Solution of Variable Coefficient Fuzzy Differential Equations by Fuzzy Laplace Transform

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Abstract

In this paper we propose a fuzzy Laplace transform to solve variable coefficient fuzzy differential equations under strongly generalized differentiability concept.The fuzzy Laplace transform of derivative was used to solve second order variable coefficient fuzzy initial value problems and fuzzy boundary value problems if t is multiplied with first or second derivative term.To illustrate applicability of proposed method we solve fuzzy differential equations using different types of fuzzy numbers i.e. triangular,trapezoidal,Gaussian etc and compare the solutions.We plot 3D plots for different values of r-level sets by mathematica software.

Keywords: Fuzzy Number.Fuzzy valued function.triangular,trapezoidal and Gaussian fuzzy numbers.Fuzzy Laplace Transform.Strongly generalized differential.Fuzzy initial value problem.Fuzzy boundary value problem..

1 Introduction

The fuzzy differential equation is very much important topic in field of science and engineering to solve dynamic problem.The concept of a fuzzy derivative was first introduced by Chang and Zadeh [56],followed up by Dubois and Prade

[17] who used the extension principle in their approach.Other fuzzy derivative concepts were proposed by Puri and Ralescu [45], and Goetschel and Vaxman [26]as an extension of the Hukuhara derivative of multivalued functions.Kandel and Byatt [33] applied the concept of fuzzy differential equation to the analysis of fuzzy dynamical problems.The fuzzy differential equations and fuzzy initial value problems are studied by Kaleva[31, 32]and Seikkala [51]

Two analytical methods for solving an nth-order fuzzy linear differential equation with fuzzy initial conditions presented by Buckley and Feuring [12, 13].Mondal and Roy [42] described the solution procedure for first order linear non-homogeneous ordinary differential equation in fuzzy environment.Existence and uniqueness of fuzzy boundary value problem has been proved by Esfahani et al.[18].Lakshikantham et al. [38] investigated the solution of two point boundary value problems associated with non-linear fuzzy differential equation by using the extension principle.Generalized differentiability concept is used by Bade et al.[11] to investigated first order linear fuzzy differential equations.Based on the idea of collocation method Allahviranloo et al. [5] solved nth order fuzzy linear differential equations.Far and Ghal-Eh [19] proposed an iterative method to solve fuzzy differential equations for the linear system of first order fuzzy differential equation with fuzzy constant coefficient.Variation of constant formula has been handle by Khastan et al.[37] to solve first order fuzzy differential equations.Akin et al.[2] developed an algorithm based on α-cut of fuzzy set for solution of second order fuzzy initial value problems.A new approach has been developed by Gasilov et al.[22] to get the solution of fuzzy initial value problem.

The concept of generalized H-differentiability is studied by Chalco-Cano and Roman Flores [14] to solve fuzzy differential equation.Hasheni et al.[29, 28] studied homotopy analysis method for solution of system of fuzzy differential equation s and obtained analytic solution of fuzzy Wave like equations with variable coefficients.As regards, methods to solve nth order fuzzy differential equation are discussed in [5, 25, 30, 35, 48, 55].the Variational iteration method (VIM) was successfully applied by Jafari et al.[30] for solving nth order fuzzy differential equation.A new result on multiple solutions for nth order fuzzy differential equations under generalized differentiability has been proposed by Khastan et al.[35].Based on idea of collocation method allahviranloo et al.[5] solved nth order fuzzy linear differential equations.Integral form of nth order fuzzy differential equations has been developed by Salahshour [48]under generalized differentiability.Mansuri and Ahmady [41] implemented characterization theorem for solving nth order fuzzy differential equations.Also Tapaswini and Chakraverty[53] implemented homotopy perturbation method for the solution of nth order fuzzy linear differential equations.Bade[10] found Solutions of fuzzy differential equations based on generalized differentiability.

Paper is organized as In section 2 preliminaries,In section 3 Examples by using fuzzy Laplace transform ,In section 4 Result and Discussion,In section 5 conclusion.

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2 Preliminaries

Definition 2.1 Fuzzy Number
A fuzzy number is a fuzzy set like $\mu : R \to I = [0, 1]$ which satisfies:

(a) μ is upper semi-continuous,

(b) μ is fuzzy convex i.e $\mu(\lambda x+(1-\lambda)y) \geq min\{\mu(x), \mu(y)\} \forall x, y \in R, \lambda \in [0, 1]$,

(c) μ is normal i.e $\exists x_0 \in R$ for which $\mu(x_0) = 1$,

(d) supp $\mu = \{x \in R \mid \mu(x) > 0\}$ is support of u, and its closure cl(supp μ) is compact.

Definition 2.2 r-cut
It is crisp set derived from its parent fuzzy set A where r-cut is defined as
$A_r = \{x \in R \mid \mu(x) \geq r\}$
Definition 2.3 Triangular Fuzzy Number
Consider triangular fuzzy number $\tilde{A} = (a, b, c)$ is depicted in Fig.1 The membership function $\mu(x)$ of $\tilde{A}$ will be defined as follows.

$$\mu(x) = \begin{cases} 0 & , x < a \\ \frac{x-a}{b-a} & , a \leq x \leq b \\ \frac{c-x}{c-b} & , b \leq x \leq c \\ 0 & , x > c \end{cases}$$

The triangular fuzzy number $\tilde{A} = (a, b, c)$ can be represented with an order pair of function of r-cut approach i.e.$[\underline{\mu}(r), \overline{\mu}(r)] = [a + (b - a)r, c - (c - b)r]$, where $r \in [0, 1]$

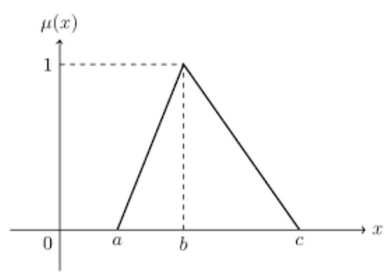

Fig.1 Triangle membership function

Definition 2.4 Trapezoidal Fuzzy Number
Consider trapezoidal fuzzy number $\tilde{A} = (a, b, c, d)$ is depicted in Fig.2 The membership function μ of $\tilde{A}$ will be defined as follows.

$$\mu(x) = \begin{cases} 0 & , x < a \\ \frac{x-a}{b-a} & , a \leq x \leq b \\ 1 & , b \leq x \leq c \\ \frac{d-x}{d-c} & , c \leq x \leq d \\ 0 & , x \geq d \end{cases}$$

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The trapezoidal fuzzy number $\tilde{A} = (a, b, c, d)$ can be represented with an order pair of function of r-cut approach i.e.$[\underline{\mu}(r), \overline{\mu}(r)] = [a + (b - a)r, d - (d - c)r]$, where $r \in [0, 1]$

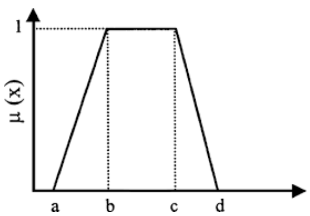

Fig.2 Trapezoidal fuzzy number

Definition 2.5 Gaussian Fuzzy Number
The asymmetric Gaussian fuzzy number $\tilde{A} = (\alpha, \sigma_l, \sigma_r)$.The membership function $\mu(x)$ of $\tilde{A}$ will be defined as follows.

$$\mu(x) = \begin{cases} e^{-\frac{(x-\alpha)^2}{2\sigma_l^2}} & , x \leq \alpha \\ e^{-\frac{(x-\alpha)^2}{2\sigma_r^2}} & , x \geq \alpha \end{cases}$$

where ,the modal value (center) denote as α and σ_l, σ_r denote left and right hand spreads(fuzziness i.e.width) corresponding to the Gaussian Distribution.For symmetric Gaussian fuzzy number the left and right-hand spreads are equal i.e.$\sigma_l = \sigma_r = \sigma$.So symmetric Gaussian fuzzy number may be written as $\tilde{A} = (\alpha, \sigma, \sigma)$ and corresponding function may be defined as $\mu(x) = e^{-\beta(x-\alpha)^2}, \forall x \in R$ where $\beta = \frac{1}{2\sigma^2}$. The symmetric Gaussian fuzzy number in parametric form can be represented as
$\tilde{A} = [\underline{\mu}(r), \overline{\mu}(r)] = \left[\alpha - \sqrt{-\frac{(log_e r)}{\beta}}, \alpha + \sqrt{-\frac{(log_e r)}{\beta}}\right]$ where $r \in [0, 1]$

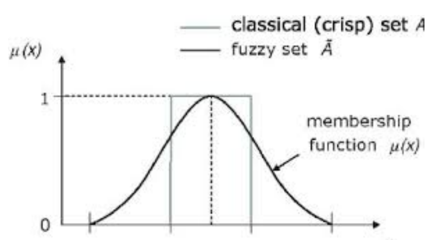

Fig.3 Gaussian fuzzy number

For all the above type of fuzzy numbers the left and right bound of fuzzy numbers satisfy the following requirements

1. $\underline{\mu}(r)$ is a bounded monotonic increasing left continuous function over $[0, 1]$,
2. $\overline{\mu}(r)$ is a bounded monotonic decreasing left continuous function $[0, 1]$,
3. $\underline{\mu}(r) \leq \overline{\mu}(r), 0 \leq r \leq 1$.

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Definition 2.6 Fuzzy arithmetic
For any arbitrary two fuzzy numbers $u = (\underline{u}(r), \overline{u}(r)), v = (\underline{v}(r), \overline{v}(r)), 0 \leq r \leq 1$ and arbitrary $k \in R$.we define addition,subtraction,multiplication,scalar multiplication by k.(see in [21])

$$u + v = (\underline{u}(r) + \underline{u}(r), \overline{v}(r) + \overline{v}(r)),$$
$$u - v = (\underline{u}(r) - \overline{v}(r), \overline{u}(r) - \underline{v}(r)),$$
$$u \cdot v =$$
$$(min\{\underline{u}(r)\overline{v}(r), \underline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r)\}, max\{\underline{u}(r)\overline{v}(r), \underline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r)\})$$

$$ku = \begin{cases} (k\underline{u}(r), k\overline{u}(r)), & k \geq 0 \\ (k\overline{u}(r), k\underline{u}(r)), & k < 0 \end{cases}$$

Definition 2.7 Hukuhara-difference
Let $x, y \in E$.If there exists $z \in E$ such that $x = y + z$,then z is called the Hakuhara-difference of fuzzy numbers x and y,and it is denoted by $z = x \ominus y$. The $\ominus$ sign stands for Hukuhara-difference,and $x \ominus y \neq x + (-1)y$.
Definition 2.8 Hukuhara-differentiability
Let $f : (a, b) \to E$ and $t_0 \in (a, b)$.We say that f is Hukuhara-differential at t_0,if there exists an element $f^{'}(t_0) \in E$ such that for all $h > 0$ sufficiently small, $\exists f(t_0 + h) \ominus f(t_0), f(t_0) \ominus f(t_0 - h)$ and the limits holds(in the metric D)

$$\lim_{h \to 0} \frac{f(t_0 + h) \ominus f(t_0)}{h} = \lim_{h \to 0} \frac{f(t_0) \ominus f(t_0 - h)}{h} = f^{'}(t_0).$$

Definition 2.9 Generalized Hukuhara differentiability
Let $f : (a, b) \to E$ and $t_0 \in (a, b)$.We say that f is (1)-differential at t_0,if there exists an element $f^{'}(t_0) \in E$ such that for all $h > 0$ sufficiently small, $\exists f(t_0 + h) \ominus f(t_0), f(t_0) \ominus f(t_0 - h)$ and the limits holds(in the metric D)

$$\lim_{h \to 0} \frac{f(t_0 + h) \ominus f(t_0)}{h} = \lim_{h \to 0} \frac{f(t_0) \ominus f(t_0 - h)}{h} = f^{'}(t_0).$$

and f is (2)-differentiable if for all $h > 0$ sufficiently small,$\exists f(t_0) \ominus f(t_0 + h)$,$\exists f(t_0 - h) \ominus f(t_0)$and the limits(in the metric D)

$$\lim_{h \to 0} \frac{f(t_0) \ominus f(t_0 + h)}{-h} = \lim_{h \to 0} \frac{f(t_0 - h) \ominus f(t_0)}{-h} = f^{'}(t_0).$$

If $f^{'}(t_0)$ exist in above cases then i.e called Generalized fuzzy derivative of $f(t)$.

Definition 2.10 Strongly generalized differentiability
Let $f : (a, b) \to E$ and $t_0 \in (a, b)$.We say that f is strongly generalized differential at t_0(Bede-Gal differential)if there exist an element $f^{'}(t_0) \in E$ such that (i)for all $h > 0$ sufficiently small,$\exists f(t_0 + h) \ominus f(t_0)$,$\exists f(t_0) \ominus f(t_0 - h)$and the

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limits(in the metric D)

$$\lim_{h\to 0}\frac{f(t_0+h)\ominus f(t_0)}{h}=\lim_{h\to 0}\frac{f(t_0)\ominus f(t_0-h)}{h}=f^{'}(t_0).$$

or
(ii)for all $h>0$ sufficiently small,$\exists f(t_0)\ominus f(t_0+h)$,$\exists f(t_0-h)\ominus f(t_0)$and the limits(in the metric D)

$$\lim_{h\to 0}\frac{f(t_0)\ominus f(t_0+h)}{-h}=\lim_{h\to 0}\frac{f(t_0-h)\ominus f(t_0)}{-h}=f^{'}(t_0).$$

or
(iii)for all $h>0$ sufficiently small,$\exists f(t_0+h)\ominus f(t_0)$,$\exists f(t_0-h)\ominus f(t_0)$and the limits(in the metric D)

$$\lim_{h\to 0}\frac{f(t_0+h)\ominus f(t_0)}{h}=\lim_{h\to 0}\frac{f(t_0-h)\ominus f(t_0)}{-h}=f^{'}(t_0).$$

or
(iv)for all $h>0$ sufficiently small,$\exists f(t_0)\ominus f(t_0+h)$,$\exists f(t_0)\ominus f(t_0-h)$and the limits(in the metric D)

$$\lim_{h\to 0}\frac{f(t_0)\ominus f(t_0+h)}{-h}=\lim_{h\to 0}\frac{f(t_0)\ominus f(t_0-h)}{h}=f^{'}(t_0).$$

(h and $-h$ at denominators mean $\frac{1}{h}$ and $\frac{1}{-h}$,respectively)
Theorem 2.1 [14].Let $f:R\to E$ be a function and denote$f(t)=(\underline{f}(t,r),\overline{f}(t,r))$,for each $r\in[0,1]$.Then

1. If f is (i)-differentiable,then $\underline{f}(t,r)$ and $\overline{f}(t,r)$ are differentiable function and
$f^{'}(t)=(\underline{f}^{'}(t,r),\overline{f}^{'}(t,r))$
2. If f is (ii)-differentiable,then $\underline{f}(t,r)$ and $\overline{f}(t,r)$ are differentiable function and
$f^{'}(t)=(\overline{f}^{'}(t,r),\underline{f}^{'}(t,r))$

Definition 2.11 Piecewise Continuous Function
$f(t)$ is piecewise continuous function in $a\le t\le b$ if there exist a finite numbers of points $t_1,t_2,......,t_N$ such that $f(t)$ is continuous on each open subinterval $a<t<t_1$,$t_1<t_2$,.......,$t_N<t<b$, and has a finite limit as t approaches each endpoint from the interior of that subinterval.
Definition 2.12 Exponential Order
$f(t)$ is of exponential order as $t\to\infty$ if there exist real constants $K,c,T\ni$ $|f(t)|\le e^{-ct},t\ge T$.

Definition 2.13 Fuzzy Laplace Transform
Fuzzy Laplace Transform is an example of integral transform relation of the form
$\tilde{F}(s) = \int_a^b K(t,s)f(t)dt$,
where t is time and $K(t,s)$ is kernel of transform which transform $f(t)$ to $\tilde{F}(s)$ i.e. which transform time domain to frequency domain.The most well known integral transform is Laplace transform
where $a = 0$ and $b = \infty$
$K(t,s) = e^{-st}$
$\tilde{F}(s) = \lim_{b\to\infty} \int_0^b e^{-st} f(t)dt$,
$\lim_{b\to\infty} \int_0^b e^{-st} f(t)dt = (\lim_{b\to\infty} \int_0^b e^{-st} \underline{f}(t)dt, \lim_{b\to\infty} \int_0^b e^{-st} \overline{f}(t)dt)$
also by using definition of classical Laplace transform:
$l[\underline{f}(t,r)] = \lim_{b\to\infty} \int_0^b e^{-st} \underline{f}(t)dt$ and
$l[\overline{f}(t,r)] = \lim_{b\to\infty} \int_0^b e^{-st} \overline{f}(t)dt$
then we follow
$L[f(t)] = (l[\underline{f}(t,r), l[\overline{f}(t,r))$
Theorem 2.2 [4].Let $f^{'}(t)$ be an integrable fuzzy-valued function,and $f(t)$ is the primitive of $f^{'}(t)$ on $[0,\infty)$.Then
$L[f^{'}(t)] = sL[f(t)] \ominus f(0)$
where f is (i)-differentiable
or
$L[f^{'}(t)] = (-f(0)) \ominus (-sL[f(t)])$
where f is (ii)-differentiable
Theorem 2.3 [4].Let $f(t)$ and $g(t)$ be continuous fuzzy-valued functions and c_1, c_2 are constants.suppose that $f(t)e^{-st}, g(t)e^{-st}$ are improper fuzzy Riemann-integrable on $[0,\infty]$,then
$L[(c_1 f(t)) + (c_2 g(t))] = (c_1 L[f(t)]) + (c_2 L[g(t)])$.
Theorem 2.4 Let $f^{''}(t)$ be integrable fuzzy-valued function,and $f(t)$,$f^{'}(t)$ are primitive of $f^{'}(t)$,$f^{''}(t)$ on $[0,\infty]$.Then
$L[f^{''}(t)] = s^2 L[f(t)] \ominus sf(0) \ominus f^{'}(0)$
where f is (i)-differentiable and $f^{'}$ is (i)-differentiable or
$L[f^{''}(t)] = s^2 L[f(t)] \ominus sf(0) - f^{'}(0)$
where f is (ii)-differentiable and $f^{'}$ is (ii)-differentiable or
$L[f^{''}(t)] = \ominus(-s^2)L[f(t)] - sf(0) - f^{'}(0)$
where f is (i)-differentiable and $f^{'}$ is (ii)-differentiable or
$L[f^{''}(t)] = \ominus(-s^2)L[f(t)] - sf(0) \ominus f^{'}(0)$
where f is (ii)-differentiable and $f^{'}$ is (i)-differentiable
Theorem 2.5 Let $f(t)$ satisfies the condition of existence theorem of Laplace transform and $L[f(t)] = \tilde{F}(s)$ then
$L[tf(t)] = -\tilde{F}^{'}(s)$
Hence if $f^{'}(t)$ satisfies the condition of existence theorem of Laplace transform then

$$L[tf^{'}(t)] = -\frac{d}{ds}L[f^{'}(t)] = -\frac{d}{ds}\{s\tilde{F}(s) - f(0)\} = -s\tilde{F}^{'}(s) - \tilde{F}(s)$$

similarly for $f^{''}$

$$L[tf^{''}(t)] = -\frac{d}{ds}L[f^{''}(t)] = -\frac{d}{ds}\{s^2\tilde{F}(s) - sf(0) - f^{'}(0)\} = -s^2\tilde{F}^{'}(s) - 2\tilde{F}(s) + f(0)$$

3 Examples

Example 3.1 Consider Variable coefficient differential equation $ty^{''} - y^{'} = -1$ subject to initial condition $y(0) = 0$
Consider Variable coefficient fuzzy differential equation where we are considering initial condition and forcing function both triangular fuzzy number

$$\begin{cases} ty^{''} - y^{'} = -\tilde{1},\ t \in [0,1] \\ y(0) = [r-1, 1-r] \\ \tilde{1} = [r, 2-r] \end{cases}$$

By using fuzzy Laplace transform method,we have:
$L[ty^{''}] - L[y^{'}] = -L[\tilde{1}]$
By using FLT of derivative and Differentiation of FLT
$l[\underline{y}(t,r)] = \frac{r-1}{s} + \frac{r}{s^2} + \frac{A}{s^3}$
$l[\overline{y}(t,r)] = \frac{1-r}{s} + \frac{2-r}{s^2} + \frac{A}{s^3}$,where A is constant.
Hence solution is as follows:
$\underline{y}(t,r) = (r-1) + rt + Bt^2$
$\overline{y}(t,r) = (1-r) + (2-r)t + Bt^2$,where B is constant.
The $\underline{y}(t,r)$ and $\overline{y}(t,r)$ at $r \in [0,1]$ are presented in Fig.4
$y(t) = \underline{y}(t,1) = \overline{y}(t,1) = t + Bt^2$

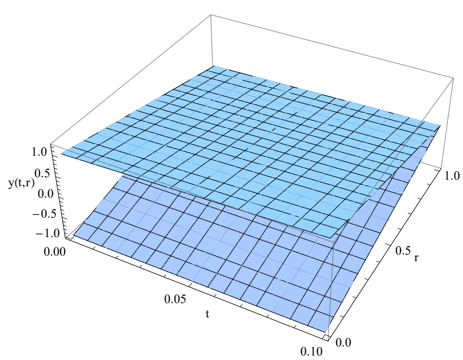

Fig.4 Solution y(t,r) by using Triangular fuzzy number

Consider Trapezoidal fuzzy number

$$\begin{cases} ty^{''} - y^{'} = -\tilde{1},\ t \in [0,1] \\ y(0) = [r-1, 2-r] \\ \tilde{1} = [r, 3-r] \end{cases}$$

By using fuzzy Laplace transform method,we have:
$L[ty^{''}] - L[y^{'}] = -L[\tilde{1}]$
By using FLT of derivative and Differentiation of FLT
$l[\underline{y}(t,r)] = \frac{r-1}{s} + \frac{r}{s^2} + \frac{A}{s^3}$
$l[\overline{y}(t,r)] = \frac{2-r}{s} + \frac{3-r}{s^2} + \frac{A}{s^3}$,where A is constant.
Hence solution is as follows:
$\underline{y}(t,r) = (r-1) + rt + Bt^2$
$\overline{y}(t,r) = (2-r) + (3-r)t + Bt^2$,where B is constant.
The $\underline{y}(t,r)$ and $\overline{y}(t,r)$ at $r \in [0,1]$ are presented in Fig.5
$\underline{y}(t,1) = t + Bt^2$, $\overline{y}(t,1) = 1 + 2t + Bt^2$

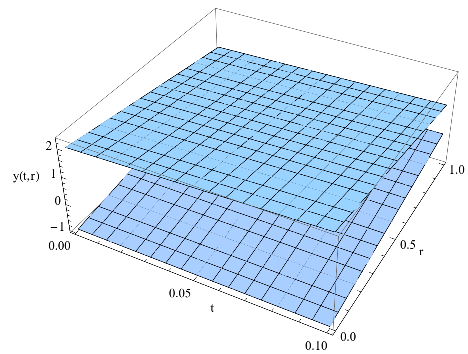

Fig.5 Solution y(t,r) by using Trapezoidal fuzzy number

Consider Gaussian fuzzy number

$$\begin{cases} ty^{''} - y^{'} = -\tilde{1},\ t \in [0,1] \\ y(0) = [-\sqrt{-(2log_e r)}, \sqrt{-(2log_e r)}] \\ \tilde{1} = [1 - \sqrt{(-2log_e r)}, 1 + \sqrt{(-2log_e r)}] \end{cases}$$

By using fuzzy Laplace transform method,we have:
$L[ty^{''}] - L[y^{'}] = -L[\tilde{1}]$
By using FLT of derivative and Differentiation of FLT
$l[\underline{y}(t,r)] = \frac{-\sqrt{-(2log_e r)}}{s} + \frac{1-\sqrt{(-2log_e r)}}{s^2} + \frac{A}{s^3}$
$l[\overline{y}(t,r)] = \frac{\sqrt{-(2log_e r)}}{s} + \frac{1+\sqrt{(-2log_e r)}}{s^2} + \frac{A}{s^3}$,where A is constant.
Hence solution is as follows:
$\underline{y}(t,r) = [-\sqrt{-(2log_e r)}] + [1 - \sqrt{(-2log_e r)}]t + Bt^2$
$\overline{y}(t,r) = [\sqrt{-(2log_e r)}] + [1 + \sqrt{(-2log_e r)}]t + Bt^2$,where B is constant.
The $\underline{y}(t,r)$ and $\overline{y}(t,r)$ at $r \in [0,1]$ are presented in Fig.6
$y(t) = \underline{y}(t,1) = \overline{y}(t,1) = t + Bt^2$

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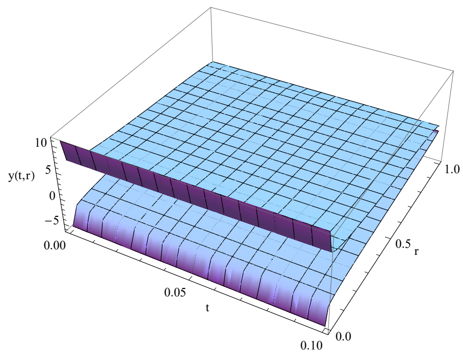

Fig.6 Solution y(t,r) by Gaussian fuzzy number

Example 3.2 Consider Variable coefficient differential equation $ty^{''}+2y^{'}+ty=0$ subject to boundary conditions
$y(0)=1$
$y(\pi)=0$
Consider Variable coefficient fuzzy differential equation where we are considering boundary conditions triangular fuzzy number

$$\begin{cases} ty^{''}+2y^{'}+ty=0, \\ y(0)=[r,2-r] \\ y(\pi)=[r-1,1-r] \end{cases}$$

By using fuzzy Laplace transform method,we have:
$L[ty^{''}]+2L[y^{'}]+L[ty]=-L[0]$
By using FLT of derivative and Differentiation of FLT $l[\underline{y}(t,r)]=(r)tan^{-1}(\frac{1}{s})$
$l[\overline{y}(t,r)]=(2-r)tan^{-1}(\frac{1}{s})$
Hence solution is as follows:
$\underline{y}(t,r)=(r)\frac{sint}{t}$
$\overline{y}(t,r)=(2-r)\frac{sint}{t}$
The $\underline{y}(t,r)$ and $\overline{y}(t,r)$ at $r\in[0,1]$ are presented in Fig.7
$y(t)=\underline{y}(t,1)=\overline{y}(t,1)=\frac{sint}{t}$

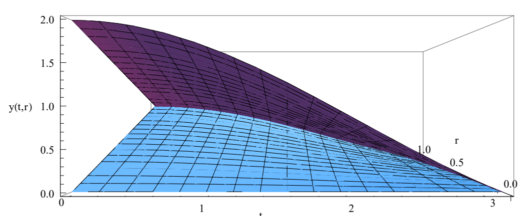

Fig.7 Solution y(t,r) by using Triangular fuzzy number

Consider Trapezoidal fuzzy number

$$\begin{cases} ty^{''}+2y^{'}+ty=0, \\ y(0)=[r,3-r] \\ y(\pi)=[r-1,2-r] \end{cases}$$

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By using fuzzy Laplace transform method,we have:
$L[ty^{''}] + 2L[y^{'}] + L[ty] = -L[0]$
By using FLT of derivative and Differentiation of FLT
$l[\underline{y}(t,r)] = (r)tan^{-1}(\frac{1}{s})$
$l[\overline{y}(t,r)] = (3-r)tan^{-1}(\frac{1}{s})$
Hence solution is as follows:
$\underline{y}(t,r) = (r)\frac{sint}{t}$
$\overline{y}(t,r) = (3-r)\frac{sint}{t}$
The $\underline{y}(t,r)$ and $\overline{y}(t,r)$ at $r \in [0,1]$ are presented in Fig.8
$\underline{y}(t,1) = \frac{sint}{t}$, $\overline{y}(t,1) = 2\frac{sint}{t}$

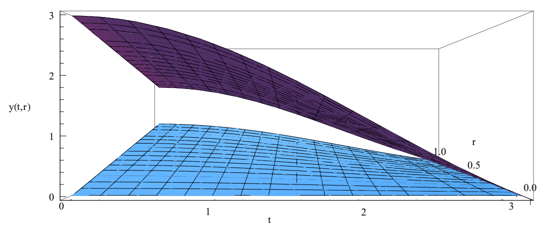

Fig.8 Solution y(t,r) by using Trapezoidal fuzzy number

Consider Gaussian fuzzy number

$$\begin{cases} ty^{''} + 2y^{'} + ty = 0, \\ y(0) = [1-\sqrt{-(2log_e r)}, 1+\sqrt{-(2log_e r)}] \\ y(\pi) = [-\sqrt{(-2log_e r)}, \sqrt{(-2log_e r)}] \end{cases}$$

By using fuzzy Laplace transform method,we have:
$L[ty^{''}] + 2L[y^{'}] + L[ty] = 0$
By using FLT of derivative and Differentiation of FLT
$l[\underline{y}(t,r)] = [1-\sqrt{(-2log_e r)}]tan^{-1}(\frac{1}{s})$
$l[\overline{y}(t,r)] = [1+\sqrt{(-2log_e r)}]tan^{-1}(\frac{1}{s})$
Hence solution is as follows:
$\underline{y}(t,r) = [1-\sqrt{(-2log_e r)}]\frac{sint}{t}$
$\overline{y}(t,r) = [1+\sqrt{(-2log_e r)}]\frac{sint}{t}$
$y(t) = \underline{y}(t,1) = \overline{y}(t,1) = \frac{sint}{t}$

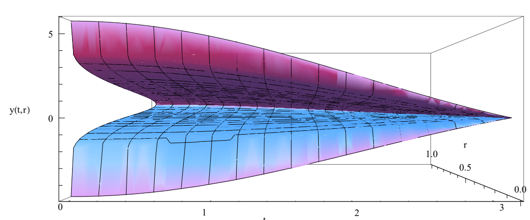

Fig.9 Solution y(t,r) by using Gaussian fuzzy number

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4 Result and Discussion

From,Examples 1,2 we see that the solution of second order FIVP and FBVP are depends on the derivative i.e.(i)-differentiable or (ii)-differentiable.Thus,as in above examples,the solution can be adequately chosen among four cases of the strongly generalize differentiability.On the other hand,In this new procedure unicity of the solution is lost because we have four possibilities,but flexibility is gained in fuzzy context.In above Examples,for Triangular and Gaussian fuzzy numbers at $r = 1$ upper bound and lower bounds are same and that is same as Exact solution of given differential equation but for Trapezoidal fuzzy number,it is some interval that contain Exact solution as lower bound or upper bound.

5 Conclusion

The Fuzzy Laplace transform method provided solutions to variable coefficient second order FIVPs and FBVPs by using the strongly generalize differentiability concept.Here we solved FIVPs and FBVPs by using different types of fuzzy numbers like Triangular,Trapezoidal and Gaussian.In that Triangular fuzzy number is easy to use in conclusion where as Trapezoidal fuzzy number required long calculations and Gaussian fuzzy number include logarithmic function i.e. again difficult to deal with it if asymmetric Gaussian fuzzy number occurs in calculation.The efficiency of method was described by solving numerical examples.

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