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Intuitionistic Hesitant Fuzzy Filters in BE-Algebras

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Abstract:- The notions of hesitant fuzzy filters and hesitant implicative filter was introduced. In this paper, we introduce the notion of intuitionistic hesitant fuzzy filters (IHFF) and intuitionistic hesitant implicative filters (IHIFB) and several properties are investigated. Also, we defined γ-level sets, and we show the relation between IHFF, IHIFF and γ-Level.

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1 Introduction and Preliminaries

Zadeh [10] proposed fuzzy set theory in 1965. The applications of fuzzy set theory appear in various fields. The intuitionistic fuzzy sets [1,2] was an extensions of fuzzy sets. Atanassov [1,2] proposed the Concept of intuitionistic fuzzy sets (IFS). It is useful in various application of science and engineering. IFS are associated for define in functions like non- membership function, membership function and hesitancy function. Liu and Wang [5] shows the basic concepts of intuitionistic fuzzy set theory and its practical applications. Recently Torra [8] and Torra, Narukawa [9] proposed the hesitant fuzzy sets (HFS). In 2011 Xia and Xu [11] introduced hesitant fuzzy information aggregation techniques and their applications for decision making. Also they defined some new operations on hesitant fuzzy sets and intuitionistic fuzzy sets. The hesitant fuzzy sets are an emerging and useful tool for dealing with uncertainty a vagueness. Interestingly, in motivates us to define new operators and properties of these sets, which must make them more important and applicable. A. Rezaei and A. Broumand [6] introduce the nation of hesitant fuzzy (implicative) filter and some properties. In this paper we introduce the notion of intuitionistic hesitant fuzzy (implicative, fantastic) filters and get some properties.

In this section, we cite the fundamental definitions that will be used in this paper:

Definitions 1.1. (Kim and Kim [4]) By a BE-algebra we shall mean an algebra $(X,*,1)$ of type (2,0) satisfying the following axioms:

(BE1) $x * x = 1$,

(BE2) $x * 1 = 1$,

(BE3) $1 * x = x$,

(BE4) $x * (y * z) = y * (x * z)$, for all $x, y, z \in X$.

From now on X is a BE-algebra, unless otherwise is stated. We introduce a relation "$\leq$" on X by $x \leq y$ if and only if $x * y = 1$.

Definitions 1.2. A BE-algebra X is said to be a self distributive if $x * (y * z) = (x * y) * (x * z)$, for all $x, y, z \in X$. A BE-algebra X is said to be commutative if $(x * y) * y = (y * x) * x$, for all $x, y \in X$.

Lemma 1.3. (Walendziak [10]) If X is a commutative BE-algebra, then for all $x, y \in X$, we have

$x * y = 1$ and $y * x = 1$ imply $x = y$.

We note that

"$\leq$" is reflexive by (BE1). If X is self distributive then relation "$\leq$" is a transitive ordered set on X, because if $x \leq y$ and $y \leq z$, then

$x * z = 1 * (x * z) = (x * y) * (x * z) = x * (y * z) = x * 1 = 1$. Hence $x \leq z$.

If X is commutative then by Lemma 2.3, relation "$\leq$" is antisymmetric. Hence if X is a commutative self distributive BE-algebra, then relation "$\leq$" is a partial ordered set on X.

Theorem 1.4. (Kim and Kim [4]) In a BE-algebra X, the following holds:

(i) $x * (y * x) = 1$,

(ii) $y * ((y * x) * x) = 1$, for all $x, y \in X$.

Definition 1.5. A subset F of X is called a filter of X if it satisfies:

(F1) $1 \in F$.

(F2) if $x \in F$ and $x * y \in F$ imply $y \in F$.

We define $A(x,y) = \{z \in X: x * (y * z) = 1\}$, which is called an upper set of x and y. It is easy to see that $1, x, y \in A(x,y)$, for every $x, y \in X$.

Definition 1.6. (Borumand and Rezaei [6]) A nonempty subset F of X is called an implicative filter if satisfies the following conditions:

(IF1) $1 \in F$,

(IF2) $x * (y * z) \in F$ and $x * y \in F$ imply $x * z \in F$, for all $x, y, z \in X$.

If we replace x of the condition (IF2) by the element 1, then it can be easily observed that every implicative filter is a filter. However, every filter is not an implicative filter as shown the following example.

Example 1.7. Let X = {1, a, b} be a BE-algebra with the following table:

*	1	a	b
1	1	a	b
a	1	1	a
b	1	a	1

Then $F = \{1, a\}$ is a filter of X, but it is not an implicative filter, since $1 * (a * b) = 1 * a = a \in F$ and $1 * a = a \in F$ but $1 * b = b \notin F$.

Definition 1.8. Let $(X, *, 1)$ and $(Y, o, 1')$ be two BE-algebras. Then a mapping $f : X \to Y$ is called a homomorphism if $(x_1 * x_2) = f(x_1) \circ f(x_2)$, for all $x_1, x_2 \in X$. It is clear that if $f : X \to Y$ is a homomorphism, then $f(1) = 1'$.

Definition 1.9. [2] A fuzzy set μ of X is called a fuzzy filter if satisfies the following conditions:

(FF1) $\mu(1) \geq \mu(x)$,

(FF2) $\mu(y) \geq min\{\mu(x * y), \mu(x)\}$, for all $x, y \in X$.

Definition 1.10. (Rao [7]) A fuzzy set μ of X is called a fuzzy implicative filter of X if satisfies the following conditions:

(FIF1) $\mu(1) \geq \mu(x)$,

(FIF2) μ (x*z) ≥ min{μ(x*(y*z)), μ (x*y)}, for all $x, y, z \in X$.

If we replace x of the condition (FIF2) by the element 1, then it can be easily observed that every fuzzy implicative filter is a fuzzy filter. However, every fuzzy filter is not a fuzzy implicative filter as shown the following example.

Example 1.11. (Rao [7]) Let $X = \{1, a, b, c, d\}$ be a BE-algebra with the following table.

*	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	b
b	1	a	1	b	a
c	1	a	1	1	a
d	1	1	1	b	1

Then it can be easily verified that $(X, *, 1)$ is a BE-algebra. Definition a fuzzy set μ on X as follows:

$$\mu(x) = \begin{cases} 0.9 & if\ x = 1, a \\ 0.2 & otherwise \end{cases}$$

Then clearly μ is a fuzzy filter of X, but it is not a fuzzy implicative filter of X, since

$$\mu(b * c) \ngeq \min\{\mu(b * (d * c)), \mu(b * d)\}.$$

Definition 1.12. (Torra [8]) Let X be a reference set. Then a hesitant fuzzy set HFSA of X is represented mathematical as:

$$A = \{\langle x, \hbar_A(x)\rangle : \hbar_A(x) \in \rho([0,1]), x \in X\},$$ where $\rho([0,1])$ is the power set of [0,1].

So, we can define a set of fuzzy sets an HFS by union of their membership functions.

Definition 1.13. (Torra [8]) Let $A = \{\mu_1, \mu_2, \ldots, \mu_n\}$ be a set of n membership functions. The HFS that is associated with $A, \hbar_A$, is define as:

$$\hbar_A : X \to \rho([0,1]), \qquad \hbar_A(x) = \cup_{\mu \in A}\{\mu(x)\}.$$

It is remarkable that this Definition is quite suitable to decision making, when experts have to assess a set of alternatives. In such a case, A represents the assessments of the experts for each alternative and $\hbar_A$ the assessments of the set of experts. However, note that it only allows to recover those HFSs whose memberships are given by sets of cardinality less than or equal to n.

For convenience, Xia and Xu in [11] named the set $\hbar = \hbar_A(x)$ as a hesitant fuzzy element HFE. The family of all hesitant fuzzy elements Definition on X show by HFE(X).

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2 Main Result

2.1 Intuitionistic Hesitant Fuzzy Filters

Definition 2.1.1. Let X be a reference set. An Intuitionistic hesitant fuzzy set $(IHFS)$ I in X is,

$I = \{< x, \hbar_I(x), k_I(x) >: x \in X\}$ such that; $\hbar_I(x), k_I(x)$: X → ρ([0,1]) are maps.

Definition 2.1.2. An Intuitionistic hesitant fuzzy set I of X is called an Intuitionistic hesitant fuzzy filter (IHFF), if satisfies the following conditions:

(IHFF1) $\hbar_I(x) \sqsubseteq \hbar_I(1)$ and $k_I(x) \sqsupseteq k_I(1)$,

(IHFF2) $\hbar_I(x) \sqcap \hbar_I(x * y) \sqsubseteq \hbar_I(y)$ and $k_I(x * y) \sqcup k_I(x) \sqsupseteq k_I(y)$, for all $x, y \in X$.

Lemma 2.1.3. Let I be an IHFF, than for all $x, y \in X$, if $x \leq y$ then;$\hbar_I(x) \sqsubseteq \hbar_I(y)$ and $k_I(x) \sqsupseteq k_I(y)$.

Proof. Let $x \leq y$. Then $x * y = 1$ and so $\hbar_I(x * y) = \hbar_I(1) \sqsubseteq \hbar_I(1)$ and $k_I(x * y) \sqcup k_I(1) \sqsupseteq k_1(1)$. By (IHFF2), we have $\hbar_I(x) = \hbar_I(x) \sqcap \hbar_I(1) = \hbar_I(x) \sqcap \hbar_I(x * y) \sqsubseteq \hbar_I(y)$ and $k_I(x) = k_I(x) \sqcup k_I(I) = k_I(x) \sqcup k_I(x * y) \sqsupseteq k_I(y)$.

Example 2.1.4. Let X={1,a,b} with the following table.

*	1	A	b
1	1	A	b
a	1	1	b
b	1	1	1

Then *(X,*,1)* is a BE-algebra. Let $\hbar_I(a) = \{0.4,0.5\}$, $\hbar_I(b) = \{0.2,0.3\}$, $\hbar_I(1) = \{0.6,0.8\}$ and

$k_I(a) = \{0.7,0.9\}, k_I(b) = \{0.2,0.25\}$, $k_I(1) = \{0.1,0.14\}$. Then I is an intuitionistic hesitant fuzzy filter.

Definition 2.1.5. let A and B be two IHFF(X) and $x \in X$. Then we defined

(i) $A \sqsubseteq B \Leftrightarrow\neg\ \hbar_A(x) \sqsubseteq \hbar_B(x) and\ k_A(x) \sqsupseteq k_B(x)$,

(ii) $A = B \leftrightarrow \hbar_A(x) = \hbar_B(x) and\ k_A(x) = k_B(x)$,

(iii) $A^c = \{\langle x, k_A(x), h_A(x)\rangle : x \in X\}$,

(iv)
$A \sqcap B = \{\langle x, inf\{\hbar_A(x), \hbar_B(x)\},\ sup\{k_A(x), k_B(x)\}\rangle : x \in X,$

(v)
$A \sqcup B = \{\langle x, sup\{\hbar_A(x), \hbar_B(x)\}, inf\{k_A(x), k_B(x)\rangle : x \in X\}$

Proposition 2.1.6. Let A ∈ IHFF(X) and $x, y, z \in X$. Then

(i) $\hbar_A(x) \sqsubseteq \hbar_A(x * y)$ and $k_A(x) \sqsupseteq k_A(x * y)$,

(ii) $\hbar_A(x) \sqcap \hbar_A(y) \sqsubseteq \hbar_A(x * y)$ and $k_A(x) \sqcup k_A(y) \sqsupseteq k_A(x * y)$,

(iii) $\hbar_A(x) \sqsubseteq \hbar_A\big((x * y) * y\big)$ and $k_A(x) \sqsupseteq k_A((x * y) * y)$,

(iv) $\hbar_A(x) \sqcap \hbar_A(y) \sqsubseteq \hbar_A((x * (y * z)) * z)$ and $k_A(x) \sqcup k_A(y) \sqsupseteq k_A((x * (y * z)) * z)$.

Proof. (i). Since x ≤ y ∗ x, by Lemma 1, (i) holds.

(ii). Since $\hbar_A(x) \sqcap \hbar_A(y) \sqsubseteq \hbar_A(y)$ .By (i), $\hbar_A(y) \sqsubseteq \hbar_A(x * y)$, so $\hbar_A(x) \sqcap \hbar_A(y) \sqsubseteq \hbar_A(x * y)$.

And, $k_A(x) \sqcup k_A(y) \sqsupseteq k_A(y)$, by (i) , $k_A(y) \sqsupseteq k_A(x * y)$, so $k_A(x) \sqcup k_A(y) \sqsupseteq k_A(x * y)$.

(iii). We have $\hbar_A(x) = \hbar_A(x) \sqcap \hbar_A(1) = \hbar_A(x) \sqcap \hbar_A\big((x * y) * (x * y)\big) = \hbar_A(x) \sqcap \hbar_A\Big(x * \big((x * y) * x\big)\Big) \sqsubseteq \hbar_A\big((x * y) * y\big)$.

And, $k_A(x) = k_A(x) \sqcup k_A(1) = k_A(x) \sqcup k_A\big((x * y) *$ *x∗y= kAx⊔kAx∗x∗y∗x⊒kAx∗y∗y.*

(iv). From (iii) we have

$$\hbar_A(x) \sqcap \hbar_A(y) \sqsubseteq \hbar_A\big(\big(x * (y * x)\big) * (y * x)\big) \sqcap \hbar_A(y) \sqsubseteq \hbar_A\big(\big(x * (y * z)\big) * z\big).$$

$$k_A(x) \sqcup k_A(y) \sqsupseteq k_A\big(\big(x * (y * x)\big) * (y * x)\big) \sqcap \hbar_A(y) \sqsupseteq \hbar_A\big(\big(x * (y * z)\big) * z\big).$$

Lemma 2.1.7. Let $F \in IHF(X)$ and x,y,z,$a_i \in X$ for i=1,2,…,n. Then;

(i) If $z \in A(x, y)$, then $\hbar_F(x) \sqcap \hbar_F(y) \sqsubseteq \hbar_F(z)$ and $k_F(x) \sqcup k_F(y) \sqsupseteq k_F(z)$.

(ii) If $\prod_{i=1}^{n} a_i * x = 1$, then $\prod_{i=1}^{n} \hbar_F(a_i) \sqsubseteq \hbar_F(x)$ and $\sqcap\ k_F(a_i) \sqsupseteq k_F(x)$, where

$$\prod_{i=1}^{n} a_i * x = a_n * (a_{n-1} * \dots .* (a_1 * x) \dots).$$

Proof. (i) Let $z \in A(x, y)$. Then $x * (y * z) = 1$. Hence $\hbar_F(x) \sqcap \hbar_F(y) = \hbar_F(x) \sqcap \hbar_F(y) \sqcap \hbar_F(1) = \hbar_F(x) \sqcap \hbar_F(y) \sqcap \hbar_F\big(x * (y * z)\big) \sqsubseteq \hbar_F(y) \sqcap \hbar_F(y * z) \sqsubseteq \hbar_F(z)$.

And, $k_F(x) \sqcup k_F(y) = k_F(x) \sqcup k_F(y) \sqcup k_F(1) = k_F(x) \sqcup k_F(y) \sqcup k_F(x*(y*z)) \sqsupseteq k_F(y) \sqcup k_F(y*z) \sqsupseteq k_F(z)$.

(ii). By (i) it is true for $n = 1,2$. We proof it by induction on n. Assume that it satisfies for $n = k$.

Suppose that $\prod_{i=1}^{k+1} a_i * x = 1$, for $a_1, \dots, a_k, a_{k+1} \in X$. By induction hypothesis, $\prod_{i=2}^{k+1} \hbar_F(a_i * x) \sqsubseteq \hbar_F(a_1 * x)$. Since F is a hesitant fuzzy filter, we have;

$$\prod_{i=1}^{k+1} \hbar_F(a_i) = \left(\prod_{i=2}^{k+1} \hbar_F(a_i)\right) \sqcap \hbar_F(a_1) \sqsubseteq \hbar_F(a_1 * x) \sqcap \hbar_F(a_1) \sqsubseteq \hbar_F(x).$$

And also by induction hypothesis $\coprod_{i=2}^{k+1} k_F(a_i) \sqsupseteq$ *kFa1∗x*. Since *F* is a hesitant fuzzy filter, we have;

$$\coprod_{i=1}^{k+1} k_F(a_i) = \coprod_{i=1}^{k+1} k_F(a_i) \sqcup k_F(a_1) \sqsupseteq k_A(a_1 * x) \sqcup k_A(a_1) \sqsupseteq k_A(x).$$

Theorem 2.1.8. Let $I \in IHFF(X)$. Then the set $\chi_I = \{x \in X : \hbar_I(X) = \hbar_I(1), k_I(x) = k_I(1)\}$ is a filter of X.

Proof. Obviously, $1 \in \chi_I$. Let x, $x*y \in \chi_I$. Then $\hbar_I(x) = \hbar_I(x*y) = \hbar_I(1), k_I(x) = k_I(x*y) = k_I(1)$. Thus $\hbar_I(1) = \hbar_I(x) \sqcap \hbar_I(x*y) \sqsubseteq \hbar_I(y) \sqsubseteq \hbar_I(1)$. Hence $\hbar_I(y) = \hbar_I(1)$, and $k_I(1) = k_I(x) \sqcup k_I(x*y) \sqsupseteq k_I(y) \sqsupseteq k_I(1)$. Hence $k_I(y) = k_I(1)$,So , $y \in \chi_I$.

Definition 2.1.9. Let $\gamma \in \rho([0,1])$ and $I \in IHFF(X)$. $\gamma - level$ set of I, is defined by

$$L_I(\hbar_I, k_I, \gamma) = \{x \in I : \gamma \sqsubseteq \hbar_I(x), k_I(x) \sqsubseteq \gamma^c\}.$$

Lemma 2.1.10. Let $\beta, \gamma \in \rho([0,1])$. If $\beta \sqsubseteq \gamma$, then $L(\hbar_I, k_I, \gamma) \sqsubseteq L(\hbar_I, k_I, \beta)$.

Proof. Let $x \in L(\hbar_I, k_I, \gamma)$. Then $\gamma \sqsubseteq \hbar_I(x)$. Since $\beta \sqsubseteq \gamma$, we have, $\beta \sqsubseteq \hbar_I(x)$. Also, since $k_I(x) \sqsubseteq \gamma^c$ and $\beta^c \sqsupseteq \gamma^c$, we have $\hbar_I(x) \sqsubseteq \beta^\gamma$. Therefore, $x \in L(\hbar_I, k_I, \beta)$.

Theorem 2.1.11. Let $I \in IHFF(X)$ and $\gamma \in \rho([0,1])$. If $L(\hbar_I, k_I, \gamma) \neq \emptyset$, then $L(\hbar_I, k_I, \gamma)$ is a filter of X.

Proof. Let $x, x*y \in L(\hbar_I, k_I, \gamma)$ for any $\gamma \in \rho([0,1])$. Therefore $\gamma \sqsubseteq \hbar_I(x), \gamma \sqsubseteq \hbar_I(x*y)$ and $\hbar_I(x) \sqsubseteq \gamma^c, \hbar_I(x*y) \sqsubseteq \gamma^c$. Hence $\gamma \sqsubseteq \hbar_I(x) \sqcap \hbar_I(x*y) \sqsubseteq \hbar_I(y)$ and $\gamma^c \sqsupseteq k_I(x) \sqcup k_I(x*y) \sqsupseteq k_I(y)$ therefore $\gamma \sqsubseteq \hbar_I(y)$ and $k_I(y) \sqsubseteq \gamma^c$ hence $y \in L(\hbar_I, k_I, \gamma)$. Obviously 1$\in L(\hbar_I, k_I, \gamma)$.

Theorem 2.1.12. Let $A \in IHFS(X)$ and $\gamma \in \rho([0,1])$. If $L(\hbar_A, k_A, \gamma)$ be a filter of X, then $A \in IHFF(X)$.

Proof. Let $L(\hbar_A, k_A, \gamma)$ be a filter for all $\gamma \in \rho([0,1])$. If for any $x \in X$ we consider $\hbar_A(x) = \gamma$ and $k_A(x) = \gamma^c$ then $x \in L(\hbar_A, k_A, \gamma)$. Since $L(\hbar_A, k_A, \gamma)$ is a filter of X, we have $1 \in L(\hbar_A, k_A, \gamma)$. Therefore $\hbar_A(x) = \gamma \sqsubseteq \hbar_A(1)$ and $k_A(x) = \gamma^c \sqsupseteq k_A(1)$. Now, for any $,y \in X$, let $\hbar_A(x*y) = \gamma_1, k_A(x*y) = {\gamma_1}^c$ and $\hbar_A(x) = \gamma_2, k_A(x) = {\gamma_2}^c$. Assume that $= \gamma_1 \sqcap \gamma_2, (\gamma^c = {\gamma_1}^c \sqcup {\gamma_2}^c)$. We see that $x, x*y \in L(\hbar_A, k_A, \gamma)$, so, $y \in L(\hbar_A, k_A, \gamma)$. Hence $\gamma \sqsubseteq \hbar_A(y)$ and $k_A(y) \sqsubseteq \gamma^c$. We have $\hbar_A(x) \sqcap \hbar_A(x*y) = \gamma_1 \sqcap \gamma_2 = \gamma \sqsubseteq \hbar_A(y)$ and $k_A(x) \sqcup k_A(x*y) = {\gamma_1}^c \cup {\gamma_2}^c = \gamma^c \sqsupseteq k_A(y)$. Therefore $A \in IHFF(X)$.

Theorem 2.1.13. Let $I \in IHFF(X)$, $\gamma \in \rho([0,1])$ and $u, v \in X$. If $u, v \in L(\hbar_I, k_I, \gamma)$, then $A(u,v) \sqsubseteq L(\hbar_A, k_A, \gamma)$.

Proof. Since $u , v \in L(\hbar_I, k_I, \gamma)$, we have $\gamma \sqsubseteq \hbar_I(u), k_I \sqsubseteq \gamma^c$ and $\gamma \sqsubseteq \hbar_I(v), k_I(u) \sqsubseteq \gamma^c$.

Let $z \in A(u,v)$. Then $u*(v*z) = 1$ and so we have, $\gamma \sqsubseteq \hbar_I(u) \sqcap \hbar_I(v) \sqsubseteq \hbar_I(u*(v*z)*z) = \hbar_I(1*z) = \hbar_I(z)$ and $\gamma^c \sqsupseteq k_I(u) \sqcup k_I(v) \sqsupseteq k_I(u*(v*z)*z) = k_I(1*z) = k_I(z)$.

We obtain, $z \in L(\hbar_I, k_I, \gamma)$. Therefore $(u,v) \sqsubseteq L(\hbar_I, k_I, \gamma)$.

Corollary 2.1.14. Let $I \in IHFF(X)$ and $\gamma \in \rho([0,1])$. If $L(\hbar_A, k_A, \gamma) \neq \emptyset$, then $L(\hbar_I, k_I, \gamma) = \coprod_{u,v \in L(\hbar_I, k_I, \gamma)} A(u,v)$.

Proof. It is obviously $\coprod_{u,v \in L(\hbar_I, k_I, \gamma)} A(u,v) \sqsubseteq L(\hbar_I, k_I, \gamma)$. Now, assume that $u \in L(\hbar_I, k_I, \gamma)$. Since

$u*(1*u) = 1$, we have $u \in A(u,1)$. Therefore, $L(\hbar_A, k_A, \gamma) \sqsubseteq A(u,1) \sqsubseteq \coprod_{u \in L(\hbar_A, k_A, \gamma)} A(u,v) \sqsubseteq$ *u,v∈Lh A,kA,γA(u,v).*

2.2 Intuitionistic Hesitant Fuzzy Implicative Filters

Definition 2.2.1. Let $A \in IHF(X)$. A is called an Intuitionistic hesitant fuzzy implicative filter if satisfies the following condition holds for every x,y,z$\in X$.

(IHFIF1) $\hbar_A(x) \sqsubseteq \hbar_A(1), k_A(x) \sqsupseteq k_A(1)$,

(IHFIF2)$\hbar_A(x*(y*z)) \sqcap \hbar_A(x*y) \sqsubseteq \hbar_A(x*z), k_A(x*(y*z)) \sqcup k_A(x*y) \sqsupseteq k_A(x*z)$.

Theorem 2.2.2. Let X be a self distributive BE-algebra. Then every $IHFF$ is a $IHFIF$ of X.

Proof. Let $I \in IHFF(X)X$. Obvious for every $x \in X$, we have $\hbar_I(x) \sqsubseteq \hbar_I(1), K_I(x) \sqsupseteq K_I(1)$. Now, let $x, y, z \in X$. Since X is self distributive, we have, $x*(y*z) = (x*y)*(x*z)$, by (IHFIF2), we have,

$$\hbar_I(x*(y*z)) \sqcap \hbar_I(x*y) = \hbar_I((x*y)*(x*z)) \sqcap \hbar_I(x*y) \sqsubseteq \hbar_I(x*z),$$

And, $k_I(x*(y*z)) \sqcup k_I(x*y) = k_I((x*y)*(x*z)) \sqcup k_I(x*y) \sqsupseteq k_I(x*z)$.

Therefore, $I \in LHFIF(X)$.

Note. The converse of the above theorem may be not true. The following example show this.

Example 2.2.3. Let $X = \{1, a, b, c, d\}$ with the following table.

*	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	b
b	1	a	1	b	a
c	1	a	1	1	a
d	1	1	1	b	1

Then $(X,*,1)$ is a BE-algebra. But, $a*(b*d) = a*a = 1$ and $(a*b)*(a*d) = b*d = a$. This shows that X is not self distributive. Let $h_I(1) = h_I(a) = \{0.8\,, 0.7\}$ and $h_I(b) = h_I(c) = h_I(d) = \{0.4\}$ and, $K_I(1) = K_I(a) = \{0.2\,, 0.3\}$, $K_I(b) = K_I(c) = K_I(d) = \{0.6\}$.

Then $I \in IHFF(X)$. But, we have, $h_I(b*c) = h_I(b) = \{0.4\}$, this shows that $h_I(b*(d*c)) \sqcap h_I(b*d) \not\sqsubseteq h_I(b*c)$. So, $I \notin IHFIF(X)$.□

Theorem 2.2.4. Let F be a implicative filter of X. Then there exists a Intuitionistic hesitant implicative fuzzy filter $I =< x, h_I(x), K_I(x) >$ of X such that $L(h_I, K_I, \delta) = F$, For some $\delta \epsilon P([0,1])$.

Proof. Define h_I and K_I as follows

$$h_I(x) = \begin{cases} F, & x \in F \\ \emptyset & otherwise \end{cases} \quad ,$$

$$k_I(x) = \begin{cases} F^C & if\ x\epsilon F \\ X & otherwise \end{cases}$$

Where $J = [0,1]$, and γ is a fixed subset of $[0,1]$. Since $1 \in F$, we have h_I(x)⊑ F=h_I (1) and $k_I(x) \sqsupseteq F^C = k_I(1)$.

We will showed I=$< x$,h_I(x),k_I(x)>∈ $IHIFF(X)$, first now we show that

(1) $h_I(x*(y*z)) \sqcap h_I(x*y) \sqsubseteq h_I(x*z)$,

(2) $k_I(x*(y*z)) \sqcup k_I(x*y) \sqsupseteq k_I(x*z)$.

We consider the following cases:

Case 1. If $x*(y*z) \in F, x*y \in F$, then $x*z \in F$. Hence $h_I(x*(y*z)) = \gamma$, $h_I(x*y) = \gamma$. Then $h_I(x*z) = \gamma$, and (1) holds.

Also, $k_I\big(x*(y*z)\big) = \gamma^c$, $k_I(x*y) = \gamma^c$. Therefore $k_I(x*z) = \gamma^c$ and (2) holds.

Case 2. If $x*(y*z) \in F, x*y \notin F$, then $h_I(x*y) = \emptyset$. Therefore $h_I(x*(y*z)) \sqcap h_I(x*y) = \emptyset \sqsubseteq h_I(x*z)$, and (1) holds.

Also $k_I(x*y) = X$, we have $k_I(x*y) \sqcup k_I(x*(y*z)) \sqsupseteq k_I(x*z)$, and (2) holds.

Case 3. If $x*(y*z) \notin F, x*y \in F$, is similary to cases (2).

Case 4. If $x*(y*z) \notin F, x*y \notin F$, then $h_I\big(x*(y*z)\big) = h_I(x*y) = \emptyset \sqsubseteq h_I(x*z)$, obvious, (1) holds,

Also, $k_I(x*y) = k_I\big(x*(y*z)\big) = X \sqsupseteq k_I(x*z)$ and (2) holds.

Clearly $L(h_I, h_I, \gamma) = \{X \in I: \gamma \sqsubseteq h_I(X), K_I(X) \sqsubseteq \gamma\} = F$.

In the same manner we can show that if F be a filter of X. Then there exists a intuitionistic hesitant fuzzy filter $I =< x\,, h_A\,, k_A >$, such that, $L(h_I\,, k_I, \gamma) = F$.

Proposition 2.2.5. Let X be a self distributive BE-algebra and $I \in IHFF(X)$. Then the following conditions are equivalent for all $x, y, z \in X$.

(i) $I \in IHFIF\ (X)$,

(ii) $h_I(y*(y*x)) \sqsubseteq h_I(y*x)$ and $k_I\big(y*(y*x)\big) \sqsupseteq k_I(y*x)$,

(iii) $h_I(\big(z*\big(y*(y*x)\big)\big)) \sqcap h_I(z) \sqsubseteq h_I(y*z)$ and $k_I(\big(z*\big(y*(y*x)\big)\big)) \sqcup k_I(z) \sqsupseteq k_I(y*z)$.

Proof. (i)⇒(ii). Since $I \in IHFIF(X)$. By (BE1) we have:

$h_I(y*(y*x)) =) h_I(y*(y*x)) \sqcap \mathrm{h_I}(1) =) h_I(y*(y*x)) \sqcap \mathrm{h_I}(y*y) \sqsubseteq \mathrm{h_I}(y*x)$ and $\mathrm{k_I}\big(y*(y*x)\big) = \mathrm{k_I}\big(y*(y*x)\big) \sqcup \mathrm{k_I}(1) = \mathrm{k_I}\big(y*(y*x)\big) \sqcup \mathrm{k_I}(y*y) \sqsupseteq \mathrm{k_I}(y*x)$

(ii)⇒(iii). Let $I \in IHFF(X)$, that satisfying (ii). We have

$\mathrm{h_I}(z*(y*(y*x))) \sqcap \mathrm{h_I}(z) \sqsubseteq \mathrm{h_I}\big(y*(y*x)\big) \sqsubseteq \mathrm{h_I}(y*$
x and kIz∗y∗y∗x⊔hIz⊒kIy∗y∗x⊒kIy∗x.

(iii)⇒(i). Since $x*(y*z) = \big(y*(x*z)\big) \leq (x*y)*\big(x*(x*z)\big)$, we haveTherefore, $\mathrm{h_I}\big(x*(x*z)\big) \sqsubseteq \mathrm{h_I}\,\Big(\big(x*y\big)*\big(x*(x*z)\big)\Big)$. Thus $\mathrm{h_I}\big(x*(y*z)\big) \sqcap \mathrm{h_I}(x*y) \sqsubseteq h_I(((x*y)*(x*(x*z))) \sqcap h_I(x*y) \sqsubseteq h_I(x*z)$ and $k_I((x*(x*z) \sqcup k_I\,(x*y) \sqsupseteq k_I(((x*y)*(x*(x*z))) \sqcup k_I(x*y) \sqsupseteq k_I(x*z)$

Therefore, $I \in IHFIF(X)$.

Let $f: X \to Y$ be a homomorphism of BE-algebra X onto BE-algebra Y and $A \in IHFS(Y)$. Define mapping $\hbar_{A(f)}$ and $k_{A(f)}: X \to p([0,1])$, by $\hbar_{A(f)}(x) = \hbar_A(f(x))$ and $k_{A(f)}\ (x) = k_A(f(x))$. This mapping are composition of two mapping, thus are well define and $A(f\ ) = \{x \in X\ :\ f(x) \in\ A\}$.

Proposition 2.2.6. Let $f: X \to Y$ be onto homomorphism of BE-algebras. $I\ \in IHFS(Y)$ (resp., $I \in IHFIF(Y)$) if and only if I(F)∈IHFF(X) (resp., $I \in\ IHFIF(X)$).

Proof. Assume that $I\ \in\ IHFIF(Y)$. For any $x \in X$, we have;

$\hbar_{(A(f))}\ (x) =\ \hbar_A\ (f(x))\ \sqsubseteq \hbar_A(\ 1_y)\ = \hbar_A(f(\ 1_y))\ = \hbar_{(A(f))}\ (1_x)$.

And

$k_{(A(f))}\ (x) =\ k_A\ (f(x))\ \sqsupseteq k_A(\ 1_y)\ = k_A(f(\ 1_y))\ = k_{(A(f))}\ (1_x)$.

Therefore, $A(f)\ \in\ IHFIF(X)$.

Conversely, assume that $\boldsymbol{A(f)} \in \boldsymbol{IHFF}$ and $\boldsymbol{y} \in \boldsymbol{Y}$. Since $\boldsymbol{f}$ is onto, there exists $\boldsymbol{x} \in \boldsymbol{X}$ such that $\boldsymbol{f(x) = y}$. Therefore, $\boldsymbol{h_A(y) = h_A(f(x)) = h_{f(A)}(y) \sqsubseteq h_{f(A)}(1_x) = h_A(f(1_x)) = h_A(1_y)}$.
And $\boldsymbol{k_A(y) = k_A(f(x)) = k_{f(A)}(y) \sqsupseteq k_{f(A)}(1_x) = k_A(f(1_x)) = k_A(1_y)}$.
Now let $x, y \in Y$. Then there exists $u, v \in X$ such that $f(u) = x, f(v) = y$, and we have;

$\hbar_A(x * y) \sqcap\ \hbar_A(x) =\ \hbar_A(f(u) * f(v)) \sqcap \hbar_A(f(u)) = \hbar_A(f(u * v))\ \sqcap \hbar_A(f(u)) =\ \hbar_{f(A)}(u * v) \sqcap \hbar_{f(A)}(u) \sqsubseteq \hbar_{f(A)}(v) =\ \hbar_A(f(v)) = \hbar_A(y)$.

And, $k_A(x * y) \sqcup\ k_A(x) =\ k_A(f(u) * f(v)) \sqcup k_A(f(u)) = k_A(f(u * v))\ \sqcup \hbar_A(f(u)) =\ k_{f(A)}(u * v) \sqcup k_{f(A)}(u) \sqsupseteq k_{f(A)}(v) =\ k_A(f(v)) = k_A(y))$. Hence $A \in IHFF(Y)$.

2.3 Fantastic Hesitant Intuitionistic Fuzzy Filters

Definition 2.3.1. Hesitant fuzzy set A is called a fantastic hesitant intuitionistic fuzzy filters of X if and only if satisfies the following conditions:

(FHIFF2) $\hbar_F(x * (y * z)) \sqcap \hbar_F(x) \sqsubseteq \hbar_F(((\left((z * y) * y\right) * z)$,

(FHIFF2) $k_F(x * (y * z)) \sqcup k_F(x) \sqsupseteq (((\left((z * y) * y\right) * z)$.

Example 2.3.2. Let $X = \{1, a, b, c, d\}$. Define binary operation $*$ on X as follows:

*	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	b
b	1	a	1	b	a
c	1	a	1	1	a
d	1	1	1	b	1

Then it can easily verified that $(X,*, 1)$ is BE-algebra. Now let;

$\hbar_F(1) = \hbar_F(c) = \hbar_F(d) = \{0.8, 0.9\}, \hbar_F(b) = \hbar_F(a) = \{0.6, 0.7\}$ and $k_F(1) = k_F(c) = k_F(d) = \{0.2, 0.3\}$, $k_F(a) = k_F(b) = \{0.4, 0.5\}$.

Obviously, we can check $< x, \hbar_F(x), k_F(x) >$ is an (IFHFF) in X.

Theorem 2.3.3. Every IFHFF in a BE-Algebra X is an IHFF in X.

Proof. Let $x, y, z \in X$. We have, $\hbar_F(x) = \hbar_F(x) \sqcap \hbar_F(x * (1 * 1)) \sqsubseteq \hbar_F(\left(\left((1 * 1) * 1\right) * 1\right) = \hbar_F(1)$.

Also, $k_F(x) = k_F(x) \sqcap k_F(x * (1 * 1)) \sqsupseteq k_F(\left(\left((1 * 1) * 1*1=kF1\right.\right.$.

Put $y := 1$ in IFHFF2, we obtain;

$$\hbar_F(x * z) \sqcap \hbar_F(x) \sqsubseteq\ \hbar_F(x * (1 * z)) \sqcap \hbar_F(x) \sqsubseteq \hbar_F(\left(\left((z * 1) * 1\right) * z\right) =\ \hbar_F(1 * z) = \hbar_F(z),$$

And, $k_F(x * z) \sqcup k_F(x) \sqsupseteq k_F(x * (1 * z)) \sqcup k_F(x) \sqsupseteq k_F(\left(\left((z * 1) * 1\right) * z\right) =\ k_F(1 * z) = k_F(z)$.

Therefore, $F \in IHFF(X)$.

Since every IFHFF is IHFF, by Theorem 5.3, we have the following corollary.

Corollary 2.3.4. Let $F \in IFHFF(X)$. Then for all $x, y \in X$ the following holds:
$(\ i)$ If $x \le y$, then $\hbar_F(x) \sqsubseteq \hbar_F(y)$, $k_F(x) \sqsupseteq k_F(y)$.
(ii) $\hbar_F(x) \sqsubseteq \hbar_F(x * y)$, $k_F(x) \sqsupseteq k_F(x * y)$.
The converse of the Theorem 2.3.3 atmospheric is not true. It can be seen the following example.

Example 2.3.5. Let $X = \{1, a, b\}$ be a BE-algebra with the following table:

Let $\hbar_A(a) = \{0.4,0.5\}$, $\hbar_A(b) = \{0.2,0.3\}$, $\hbar_A(1) = \{0.6,0.8\}$ and $k_A(a) = \{0.4,0.5\}$, $k_A(b) = \{0.6,0.8\}$, $k_A(1) = \{0.2,0.3\}$. Then $A \in IHFF(X)$. But $\hbar_A(1 * (b * a)=\hbar Ax* y=\hbar A1$ and $\hbar_A(((a * b) * b) * a) = \hbar_A\big((b * b) * a\big) = \hbar_A(1 * a) = \hbar Aa=0.4,0.5$. If we put $x:=1, y:=b, z:=a$ in (IFHFF2) we have, $\hbar_A(1 * (b * a)) \sqcap \hbar_A(1) = \hbar_A(1) = \{0.6,0.8\} \not\sqsubseteq \{0.4,0.5\} = \hbar_A(((a * b) * b) * a)$.

Therefore, $F \notin IFHFF(X)$.

Theorem 2.3.6. Let $F \in IHFF(X)$. $F \in IFHFF$ if and only if the following condition holds for all $x, y \in X$:

$$\hbar_F(y * x) \sqsubseteq \hbar_F\big(\big((x * y) * y\big) * x\big), k_F(y * x) \sqsupseteq k_F\big(\big((x * y) * y\big) * x\big) \qquad (1)$$

Proof: Assume that $F \in IFHFF(X)$. Let $x, y \in X$, since $1 * (y * x) = y * x$ and $F \in IFHFF(X)$, by (IFHFF2), we have

$$\hbar_F(y * x) = \hbar_F\big(1 * (y * x)\big) = \hbar_F\big(1 * (y * x)\big) \sqcap \hbar_F(1) \sqsubseteq \hbar_F\Big(\big((x * y) * y\big) * x\Big),$$

$$k_F(y * x) = k_F\big(1 * (y * x)\big) = k_F\big(1 * (y * x)\big) \sqcap k_F(1) \sqsupseteq k_F\Big(\big((x * y) * y\big) * x\Big).$$

Conversely, assume (1) holds. Since $F \in IHFF(X)$, we have;

$$\hbar_F(x) \sqcap \hbar_F(x * y) \sqsubseteq \hbar_F(y), k_F(x) \sqcup k_F((y * x)) \sqsupseteq k_F(y). \qquad (2)$$

And by (1) we obtain;

$$\hbar_F(y * x) \sqsubseteq \hbar_F\big(\big((z * y) * y\big) * z\big), k_F(y * x) \sqsupseteq k_F\big(\big((z * y) * y\big) * z\big). \qquad (3)$$

From (2) and (3) we have:

$$\hbar_F(x) \sqcap \hbar_F(x * (y * z)) \sqsubseteq \hbar_F(y * z) \sqsubseteq \hbar_F(((z * y) * y) * z),$$

and

$$k_F(x) \sqcup k_F(x * (y * z)) \sqsupseteq k_F(y * z) \sqsupseteq k_F(((z * y) * y) * z).$$

Therefore, $F \in IFHFF(X)$.

Conclusions and further

In this paper, we introduce the notion of Intuitionistic Hesitant Fuzzy Filters on a BE-algebra . I give some results and examples about Intuitionistic Hesitant Fuzzy Filters (IHFF), and the relationship between an IHFF of a BE-algebra X and filters of X is showed by define $\chi_I = \{x \in X: \hbar_I(X) = \hbar_I(1), k_I(x) = k_I(1)\}$. By define γ-Level set of a IHFF we show the relation between IHFF, IHIFF and γ-Level. Finally we defined fantastic hesitant intuitionistic fuzzy filters and study it's properties. In other paper we are study about Intuitionistic Hesitant Fuzzy Filters on residuated algebra.

*	1	a	b
1	1	a	b
a	1	1	b
b	1	1	1

References:

[1] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20(1) (1986) 87-96.

[2] K. T. Atanassov, Intuitionistic fuzzy sets, Springer Physica-verlag, Hedielberg 1999.

[3] A. Boromand Saeid, A. Rezaei, R. A. Borzoei, Some type of filters in BE-algebra, Math, Comput, sci, 7(3) (2013), 341-352

[4] H. S. Kim, Y. H, Kim, On BE-algebra, Sci, Math, Jpn, 66(1) (2007), 113-116.

[5] H. W. Liu, G. J. Wang, Multi-criteria decision-making methods based on Intuitionistic fuzzy sets, European Journal of Operation Research 179(1) (2007) 220-223

[6] A. Rezaei, A. Boromand Saeid, Hesitant fuzzy filters in BE-algebras, International Journal of Computatational Intelligence systems, 9(2016), 110-119.

[7] S. M. Roa, Fuzzy implicative filters of BE-algebra, Annals of Fuzzy mathematics and informatics, 6(3) (2013), 129-146.

[8] V. Torra, Hesitant fuzzy sets, International Journal of Intelligent Systems 25(6) (2010) 529-539.

[9] V. Torra, Y. Narukwa, On hesitant fuzzy sets and decision, The 18th International Conference of Systems Island, Korea, Jeju (2009) 1378-1382.

[10] A. Walendziak, On normal filters and congruence relations in BE-algebra, Commentationes mathematicae, 52 (2) (2012), 199-205.

[11] M. M. Xia, Z, S, Xu, Hesitant fuzzy information aggregation in decision making, International Journal of Approximate Reasoning, 52(3) (2011) 395-407.

[12] L. A. Zadeh, Fuzzy sets, Information and Control, 8 (3) (1965) 338-353.