Stability Analysis of a Discrete Time Model on Plant-Soil Interactions

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Abstract— In this paper, a nonlinear mathematical model on plant soil interactions is proposed and analyzed. We consider the variables namely, density of the plant species, nutrient concentration in the soil and in the plant for metabolic activity. We consider that the growth rate of the plant species is dependent on the density of the nutrient concentration in the plant. The relationship between the concentration and the rate of uptake is often described quantitatively by Michaelis–Menten kinetics. We discretize the model by applying Backward Euler method and analyse the stability of the model both locally and globally. We analyse the nutrient concentration in Tomato plant and provide numerical simulations for the dynamical behaviour of plant soil interactions for each nutrient. The numerical simulations are provided using MATLAB.

Keywords-: *Difference equations, Local stability, Global stability, Plant species, Nutrient concentration.*

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I. Introduction

Plants grow in the thin upper layer of the Earth's crust known as soil. Plants need water, carbon dioxide and a range of trace minerals known as 'nutrients' to grow. They obtain these nutrients, and most of their water, from the soil. Soil is formed over very long times from igneous or sedimentary rock, volcanic ash, sand or peat. It is a highly complex and heterogeneous system with many different components that provide plants with water and nutrients. Nutrients exist in the soil in gaseous, liquid, and solid forms.

Plant nutrients in soil come originally from the parent material from which the soil was formed. In addition to water, plants require thirteen essential mineral nutrients to complete their full lifecycle. Essential mineral nutrients can be divided into macro-nutrients and micro-nutrients depending on the amount required by plants. The macro-nutrients are: nitrogen, potassium, sulphur, phosphorus, magnesium and calcium, and micro-nutrients are: iron, zinc, manganese, copper, molybdenum, boron, and chlorine[1].

Modelling Plant Uptake of dissolved soil constituents is essential to predicting plant growth under nutrient limitation. Solute uptake by plants can also be important in explaining changes in the chemistry of soil. Under natural conditions most nutrients are recycled from plant to soil to plant. Loss of nutrient from soil by leaching by rainwater depend on how strongly the nutrient is bound to the soil constituents.

Nutrients are typically distributed heterogeneously throughout the soil and plants are adept at assessing and responding to this nutrient heterogeneity. Generally, plants respond to nutrient-rich patches by preferentially proliferating roots into those patches. Absorption of elemental nutrients by plants and ingestion of food by animals are two crucial processes in the understanding of ecosystem functioning. Nutrient uptake of plants from soil is the result of interactions between plant and soil. The rate of uptake of a nutrient depends on the concentration of this nutrient in soil solution at the root surface. The relation between the concentration and the rate of uptake can often be described quantitatively by Michaelis–Menten kinetics as has been published by Epstein and Nielsen[2]. Solute uptake is assumed to be independent of water uptake; only active uptake is considered.

In the broader ecological literature on foraging and forager functional responses, Holling’s disc equation (Holling 1959) provides one commonly used framework for modelling resource capture. In the plant literature on nutrient uptake kinetics, the Michaelis–Menten equation (Michaelis and Menten 1913) provides the framework for modelling nutrient capture. The resource harvest rate (units of resource uptake per time per gram of root) is given by

$$\frac{abN}{1+ahN}$$

Where b is the biomass of the roots possessed by the plant , N is the available nutrient concentration in the environment (units of resources per unit volume of soil), the encounter rate between a unit of root and a nutrient molecule is given by a and the cost in time associated with handling a given amount of nutrient molecules is given by h [4].

II. The Mathematical Model

The Mathematical model of Plant soil interaction is given below

$$\begin{aligned}
\frac{dP}{dt} &= r\left(1-\frac{P}{K}\right)P + r_0PM - r_1P \\
\frac{dN}{dt} &= Q - \mu N - \sigma PN \\
\frac{dM}{dt} &= \sigma PN - r_0PM
\end{aligned} \qquad (1)$$

where

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P - Density of plant species.

N - Concentration of the nutrient in soil.

M - Concentration of the nutrient in plant species that is used for metabolic activities.

r - Intrinsic growth rate of the plant species.

r_0 - Growth rate of plant species due to the metabolism of the nutrients.

Q - Rate at which a nutrient is applied to the soil.

μ - Leaching rate of a nutrient from the soil.

σ - The nutrient harvest rate by the plant species.

K - Carrying Capacity of the Plant species.

By applying Michaelis–Menten equation for the nutrient harvest rate by the plant species we obtain the following system of equations

$$\begin{aligned}\frac{dP}{dt} &= r(1-\frac{P}{K})P + r_0 PM - r_1 P\\ \frac{dN}{dt} &= Q-\mu N - \frac{abN}{1+ahN}PN\\ \frac{dM}{dt} &= \frac{abN}{1+ahN}PN - r_0 PM\end{aligned} \tag{2}$$

By applying Backward Euler Method, we obtain the following system of difference equations

$$\begin{aligned}P_{t+1} &= P_t + r(1-\frac{P_{t+1}}{L})P_{t+1} + r_0 P_{t+1}M_{t+1} - r_1 P_{t+1}\\ N_{t+1} &= N_t + Q - \mu N_{t+1} - \frac{abN}{1+ahN}P_{t+1}N_{t+1}\\ M_{t+1} &= M_t + \frac{abN}{1+ahN}P_{t+1}N_{t+1} - r_0 P_{t+1}M_{t+1}\end{aligned} \tag{3}$$

III. Equilibrium Points

We list the possible equilibrium points of the model (3):

i. $E_1 = (\bar{P}, 0, \bar{M})$ - Nutrient Free Equilibrium

Where

$\bar{P} = \frac{1}{r_0}$,

$\bar{M} = \frac{1}{Kr_0{}^2}(Kr_0(r_1 - r - 1) + r)$ =0 and $r_1 > r+1$.

ii. $E_2 = (\tilde{P}, \tilde{N}, 0)$

Where $\tilde{P} = \frac{K(r-r_1)}{r}$ and $r > r_1$

$\tilde{N}$ is the positive root of the characteristic equation

$$\tilde{N}^2\left[\frac{abK(r-r_1)}{r} + ah\mu\right] + \tilde{N}(\mu - ahQ) - Q = 0.$$

iii. $E_3 = (P^*, N^*, M^*)$

Where P^*, N^*, M^* satisfy

$$\begin{aligned}&r(1-\frac{P^*}{L})P^* + r_0 P^* M^* - r_1 P^* = 0\\ &Q - \mu N^* - \sigma P^* N^* = 0\\ &\sigma P^* N^* - \sigma_0 M^* = 0\end{aligned}$$

The jacobian matrix of the system (3) is given by

$$J = \begin{pmatrix} 1+(r-r_1)-\frac{2rP}{K}+r_0 M & 0 & r_0 P\\ \frac{-abN^2}{1+ahN} & 1-\mu-\frac{abNP(2+ahN)}{(1+ahN)^2} & 0\\ \frac{abN^2}{(1+ahN)} & \frac{abNP(2+ahN)}{(1+ahN)^2} & 1-r_0 P\end{pmatrix} \tag{4}$$

IV. Stability Analysis Of The Model

Theorem 1:

The fixed point $E_1 = (\bar{P}, 0, \bar{M})$ is non-hyperbolic.

Proof:

Consider the jacobian matrix of the system (3) with respect to the fixed point E_1.

$$J_1 = \begin{pmatrix} 1+(r-r_1)-\frac{2r\bar{P}}{K} & 0 & r_0\bar{P}\\ 0 & 1-\mu & 0\\ 0 & 0 & 1-r_0\end{pmatrix} \tag{5}$$

The eigen values of the above matrix is given by $|\lambda_1| = \frac{r}{Kr_0}$, $|\lambda_2| = \mu - 1$ and $|\lambda_3| = 1$. Therefore the fixed point E_1 is non-hyperbolic.

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Theorem 2:

The fixed point $E_2 = (\tilde{P}, \tilde{N}, 0)$ is stable if

$$\left\{1 + r < \frac{2r\tilde{P}}{K} + r_1\right\} \tag{6}$$

$$\left\{\mu + \frac{ab\tilde{N}\tilde{P}(2 + ah\tilde{N})}{(1 + ah\tilde{N})^2}, r_0\tilde{P}\right\} > 1 \tag{7}$$

and $(\phi_1\phi_2 + \phi_4) - (1 + \phi_1\phi_2(1 - \phi_3)) < \phi_1\phi_2\phi_3 < 1 + \phi_4$,

otherwise unstable.

Proof:

Consider the jacobian matrix of the system (3) with respect to the fixed point E_2.

$$J_2 = \begin{pmatrix} 1 + (r - r_1) - \frac{2r\tilde{P}}{K} & 0 & r_0\tilde{P} \\ \frac{-ab\tilde{N}^2}{1 + ah\tilde{N}} & 1 - \mu - \frac{ab\tilde{N}\tilde{P}(2 + ah\tilde{N})}{(1 + ah\tilde{N})^2} & 0 \\ \frac{ab\tilde{N}^2}{(1 + ah\tilde{N})} & \frac{ab\tilde{N}\tilde{P}(2 + ah\tilde{N})}{(1 + ah\tilde{N})^2} & 1 - r_0\tilde{P} \end{pmatrix} \tag{8}$$

The characteristic equation of the above matrix is given by,

$$\varphi(\lambda) = \lambda^3 - \Omega_1\lambda^2 + \Omega_2\lambda - \Omega_3 = 0 \tag{9}$$

Where

$$\begin{aligned} \Omega_1 &= \phi_1 + \phi_2 + \phi_3 \\ \Omega_2 &= (\phi_1 + \phi_2)\phi_3 + \phi_1\phi_2 \\ \Omega_3 &= (\phi_1\phi_2\phi_3 - \phi_4) \end{aligned} \tag{10}$$

$$\begin{aligned} \phi_1 &= 1 + (r - r_1) - \frac{2r\tilde{P}}{K} \\ \phi_2 &= (1 - \mu - \frac{ab\tilde{N}\tilde{P}(2 + ah\tilde{N})}{(1 + ah\tilde{N})^2}) \\ \phi_3 &= (1 - r_0\tilde{P}) \\ \phi_4 &= \left\{\frac{r_0(1 - \mu)ab\tilde{N}\tilde{P}}{(1 + ah\tilde{N})}\right\} \end{aligned} \tag{11}$$

It follows from the well-known Jury conditions that the modulus of all the roots of the above characteristic equation is less than 1 if and only if the conditions $\varphi(1) > 0, \varphi(-1) < 0$ and $|\det J_2| < 1$ hold[3].

$$\begin{aligned} \varphi(1) = 1 &- \left\{(1 + (r - r_1) - \frac{2r\tilde{P}}{K}) + (1 - \mu - \frac{ab\tilde{N}\tilde{P}(2 + ah\tilde{N})}{(1 + ah\tilde{N})^2}) + (1 - r_0\tilde{P})\right\} + \\ &\left\{\begin{array}{l} \left[(1 + (r - r_1) - \frac{2r\tilde{P}}{K}) + (1 - \mu - \frac{ab\tilde{N}\tilde{P}(2 + ah\tilde{N})}{(1 + ah\tilde{N})^2})\right]\left\{(1 - r_0\tilde{P})\right\} \\ + (1 + (r - r_1) - \frac{2r\tilde{P}}{K})(1 - \mu - \frac{ab\tilde{N}\tilde{P}(2 + ah\tilde{N})}{(1 + ah\tilde{N})^2}) \end{array}\right\} \\ &- \left\{(1 + (r - r_1) - \frac{2r\tilde{P}}{K})(1 - \mu - \frac{ab\tilde{N}\tilde{P}(2 + ah\tilde{N})}{(1 + ah\tilde{N})^2})(1 - r_0\tilde{P}) - \left\{\frac{r_0(1 - \mu)ab\tilde{N}\tilde{P}}{(1 + ah\tilde{N})}\right\}\right\} > 0 \end{aligned} \tag{12}$$

Assume the conditions (6) and (7). From (12) we can see that $\varphi(1) > 0$.

We can see that $\varphi(-1) < 0$ implies

$$\begin{aligned} &\left\{\begin{array}{l} \left[(r_1 + \frac{2r\tilde{P}}{K} - 1 - r) + (\mu + \frac{ab\tilde{N}\tilde{P}(2 + ah\tilde{N})}{(1 + ah\tilde{N})^2} - 1)\right](2 - r_0\tilde{P}) + (r_0\tilde{P} - 2) \\ - \left[(1 + (r - r_1) - \frac{2r\tilde{P}}{K})(1 - \mu - \frac{ab\tilde{N}\tilde{P}(2 + ah\tilde{N})}{(1 + ah\tilde{N})^2})\right] \end{array}\right\} + \\ &\left\{\frac{r_0(1 - \mu)ab\tilde{N}\tilde{P}}{(1 + ah\tilde{N})}\right\} < \left\{(1 + (r - r_1) - \frac{2r\tilde{P}}{K})(1 - \mu - \frac{ab\tilde{N}\tilde{P}(2 + ah\tilde{N})}{(1 + ah\tilde{N})^2})(1 - r_0\tilde{P})\right\} \end{aligned} \tag{13}$$

$$\Rightarrow \phi_4 - (\phi_1\phi_2 + (\phi_1 + \phi_2 + 1)(\phi_3 + 1)) < \phi_1\phi_2\phi_3$$

We can see that $|\det J_2| < 1$ implies

$$\begin{aligned} &\left\{(1 - r + r_1)(1 - \mu - \frac{ab\tilde{N}\tilde{P}(2 + ah\tilde{N})}{(1 + ah\tilde{N})^2})(1 - r_0\tilde{P})\right\} \\ &< 1 + \left\{\frac{r_0(1 - \mu)ab\tilde{N}\tilde{P}}{(1 + ah\tilde{N})}\right\} \end{aligned} \tag{14}$$

$$\Rightarrow \phi_1\phi_2\phi_3 < 1 + \phi_4$$

Using the conditions, (6) and (7), we can say that the fixed point E_2 is stable if

$$\phi_4 - (\phi_1\phi_2 + (\phi_1 + \phi_2 + 1)(\phi_3 + 1)) < \phi_1\phi_2\phi_3 < 1 + \phi_4 \tag{15}$$

Theorem 3:

The fixed point $E_3 = (P^*, N^*, M^*)$ is stable if

$$\left\{1 + (r - r_1) + r_0 M^* < \frac{2rP^*}{K}\right\} \tag{16}$$

$$\left\{\mu + \frac{abN^*P^*(2 + ahN^*)}{(1 + ahN^*)^2}, r_0 P^*\right\} > 1 \tag{17}$$

and $(\varepsilon_1\varepsilon_2 + \varepsilon_4) - (1 + \varepsilon_1\varepsilon_2(1 - \varepsilon_3)) < \varepsilon_1\varepsilon_2\varepsilon_3 < 1 + \varepsilon_4$, otherwise unstable.

Proof:

Consider the jacobian matrix of the system (3) with respect to the fixed point E_3.

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$$J_3 = \begin{pmatrix} 1+(r-r_1)-\frac{2rP^*}{K}+r_0M^* & 0 & r_0P^* \\ \frac{-abN^{*2}}{1+ahN^*} & 1-\mu-\frac{abN^*P^*(2+ahN^*)}{(1+ahN^*)^2} & 0 \\ \frac{abN^{*2}}{(1+ahN^*)} & \frac{abN^*P^*(2+ahN^*)}{(1+ahN^*)^2} & 1-r_0P^* \end{pmatrix} \quad (18)$$

The characteristic equation of the above matrix is given by,

$$\omega(\lambda) = \lambda^3 - \alpha_1\lambda^2 + \alpha_2\lambda - \alpha_3 = 0 \quad (19)$$

Where

$$\begin{aligned} \alpha_1 &= \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \\ \alpha_2 &= (\varepsilon_1 + \varepsilon_2)\varepsilon_3 + \varepsilon_1\varepsilon_2 \\ \alpha_3 &= (\varepsilon_1\varepsilon_2\varepsilon_3 - \varepsilon_4) \end{aligned} \quad (20)$$

$$\begin{aligned} \varepsilon_1 &= 1+(r-r_1)-\frac{2rP^*}{K}+r_0M^* \\ \varepsilon_2 &= 1-\mu-\frac{abN^*P^*(2+ahN^*)}{(1+ahN^*)^2} \\ \varepsilon_3 &= (1-r_0P^*) \\ \varepsilon_4 &= \left\{\frac{r_0(1-\mu)abN^*P^*}{(1+ahN^*)}\right\} \end{aligned} \quad (21)$$

It follows from the well-known Jury conditions that the modulus of all the roots of the above characteristic equation is less than 1 if and only if the conditions $\omega(1) > 0, \omega(-1) < 0$ and $|DetJ_3| < 1$ hold[3].

$$\omega(1) = 1 - \left\{ \begin{array}{l} \left(1+(r-r_1)-\frac{2rP^*}{K}+r_0M^*\right) \\ +\left(1-\mu-\frac{abN^*P^*(2+ahN^*)}{(1+ahN^*)^2}\right)+(1-\sigma_0) \end{array} \right\}$$
$$+\left\{ \begin{array}{l} \left[\left(1+(r-r_1)-\frac{2rP^*}{K}+r_0M^*\right)+\left(1-\mu-\frac{abN^*P^*(2+ahN^*)}{(1+ahN^*)^2}\right)\right](1-\sigma_0) \\ +\left(1+(r-r_1)-\frac{2rP^*}{K}+r_0M^*\right)\left(1-\mu-\frac{abN^*P^*(2+ahN^*)}{(1+ahN^*)^2}\right) \end{array} \right\}$$
$$-\left\{ \begin{array}{l} \left(1+(r-r_1)-\frac{2rP^*}{K}+r_0M^*\right)\left(1-\mu-\frac{abN^*P^*(2+ahN^*)}{(1+ahN^*)^2}\right)(1-\sigma_0) \\ -\left\{\frac{r_0(1-\mu)ab\tilde{N}\tilde{P}}{(1+ah\tilde{N})}\right\} \end{array} \right\} > 0 \quad (22)$$

Assume the conditions (16) and (17). We can see that

$\omega(1) > 0$.

We can see that $\omega(-1) < 0$ implies

$$\left(\frac{2rP^*}{K}-(r-r_1)-r_0M^*-1\right)\times(2-r_0P^*)$$
$$+\left(\mu+\frac{abN^*P^*(2+ahN^*)}{(1+ahN^*)^2}-1\right)\times(2-r_0P^*)$$
$$+(r_0P^*-2)-\left[(1+(r-r_1)-\frac{2rP^*}{K})(1-\mu-\frac{abN^*P^*(2+ahN^*)}{(1+ahN^*)^2})\right]$$
$$+\left\{\frac{r_0(1-\mu)abN^*P^*}{(1+ahN^*)}\right\} < \left(1+(r-r_1)-\frac{2rP^*}{K}+r_0M^*\right)$$
$$\times\left(1-\mu-\frac{abN^*P^*(2+ahN^*)}{(1+ahN^*)^2}\right)(1-r_0P^*) \quad (23)$$

$$\Rightarrow \varepsilon_4 - (\varepsilon_1\varepsilon_2 + (\varepsilon_1+\varepsilon_2+1)(\varepsilon_3+1)) < \varepsilon_1\varepsilon_2\varepsilon_3 \quad (24)$$

We can see that $|DetJ_3| < 1$ implies

$$\left\{\left(1+(r-r_1)-\frac{2rP^*}{K}+r_0M^*\right)\left(1-\mu-\frac{abN^*P^*(2+ahN^*)}{(1+ahN^*)^2}\right)(1-r_0P^*)\right\}$$
$$< 1+\left\{\frac{r_0(1-\mu)abN^*P^*}{(1+ahN^*)}\right\} \quad (25)$$

$$\Rightarrow \varepsilon_1\varepsilon_2\varepsilon_3 < 1+\varepsilon_4$$

Using the conditions, (16) and (17) , we can say that the fixed point E_3 is stable if

$$\varepsilon_4 - (\varepsilon_1\varepsilon_2 + (\varepsilon_1+\varepsilon_2+1)(\varepsilon_3+1)) < \varepsilon_1\varepsilon_2\varepsilon_3 < 1+\varepsilon_4 \quad (26)$$

V. Global Stability

We define the Lyapunov function,

$$U^*(t) = g\left(\frac{P_t}{P^*}\right)+g\left(\frac{M_t}{M^*}\right)+g\left(\frac{N_t}{N^*}\right) \quad (27)$$

Where $g(x) = x-1-In(x)$ is defined for $x > 0$. We consider the Lyapunov function $U^*(t)$ to prove the global asymptotic stability of the equilibrium $E_3 = (P^*, N^*, M^*)$.

$$U^*(t) = \lim_{P\to P^*, N\to N^*, M\to M^*} U(t) \quad (28)$$

We have $U^*(t+1)-U^*(t) \le 0$ with the equality iff $P_{t+1} = P^*, N_{t+1} = N^*, M_{t+1} = M^*$.

Theorem 4:
For system of equations (3), the equilibrium $E_3 = (P^*, N^*, M^*)$ is globally asymptotically stable.
Proof:
For the equilibrium $E_3 = (P^*, N^*, M^*)$

$$U^*(t) = U_1^*(t)+U_2^*(t)+U_3^*(t) \quad (29)$$

Where $U_1^*(t) = g\left(\frac{P_t}{P^*}\right), U_2^*(t) = g\left(\frac{N_t}{N^*}\right), U_3^*(t) = g\left(\frac{M_t}{M^*}\right)$

We have,

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$$
\begin{aligned}
U_1^*(t+1)-U_1^*(t) &= \left(\frac{P_{t+1}}{P^*}-1-In\frac{P_{t+1}}{P^*}\right)-\left(\frac{P_t}{P^*}-1-In\frac{P_t}{P^*}\right)\\
&\le \frac{1}{P^*}\left[1-\frac{P^*}{P_{t+1}}\right]\left[P_{t+1}-P_t\right]\\
&= \frac{1}{P^*}\left[1-\frac{P^*}{P_{t+1}}\right]\left[r\left[1-\frac{P_{t+1}}{K}\right]P_{t+1}+r_0P_{t+1}M_{t+1}-r_1P_{t+1}\right]\\
&= \frac{1}{P^*}\left[1-\frac{P^*}{P_{t+1}}\right]\left[rP_{t+1}-r\frac{P_{t+1}{}^2}{K}+r_0P_{t+1}M_{t+1}-r_1P_{t+1}\right]\\
&= \frac{1}{P^*}\left[1-\frac{P^*}{P_{t+1}}\right]\left[-r\frac{P_{t+1}{}^2}{K}\left[1-\frac{P^*}{P_{t+1}}\right]+r_0P_{t+1}M_{t+1}\left[1-\frac{M^*}{M_{t+1}}\right]\right]\\
&= -\frac{1}{P^*}r\frac{P_{t+1}{}^2}{K}\left[1-\frac{P^*}{P_{t+1}}\right]^2+r_0P_{t+1}M_{t+1}\left[1-\frac{P^*}{P_{t+1}}\right]\left[1-\frac{M^*}{M_{t+1}}\right]\\
&= -\frac{rP^*}{K}\left[\frac{P_{t+1}}{P^*}-1\right]^2+r_0M^*\left[\frac{P_{t+1}}{P^*}-1\right]\left[\frac{M_{t+1}}{M^*}-1\right]
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
U_2^*(t+1)-U_2^*(t) &= \left(\frac{N_{t+1}}{N^*}-1-In\frac{N_{t+1}}{N^*}\right)-\left(\frac{N_t}{N^*}-1-In\frac{N_t}{N^*}\right)\\
&\le \frac{1}{N^*}\left[1-\frac{N^*}{N_{t+1}}\right]\left[N_{t+1}-N_t\right]\\
&= \frac{1}{N^*}\left[1-\frac{N^*}{N_{t+1}}\right]\left[Q-\mu N_t-\frac{abN_{t+1}}{1+ahN_{t+1}}P_{t+1}N_{t+1}\right]\\
&= \frac{1}{N^*}\left[1-\frac{N^*}{N_{t+1}}\right]\left[Q\left[1-\frac{N_{t+1}}{N^*}\right]-N_{t+1}\left[\frac{abN_{t+1}P_{t+1}}{1+ahN_{t+1}}-\frac{abN^*P^*}{1+ahN^*}\right]\right]\\
&= \frac{-QN_{t+1}}{N^{*2}}\left[1-\frac{N^*}{N_{t+1}}\right]^2-\frac{N_{t+1}}{N^*}\left[1-\frac{N^*}{N_{t+1}}\right]\\
&\quad\times\left[\frac{abN_{t+1}P_{t+1}}{1+ahN_{t+1}}-\frac{abN^*P^*}{1+ahN^*}\right]
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
U_3^*(t+1)-U_3^*(t) &= \left(\frac{M_{t+1}}{M^*}-1-In\frac{M_{t+1}}{M^*}\right)-\left(\frac{M_t}{M^*}-1-In\frac{M_t}{M^*}\right)\\
&\le \frac{1}{M^*}\left[1-\frac{M^*}{M_{t+1}}\right]\left[M_{t+1}-M_t\right]\\
&= \frac{1}{M^*}\left[1-\frac{M^*}{M_{t+1}}\right]\left[\frac{abN_{t+1}}{1+ahN_{t+1}}P_{t+1}N_{t+1}-\sigma_0M_{t+1}\right]\\
&= \frac{1}{M^*}\left[1-\frac{M^*}{M_{t+1}}\right]\frac{abN_{t+1}}{1+ahN_{t+1}}P_{t+1}N_{t+1}\\
&\quad -\left[1-\frac{M^*}{M_{t+1}}\right]\frac{abN^*}{1+ahN^*}P^*N^*\frac{M_{t+1}}{M^*}
\end{aligned}
\tag{32}
$$

Substituting (30), (31), (32) in (29), we get $U^*(t+1)-U^*(t)\le 0$, for all $t\ge 0$. We see that $U^*(t)\ge 0$ is a monotone decreasing function. Therefore, $U^*(t+1)-U^*(t)\le 0$. Then $\lim_{t\to\infty} U^*(t+1)-U^*(t)=0$, which implies that $\lim_{t\to\infty} P_{t+1}=P^*, \lim_{t\to\infty} N_{t+1}=N^*, \lim_{t\to\infty} M_{t+1}=M^*$.

Therefore $E_3=(P^*,N^*,M^*)$ is globally asymptotically stable.

VI. MINERAL NUTRIENT IN TOMATO PLANTS

The tomato is the edible, red fruit of Solanum lycopersicum, commonly known as a tomato plant, which belongs to the nightshade family, Solanaceae. The species originated in Central and South America. The Nahuatl (Aztec language) word tomatl gave rise to the Spanish word "tomate", from which the English word tomato originates. Tomato requires at least twelve nutrients, also called “essential elements”, for normal growth and reproduction. These are nitrogen(N), phosphorus (P), potassium (K), calcium (Ca), magnesium(Mg), sulfur (S), boron (B), iron (Fe), manganese (Mn), copper(Cu), zinc (Zn), and molybdenum (Mo).

We now list the five major nutrients, their availability in the soil and their utilisation by the plant:

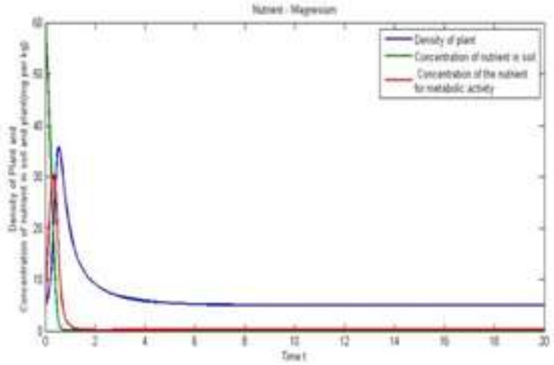

Figure 1. Plant Soil interactions of Magnesium nutrient in Tomato Plant

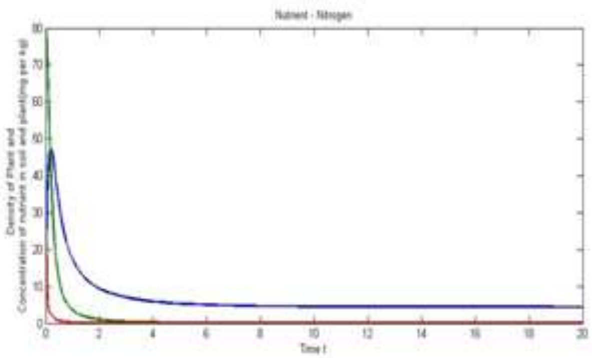

Figure 2. Plant Soil interactions of Nitrogen nutrient in Tomato Plant

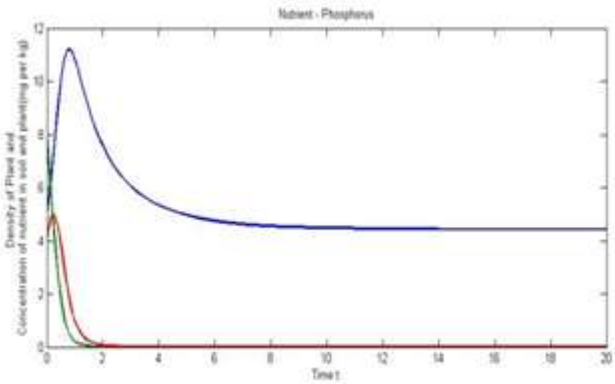

Figure 3. Plant Soil interactions of Phosphorus nutrient in Tomato Plant

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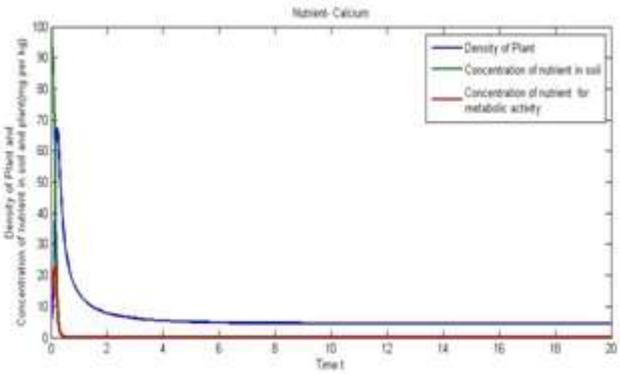

Figure 4. Plant Soil interactions of Calcium nutrient in Tomato Plant

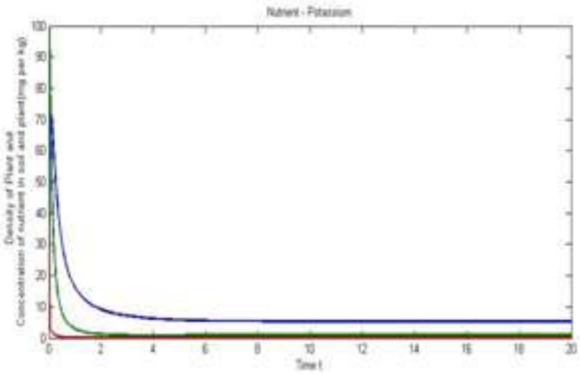

Figure 5. Plant Soil interactions of Potassium nutrient in Tomato Plant

VII. Discussion

From these numerical simulations we can see that, the metabolic activity of the plant decreases as the nutrient concentration in the soil decreases. And once the nutrient concentration in the soil decreases the growth of the plant is constant. Here we can see that there is a high metabolic rate for the nutrients Magnesium and Calcium. And there is low metabolic rate for Potassium, Phosphorus and Nitrogen. Hence we see that there must be an increase in the Potassium, Phosphorus and Nitrogen content in the soil through fertilizers. Higher the metabolic rate, higher the growth rate of plants which leads to higher productivity.

Here we can see that for the nutrients Nitrogen and Potassium the growth of the plant is high even though the metabolic rate of the plant for these nutrients is low. Therefore, we can see that Nitrogen and Potassium are essential for the growth of the tomato plant.

VIII. Conclusion

In this paper, we have constructed a discrete time model on plant soil interactions. We list the possible equilibrium points of the model and analyze both the local and global stability conditions of the model. We analyze the nutrient concentration of a tomato plant in both the soil and the plant that is used for the metabolic activity using MATLAB. We provide the results for five major nutrients that are essential for the growth of a tomato plant. From the results, we can see that the growth rate of the plant depends on the nutrient concentration in the plant and in the soil.

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TABLE I. List of the five major nutrients of the tomato plant

Nutrition	**Rate of usage by plants(%)** r_0	**Nutrient content applied to the soil(gm per kg)** Q	**Resource harvest rate(Michealis Menten equation)**	**Initial concentration of nutrient in the soil -** N_0 **(ppm)**	**Initial concentration of nutrient used by the plant -** M_0 **(gm per kg)**	**Source**
Nitrogen(N)	1.79	0.05	0.1	80	30	[11,12,13]
Phosphorus(P)	0.37	0.006	0.4	8	4	[11,12,13]
Potassium(K)	2.67	0.6	0.1	100	60	[11,12,13]
Calcium(Ca)	0.84	0.1	0.4	100	12.5	[11,12,13]
Magnesium(Mg)	0.23	0.34	0.4	60	5	[11,12,13]