Solve Non-Linear Parabolic Partial Differential Equation by Spline Collocation Method

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Abstract:- This paper provides an overview of the formulation, analysis and implementation of Spline collocation method for the numerical solution of partial differential equation with two space variable which is of parabolic type. The method includes the solution of non-linear equation which can be expressed as in matrix form. The use of spline collocation methods in the solution of initial-boundary value problems for parabolic-type system id described, with emphasis on alternating direction implicit methods. Problem of vertical groundwater recharge solve by spline collocation method. Finally, recent applications of spline collocation method are outlined.

Keywords: *Spline collocation, Partial differential equation.*

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1. Introduction:

Non-Linear parabolic partial differential equation solve by spline collocation method while numerical solution obtained. Therefore, two common questions are encountered, first is about its acceptance whether it is sufficiently close to true solution or not. If one has an analytic solution then this can be answered very clearly but in either case it is not so easy. One has to be careful while concluding that a particular numerical solution is acceptable when an analytic solution is not available. Normally a method is selected which does not produce an excessive errors.

2. Spline collocation method:

For solving linear and nonlinear differential equation with the help of the numerical methods required much computational work and time. Brickley [1968] suggested the method of spline function containing truncated power polynomials to solve a linear boundary value problem. Ahlberg et al [1967] used cardinal splines for solving differential equations. Doctor et al [1983, 1984] have shown that the method of spline collocation is quite useful for the solution of physical phenomena which give rise to linear parabolic one dimensional partial differential equation. The method demonstrates the use of spline function. Spline functions are piecewise polynomial and their successive derivatives are continuous. They were used for data interpolation initially. In 1967, Blue [1969] suggested the use of spline function for the solution of B.V.P.

$y'' = f(x, y, y')$

With boundary conditions

$G1[Y(0), Y'(0)]$

$G2\ [\ Y(1), Y'(1)]$

The following recurrence relations were used.

$S''(x_{i-1}) + 4\,S''(x_i) + s''(x_{i+1}) = 6/h^2(f(x_{i-1}) - 2f(x_i) + f(x_{i+1}))$------------- (I)

3. Non-Linear Parabolic Partial Differential Equation:

Consider a nonlinear parabolic partial differential equation

$\frac{\partial u}{\partial t} + Au\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}$.......... (1)

Where A is constant.

Subject to certain initial and boundary conditions.

Initial condition: u (u, 0) =f(x)

Boundary condition: **u (0, t) = u (a,t) = 0**

4. Spline formula to solve nonlinear parabolic partial differential equation:

Divide the region [0, a] into say N- subintervals of equal length. Denote the points of subdivisions by x_0, x_1, $x_{2\,,\ldots\ldots\,,}$ xn. Let u_{ij} denote the value of u at ith mesh point at time jΔt. We approximate the function u at time jΔt by cubic spline s(x). Discretizing the left side of equation (1) by forward and central difference formula and replace right side by the second derivative $s''(x_i)$at jth level like explicit scheme in finite difference, we get,

$\frac{u_{i,j+1}-u_{i,j}}{\Delta t} + Au_{i,j}\frac{u_{i+1,j}-u_{i-1,j}}{2\Delta x} = s''_{i,j}$ ---------- (2)

Where $s''_{i,j}$ denotes s'' (x_i) at jth level

Now with the help of equation (2) and spline recurrence relation (I) we get,

$u_{i+1,j+1} + 4u_{i,j+1} + u_{i-1,j+1} = u_{i+1,j}(1+6r) + u_{i,j}(4-12r) + u_{i-1,j}(1+6r) - \left(\frac{AB}{2}\right)\{u_{i+1,j}(u_{i+2,j}-u_{i,j}) + 4u_{i,j}(u_{i+1,j} - u_{i-1,j}) + u_{i-1,j}(u_{i,j}-u_{i-2,j})\}$ ----------- (3)

Where r = k/h^2& B = k/h

The above set of simultaneous equation gives a square matrix. The equation (3) known as cubic spline explicit formula to solve equation (1).

Implicit method finite difference replacement of equation (1) is

$\frac{u_{i,j+1}-u_{i,j}}{\Delta t} + Au_{i,j}\frac{u_{i+1,j}-u_{i-1,j}}{2\Delta x} = \frac{1}{2}(s''_{i,j} + s''_{i,j+1})$--------------- (4)

Where $s''_{i,j}$ and $s''_{i,j+1}$ denotes the second order derivatives of s(x) at x = x_i at the time interval j & j+1 respectively. Using equation (4) and spline recurrence relation (I) we get,

$$u_{i+1,j+1}(6r-2) + u_{i,j+1}(-12r-8) + u_{i-1,j+1}(6r-2) = AB\{u_{i+1,j}(u_{i+2,j}-u_{i,j}) + 4u_{i,j}(u_{i+1,j}-u_{i-1,j}) + u_{i-1,j}(u_{i,j}-u_{i-2,j})\} - \{u_{i+1,j}(6r+2) + u_{i,j}(8-12r) + u_{i-1,j}(6r+2)\}$$

------------ (5)

Above equation (5) is known as cubic spline implicit formula to solve equation (1). Now $u_{0,,j+1}$ and $u_{n,j+1}$ are known due to the prescribed boundary conditions. The set of simultaneous equations obtained in explicit as well as implicit scheme contains (n-1) unknowns. These (n-1) equations in (n-1) unknowns can be solved by any standard method.

Once the value of u are known at (j+1)th level, we can proceed to compute next level j+2 by repeating the same process. Each set of equations give tri-diagonal matrix. It can be solved by any standard method. Thus, the method can proceed by steps.

The convergence and stability of these methods totally depend upon value of r. Convergence and stability along with small values of r is more accurate. Values much larger than unity are not recommended. These two methods will be discussed later on by taking its actual approximation to a problem.

5. Problem of Vertical Groundwater Recharge:

The problem of flow of water through partially saturated porous media has been discussed by Klute [1952] and Verma [1969]. We have obtained a numerical solution of the problem by using spline collocation technique. In the investigated mathematical model

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we consider that the ground water recharge takes place over the large basin of such geological location that the sides are limited by rigid boundaries and the bottom by a thick layer of water table. In this case the flow may be assumed vertically downwards through unsaturated porous media. Here the average diffusivity coefficient of the whole range of moisture content is regarded as constant and the permeability of the moisture content is assumed to have a parabolic distribution. The theoretical formulation of the problem yields a non-linear partial differential equation for the moisture content.

6. Formulation of the problem:

The equation of continuity for an unsaturated medium is given by

$$\frac{\partial}{\partial t}(\rho s,\ \theta) = -\nabla M;$$

The boundary value problem is

$$\frac{\partial \theta}{\partial T} = \frac{\partial^2 \theta}{\partial \xi^2} - \beta_0 \theta \frac{\partial \theta}{\partial \xi} ; \theta\,(\xi\,,0\,) = 0\,;\ \frac{\partial \theta}{\partial \xi}\,(1, T) = 0 \text{ ------------------- } (6)$$

Where ρs the bulk density of the medium is, θ is its moisture content on a dry weight basis, M is the mass flux of moisture, ξ is penetration depth (dimensionless) ,T is the time and β_0 is the flow parameter.

7. Explicit Spline method to solve the problem:

From equation (3) explicit spline formula is

$$\theta_{i+1,j+1} + 4\theta_{i,j+1} + \theta_{i-1,j+1} = \theta_{i+1,j}(1+6r) + \theta_{i,j}(4-12r) + \theta_{i-1,j}(1+6r) - \left(\frac{AB}{2}\right)\{\theta_{i+1,j}(\theta_{i+2,j} - \theta_{i,j}) + 4\theta_{i,j}(\theta_{i+1,j} - \theta i{-}1{,}j{+}\theta i{-}1{,}j(\theta i{,}j{-}\theta i{-}2{,}j)\} \text{ ------- } (7)$$

Where $r = k / h^2$ & $B = k/h$

Taking A = 1, r = 1/6, h =1 / 4, kΔt = 1/96 & j = 0, i = 1,2,3,4

We get set of equations which can be solved by any well-known method. The solutions θ can be obtained for different values of j = 0, 1, 2, 3… i = 1,2,3,4

The results are presented in the table (I) and are plotted in figure (I)

Table I
Numerical Results for Vertical Ground Water Recharge
Spline solutions by explicit method

θ →						
ξ	t = 1/96	t = 7/96	t = 8/96	t = 9/96	t = 10/96	t = 11/96
0.0	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000
0.25	0.0406	0.0622	0.0642	0.0653	0.0669	0.0678
0.50	0.0133	0.0337	0.0352	0.0379	0.0393	0.0415
0.75	0.0055	0.0161	0.0190	0.0201	0.0227	0.0241
1.00	0.0137	0.0121	0.0126	0.0152	0.0163	0.0188

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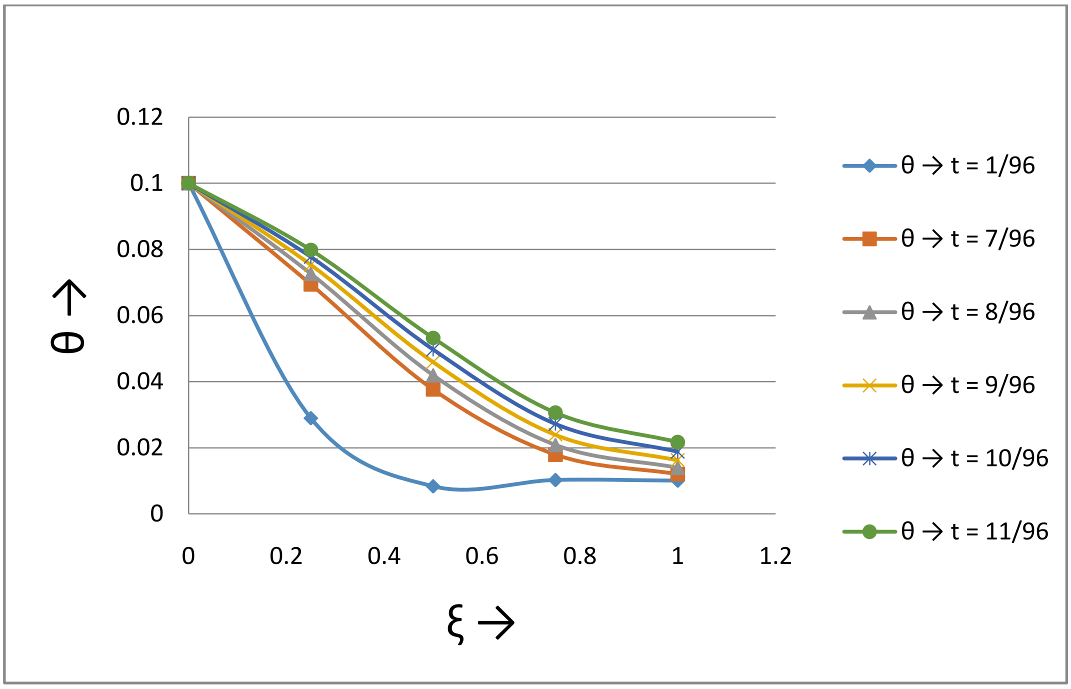

Figure I
Behavior of Moisture Content

8. Implicit Spline method to solve the problem:

From equation (5) implicit spline formula is

$$\theta_{i+1,j+1}(6r-2)+\theta_{i,j+1}(-12r-8)+\theta_{i-1,j+1}(6r-2) = AB\{\theta_{i+1,j}(\theta_{i+2,j}-\theta_{i,j})+4\theta_{i,j}(\theta_{i+1,j}-\theta_{i-1,j})+\theta_{i-1,j}(\theta_{i,j}-\theta_{i-2,j})\}-\{\theta_{i+1,j}(6r+2)+\theta_{i,j}(8-12r)+\theta_{i-1,j}(6r+2)\}$$

---------------- (8)

Where $r = k/h^2$ & $B = k/h$

Taking A = 1, r = 1/6, h =1 / 4, kΔt = 1/96 & j = 0, i = 1,2,3,4

We get set of equations which can be solved by any well-known method. The solutions θ can be obtained for different values of j = 0, 1, 2, 3…, i = 1,2,3,4

The results are presented in the table (II) and are plotted in figure (II)

Table II
Numerical Results for Vertical Ground Water Recharge
Spline solutions by implicit method

θ →						
ξ	t = 1/96	t = 7/96	t = 8/96	t = 9/96	t = 10/96	t = 11/96
0.0	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000
0.25	0.0289	0.0694	0.0726	0.0753	0.0777	0.0798
0.50	0.0083	0.0376	0.0419	0.0459	0.0497	0.0532
0.75	0.0102	0.0179	0.0208	0.0239	0.0272	0.0306
1.00	0.0100	0.0120	0.0139	0.0162	0.0188	0.0217

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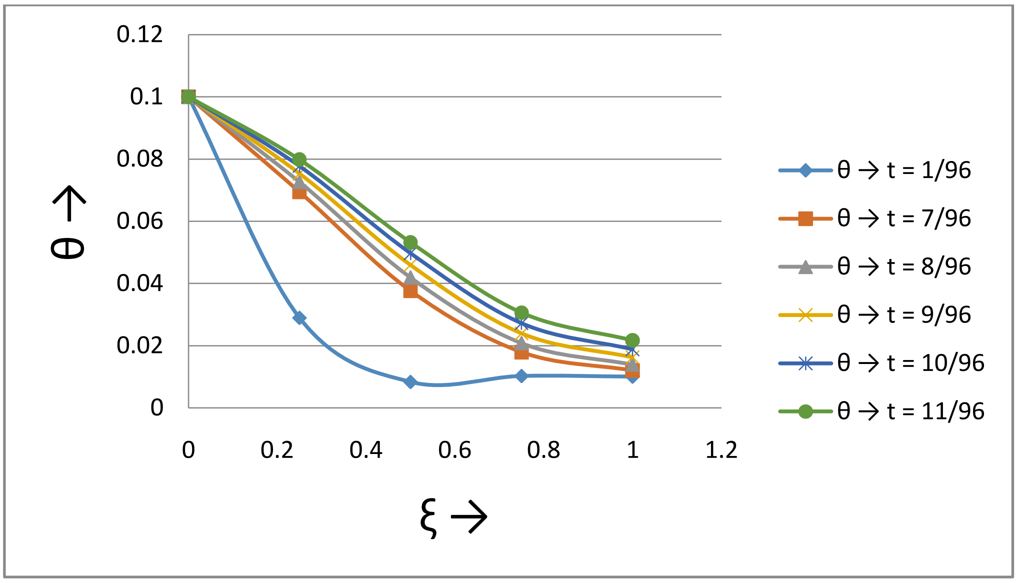

Figure (II)
Behaviour of Moisture Content
(Spline implicit solutions)

9. Conclusion:

We can conclude that from the graph the curves indicate the behavior of moisture content corresponding to different values, from the figure it can be observed that moisture content θ decreases considerably throughout the region as ξ increase but time t increases moisture content throughout region as well as at the layer increases.

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