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The Classical Sumudu Transform and its q-Image of the Most Generalized Hypergeometric and Wright-Type Hypergeometric Functions

[1]Altaf Ahmad, [1]Renu Jain, [2]D. K. Jain and [3]Farooq Ahmad
[1]School of Mathematics and Allied Sciences Jiwaji University, Gwalior, (M.P.) India.
E-mail: renujain3@rediffmail.com, altaf.u786@gmail.com
[2]Madhav Institute of Technology and Science, Gwalior (M.P.) India.
E-mail: jain_dkj@yahoo.co.in
[3]Department of Mathematics, Govt. Degree College, Kupwara (J&K), India.
E-mail: sheikhfarooq85@gmail.com

Abstract: The q- Calculus has served as a bridge between mathematics and physics, particularly in case of quantum physics. The q-generalizations of mathematical concepts like Laplace, Fourier and Sumudu transforms, Hypergeometric functions etc. can be advantageously used in solution of various problems arising in the field of physical and engineering sciences. The q-Sumudu transform, the q-image of classical Sumudu transform is the theoretical dual of the q-Laplace transform. In view of this, the present paper deals with some of the important applications of classical Sumudu and q-Sumudu transform of generalized hypergeometric function and Wright-type hypergeometric function. The results have been presented in terms of well-known Fox's H-function. Some special cases have also been discussed.

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Key Words: *Classical Sumudu transform, q-image of Sumudu transform, Fox-Wright function, Wright-type hypergeometric function.*

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1 Introduction:

Our translation of real world problems to mathematical expressions relies on calculus which has been generalized in several directions. A natural generalization of calculus, called fractional calculus was developed during eighteenth century which involved the differentiation and integration operations of arbitrary order, which is a sort of misnomer. In the beginning it did not develop sufficiently due to lack of applications. Over the years various applications of the concept were explored and the efforts were so rewarding that the subject itself has been categorized as a significant branch of applicable mathematics. It plays a significant role in number of fields such as physics, rheology, quantitative biology, electro-chemistry, scattering theory, diffusion, transport theory, probability, elasticity, control theory, engineering mathematics and many others.

In order to stimulate more interest in the subject, many research workers engaged their focus on another dimension of calculus which sometimes called calculus without limits or popularly q-calculus. The q-calculus was initiated in twenties of the last century. Kac and Cheung's book [21] entitled "Quantum Calculus" provides the basics of such type of calculus. The fractional q-calculus is the q-extension of the ordinary fractional calculus. The investigations of q-integrals and q-derivatives of arbitrary order have gained importance due to its various applications in the areas like ordinary fractional calculus, solutions of the q-difference (differential) and q-integral equations, q-transform analysis etc.

Hypergeometric functions evolved as natural unification of a host of functions discussed by analysts from the seventeenth century to the present day. Functions of this type may also be generalized using the concept of basic number. Over the last thirty years, a great resurgence of interest in q-functions has been witnessed in view of their application to number theory and other areas of mathematics and physics.

The Wright-type hypergeometric function and Fox-Wright functions are generalizations of Hypergeometric functions which appear as solution of well-known fractional differential and integral equations representing some physical and physiological phenomena like diffusion, transport theory, probability, elasticity and control theory.

The purpose of this paper is to increase the accessibility of different dimensions of q-fractional calculus and generalization of basic hypergeometric functions to the real world problems of engineering and science through various integral transforms including Laplace, Fourier and Sumudu transforms and their q-images.

The classical Laplace, Fourier, Mellin transforms and Sumudu transform have been widely used in mathematical physics and applied mathematics. The classical theory of the Laplace transform is well known [1] and its generalization was considered by many authors [2, 3, 4, 8, and 9]. Various existence conditions and the detailed study about the range and invertibility was studied by Rooney [7]. The Mellin Transform and Sumudu transform which is the theoretical dual of the Laplace transform are widely used together to solve the fractional Kinetic equations, two–parameter fractional Telegraph equation and thermonuclear equations [5].

1.1 Mathematical Preliminaries:

Classical Laplace transform: The Laplace transform is very useful in analysis and design for systems that are linear and time-invariant (LTI). Beginning in about 1910, transform techniques were applied to signal processing at Bell Labs for signal filtering and telephone long-lines communication by H. Bode and others. Transform theory subsequently provided the backbone of Classical Control Theory as practiced during the World Wars and up to about 1960, when State Variable techniques began to be used for controls design. Pierre Simon Laplace was a French mathematician who lived 1749-1827, during the age of enlightenment characterized by the French Revolution, Rousseau, Voltaire, and Napoleon

Bonaparte. Let $f(t)$ be a function piecewise continuous on $[0, A]$ (for every $A > 0$) and have an exponential order at infinity with $f(t) \leq Me^{at}$. Then, the Laplace transform L(f) is defined for s > a , that is {s > a} ⊂ Domain(L(f)) . The Laplace transform of f(t) is defined by

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\, dt . \qquad (1)$$

The Laplace transform is said to exist if the integral (1) is convergent for some values of s.

Classical Fourier Transform: Fourier analysis is named after Jean Baptiste Joseph Fourier (1768 to 1830), a French mathematician and physicist. Joseph Fourier, while studying the propagation of heat in the early 1800's, introduced the idea of a harmonic series that can describe any periodic motion regardless of its complexity. Later, the spelling of Fourier analysis gave place to Fourier transform (FT) and many methods derived from FT are proposed by researchers. In general, FT is a mathematical process that relates the measured signal to its frequency content Heideman et al. [15]. The Fourier series is described in theory and problems of advanced calculus as follows:

If f(x) be a function defined on $(-\infty, \infty)$ uniformly continuous in finite interval such that $\int_0^\infty \| f(x) \| d(x)$ converges, then the Fourier transform of f(x) is defined by

$$F(f(x)) = \bar{f}(s) = \int_{-\infty}^{\infty} e^{isx} f(x)\, d(x),$$ where e^{isx} is said to be kernel of the Fourier transform.

q- image of Laplace transform: Hahn [6] defined the q-image of classical Laplace transform as

$$L_q f(s) = \int_0^\infty e_q^{-sx} f(x)\, d(x), \qquad Re(s) > 0. \tag{2}$$

Where e_q^{-sx} is defined by $e_q^{-sx} = \frac{1}{(1+s(1-q)x)_q^\infty}$.

The Laplace transform of the power function is defined as

$$L\,(t^\mu) = \frac{\Gamma(\mu+1)}{s^{\mu+1}} \tag{3}$$

The q-Laplace transform of the power function is defined as in [10 & 11]

$$L_q\,(t^\mu) = \frac{\Gamma_q(\mu+1)(1-q)^\mu}{s^{\mu+1}} \tag{4}$$

Also, $(1-q)^{\alpha-1}\Gamma_q(\alpha) = (q;q)_{\alpha-1}$

Fox-Wright generalized hypergeometric Function:
The Fox-Wright (Psi) function is defined as follows [16].

$${}_p\psi_q\left(\begin{matrix}(a_1,A_1),(a_2,A_2),\ldots(a_p,A_p)\\ (a_1,A_1),(a_2,A_2),\ldots(a_p,A_p)\,|\,z\end{matrix}\right) = \sum_{n=0}^{\infty}\frac{\Gamma(a_1+nA_1)\Gamma(a_2+nA_2)\ldots\Gamma(a_p+nA_p)}{\Gamma(b_1+nB_1)\Gamma(b_2+nB_2)\ldots\Gamma(b_q+nB_q)}\frac{z^n}{n!} \tag{5}$$

The basic analogue of Fox-Wright hypergeometric function denoted ${}_p\psi_q\,(z;q)$ for $z \in C$ is defined in series form as [17]

$${}_p\Psi_q\,(z;q) = \sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma_q(a_i+kA_i)}{\prod_{j=1}^{q}\Gamma_q(b_j+kB_j)}\frac{z^k}{(q;q)_k}, \text{ where } |q| < 1.$$

Where $ai, bj \epsilon$ C > 0 ; Ai >0 ,Bj >0 ; $1+\sum_{j=1}^{q}B_j - \sum_{i=1}^{p}Ai \geq 0$;$a \epsilon$ R ,for suitably bounded value of |z|.

Wright-type Hypergeometric function: The generalized form of the hypergeometric function has been investigated by Dotsenko [18], Malovichko [19] and one of the special case is considered by Dotsenko [18] as

$${}_2R_1^{\omega,\mu}(z) = {}_2R_1\,(a,b,c,\omega,\mu,z) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma\left(b+\frac{\omega}{\mu}n\right)}{\Gamma\left(c+\frac{\omega}{\mu}n\right)}\frac{z^n}{n!} \tag{6}$$

And its integral representation expressed as

$${}_2R_1^{\omega,\mu}(z) = \frac{\Gamma(c)\mu}{\Gamma(c-b)\,\Gamma(b)}\int_0^1 t^{\mu b-1}(1-t)^{c-b-1}(1-zt^\omega)^{c-b-1}\,dt,$$

Where $Re(c) > Re(b) > 0$. This is the analogue of Euler's formula for Gauss's hypergeometric functions [20]. In 2001 Virchenko et al [13] defined the said Wright type Hypergeometric function by taking $\frac{\omega}{\mu} = \tau > 0$ in above equation as

$${}_2R_1^{\tau}(z) = {}_2R_1\,(a, b, c, \tau, z) = \frac{\Gamma(c)}{\Gamma(b)}\sum_{k=0}^{\infty}\frac{(a)_k\Gamma(b+\tau k)}{\Gamma(c+\tau k)}\frac{z^k}{k!};\ \tau > 0, |z| < 1.$$

If $\boldsymbol{\tau}$ =1, then (.) reduces to Gauss's hyper geometric function.

Classical Sumudu Transform:

Over the set of function

$$A = \{f(t)|\exists M,\ \tau_1, \tau_2 > 0, |f(t)| < Me^{|t|/\tau_j}, \qquad if\ \ t \in (-1)^j \times [0, \infty)\}$$

the Sumudu transform is defined by

$$G(u) = S[f(t)] = \int_0^\infty f(ut)e^{-t}\, dt,\ \ u \in (-\tau_1, \tau_2) \ldots \tag{7}$$

q-Image of Sumudu Transform:

Albayrak, Purohit and Ucar [12] defined the q-analogues of the Sumudu transform by means of the following

$$S_q\{f(t)\,;s\} = \frac{1}{(1-q)s}\int_0^\infty E_q(\tfrac{q}{s}t)f(t)d_q(t). \tag{8}$$

$$S_q\{x^{\alpha-1}\,;s\} = s^{\alpha-1}\,(1-q)^{\alpha-1}\Gamma_q(\alpha) \tag{9}$$

Also, $(1-q)^{\alpha-1}\Gamma_q(\alpha) = (q;q)_{\alpha-1}$.

q- analogue of Wright-type hypergeometric function:

In this section, we have introduced the q- analogue of Wright-type hypergeometric function named as ${}_2R_1$-function, which was first time coined by Virchenko et.al. [13] defined in (6).

If t = 1, then (6) reduces to a Gauss Hypergeometric function ${}_2F_1$(a,b;c;z). It is a direct generalisation of exponential series. In the sequel to this study, we introduced the basic analogue of ${}_2R_1$- function as follows

$${}_2R_1{}^{\tau}(q;z) = {}_2R_1(a,b;c;\tau;q;z) = \frac{\Gamma_q(c)}{\Gamma_q(b)}\sum_{k=0}^{\infty}\frac{(a;q)_k\Gamma_q(b+\tau k)}{\Gamma_q(c+\tau k)(q;q)_k}z^k, \quad \tau>0, |z|<1, |q|<1.$$

yields ${}_2R_1^t$(z) when q = 1.

The function ${}_2R_1{}^{\tau}(q;z)$ converges under the convergence conditions of basic analogue of H- function. The integral converges if *Re[slog(z)- logSinπs] < 0*, on the contour C, where *0<|q|< 1* verified by Saxena, et.al. [14].

2 Main Results:

2.1 In this section of paper, the authors have derived the classical Sumudu transform of generalized hypergeometric Function and Wright-type hypergeometric Function in terms of Fox's H – function is given by

Theorem 1: The classical Sumudu transform of generalized hypergeometric Function in terms of Fox's H – function is given by

$$S\left\{\ {}_p\psi_q\left(\begin{matrix}(a_1,A_1)(a_2,A_2)\cdots(a_p,A_p)\\(b_1,B_1)(b_2,B_2)\cdots(b_q,B_q)\end{matrix}\middle| z\right)\right\} =$$

$$H^{1,p}_{1,q}\left[\begin{matrix}(1-a_1,A_1)(1-a_2,A_2)(1-a_3,A_3)\cdots(1-a_p,A_p)\\(1-b_1,B_1)(1-b_2,B_2)(1-b_3,B_3)\cdots(1-b_q,B_q)\end{matrix}\middle| u\right],$$

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Proof: For $R(\alpha) > 0$, the classical Sumudu transform of generalized hypergeometric Function in terms of Fox's H – function is given by

$$S\left\{ {}_p\psi_q\left(\begin{matrix} (a_1,A_1)(a_2,A_2)\cdots(a_p,A_p) \\ (b_1,B_1)(b_2,B_2)\cdots(b_q,B_q) \end{matrix} \middle| z \right) \right\} =$$

$$S\left\{ \sum_{n=0}^{\infty} \frac{\Gamma(a_1+nA_1)\Gamma(a_2+nA_2)\ldots\Gamma(a_p+nA_p)}{\Gamma(b_1+nB_1)\Gamma(b_2+nB_2)\ldots\Gamma(b_q+nB_q)} \frac{z^n}{n!} \right\} \tag{10}$$

From equation (10) we have,

$$S\left\{ {}_p\psi_q\left(\begin{matrix} (a_1,A_1)(a_2,A_2)\cdots(a_p,A_p) \\ (b_1,B_1)(b_2,B_2)\cdots(b_q,B_q) \end{matrix} \middle| z \right) \right\} =$$

$$\left\{ \sum_{n=0}^{\infty} \frac{\Gamma(a_1+nA_1)\Gamma(a_2+nA_2)\ldots\Gamma(a_p+nA_p)}{\Gamma(b_1+nB_1)\Gamma(b_2+nB_2)\ldots\Gamma(b_q+nB_q)} \right\} S\left(\frac{z^n}{n!} \right)$$

On using equation (7) we get,

$$S\left\{ {}_p\psi_q\left(\begin{matrix} (a_1,A_1)(a_2,A_2)\cdots(a_p,A_p) \\ (b_1,B_1)(b_2,B_2)\cdots(b_q,B_q) \end{matrix} \middle| z \right) \right\} =$$

$$\left\{ \sum_{n=0}^{\infty} \frac{\Gamma(a_1+nA_1)\Gamma(a_2+nA_2)\ldots\Gamma(a_p+nA_p)}{\Gamma(b_1+nB_1)\Gamma(b_2+nB_2)\ldots\Gamma(b_q+nB_q)} \frac{1}{\Gamma(n+1)} \right\} u^n \Gamma(n+1)$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(1-(1-a_1)+nA_1)\Gamma(1-(1-a_2)+nA_2)\ldots\Gamma(1-(1-a_p)+nA_p)}{\Gamma(1-(1-b_1)+nB_1)\Gamma(1-(1-b_2)+nB_2)\ldots\Gamma(1-(1-b_q)+nB_q)} u^n$$

$$= H_{1,q}^{1,p}\left[\begin{matrix} (1-a_1,A_1)(1-a_2,A_2)(1-a_3,A_3)\ldots(1-a_p,A_p) \\ (1-b_1,B_1)(1-b_2,B_2)(1-b_3,B_3)\ldots(1-b_q,B_q) \end{matrix} \middle| u \right].$$

This is the proof of theorem.

Theorem 2: The classical Sumudu transform of Wright-type hypergeometric Function in terms of Fox's H – function is given by

$$S({}_2R_1(a,b;c;\tau;z)) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} H_{2,1}^{1,2}\left[\begin{matrix} (1-a,1),(b,\tau) \\ (0,1),(1-c,\tau) \end{matrix} \middle| -u \right]$$

Proof: The classical Sumudu transform of Wright-type hypergeometric Function in terms of Fox's H – function is given by

$$S({}_2R_1(a,b;c;\tau;z)) = S\left(\frac{\Gamma(c)}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{(a)_k \Gamma(b+\tau k)}{\Gamma(c+\tau k) k!} z^k \right) \tag{11}$$

Since, $(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}$.

From equation (11) we have,

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$$S({}_2R_1(a,b;c;\tau;z))=\frac{\Gamma(c)}{\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+\tau k)}{\Gamma(a)\Gamma(c+\tau k)k!}S\left(z^k\right)$$

We know that the Laplace transform of the power function is,

$$L(t^{\mu})=\frac{\Gamma(\mu+1)}{s^{\mu+1}}$$

Therefore $$S({}_2R_1(a,b;c;\tau;z))=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+\tau k)}{\Gamma(c+\tau k)\Gamma(1+k)}u^k\Gamma(k+1)$$

$$=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+\tau k)}{\Gamma(c+\tau k)}u^k$$

$$=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma(1-(1-a)+k)\Gamma(1-(1-b)+\tau k)}{\Gamma(1-(1-c)+\tau k)}u^k$$

$$\Rightarrow \qquad S({}_2R_1(a,b;c;\tau;z))=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}H_{2,1}^{1,2}\left[\begin{matrix}(1-a,1),(b,\tau)\\(0,1),(1-c,\tau)\end{matrix}\Big|-u\right].$$

This completes the proof of theorem.

Observation:

(1.1): if $\tau=1$ then from above theorem (2)

$$S({}_2R_1(a,b;c;z))=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}H_{2,1}^{1,2}\left[\begin{matrix}(1-a,1),(b,1)\\(0,1),(1-c,1)\end{matrix}\Big|-u\right].$$

2.2 In this section of paper, the authors have derived the q-image Sumudu transform of basic analogue of generalized hypergeometric Function and basic analogue of Wright-type hypergeometric Function in terms of Fox's q-analogue of H – function, which is given by

Theorem 3: The q- Sumudu transform of q-analogue of generalized hypergeometric Function in terms of q-analogue of H – function is given by

$$S_q\left\{ {}_p\psi_q\left(\begin{matrix}(a_1,A_1)(a_2,A_2)\cdots(a_p,A_p)\\(b_1,B_1)(b_2,B_2)\cdots(b_q,B_q)\end{matrix}\Big| z;q\right)\right\}=$$

$$H_{1,q}^{1,p}\left[\begin{matrix}(1-a_1,A_1)(1-a_2,A_2)(1-a_3,A_3)\cdots(1-a_p,A_p)\\(1-b_1,B_1)(1-b_2,B_2)(1-b_3,B_3)\cdots(1-b_q,B_q)\end{matrix}\Big|u;q\right]$$

Proof: For $q>0$, $R(\alpha)>0$, the q-image of Sumudu transform of q- generalized hypergeometric Function in terms of basic analogue of H – function is given by

$$S_q\left\{ {}_p\psi_q\left(\begin{matrix}(a_1,A_1)(a_2,A_2)\cdots(a_p,A_p)\\(b_1,B_1)(b_2,B_2)\cdots(b_q,B_q)\end{matrix}\Big| z;q\right)\right\}=$$

$$S_q\left\{\sum_{n=0}^{\infty}\frac{\Gamma_q(a_1+nA_1)\Gamma_q(a_2+nA_2)\cdots\Gamma_q(a_p+nA_p)}{\Gamma_q(b_1+nB_1)\Gamma_q(b_2+nB_2)\cdots\Gamma_q(b_q+nB_q)}\frac{z^n}{(q;q)_n}\right\} \qquad (12)$$

From equation (12) we have,

$$S_q\left\{ \ {}_p\psi_q\left(\begin{matrix} (a_1,A_1)(a_2,A_2)\ldots(a_p,A_p) \\ (b_1,B_1)(b_2,B_2)\ldots(b_q,B_q) \end{matrix} \middle| z;q \right)\right\} =$$
$$\left\{\sum_{n=0}^{\infty}\frac{\Gamma_q(a_1+nA_1)\Gamma_q(a_2+nA_2)\ldots\Gamma_q(a_p+nA_p)}{\Gamma_q(b_1+nB_1)\Gamma_q(b_2+nB_2)\ldots\Gamma_q(b_q+nB_q)}\right\}S_q\left(\frac{z^n}{(q;q)_n}\right)$$

On using equation (9) we get,

$$S_q\left\{ \ {}_p\psi_q\left(\begin{matrix} (a_1,A_1)(a_2,A_2)\ldots(a_p,A_p) \\ (b_1,B_1)(b_2,B_2)\ldots(b_q,B_q) \end{matrix} \middle| z;q \right)\right\} =$$
$$\left\{\sum_{n=0}^{\infty}\frac{\Gamma_q(a_1+nA_1)\Gamma_q(a_2+nA_2)\ldots\Gamma_q(a_p+nA_p)}{\Gamma_q(b_1+nB_1)\Gamma_q(b_2+nB_2)\ldots\Gamma_q(b_q+nB_q)}\frac{1}{(q;q)_n}\right\}u^n(q;q)_n$$
$$=\left\{\sum_{n=0}^{\infty}\frac{\Gamma_q(1-(1-a_1)+nA_1)\Gamma_q(1-(1-a_2)+nA_2)\ldots\Gamma_q(1-(1-a_p)+nA_p)}{\Gamma_q(1-(1-b_1)+nB_1)\Gamma_q(1-(1-b_2)+nB_2)\ldots\Gamma_q(1-(1-b_q)+nB_q)}\right\}u^n$$

$$S_q\left\{ \ {}_p\psi_q\left(\begin{matrix} (a_1,A_1)(a_2,A_2)\ldots(a_p,A_p) \\ (b_1,B_1)(b_2,B_2)\ldots(b_q,B_q) \end{matrix} \middle| z;q \right)\right\} =$$

$$H_{1,q}^{1,p}\left[\begin{matrix}(1-a_1,A_1)(1-a_2,A_2)(1-a_3,A_3)\ldots(1-a_p,A_p) \\ (1-b_1,B_1)(1-b_2,B_2)(1-b_3,B_3)\ldots(1-b_q,B_q)\end{matrix}\middle| u;q\right].$$

This is the proof of theorem.

Theorem 4: The q- Sumudu transform of q-analogue of Wright-type hypergeometric Function in terms of q-analogue of H – function is given by

$$S_q({}_2R_1(a,b;c;\tau;q;z)) = \frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(b)}H_{2,1}^{1,2}\left[\begin{matrix}(1-a,1),(b,\tau) \\ (0,1),(1-c,\tau)\end{matrix}\middle| -u;q\right]$$

Proof: The q-image of Sumudu transform of q-type of Wright type hypergeometric Function in terms of basic analogue of H – function is given by

$$S_q({}_2R_1(a,b;c;\tau;q;z)) = S_q\left(\frac{\Gamma_q(c)}{\Gamma_q(b)}\sum_{k=0}^{\infty}\frac{(a;q)_k\Gamma_q(b+\tau k)}{\Gamma_q(c+\tau k)(q;q)_k}z^k\right)$$

$$S_q({}_2R_1(a,b;c;\tau;q;z)) = \frac{\Gamma_q(c)}{\Gamma_q(b)}\sum_{k=0}^{\infty}\frac{(a;q)_k\Gamma_q(b+\tau k)}{\Gamma_q(c+\tau k)(q;q)_k}S_q\left(z^k\right)$$

Since, $(\gamma;q)_n = \dfrac{\Gamma_q(\gamma+n)}{\Gamma_q(\gamma)}$

Albayrak, Purohit and Ucar [12] defined the q-analogues of the Sumudu transform by means of the following

$$S_q\{f(t);s\} = \frac{1}{(1-q)s}\int_0^{\infty}\mathbf{E}_q\left(\frac{q}{s}\mathbf{t}\right)f(t)\mathbf{d}_q(\mathbf{t}).$$
$$\boldsymbol{S}_q\{x^{\alpha-1};s\} = s^{\alpha-1}(1-q)^{\alpha-1}\boldsymbol{\Gamma}_q(\alpha)$$

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Also, $(1-q)^{\alpha-1}\Gamma_q(\alpha) = (q;q)_{\alpha-1}$

$$S_q({}_2R_1(a,b;c;\tau;q;z)) = \frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(b)}\sum_{k=0}^{\infty}\frac{\Gamma_q(a+k)\Gamma_q(b+\tau k)}{\Gamma_q(c+\tau k)(q;q)_k}u^k(q;q)_k$$

$$= \frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(b)}\sum_{k=0}^{\infty}\frac{\Gamma_q(a+k)\Gamma_q(b+\tau k)}{\Gamma_q(c+\tau k)}u^k$$

$$= \frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(b)}\sum_{k=0}^{\infty}\frac{\Gamma_q(1-(1-a)+k)\Gamma_q(1-(1-b)+\tau k)}{\Gamma_q(1-(1-c)+\tau k)}u^k$$

$$\Rightarrow\ S_q({}_2R_1(a,b;c;\tau;q;z)) = \frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(b)}H_{2,1}^{1,2}\left[\begin{matrix}(1-a,1),(b,\tau)\\(0,1),(1-c,\tau)\end{matrix}\middle| -u;q\right]$$

This completes the proof.

Observation:

(1.2): if $t = 1$ then from above theorem

$$S_q({}_2R_1(a,b;c;q;z)) = \frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(b)}H_{2,1}^{1,2}\left[\begin{matrix}(1-a,1),(b,1)\\(0,1),(1-c,1)\end{matrix}\middle| -u;q\right]$$

Special cases:

Taking q=1, we get following as special cases of theorem (4)

$$S({}_2R_1(a,b;c;\tau;z)) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}H_{2,1}^{1,2}\left[\begin{matrix}(1-a,1),(b,\tau)\\(0,1),(1-c,\tau)\end{matrix}\middle| -u\right]$$

if $t = 1$ then from above theorem

$$S({}_2R_1(a,b;c;z)) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}H_{2,1}^{1,2}\left[\begin{matrix}(1-a,1),(b,1)\\(0,1),(1-c,1)\end{matrix}\middle| -u\right]$$

3. Conclusion:

The results proved in this paper give some contributions to the theory of the q- series, especially q- analogue of generalized hypergeometric function and Wright-type hypergeometric Function and may find applications to solutions of certain q-difference, q-integral and q-differential equations with the help of q- images of transforms like Sumudu, Laplace and Fourier transforms. The results proved in this paper appear to be new and likely to have useful applications to a wide range of problems of mathematics, statistics and physical sciences.

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