An Approximate Analytical Solution of the Burger's Equation for Longitudinal Dispersion Phenomenon Arising in Fluid Flow through Porous Medium

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Abstract—The present paper discusses the longitudinal dispersion phenomenon in miscible fluid flow through porous medium. The mathematical formulation yields a nonlinear partial differential equation in the form of Burger's equation. An approximate analytical solution of the Burger's equation for longitudinal dispersion phenomenon has been obtained. Homotopy analysis method is adopted to solve this equation with suitable boundary conditions. The numerical interpretation of solution has been obtained at distance *x* for a given time *t*. The graphical interpretation of solution has been also given by Mathematica software.

Keywords-*Fluid flow; Homogeneous porous medium; Longitudinal dispersion phenomenon; Homotopy analysis method.*

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I. INTRODUCTION

The main goal of the present paper is to examine the concentration of the longitudinal dispersion phenomenon occurring in miscible fluid flow through porous medium. The problems involving the flow of two or more fluids are frequently encountered in groundwater hydrology, oil recovery engineering, agriculture engineering, environmental science and soil mechanics [1-21]. Two types of flow are possible when two or more fluids flow in porous medium domain [1]: 1. Miscible fluid flow 2. Immiscible fluid flow. Miscible fluid flow is a type of flow in which two fluids are completely soluble in each other. In the immiscible fluid flow we have a simultaneous flow of two or more immiscible fluids in the porous medium domain.

The longitudinal dispersion phenomenon is the process by which miscible fluid flow disperse in the direction of flow. The problem of miscible fluid flow can be seen in coastal areas, where the groundwater beds are gradually displaced by seawater. Longitudinal dispersion phenomenon plays an important role to control salinity of the soil in coastal areas.

Many researchers have discussed the longitudinal dispersion phenomenon with different point of views like as Carrier [2] studied the subterranean mixing in permeable media of seawater and groundwater. Scheidegger [3] investigated the possibilities of generalizing the dispersion equations of flow through porous media. Al-Niami and Rushton [4] studied the analysis of flow against dispersion in a porous media. Hunt [5] applied the perturbation method to longitudinal and lateral dispersion in non-uniform seepage flow through heterogeneous aquifers. Patel and Mehta [6] worked on Burger's equation for longitudinal dispersion of miscible fluid flow through porous media. Meher and Mehta [7] obtained solution of Burger's equation for longitudinal dispersion phenomenon of miscible fluid flow through porous media by Backlund transformation. Meher at al. [8] applied adomian decomposition method for dispersion phenomenon arising in longitudinal dispersion of miscible fluid flow through porous media. Joshi at al. [9] discussed the group theoretic approach for longitudinal dispersion phenomenon in miscible fluid flow through porous media. Borana at al. [10] obtained numerical solution of Burger’s equation arising in longitudinal dispersion phenomenon in fluid flow through porous media by Crank-Nicolson scheme. Olayiwola [11] applied modified variational iteration method for the solution of nonlinear Burger's equation arising in longitudinal dispersion phenomenon in fluid flow through porous media.

The problem is to describe the growth of the region occupied by mixture of miscible fluids, i.e. to find the concentration of contaminated water as function of position x and time t, as the mixture flow through homogeneous porous medium. Outside of the mixed region, the single fluid equation describe the motion of fluid. The flow of mixture takes place both longitudinal as well as transversely but the spreading caused by dispersion is more in the direction of flow than the transversely direction. Thus the longitudinal dispersion of the contaminated water flowing in the x-direction has been considered.

The present paper discusses the solution of Burger's equation for longitudinal dispersion phenomenon arising in miscible fluid flow through homogeneous porous medium. Homotopy analysis method is applied to solve Burger's equation with appropriate boundary conditions. The homotopy series solution represents the concentration of contaminated water in longitudinal dispersion.

II. Mathematical Formulation

The mixture of miscible fluids could be thought to behave as a single phase fluid, therefore it will obey the Darcy's law. The equation of continuity for the mixture is given by [1, 12]:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{V}) = 0 \tag{1}$$

where ρ is the density for mixture and $\bar{V}$ is the pore seepage velocity vector.

The equation of diffusion for a fluid flow through a homogeneous porous medium without increasing or decreasing the dispersing material is given by

$$\frac{\partial C}{\partial t} + \nabla \cdot (C\bar{V}) = \nabla \cdot \left[\rho \bar{D} \nabla \left(\frac{C}{\rho}\right)\right] \tag{2}$$

where C is the concentration of contaminated water in a porous medium and $\bar{D}$ is the tensor coefficient of dispersion with nine components D_{ij}.

In a laminar flow for an incompressible fluid through a homogeneous porous medium at constant temperature, ρ is considered as constant. Then

$$\nabla \cdot \bar{V} = 0. \tag{3}$$

Using (3) in (2) we get

$$\frac{\partial C}{\partial t} + \bar{V} \cdot \nabla C = \nabla \cdot (\bar{D} \nabla C). \tag{4}$$

Consider that the seepage velocity $\bar{V}$ is along the x-axis then $\bar{V} = u(x,t)$ and the non-zero component will be $D_{11} = D_L = \gamma$ (coefficient of longitudinal dispersion) and other components D_{ij} are zero [12]. Thus the Eq. (4) becomes

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = D_L \frac{\partial^2 C}{\partial x^2} \tag{5}$$

where u is the component of velocity along the x-axis which is dependent on time as well as concentration along the positive x-axis and $D_L > 0$, and it is the cross-sectional flow velocity in porous medium. The seepage velocity u is related with the concentration of the dispersion material as $u = \frac{C(x,t)}{C_0}$ where $x > 0$ where the concentration of contaminated water at $x = 0$ is very high and for $C_0 \cong 1$ [13], the Eq. (5) becomes [14, 15]

$$\frac{\partial C}{\partial t} + C\frac{\partial C}{\partial x} = \gamma \frac{\partial^2 C}{\partial x^2}. \tag{6}$$

The boundary conditions in longitudinal direction are

$$C(0,t) = 1 \text{ and } C(1,t) = b, 0 \le t \le 1, b \text{ is constant.} \tag{7}$$

Thus the Eq. (6) together with boundary conditions (7) represents the boundary value problem for the longitudinal dispersion. This equation is the nonlinear Burger's equation for longitudinal dispersion phenomenon of miscible fluid flow through porous medium.

III. Homotopy Analysis Method

In 1992, Liao [22] has proposed the homotopy analysis method in his Ph.D. thesis for solving nonlinear differential equations. This technique has successfully employed to solve many types of nonlinear differential equations. The homotopy analysis method has applied to various types of ordinary differential equations as well as partial differential equations [17-27].

Define a nonlinear operator $\mathcal{N}$ as

$$\mathcal{N}(\varphi(x,t;q)) = \gamma \frac{\partial^2 \varphi(x,t;q)}{\partial x^2} - \varphi(x,t;q)\frac{\partial \varphi(x,t;q)}{\partial x} - \frac{\partial \varphi(x,t;q)}{\partial t}. \tag{8}$$

According to boundary conditions (7), we choose the initial approximation of $C(x,t)$ as

$$v_0(x,t) = (1-x)^2(1+xt) + bx. \tag{9}$$

We choose the auxiliary linear operator

$$\mathcal{L}[\varphi(x,t;q)] = \frac{\partial^2 \varphi(x,t;q)}{\partial x^2} \tag{10}$$

which satisfies the property $\mathcal{L}[Ax + B] = 0$ where A and B are constants.

Liao [22] constructed the zeroth-order deformation equation

$$(1-q)\mathcal{L}[\varphi(x,t;q) - v_0(x,t)] = qc_0 H(x,t)\mathcal{N}(\varphi(x,t;q)) \tag{11}$$

where $q \in [0,1]$ the embedding-parameter, $c_0 \neq 0$ the convergence control parameter, $H(x,t) \neq 0$ the auxiliary function.

When $q = 0$ and $q = 1$, Eq. (11) provided

$$\varphi(x,t;0) = v_0(x,t) \text{ and } \varphi(x,t;1) = C(x,t). \tag{12}$$

Thus as q increases from 0 to 1, $\varphi(x,t;q)$ continuously deforms from the initial approximation $v_0(x,t)$ to the exact solution $C(x,t)$ of the Eq. (6). Assume that the auxiliary linear operator, the initial approximation, the convergence control parameter and the auxiliary function are chosen so properly that the Maclaurin series of $\varphi(x,t;q)$ with respect to q

$$\varphi(x,t;q) = v_0(x,t) + \sum_{m=1}^{\infty} v_m(x,t)q^m \qquad (13)$$

where

$$v_m(x,t) = \frac{1}{m!}\frac{\partial^m \varphi(x,t;q)}{\partial q^m}\bigg|_{q=0} \qquad (14)$$

converges at $q = 1$. Thus the homotopy series solution is

$$C(x,t) = v_0(x,t) + \sum_{m=1}^{\infty} v_m(x,t). \qquad (15)$$

Define $\overrightarrow{v_m(x,t)} = \{v_0, v_1, \dots, v_m\}$. Differentiating the Eq. (11) m times with respect to the embedding-parameter q and dividing them by $m!$ and then finally setting $q = 0$, we have the high-order deformation equation

$$\mathcal{L}[v_m(x,t) - \chi_m v_{m-1}(x,t)] = c_0 H(x,t)\mathcal{R}_m(\overrightarrow{v_{m-1}}) \qquad (16)$$

subject to the boundary conditions

$$v_m(0,T) = 0, \;\; v_m(1,T) = 0, \;\; m \geq 1 \qquad (17)$$

where

$$\mathcal{R}_m(\overrightarrow{v_{m-1}}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}(\varphi(x,t;q))}{\partial q^{m-1}}\bigg|_{q=0} \qquad (18)$$

or

$$\mathcal{R}_m(\overrightarrow{v_{m-1}}) = \gamma\frac{\partial^2 v_{m-1}}{\partial x^2} - \sum_{i=0}^{m-1} v_i \frac{\partial v_{m-1-i}}{\partial x} - \frac{\partial v_{m-1}}{\partial t}$$

and

$$\chi_m = \begin{cases} 0\,, & \text{when} \;\; m \leq 1 \\ 1\,, & \text{when} \;\; m > 1. \end{cases} \qquad (19)$$

For simplicity, we assume that $H(x,t) = 1$. Thus it is easy to solve the linear ordinary differential equations (16). Then the general solution of the high-order deformation equation (16) is

$$v_m(x,t) = \chi_m v_{m-1}(x,t) + c_0 \iint \mathcal{R}_m(\overrightarrow{v_{m-1}})dxdx + Ax + B \qquad (20)$$

where A and B are constants or functions of t. Hence the approximate analytical solution of the Eq. (6) is as

$$\begin{aligned} C(x,t) &= (1-x)^2(1+xt) + bx \\ &+ c_0\Big[-\frac{7x}{20} + \frac{bx}{12} + \frac{b^2x}{6} + \frac{tx}{30} + \frac{btx}{30} + \frac{t^2x}{210} \\ &+ x^2 - \frac{bx^2}{2} - \frac{tx^2}{2} - \frac{7x^3}{6} + \frac{2bx^3}{3} - \frac{b^2x^3}{6} \\ &+ \frac{4tx^3}{3} - \frac{btx^3}{3} - \frac{t^2x^3}{6} + \frac{2x^4}{3} - \frac{bx^4}{4} - \frac{3tx^4}{2} \\ &+ \frac{btx^4}{2} + \frac{t^2x^4}{2} - \frac{3x^5}{20} + \frac{4tx^5}{5} - \frac{btx^5}{5} \\ &- \frac{3t^2x^5}{5} - \frac{tx^6}{6} + \frac{t^2x^6}{3} - \frac{t^2x^7}{14} - \gamma x + \gamma tx \\ &+ \gamma x^2 - 2\gamma tx^2 + \gamma tx^3\Big] + \cdots. \end{aligned} \qquad (21)$$

The solution represents the concentration of contaminated water in longitudinal dispersion. The solution expression (21) contains the convergence control parameter c_0 which plays an important role for obtaining convergent homotopy analysis solution. Thus the convergent homotopy analysis solution strongly dependent on the convergence control parameter c_0. The proper value of c_0 is chosen with the help of c_0-curve [18-21, 23-25, 27]. The c_0-curve of $C_{xx}(0,0)$ is plotted using Mathematica package for BVPh [26]; (see figure 1). The line segment almost parallel to horizontal axis gives us the valid region $-1.6 \leq c_0 \leq -0.4$ for c_0. The following constant values are considered as $\gamma = 1, b = 0.001$ for obtaining the numerical values and graphical presentation of solution.

IV. Numerical Interpretation of Solution

The solution expression (21) with proper value of $c_0 = -0.9$ is used to interpret numerically. Table 1 indicates the numerical values of the solution $C(x,t)$ at different x for $t = 0.1, 0.2, \dots, 1$ for 20th order of approximation. Tabular values show that the concentration $C(x,t)$ of contaminated water decreases when x increases for a given time t.

V. Graphical Interpretation of Solution

The graphical interpretation of the solution (21) is given in figures 2-4. These figures are plotted using Mathematica package for BVPh [26]. Figures 2 and 3 show the graph of concentration $C(x,t)$ v/s x for fixed time $t = 0.3$ and for fixed time $t = 0.7$ respectively. Figure 4 represents the graph of concentration $C(x,t)$ v/s distance x and time t.

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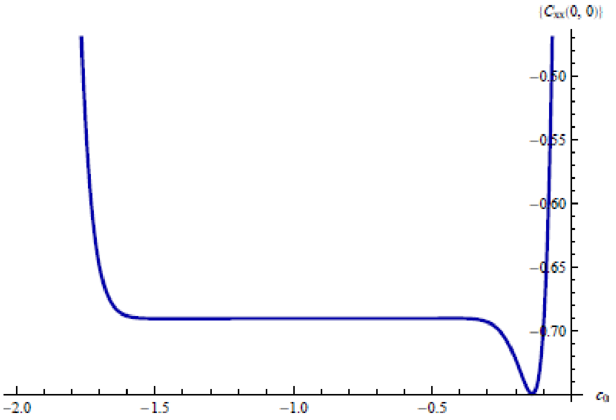

Figure 1: The c_0-curve of $C_{xx}(0,0)$ for 20thorder of approximation.

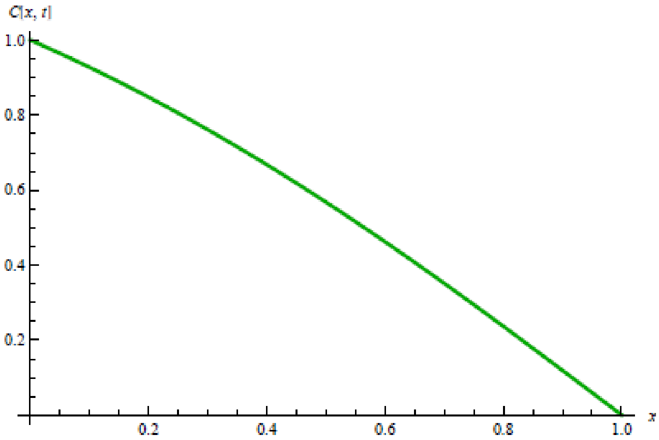

Figure 2: The graph of $C(x,t)$ v/s x for fixed $t = 0.3$.

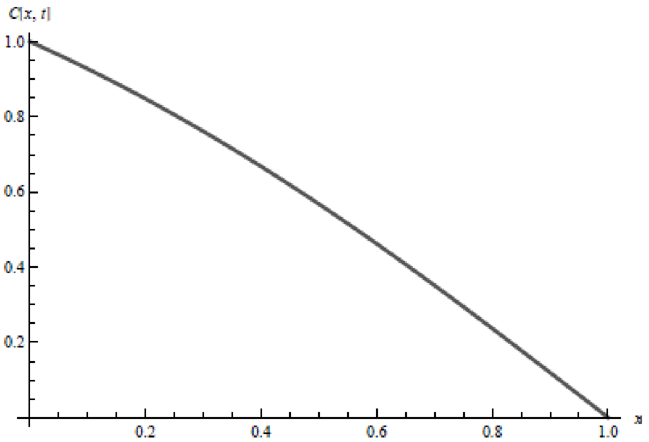

Figure 3: The graph of $C(x,t)$ v/s x for fixed $t = 0.7$.

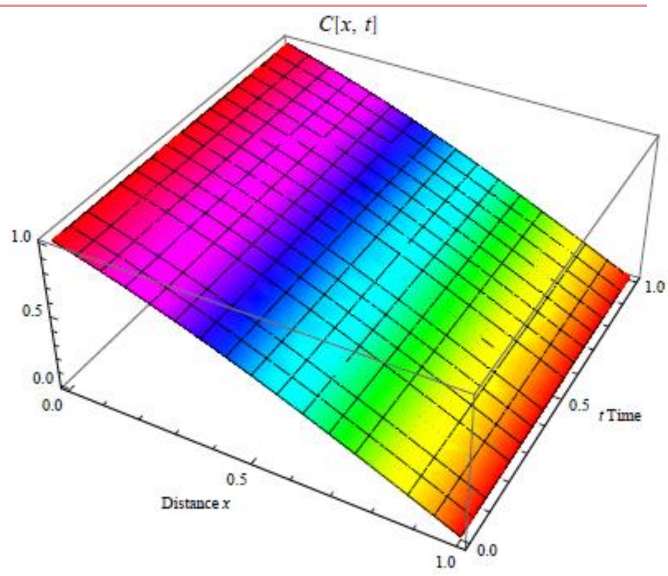

Figure 4: The graph of $C(x,t)$ v/s x and t.

VI. Conclusions

We have discussed the nonlinear Burger's equation for longitudinal dispersion phenomenon in miscible fluid flow through porous medium. Homotopy analysis solution is obtained for longitudinal dispersion phenomenon with appropriate boundary conditions. The solution satisfies both the boundary conditions. We have discussed the numerical interpretation and graphical interpretation of solution. The results show that the concentration of contaminated water decreases when the distance increases for a given time t.

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TABLE 1: NUMERICAL VALUES OF THE CONCENTRATION OF CONTAMINATED WATER

x	$t = 0.1$	$t = 0.2$	$t = 0.3$	$t = 0.4$	$t = 0.5$	$t = 0.6$	$t = 0.7$	$t = 0.8$	$t = 0.9$	$t = 1.0$
0	1.00000000000	1.00000000000	1.00000000000	1.00000000000	1.00000000000	1.00000000000	1.00000000000	1.00000000000	1.00000000000	1.00000000000
0.1	0.92749536169	0.92749536169	0.92749536169	0.92749536169	0.92749536169	0.92749536169	0.92749536169	0.92749536169	0.92749536169	0.92749536169
0.2	0.84795355919	0.84795355918	0.84795355918	0.84795355918	0.84795355918	0.84795355918	0.84795355918	0.84795355917	0.84795355917	0.84795355917
0.3	0.76138291228	0.76138291227	0.76138291227	0.76138291226	0.76138291226	0.76138291226	0.76138291225	0.76138291225	0.76138291225	0.76138291225
0.4	0.66797416303	0.66797416302	0.66797416301	0.66797416300	0.66797416300	0.66797416299	0.66797416299	0.66797416299	0.66797416298	0.66797416298
0.5	0.56812357531	0.56812357530	0.56812357529	0.56812357528	0.56812357527	0.56812357527	0.56812357526	0.56812357526	0.56812357526	0.56812357525
0.6	0.46244627975	0.46244627974	0.46244627973	0.46244627973	0.46244627972	0.46244627972	0.46244627971	0.46244627971	0.46244627971	0.46244627970
0.7	0.35177643762	0.35177643762	0.35177643761	0.35177643761	0.35177643760	0.35177643760	0.35177643760	0.35177643760	0.35177643759	0.35177643759
0.8	0.23715177269	0.23715177269	0.23715177268	0.23715177268	0.23715177268	0.23715177268	0.23715177268	0.23715177268	0.23715177268	0.23715177267
0.9	0.11978165310	0.11978165310	0.11978165309	0.11978165309	0.11978165309	0.11978165309	0.11978165309	0.11978165309	0.11978165309	0.11978165309
1.0	0.00100000000000	0.00100000000000	0.00100000000000	0.00100000000000	0.00100000000000	0.00100000000000	0.00100000000000	0.00100000000000	0.00100000000000	0.00100000000000