = u_2$ and $v_1 v_2 \in E(H)$. In [7], it is proved that if (u_1, v_1) and (u_2, v_2) are two vertices in $G[H]$, then

$$d_{G[H]}((u_1, v_1), (u_2, v_2)) = \begin{cases} d_G(u_1, u_2), & if\ \ u_1 \neq u_2, \\ d_H(v_1, v_2), & if\ \ u_1 = u_2\ and\ d_G(u_1) = 0, \\ \min\{d_H(v_1, v_2), 2\}, & if\ \ u_1 = u_2\ and\ d_G(u_1) \neq 0. \end{cases}$$

The double graph $D(G)$ [10] of a graph G is the lexicographic product of G and K_2', where K_2' denotes the complement of K_2.

For any graph theoretic terminology and notations not mentioned here, the readers may refer [4].

1.2 Our Results

In this paper, we prove that the homometric number of Mycielskian of a graph G is at least twice as that of G. We also obtain lower bounds for the homometric number of the join and the lexicographic product of two graphs. Further, we find the homometric number of the double graph of a graph on n vertices, the cartesian product of any graph on n vertices with K_2, and the complete bipartite graph. Finally we introduce a new concept called weak homometric number and find weak homometric number of some graphs.

II. Lower Bounds of Homometric Number

In this section we find lower bounds for the homometric number of the Mycielskian of a graph G. We also obtain lower bounds for the homometric number of the join and the lexicographic product of two graphs.

Theorem 2.1. For any connected graph G, $h(M(G)) \geq 2h(G)$.

Proof. Let $\{u_1, u_2, \ldots, u_n\}$ be the vertex set of G. In $M(G)$, for $i = 1, 2, \ldots, n$, let v_i be the vertex corresponding to u_i and w be the vertex adjacent to all the v_i's. Let $S_1 = \{u_{11}, u_{12}, \ldots, u_{1h}\}$ and $S_2 = \{u_{21}, u_{22}, \ldots, u_{2h}\}$ be two disjoint homometric subsets of $V(G)$ such that $|S_1| = |S_2| = h(G)$.

Consider two subsets $S_1' = \{u_{11}, u_{12}, \ldots, u_{1h}, v_{11}, v_{12}, \ldots v_{1h}\}$ and $S_2' = \{u_{21}, u_{22}, \ldots, u_{2h}, v_{21}, v_{22}, \ldots, v_{2h}\}$ of $V(M(G))$. Clearly, $|S_1'| = |S_2'| = 2h(G)$.

Case 1: Consider $u_{1i}, u_{1j} \in S_1'$.

Since S_1 and S_2 are two disjoint homometric subsets of $V(G)$, there exist $u_{2k}, u_{2l} \in S_2$ such that $d_G(u_{1i}, u_{1j}) = d_G(u_{2k}, u_{2l})$. If $d_G(u_{1i}, u_{1j}) \leq 4$, then $d_{M(G)}(u_{1i}, u_{1j}) = d_G(u_{1i}, u_{1j}) = d_G(u_{2k}, u_{2l}) = d_{M(G)}(u_{2k}, u_{2l})$. If $d_G(u_{1i}, u_{1j}) > 4$, then $d_{M(G)}(u_{1i}, u_{1j}) = d_{M(G)}(u_{2k}, u_{2l}) = 4$.

Case 2: Consider $u_{1i}, v_{1j} \in S_1'$.

Corresponding to every $v_{1j} \in S_1'$ there exists $u_{1j} \in S_1'$. By Case 1, there exist $u_{2k}, u_{2l} \in S_2'$ such that $d_{M(G)}(u_{1i}, u_{1j}) = d_{M(G)}(u_{2k}, u_{2l})$. Corresponding to every $u_{2l} \in S_2'$ there exists $v_{2l} \in S_2'$. Clearly, $d_{M(G)}(u_{1i}, v_{1j}) = d_{M(G)}(u_{2l}, v_{2k})$.

Case 3: Consider $v_{1i}, v_{1j} \in S_1'$.

Clearly, $d_{M(G)}(v_{1i}, v_{1j}) = 2$. Choose $v_{2i}, v_{2j} \in S_2'$. By the construction of $M(G)$, $d_{M(G)}(v_{2i}, v_{2j})$ is also two.

Thus, S_1' and S_2' are disjoint homometric subsets of $V(M(G))$ each of cardinality $2h(G)$. Hence, $h(M(G)) \geq 2h(G)$.

□

Theorem 2.2. For any two connected graphs G and H, the homometric number of $G \vee H$, $h(G \vee H) \geq \max\{\min\{\alpha(G), \alpha(H)\}, \min\{diam(G), diam(H)\}\}$.

Proof. If S_1 and S_2 are independent subsets of $V(G)$ and $V(H)$ respectively and $|S_1| = |S_2|$, then $DM(S_1)$ and $DM(S_2)$ in $G \vee H$ contains only the element 2, repeated the same number of times. Hence S_1 and S_2 are two disjoint homometric subsets of $V(G \vee H)$. Therefore, $h(G \vee H) \geq \min\{\alpha(G), \alpha(H)\}$.

Let $d = \min\{diam(G), diam(H)\}$. Let S_1 and S_2 be the vertices in an induced path of length d in G and H respectively. Then, in $G \vee H$, $DM(S_1) = DM(S_2) = \{1, \ldots, 1, 2, \ldots, 2\}$, where 1 is repeated d times and 2 is repeated ${}^{d+1}C_2 - d$ times. Thus, S_1 and S_2 are disjoint homometric subsets of $V(G \vee H)$ of cardinality $\min\{diam(G), diam(H)\}$. Therefore, $h(G \vee H) \geq \min\{diam(G), diam(H)\}$.

Hence, $h(G \vee H) \geq \max\{\min\{\alpha(G), \alpha(H)\}, \min\{diam(G), diam(H)\}\}$.
□

Theorem 2.3. For any two connected graphs G and H, $h(G[H]) \geq h(G)n(H)$.
Proof. Let $S_1 = \{u_{11}, u_{12}, \ldots, u_{1h}\}$ and $S_2 = \{u_{21}, u_{22}, \ldots, u_{2h}\}$ be two disjoint homometric subsets of $V(G)$. Let $V(H) = \{v_1, v_2, \ldots, v_n\}$. Consider two subsets of $V(G[H])$, $S_1' = \{(u_{1i}, v_j) \,/\, i = 1, 2, \ldots, h \text{ and } j = 1, 2, \ldots, n\}$ and $S_2' = \{(u_{2i}, v_j) \,/\, i = 1, 2, \ldots, h \text{ and } j = 1, 2, \ldots, n\}$. Clearly, $|S_1'| = |S_2'| = h(G)n(H)$. Let (u_{1i}, v_x) and (u_{1j}, v_y) be any two vertices in S_1'.

Case 1: i ≠ j.
In this case, $d_{G[H]}\big((u_{1i}, v_x), (u_{1j}, v_y)\big) = d_G(u_{1i}, u_{1j})$. Since S_1 and S_2 are two homometric subsets of $V(G)$ there exist two vertices $u_{2k}, u_{2l} \in S_2$ such that $d_G(u_{1i}, u_{1j}) = d_G(u_{2k}, u_{2l})$. So, $d_{G[H]}\big((u_{2k}, v_x), (u_{2l}, v_y)\big) = d_{G[H]}\big((u_{1i}, v_x), (u_{1j}, v_y)\big)$. Also, $(u_{2k}, v_x), (u_{2l}, v_y) \in S_2'$.

Case 2: i = j.
In this case, $d_{G[H]}\big((u_{1i}, v_x), (u_{1j}, v_y)\big) = \min\{d_H(v_x, v_y), 2\} =$ *dGHu2i, vx, u2j, vy* . Also, *u2i, vx,(u2j, vy)∈S2′.*

Hence, we have proved that corresponding to any two vertices in S_1', there exists a pair of vertices in S_2' such that the distance is preserved. Thus S_1' and S_2' are two disjoint homometric subsets of $V(G[H])$. Hence, $h(G[H]) \geq h(G)n(H)$. □

III. Homometric Number of Some Graphs

In this section we obtain the homometric number of the double graph of a graph, the cartesian product of a graph with K_2 and the complete bipartite graph.

Theorem 3.1. For any connected graph G, $h(D(G)) = n$, where n denotes the number of vertices of G.
Proof. Let G be any connected graph with vertex set $\{u_1, u_2, \ldots, u_n\}$ and let the vertices of K_2' be v_1 and v_2. Consider the subsets $S_1 = \{(u_1, v_1), (u_2, v_1), \ldots, (u_n, v_1)\}$ and $S_2 = \{(u_1, v_2), (u_2, v_2), \ldots, (u_n, v_2)\}$ of $V(D(G))$. Clearly, $DM(S_1) = DM(S_2) = DM(V(G))$. Therefore, $h(D(G)) \geq n$. But, the maximum value of the homometric number of any graph cannot exceed the half of its order. Thus, $h(D(G)) \leq \left\lfloor \frac{2n}{2} \right\rfloor = n$ always. Hence the result. □

Theorem 3.2. For any connected graph G, $h(G \,\square\, K_2) = n$, where n denotes the number of vertices of G.
Proof. Let G be any connected graph with vertex set $\{u_1, u_2, \ldots, u_n\}$ and the vertices of K_2 be v_1 and v_2. Consider the subsets $S_1 = \{(u_1, v_1), (u_2, v_1), \ldots, (u_n, v_1)\}$ and $S_2 = \{(u_1, v_2), (u_2, v_2), \ldots, (u_n, v_2)\}$ of $V(G \,\square\, K_2)$. Clearly, $DM(S_1) = DM(S_2) = DM(V(G))$. Therefore, $h(G \,\square\, K_2) \geq n$. But, the maximum value of the homometric number of any graph cannot exceed the half of its order. Thus, $h(G \square H) \leq \left\lfloor \frac{2n}{2} \right\rfloor = n$ always. Hence the result.
□

Theorem 3.3. For any complete bipartite graph $K_{m,n}$,
$$h(K_{m,n}) = \begin{cases} m, & if\ m = n, \\ \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor, & if\ m \neq n. \end{cases}$$
Proof. Let $X = \{u_1, u_2, \ldots, u_m\}$ and $Y = \{v_1, v_2, \ldots, v_n\}$ be a bipartition of $K_{m,n}$. If $m = n$, then X and Y itself are disjoint homometric sets. Now without loss of generality let $m > n$. Let $S_1 = \left\{u_1, u_2, \ldots, u_{\lfloor \frac{m}{2} \rfloor}, v_1, v_2, \ldots, v_{\lfloor \frac{n}{2} \rfloor}\right\}$ and $S_2 = \left\{u_{\lfloor \frac{m}{2} \rfloor + 1}, \ldots, u_{2\lfloor \frac{m}{2} \rfloor}, v_{\lfloor \frac{n}{2} \rfloor + 1}, \ldots v_{2\lfloor \frac{n}{2} \rfloor}\right\}$. Then S_1 and S_2 are disjoint homometric sets with distance multi set containing ${}^{\lfloor m/2 \rfloor}C_2 + {}^{\lfloor n/2 \rfloor}C_2$ two's and $\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor$ one's. Hence homometric number is at least $\left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$. If m or n is even then $\left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{m+n}{2} \right\rfloor$, which is the maximum possible value. Therefore, we need only consider the case where both m and n are odd. If possible assume that S_1 and S_2 are disjoint homometric sets such that $S_1 \cup S_2 = V(K_{m,n})$. Let $|S_1 \cap X| = m_1$, $|S_1 \cap Y| = n_1$, $|S_2 \cap X| = m_2$ and $|S_2 \cap Y| = n_2$. Since S_1 and S_2 are homometric, $|S_1| = m_1 + n_1 = m_2 + n_2 = |S_2|$. Also, since the number of one's in

$DM(S_1)$ is m_1n_1 and that in $DM(S_2)$ is $m_2n_2, m_1n_1 = m_2n_2$. Using these two equations and the fact that, $m_1 \neq m_2$, we get $m_1 = n_2$ and $m_2 = n_1$. But this contradicts the fact that $m_1 + m_2 = m \neq n = n_1 + n_2$. Hence we can conclude that the homometric number in this case is $\left\lfloor\frac{m}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor$.
□

IV. Weak Homometric Number

In the definition of homometric number we consider distance multi set of subsets of the vertex set. In some practical situations only distances are important and not the number of pairs of vertices at a particular distance. For example, while considering communication delay we are interested in how far the communication centers are, but not in how many communication centers are there at a fixed distance. With this in mind, we are introducing a new concept called weak homometric number.

For any set $S \subseteq V$, the distance set of S, denoted by $D(S)$, is the set of all pair-wise distances between any two vertices of S. Two subsets S_1 and S_2 of the vertex set V are said to be weakly homometric if their distance sets are equal [9]. The weak homometric number of a graph G is the largest integer k such that there exist two disjoint weakly homometric subsets S_1 and S_2 of the vertex set V each of cardinality k and it is denoted by $h_w(G)$. Clearly $h(G) \leq h_w(G) \leq \left\lfloor\frac{n}{2}\right\rfloor$.

Theorem 4.1. If G is a connected graph with n vertices, $g(G) \geq 5$ and $d(v) \neq 2, \forall\, v \in V(G)$, then $h_w(G) = \left\lfloor\frac{n}{2}\right\rfloor$.

Proof. Let $v_1v_2 \dots v_{d+1}$ be a diametral path in G.

Case 1. d is even.

Let $v_1, v_2, \dots, v_{\frac{d}{2}+1} \in S_1$ and $v_{\frac{d}{2}+2}, \dots, v_{d+1} \in S_2$. Since $d(v_i) \neq 2$, every v_i, $i = 2, \dots, d$, has at least one neighbour other than v_{i-1} and v_{i+1}. For each $i = 2, \dots, d$, let u_i be adjacent to v_i. Put $u_2, \dots, u_{\frac{d}{2}+1}$ in S_2 and $u_{\frac{d}{2}+2}, \dots, u_d$ in S_1. Clearly, $d(v_1, v_i) = i - 1, \forall i = 2, \dots, \frac{d}{2} + 1$. Hence, $1, 2, \dots, \frac{d}{2} \in D(S_1)$. Also, $d(v_1, u_i) = i, \forall i = \frac{d}{2} + 2, \dots, d$. Hence, $\frac{d}{2} + 2, \dots, d \in D(S_1)$. Again, $d\left(v_2, u_{\frac{d}{2}+2}\right) = \frac{d}{2} + 1$. Therefore, $\frac{d}{2} + 1 \in D(S_1)$. Hence, $D(S_1) = \{1, 2, \dots, d\}$.

Thus, $D(S_1) = D(S_2)$. Now there are $n - 2d$ vertices remaining in V. Put $\left\lfloor\frac{n-2d}{2}\right\rfloor$ vertices in S_1 and S_2 so that $|S_1| = |S_2| = \left\lfloor\frac{n}{2}\right\rfloor$.

Case 2. d is odd.

Let $v_1, v_2, \dots, v_{\frac{d+1}{2}} \in S_1$ and $v_{\frac{d+3}{2}}, \dots, v_{d+1} \in S_2$. Since $d(v_i) \neq 2$, every v_i, $i = 2, \dots, d$, has at least one neighbour other than v_{i-1} and v_{i+1}. For each $i = 2, \dots, d$, let u_i be adjacent to v_i. Put $u_2, \dots, u_{\frac{d+1}{2}}$ in S_2 and $u_{\frac{d+3}{2}}, \dots, u_d$ in S_1. Clearly, $d(v_1, v_i) = i - 1, \forall i = 2, \dots, \frac{d+1}{2}$. Hence, $1, 2, \dots, \frac{d-1}{2} \in D(S_1)$. Also, $d(v_1, u_i) = i, \forall i = \frac{d+3}{2}, \dots, d$. Hence, $\frac{d+3}{2}, \dots, d \in D(S_1)$. Again, $d\left(v_2, u_{\frac{d+3}{2}}\right) = \frac{d+1}{2}$. Therefore, $\frac{d+1}{2} \in D(S_1)$. Hence, $D(S_1) = \{1, 2, \dots, d\}$.

Now, $d(v_{d+1}, v_i) = d + 1 - i, \forall i = \frac{d+3}{2}, \dots, d$. Hence, $1, 2, \dots, \frac{d-1}{2} \in D(S_2)$. Also, $d(v_{d+1}, u_i) = d + 2 - i$, $\forall i = 2, \dots, \frac{d+1}{2}$. Hence, $\frac{d+3}{2}, \dots, d \in D(S_2)$. Again, $d\left(v_d, u_{\frac{d+1}{2}}\right) = \frac{d+1}{2}$. Therefore, $\frac{d+1}{2} \in D(S_2)$. Hence, $D(S_2) = \{1, 2, \dots, d\}$. Thus, $D(S_1) = D(S_2)$. Now there are $n - 2d$ vertices remaining in V. Put $\left\lfloor\frac{n-2d}{2}\right\rfloor$ vertices in S_1 and S_2 so that $|S_1| = |S_2| = \left\lfloor\frac{n}{2}\right\rfloor$.

Thus, in both the cases we have proved that S_1 and S_2 are two disjoint weakly homometric subsets of $V(G)$ and hence $h_w(G) = \left\lfloor\frac{n}{2}\right\rfloor$. □

Theorem 4.2. If G and H are any two noncomplete graphs with n_1 and n_2 vertices respectively, then $h_w(G \vee H) = \left\lfloor\frac{n_1+n_2}{2}\right\rfloor$.

Proof. Let $V(G) = \{u_1, u_2, \dots, u_{n_1}\}$ and $V(H) = \{v_1, v_2, \dots, v_{n_2}\}$. Suppose that u_1 and u_2 are two nonadjacent vertices in G and v_1 and v_2 are two nonadjacent vertices in H.

Case 1. $n_1, n_2 \geq 3$.

Put u_1, u_2 and v_3 in S_1 and v_1, v_2 and u_3 in S_2. Hence $D(S_1) = D(S_2) = \{1, 2\}$. Now there are $n_1 + n_2 - 6$ vertices remaining in $V(G \vee H)$. Distribute these vertices in S_1 and S_2 so that $|S_1| = |S_2| = \left\lfloor\frac{n_1+n_2}{2}\right\rfloor$.

Now, $d(v$

Case 2. Either n_1 or n_2 (but not both) is 2.

Let $n_1 = 2$. (The other case follows similarly.) If $n_2 = 3$, then take $S_1 = \{u_1, u_2\}$ and $S_2 = \{v_1, v_2\}$. Now, suppose $n_2 \geq 4$. If v_i and v_j, $i, j \neq 1,2$, are non adjacent in H, then put v_1, v_2 and u_1 in S_1 and v_i, v_j and u_2 in S_2. Hence, $D(S_1) = D(S_2) = \{1, 2\}$. Distribute the remaining $n_2 - 4$ vertices in S_1 and S_2 so that $|S_1| = |S_2| = \left\lfloor \frac{n_1+n_2}{2} \right\rfloor$. Otherwise, $\{v_3, v_4, \dots, v_{n_2}\}$ will induce a complete subgraph H_1 in H. If there is no $v_1 v_i$ and $v_2 v_i$, $i = 3, 4, \dots, n_2$, edge, then put u_1 and v_1 in S_1 and u_2 and v_2 in S_2. Thus, $D(S_1) = D(S_2) = \{1\}$. Distribute the remaining $n_2 - 2$ vertices in S_1 and S_2 so that $|S_1| = |S_2| = \left\lfloor \frac{n_1+n_2}{2} \right\rfloor$. Then $D(S_1) = D(S_2) = \{1, 2\}$. Now, suppose there is an edge from v_1 (or v_2 or both) to some vertex v_i in H_1. Then put u_1 and u_2 in S_1 and v_1, v_2 and v_i in S_2. Distribute the remaining $n_2 - 3$ vertices in S_1 and S_2 so that $|S_1| = |S_2| = \left\lfloor \frac{n_1+n_2}{2} \right\rfloor$. Then $D(S_1) = D(S_2) = \{1, 2\}$.

If $n_1 = n_2 = 2$, then $G \vee H$ will be the complete bipartite graph $K_{2,2}$ which is discussed earlier.

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