Notes on Interval-Valued Hesitant Fuzzy Soft Topological Space

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Abstract: In this paper we introduce the notion of interval valued hesitant fuzzy soft topological space. Also the concepts of interval valued hesitant fuzzy soft closure; interior and neighbourhood are introduced here and established some important results.

Keywords: Fuzzy soft sets; Interval-valued hesitant fuzzy sets; Interval-valued hesitant fuzzy soft sets; Interval-valued hesitant fuzzy soft topological space.
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1. Introduction

The concept of interval arithmetic was first suggested by Dwyer [10] in 1951. Chiao.[9] introduced sequence of interval numbers and defined usual convergence of sequences of interval number. A set is called an interval number if it consisting of a closed interval of real numbers x such that $a \le x \le b$. A real interval can also be considered as a set. Thus we can investigate some properties of interval numbers, for instance arithmetic properties or analysis properties. We denote the set of all real valued closed intervals by IR. Any elements of IR is called closed interval and denoted by x. That is $\bar{x} = \{x \in \Re : a \le x \le b\}$. An interval number $\bar{x}$ is a closed subset of real numbers (see [9]). Let x_1 and x_r be first and last points of $\bar{x}$ interval number, respectively. For $\bar{x}_1, \bar{x}_2 \in IR,$ we have $\bar{x}_1 = \bar{x}_2 \Leftrightarrow x_{1l} = x_{2l}, x_{1r} = x_{2r}$. $\bar{x}_1 + \bar{x}_2 = \{x \in \Re : x_{1l} + x_{2l} \le x \le x_{1r} + x_{2r}\}$, and if $\alpha \ge 0,$ then $\alpha\bar{x} = \{x \in \Re : \alpha x_{1l} \le x \le \alpha x_{1r}\}$ and if $\alpha < 0,$ then $\alpha\bar{x} = \{x \in \Re : \alpha x_{1r} \le x \le \alpha x_{1l}\}$,

$$\bar{x}_1.\bar{x}_2 = \{x \in \Re : \min\{x_{1l}.x_{2l}, x_{1l}.x_{2r}, x_{1r}.x_{2l}, x_{1r}.x_{2r}\} \le x \le \max\{x_{1l}.x_{2l}, x_{1l}.x_{2r}, x_{1r}.x_{2l}, x_{1r}.x_{2r}\}\}.$$

The most appropriate theory for dealing with uncertainties is the theory of fuzzy sets, introduced by L.A. Zadeh [31] in 1965. This theory brought a paradigmatic change in mathematics. But there arise difficulty that how to set the membership function in each particular case. The Hesitant fuzzy set, as one of the extensions of Zadeh [31] fuzzy set, allows the membership degree that an element to a set presented by several possible values, and itcan express the hesitant information more comprehensively than other extensions of fuzzyset. Torra and Narukawa [24] introduced the concept of hesitant fuzzy set. Xu and Xia [30] defined the concept of hesitant fuzzy element, which can be considered as the basic unit ofa hesitant fuzzy set, and is a simple and effective tool used to express the decision makers' hesitant preferences in the process of decision making. So many researchers has done lots of research work on aggregation, distance, similarity and correlation measures, clustering analysis, and decision making with hesitant fuzzy information. Babitha and John [3] defined another important soft set i.e. Hesitant fuzzy soft sets. They introduced basic operations such as intersection, union, compliment and De Morgan's law was proved. Chen et al. [8]extended hesitant fuzzy sets into interval-valued hesitant fuzzy environment and introduced the concept of interval-valued hesitant fuzzy sets. Zhang et al. [32] introduced some operations such as complement, "AND","OR", ring sum and ring product on interval-valuedhesitant fuzzy soft sets.

There are many theories like theory of probability, theory of fuzzy sets, and theory of intuitionistic fuzzy sets, theory of rough sets etc. which can be considered as mathematical tools for dealing with uncertain data, obtained in various fields of engineering, physics, computer science, economics, social science, medical science, and of many other diverse fields. But all these theories have their own difficulties. The theory of intuitionistic fuzzy sets (see[1, 2]) is a more generalized concept than the theory of fuzzy sets, but this theory has the some difficulties. All the above mentioned theories are successful to some extent in dealing with problems arising due to vagueness present in the real world. But there are also cases where these theories failed to give satisfactory results, possibly due to inadequacy of the parameterization tool in them. As a necessary supplement to the existing mathematical tools for handling uncertainty, Molodtsov [16] introduced the theory of soft sets as a new mathematical tool to deal with uncertainties while modelling the problems in engineering, physics, computer science, economics, social sciences, and medical sciences. Molodtsov et.al [17] successfully applied soft sets in directions such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability, and theory of measurement. Maji et al [13] gave the first practical application of soft sets in decision-making problems. Maji et al [14] defined and studied several basic notions of the soft set theory. Also Cagman et al [6] studied several basic notions of the soft set theory. V. Torra[23, 24] and Verma and Sharma [25] discussed the relationship between hesitant fuzzy setand showed that the envelope of hesitant fuzzy set is an intuitionistic fuzzy set. Zhang et.al [32] introduced weighted interval-valued hesitant fuzzy soft sets and finally applied it indecision making problem.

The notion of topological space is defined on crisp sets and hence it is affected by differentgeneralizations of crisp sets like fuzzy sets and soft sets. In 1968, C. L. Chang [7] introduced fuzzy topological space and in 2011, subsequently Cagman et al. [6] and Shabir et al. [21]introduced fuzzy soft topological spaces and studied neighborhood. Mahanta et al. [12],Neog et al. [18] and Ray et al. [20] introduced fuzzy soft topological spaces in different direction.

In this paper, in section 3, first we give a counter example of equality of IVHFSSs proposed by Zhang et al. [32]. Secondly we point out that proposition 3.11 in a previous paper by Borah and Hazarika [4] true in general by counter example. Thirdly we introduce about notion of topological space.

2. Preliminaries and Definitions

In this section we recall some basic concepts and definitions regarding fuzzy soft sets, hesitant fuzzy set and hesitant fuzzy soft set.

Definition2.1. [15] Let U be an initial universe and F be a set of parameters. Let $\widetilde{P}(U)$ denote the power set of U and A be a non-empty subset of F. Then F_A is called a fuzzy softset over U, where $F: A \to \widetilde{P}(U)$ is a mapping from A into $\widetilde{P}(U)$.

Definition2.2. [16] F_E is called a soft set over U if and only if F is a mapping of E intothe set of all subsets of the set U.

In other words, the soft set is a parameterized family of subsets of the set U. Every set $F(\varepsilon), \varepsilon \tilde{\in} E,$ from this family may be considered as the set of $\varepsilon -$ element of the soft set F_E oras the set of $\varepsilon -$ approximate elements of the soft set.

Definition2.3. [2, 28] Let intuitionistic fuzzy value IFV(X) denote the family of all IFVs defined on the universe X and let $\alpha, \beta \in IFV(X)$ be given as:

$\alpha = (\mu_\alpha, \upsilon_\alpha), \beta = (\mu_\beta, \upsilon_\beta),$

(i) $\alpha \cap \beta = (\min(\mu_\alpha, \mu_\beta), \max(\upsilon_\alpha, \upsilon_\beta))$

(ii) $\alpha \cup \beta = (\max(\mu_\alpha, \mu_\beta), \min(\upsilon_\alpha, \upsilon_\beta))$

(iii) $\alpha * \beta = (\frac{\mu_\alpha + \mu_\beta}{2(\mu_\alpha, \mu_{\beta+1})}, \frac{\upsilon_\alpha + \upsilon_\beta}{2(\upsilon_\alpha, \upsilon_{\beta+1})})$

Definition2.4. [23] Given a fixed set X, then a hesitant fuzzy set (shortly HFS) in X isin terms of a function that when applied to X returns a subset of [0, 1]. We express the HFSby a mathematical symbol:

$F = \{< h, \mu_F(x) >: h \in X\}$, where $\mu_F(x)$ is a set of some values in [0, 1], denoting thepossible membership degrees of the element $h \in X$ to the set F. $\mu_F(x)$ is called a hesitantfuzzy element (HFE) and H is the set of all HFEs.

Definition2.5. [23] Let $\mu_1, \mu_2 \in H$ and three operations are defined as follows:

(i) $\mu_1^C = \cup_{\gamma_1 \in \mu_1} \{1 - \gamma_1\};$

(ii) $\mu_1 \cup \mu_2 = \cup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} \max\{\gamma_1, \gamma_2\};$

(iii) $\mu_1 \cap \mu_2 = \cap_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} \min\{\gamma_1, \gamma_2\};$

Definition2.6. [8] Let X be a reference set, and D[0, 1] be the set of all closed subintervalsof [0, 1]. An IVHFS on X is $F = \{< h_i, \mu_F(h_i) >: h_i \in X, i = 1,2,...,n\}$, where $\mu_F(h_i): X \to D[0,1]$ denotes all possible interval-valued membership degrees of the element $h_i \in X$ to theset F. For convenience, we call $\mu_F(h_i)$ an interval-valued hesitant fuzzy element (IVHFE),which reads $\mu_F(h_i) = \{\gamma : \gamma \in \mu_F(h_i)\}$.

Here $\gamma = [\gamma^L, \gamma^U]$ is an interval number. $\gamma^L = \inf \gamma$ and $\gamma^U = \sup \gamma$ represent the lower and upper limits of γ respectively. An IVHFE is the basic unit of an IVHFS and it can be considered as a special case of the IVHFS. The relationship between IVHFE and IVHFS is similar to that between interval-valued fuzzy number and interval-valued fuzzy set.

Example2.7. Let $U = \{h_1, h_2\}$ be a reference set and let $\mu_F(h_1) = \{[0.6,0.8],[0.2,0.7]\}$ $\mu_F(h_2) = \{[0.1,0.4]\}$ be the IVHFEs of $h_i (i = 1,2)$ to a set F, respectively. Then IVHFS F can be written as $F = \{< h_1, \{[0.6,0.8],[0.2,0.7]\} >, < h_2, \{[0.1,0.4]\} >\}$.

Definition2.8. [29] Let $\tilde{a} = [\tilde{a}^L, \tilde{a}^U]$ and $\tilde{b} = [\tilde{b}^L, \tilde{b}^U]$ be two interval numbers and $\lambda \geq 0$, then

(i) $\tilde{a} = \tilde{b} \Leftrightarrow \tilde{a}^L = \tilde{b}^L and \tilde{a}^U = \tilde{b}^U$;

(ii) $\tilde{a} + \tilde{b} \Leftrightarrow [\tilde{a}^L + \tilde{b}^L, \tilde{a}^U + \tilde{b}^U]$;

(iii) $\lambda\tilde{a} = [\lambda\tilde{a}^L, \lambda\tilde{a}^U]$,especially $\lambda\tilde{a} = 0, if \lambda = 0$.

Definition2.9. [29] Let $\tilde{a} = [\tilde{a}^L, \tilde{a}^U]$ and $\tilde{b} = [\tilde{b}^L, \tilde{b}^U]$ and let $l_a = \tilde{a}^U - \tilde{a}^L$ and $l_b = \tilde{b}^U - \tilde{b}^L$; then the degree of possibility of $\tilde{a} \geq \tilde{b}$ is formulated by

$$p(\tilde{a} \geq \tilde{b}) = \max\{1 - \max(\frac{\tilde{b}^U - \tilde{a}^L}{l_{\tilde{a}} + l_{\tilde{b}}}, 0), 0\}$$

Above equation is proposed in order to compare two interval numbers and to rank all theinput arguments.

Definition2.10. [8] For an IVHFE $\tilde{\mu}, s(\tilde{\mu}) = \frac{1}{l_{\tilde{\mu}}} \sum_{\tilde{\gamma} \in \tilde{\mu}} \tilde{\gamma}$ is called the score function of μ with $l_{\tilde{\mu}}$ being the number of the interval values in $\tilde{\mu}$, and $s(\tilde{\mu})$ is an interval value belongingto [0, 1]. For two IVHFEs μ_1 and μ_2, if $s(\tilde{\mu}_1) \geq s(\tilde{\mu}_2)$,then $\mu_1 \geq \mu_2$ .We can judge the magnitude of two IVHFEs using above equation.

Definition2.11. [8] Let μ , μ_1 and μ_2 be three IVHFEs, then

(i) $\tilde{\mu}^C = \{[1 - \tilde{\gamma}^U, 1 - \tilde{\gamma}^L] : \tilde{\gamma} \in \tilde{\mu}\}$;

(ii) $\tilde{\mu}_1 \cup \tilde{\mu}_2 = \{[\max(\gamma_1{}^L, \gamma_2{}^L), \max(\gamma_1{}^U, \gamma_2{}^U)] : \tilde{\gamma}_1 \in \tilde{\mu}_1, \tilde{\gamma}_2 \in \tilde{\mu}_2\}$;

(iii) $\tilde{\mu}_1 \cap \tilde{\mu}_2 = \{[\min(\gamma_1{}^L, \gamma_2{}^L), \min(\gamma_1{}^U, \gamma_2{}^U)] : \tilde{\gamma}_1 \in \tilde{\mu}_1, \tilde{\gamma}_2 \in \tilde{\mu}_2\}$;

(iv) $\tilde{\mu}_1 \oplus \tilde{\mu}_2 = \{[\gamma_1{}^L + \gamma_2{}^L - \gamma_1{}^L.\gamma_2{}^L, \gamma_1{}^U + \gamma_2{}^U - \gamma_1{}^U.\gamma_2{}^U] : \tilde{\gamma}_1 \in \tilde{\mu}_1, \tilde{\gamma}_2 \in \tilde{\mu}_2\}$;

(v) $\tilde{\mu}_1 \otimes \tilde{\mu}_2 = \{[\gamma_1{}^L.\gamma_2{}^L, \gamma_1{}^U.\gamma_2{}^U] : \tilde{\gamma}_1 \in \tilde{\mu}_1, \tilde{\gamma}_2 \in \tilde{\mu}_2\}$;

Proposition2.12. [8] For three IVHFEs μ , μ_1 and μ_2, we have

(i) $\tilde{\mu}_1{}^C \cup \tilde{\mu}_2{}^C = (\tilde{\mu}_1 \cap \tilde{\mu}_2)^C$;

(ii) $\tilde{\mu}_1{}^C \cap \tilde{\mu}_2{}^C = (\tilde{\mu}_1 \cup \tilde{\mu}_2)^C$;

Definition2.13. [26] Let U be an initial universe and E be a set of parameters. Let $\tilde{F}(U)$ be the set of all hesitant fuzzy subsets of U. Then F_E is called a hesitant fuzzy soft set (HFSS)over U,where $F : E \to \tilde{F}(U)$.

A HFSS is a parameterized family of hesitant fuzzy subsets of U, that is $\tilde{F}(U)$. For all $\varepsilon \tilde{\in} E, F(\varepsilon)$ is referred to as the set of $\varepsilon -$ approximate elements of the HFSS F_E . It can be writtenas

$\widetilde{F}(\varepsilon)=\{<h,\mu_{F(\varepsilon)(x)}>: h\in U\}$,

Since HFE can represent the situation, in which different membership function are consideredpossible (see [23]), $\mu_{F(\varepsilon)(x)}$ is a set of several possible values, which is the hesitant fuzzymembership degree. In particular, if $\widetilde{F}(\varepsilon)$ has only one element, $\widetilde{F}(\varepsilon)$ can be called a hesitantfuzzy soft number. For convenience, a hesitant fuzzy soft number (HFSN) is denoted by $\{<h,\mu_{F(\varepsilon)(x)}>\}$.

Example2.14. Suppose $U=\{h_1,h_2\}$ be an initial universe and $E=\{e_1,e_2,e_3,e_4\}$ be aset of parameters. Let $A=\{e_1,e_2\}$ .Then the hesitant fuzzy soft set F_A is given as

$F_A=\{F(e_1)=\{<h_1,\{0.6,0.8\}>,<h_2,\{0.8,0.4,0.9\}>\}$
$F(e_2)=\{<h_1,\{0.9,0.1,0.5\}>,<h_2,\{0.2\}>\}\}$.

Definition2.15. [32] Let (U,E) be a soft universe and $A\subseteq E$. Then F_A is called an intervalvalued hesitant fuzzy soft set over U, where F is a mapping given by $F: A\rightarrow IVHF(U)$. An interval-valued hesitant fuzzy soft set is a parameterized family of interval-valued hesitantfuzzy subset of U. That is to say, F(e) is an interval-valued hesitant fuzzy subset in U, $\forall e\in A$. Following the standard notations, F(e) can be written as

$\widetilde{F}(e)=\{<h,\mu_{F(e)(x)}>: h\in U\}$.

Example2.16. Suppose $U=\{h_1,h_2\}$ be an initial universe and $E=\{e_1,e_2,e_3,e_4\}$ be aset of parameters. Let $A=\{e_1,e_2\}$ .Then the interval valued hesitant fuzzy soft set F_A is given as

$F_A=\{e_1=\{<h_1,[0.6,0.8]>,<h_2,[0.1,0.4]>\}$
$e_2=\{<h_1,[0.2,0.6],[0.3,0.9]>,<h_2,[0.2,0.5],[0.2,0.8],[0.2,0.8]>\}\}$.

Definition2.17. [32] Let U be an initial universe and let E be a set of parameters. Supposing that $A,B\,\widetilde{\subseteq}\,E, F_A$ and F_B are two interval-valued hesitant fuzzy soft sets, one says that F_A is an interval-valued hesitant fuzzy soft subset of G_B if and only if

(i) $A\,\widetilde{\subseteq}\,B$,

(ii) $\gamma_1{}^{\sigma(k)}\,\widetilde{\leq}\,\gamma_2{}^{\sigma(k)}$

Where for all $e\,\widetilde{\in}\,A, \gamma_1{}^{\sigma(k)}$ and $\gamma_2{}^{\sigma(k)}$ stand for the kth largest interval number in theIVHFEs $\mu_{F(e)(x)}$ and $\mu_{G(e)(x)}$ respectively. In this case, we write $F_A\,\widetilde{\subseteq}\,G_A$.

Definition2.18. [32] The complement of F_A, denoted by $F_A{}^C$,is defined by

$\widetilde{F}_A{}^C(e)=\{<h,\mu_{\widetilde{F}^C(e)(x)}>: h\in U\}$.

where $\mu_F{}^C: A\rightarrow IVHF(U)$ is a mapping given by $\mu_{\widetilde{F}^C(e)}, \forall e\,\widetilde{\in}\,A$ such that $\mu_{\widetilde{F}^C(e)}$ is thecomplement of interval-valued hesitant fuzzy element $\mu_{F(e)}$ on U.

Definition2.19. [32] An interval-valued hesitant fuzzy soft set is said to be an emptyinterval-valued hesitant fuzzy soft set, denoted by $\widetilde{\phi}$, if $F: E\rightarrow IVHF(U)$ such that

$\widetilde{F}(e)=\{<h,\mu_{F(e)(x)}>: h\in U\}=\{<h,\{[0,0]\}>: h\in U\},\forall e\,\widetilde{\in}\,E$.

Definition2.20. [32] An interval-valued hesitant fuzzy soft set is said to be an fullinterval-valued hesitant fuzzy soft set, denoted by $\widetilde{E}$, if $F: E\rightarrow IVHF(U)$ such that

$\widetilde{F}(e)=\{<h,\mu_{F(e)(x)}>: h\in U\}=\{<h,\{[1,1]\}>: h\in U\},\forall e\,\widetilde{\in}\,E$.

Definition2.21. [4] The union of two interval-valued hesitant fuzzy soft sets F_A and G_B over (U,E), is the interval-valued hesitant fuzzy soft set H_C, where $C = A \cup B$ and, $\forall e \tilde{\in} C$,

$$\mu_{H(e)} = \begin{cases} \mu_{F(e)}, if e \tilde{\in} A - B; \\ \mu_{G(e)}, if e \tilde{\in} B - A; \\ \mu_{F(e)} \cup \mu_{G(e)}, if e \tilde{\in} A \cap B. \end{cases}$$

We write $F_A \tilde{\cup} G_B = H_C$.

Definition2.22. [4] The intersection of two interval-valued hesitant fuzzy soft sets F_A and G_B with $A \cap B \neq \phi$ over (U,E), is the interval-valued hesitant fuzzy soft set H_C, where $C = A \cap B$, and, $\forall e \tilde{\in} C$, $\mu_{H(e)} = \mu_{F(e)} \cap \mu_{G(e)}$. We write $F_A \tilde{\cap} G_B = H_C$.

3. Interval-valued hesitant fuzzy soft topological space

In this section, first we give a counter example of equality of IVHFSSs proposed by Zhang et al. [32]. Secondly we point out that proposition 3.11 in a previous paper by Borahand Hazarika [4] true in general by counter example. Thirdly we introduce about notion oftopological space.

Definition3.1. [32] Let F_A and G_B be two interval-valued hesitant fuzzy soft sets. Now F_A and G_B are said to be interval-valued hesitant fuzzy soft equal if and only if
(i) $F_A \tilde{\subseteq} G_B$, (ii) $G_B \tilde{\subseteq} F_A$,
This can be denoted by $F_A = G_B$.

Example3.2. Let,

$$F_A = \{e_1 = \{<h_1,[0.2,0.5]>,<h_2,[0.5,0.8],[0.4,0.9]>\},$$
$$e_2 = \{<h_1,[0.3,0.6],[0.4,0.8]>,<h_2,[0.6,0.8]>\}\}.$$
$$G_A = \{e_1 = \{<h_1,[0.2,0.5],[0.2,0.5]>,<h_2,[0.5,0.8],[0.4,0.9]>\},$$
$$e_2 = \{<h_1,[0.3,0.6],[0.4,0.8]>,<h_2,[0.6,0.8],[0.6,0.8]>\}\}.$$

Therefore
$F_A \tilde{\subseteq} G_A$, and $G_A \tilde{\subseteq} F_A$.
Hence
$F_A = G_A$.

Proposition3.3. Let F_A, G_B and H_C be three interval-valued hesitant fuzzy soft sets. Then the following are satisfied:
(i) $F_A \tilde{\cup} (G_B \tilde{\cap} H_C) = (F_A \tilde{\cup} G_B) \tilde{\cap} (F_A \tilde{\cup} H_C)$,
(ii) $F_A \tilde{\cap} (G_B \tilde{\cup} H_C) = (F_A \tilde{\cap} G_B) \tilde{\cup} (F_A \tilde{\cap} H_C)$.

Proof. We consider IVHFSSs.

$$F_A = \{e_1 = \{<h_1,[0.3,0.8]>,<h_2,[0.3,0.8],[0.5,0.6],[0.3,0.6]>\},$$
$$e_2 = \{<h_1,[0.2,0.9],[0.7,1.0]>,<h_2,[0.8,1.0],[0.2,0.6]>\}\}.$$
$$G_B = \{e_1 = \{<h_1,[0.7,0.9],[0.0,0.6]>,<h_2,[0.4,0.7],[0.4,0.5]>\},$$
$$e_2 = \{<h_1,[0.6,0.8]>,<h_2,[0.3,0.8],[0.3,0.6]>\}$$
$$e_3 = \{<h_1,[0.5,0.6],[0.3,0.6]>,<h_2,[0.1,0.6],[0.3,0.9],[0.3,0.6]>\}\}.$$

And

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$H_C = \{e_2 = \{< h_1,[0.4,0.6],[0.2,0.6],[0.7,1.0] >,< h_2,[0.3,0.8] >\},$
$e_3 = \{< h_1,[0.2,0.5],[0.3,0.5] >,< h_2,[0.6,0.8],[0.2,0.5] >\}\}.$

(i) We have

$F_A \tilde{\cup} G_B = \{e_1 = \{< h_1,[0.3,0.8],[0.7,0.9] >,< h_2,[0.4,0.6],[0.4,0.8],[0.5,0.7] >\},$
$e_2 = \{< h_1,[0.6,0.9],[0.7,1.0] >,< h_2,[0.3,0.6],[0.8,1.0] >\}$
$e_3 = \{< h_1,[0.3,0.6],[0.5,0.6] >,< h_2,[0.1,0.6],[0.3,0.9],[0.3,0.6] >\}\}.$
$F_A \tilde{\cup} H_C = \{e_1 = \{< h_1,[0.3,0.8] >,< h_2,[0.3,0.6],[0.3,0.8],[0.5,0.6] >\},$
$e_2 = \{< h_1,[0.2,0.9],[0.7,1.0],[0.7,1.0] >,< h_2,[0.3,0.8],[0.8,1.0] >\}$
$e_3 = \{< h_1,[0.2,0.5],[0.3,0.5] >,< h_2,[0.2,0.5],[0.6,0.8] >\}\}.$

$(F_A \tilde{\cup} G_B)\tilde{\cap}(F_A \tilde{\cup} H_C) = \{e_1 = \{< h_1,[0.3,0.8],[0.3,0.8] >,< h_2,[0.3,0.6],[0.3,0.8],[0.5,0.6] >\},$
$e_2 = \{< h_1,[0.2,0.9],[0.7,1.0],[0.7,1.0] >,< h_2,[0.3,0.6],[0.8,1.0] >\}$
$e_3 = \{< h_1,[0.2,0.5],[0.3,0.5] >,< h_2,[0.1,0.5],[0.3,0.8],[0.3,0.6] >\}\}$

$G_B \tilde{\cap} H_C = \{e_2 = \{< h_1,[0.2,0.6],[0.4,0.6],[0.6,0.8] >,< h_2,[0.3,0.6],[0.3,0.8] >\},$
$e_3 = \{< h_1,[0.2,0.5],[0.3,0.5] >,< h_2,[0.1,0.5][0.3,0.8],[0.3,0.6] >\}\}.$

Therefore,

$F_A \tilde{\cup}(G_B \tilde{\cap} H_C) = \{e_1 = \{< h_1,[0.3,0.8] >,< h_2,[0.3,0.6],[0.3,0.8],[0.5,0.6] >\},$
$e_2 = \{< h_1,[0.2,0.9],[0.7,1.0],[0.7,1.0] >,< h_2,[0.3,0.6],[0.8,1.0] >\}$
$e_3 = \{< h_1,[0.2,0.5],[0.3,0.5] >,< h_2,[0.1,0.5],[0.3,0.8],[0.3,0.6] >\}\}$

Hence $F_A \tilde{\cup}(G_B \tilde{\cap} H_C) = (F_A \tilde{\cup} G_B)\tilde{\cap}(F_A \tilde{\cup} H_C).$

(ii) We have, $F_A \tilde{\cap} G_B = \{e_1 = \{< h_1,[0.0,0.6],[0.3,0.8] >,< h_2,[0.3,0.5],[0.3,0.7],[0.4,0.6] >\},$
$e_2 = \{< h_1,[0.2,0.8],[0.6,0.8] >,< h_2,[0.2,0.6],[0.3,0.8] >\}\}.$
$G_B \tilde{\cup} H_C = \{e_1 = \{< h_1,[0.0,0.6],[0.7,0.9] >,< h_2,[0.4,0.5],[0.4,0.7],[0.4,0.7] >\},$
$e_2 = \{< h_1,[0.6,0.8],[0.6,0.8],[0.7,1.0] >,< h_2,[0.3,0.8],[0.3,0.8] >\}$
$e_3 = \{< h_1,[0.3,0.6],[0.5,0.6] >,< h_2,[0.2,0.6],[0.6,0.9],[0.6,0.8] >\}\}.$

Therefore,

$F_A \tilde{\cap}(G_B \tilde{\cup} H_C) = \{e_1 = \{< h_1,[0.0,0.6],[0.3,0.8] >,< h_2,[0.3,0.5],[0.3,0.7],[0.4,0.6] >\},$
$e_2 = \{< h_1,[0.2,0.8],[0.6,0.8],[0.7,1.0] >,< h_2,[0.2,0.6],[0.3,0.8] >\}\}.$

Again,

$F_A \tilde{\cap} H_C = \{e_2 = \{< h_1,[0.2,0.6],[0.4,0.6],[0.7,1.0] >,< h_2,[0.2,0.6],[0.3,0.8] >\}\}.$

Therefore,

$(F_A \tilde{\cap} G_B)\tilde{\cup}(F_A \tilde{\cap} H_C) = \{e_1 = \{< h_1,[0.0,0.6],[0.3,0.8] >,< h_2,[0.3,0.5],[0.3,0.7],[0.4,0.6] >\},$
$e_2 = \{< h_1,[0.2,0.8],[0.6,0.8],[0.7,1.0] >,< h_2,[0.2,0.6],[0.3,0.8] >\}\}.$

Hence $F_A \tilde{\cap}(G_B \tilde{\cup} H_C) = (F_A \tilde{\cap} G_B)\tilde{\cup}(F_A \tilde{\cap} H_C).$

Definition3.4. A interval-valued hesitant fuzzy soft topology τ on (U, E) is a family ofinterval-valued hesitant fuzzy soft sets over (U, E) satisfying the following properties:

(i) $\tilde{\phi}, \tilde{E} \tilde{\in} \tau$

(ii) $F_A, G_B \tilde{\in} \tau$ then $F_A \tilde{\cap} G_B \tilde{\in} \tau.$

(i) If $F_{A_\alpha} \tilde{\in} \tau$ for all $\alpha \tilde{\in} \Delta$ an index set, then $\cup_{\alpha\in\Delta} F_{A_\alpha} \tilde{\in} \tau.$

Example3.5. Let $U = \{h_1, h_2\}$ and $E = \{e_1, e_2, e_3\}$ and consider $A = \{e_1, e_2, e_3\}, B = \{e_1, e_2\} \tilde{\subseteq} E.$

Let

$$F_A = \{e_1 = \{< h_1, [0.7,0.9], [0.3,0.8] >, < h_2, [0.4,0.6], [0.5,0.7], [0.4,0.8] >\}$$
$$e_2 = \{< h_1, [0.6,0.9], [0.7,1.0] >, < h_2, [0.3,0.6], [0.8,1.0] >\}$$
$$e_3 = \{< h_1, [0.3,0.6], [0.5,0.6] >, < h_2, [0.3,0.9], [0.1,0.6] >\}\}.$$
$$G_B = \{e_1 = \{< h_1, [0.3,0.8] >, < h_2, [0.3,0.8], [0.3,0.6] >\},$$
$$e_2 = \{< h_1, [0.2,0.9], [0.7,1.0] >, < h_2, [0.8,1.0], [0.2,0.6] >\}\}.$$

Now rearrange the membership value of F_A and G_B with the help of Definitions 2.9, 2.10and assumptions given by [8], we have

$$F_A = \{e_1 = \{< h_1, [0.3,0.8], [0.7,0.9] >, < h_2, [0.4,0.6], [0.4,0.8], [0.5,0.7] >\}$$
$$e_2 = \{< h_1, [0.6,0.9], [0.7,1.0] >, < h_2, [0.3,0.6], [0.8,1.0] >\}$$
$$e_3 = \{< h_1, [0.3,0.6], [0.5,0.6] >, < h_2, [0.1,0.6], [0.3,0.9] >\}\}.$$
$$G_B = \{e_1 = \{< h_1, [0.3,0.8] >, < h_2, [0.3,0.6], [0.3,0.8] >\},$$
$$e_2 = \{< h_1, [0.2,0.9], [0.7,1.0] >, < h_2, [0.2,0.6], [0.8,1.0] >\}\}.$$

Suppose a collection τ of interval-valued hesitant fuzzy soft sets over (U,E) as $\tau = \{\tilde{\phi}, \tilde{E}, \tilde{F}_A, \tilde{G}_B\}.$

Therefore

(i) $\tilde{\phi}, \tilde{E} \tilde{\in} \tau$

(ii) $\tilde{\phi} \tilde{\cap} \tilde{E} = \tilde{\phi}, \tilde{\phi} \tilde{\cap} F_A = \tilde{\phi}, \tilde{\phi} \tilde{\cap} G_B = \tilde{\phi}, F_A \tilde{\cap} \tilde{E} = F_A, G_B \tilde{\cap} \tilde{E} = G_B and$

$$F_A \tilde{\cap} G_B = \{e_1 = \{< h_1, [0.3,0.8], [0.3,0.8] >, < h_2, [0.3,0.6], [0.3,0.8], [0.3,0.8] >\},$$
$$e_2 = \{< h_1, [0.2,0.9], [0.7,1.0] >, < h_2, [0.2,0.6], [0.8,1.0] >\}\}.$$

Hence

$$F_A \tilde{\cap} G_B = G_B.$$

(iii) $\tilde{\phi} \tilde{\cup} \tilde{E} = \tilde{E}, \tilde{\phi} \tilde{\cup} F_A = F_A, \tilde{\phi} \tilde{\cup} G_B = G_B, F_A \tilde{\cup} \tilde{E} = \tilde{E}, G_B \tilde{\cup} \tilde{E} = \tilde{E} and$

$$F_A \tilde{\cup} G_B = \{e_1 = \{< h_1, [0.3,0.8], [0.7,0.9] >, < h_2, [0.4,0.6], [0.4,0.8], [0.5,0.7] >\}$$
$$e_2 = \{< h_1, [0.6,0.9], [0.7,1.0] >, < h_2, [0.3,0.6], [0.8,1.0] >\}$$
$$e_3 = \{< h_1, [0.3,0.6], [0.5,0.6] >, < h_2, [0.1,0.6], [0.3,0.9] >\}\} = F_A,$$

$\tilde{\phi} \tilde{\cup} \tilde{E} \tilde{\cup} F_A = \tilde{E}, \tilde{\phi} \tilde{\cup} \tilde{E} \tilde{\cup} G_B = \tilde{E}, \tilde{E} \tilde{\cup} F_A \tilde{\cup} G_B = \tilde{E}, \tilde{\phi} \tilde{\cup} F_A \tilde{\cup} G_B \tilde{\cup} \tilde{E} = \tilde{E}.$ Therefore τ is a IVHFS topology on $(U, E).$

Definition3.6. If τ is a IVHFS topology on $(U, E),$ the triple (U, E, τ) is said to be ainterval-valued hesitant fuzzy soft topological space (IVHFSTS). Also each member of τ is called a interval-valued hesitant fuzzy soft open set in $(U, E, \tau).$

Example3.7. From example 3.5, The triple (U, E, τ) is a IVHFS topological space and theinterval-valued hesitant fuzzy soft open sets in (U, E, τ) are $\tilde{\phi}, \tilde{E}, \tilde{F}_A, \tilde{G}_B.$

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Example3.8. A IVHFSS F_A over (U,E) is called an interval-valued hesitant fuzzy softclosed set in (U,E,τ) if and only if its complement ${F_A}^C$ is a interval-valued hesitant fuzzysoft open set in (U,E,τ).

Definition3.9. Let (U,E,τ) be a IVHFSTS. Let F_A be a IVHFSS over (U,E). The interval-valued hesitant fuzzy soft closure of F_A is defined as the intersection of all interval-valuedhesitant fuzzy soft closed sets(IVHFSCSs) which contained F_A and is denoted by $cl(F_A)$ or $\overline{F}_A$. We write

$cl(F_A)=\tilde{\cap}\{G_B : G_B$ is IVHFSCS and $F_A \tilde{\subseteq} G_B\}$.

Example3.10. From example 3.5, we have

$$F_A=\{e_1=\{<h_1,[0.3,0.8],[0.7,0.9]>,<h_2,[0.4,0.6],[0.4,0.8],[0.5,0.7]>\}$$
$$e_2=\{<h_1,[0.6,0.9],[0.7,1.0]>,<h_2,[0.3,0.6],[0.8,1.0]>\}$$
$$e_3=\{<h_1,[0.3,0.6],[0.5,0.6]>,<h_2,[0.1,0.6],[0.3,0.9]>\}\}.$$
$$G_B=\{e_1=\{<h_1,[0.3,0.8]>,<h_2,[0.3,0.6],[0.3,0.8]>\},$$
$$e_2=\{<h_1,[0.2,0.9],[0.7,1.0]>,<h_2,[0.2,0.6],[0.8,1.0]>\},$$
$$e_3=\{<h_1,[0.0,0.0],[0.0,0.0]>,<h_2,[0.0,0.0],[0.0,0.0]>\}\}.$$

Then interval-valued hesitant fuzzy soft closed sets are

$${F_A}^C=\{e_1=\{<h_1,[0.1,0.3],[0.2,0.7]>,<h_2,[0.3,0.5],[0.2,0.6],[0.4,0.6]>\}$$
$$e_2=\{<h_1,[0.0,0.3],[0.1,0.4]>,<h_2,[0.0,0.2],[0.4,0.7]>\}$$
$$e_3=\{<h_1,[0.4,0.5],[0.4,0.7]>,<h_2,[0.1,0.7],[0.4,0.9]>\}\}.$$

$${G_B}^C=\{e_1=\{<h_1,[0.2,0.7]>,<h_2,[0.2,0.7],[0.4,0.7]>\},$$
$$e_2=\{<h_1,[0.0,0.3],[0.1,0.8]>,<h_2,[0.0,0.2],[0.4,0.8]>\},$$
$$e_3=\{<h_1,[1.0,1.0],[1.0,1.0]>,<h_2,[1.0,1.0],[1.0,1.0]>\}\}.$$

Suppose interval-valued hesitant fuzzy soft set I_C over (U,E) as

$$I_C=\{e_1=\{<h_1,[0.1,0.2],[0.1,0.7]>,<h_2,[0.3,0.4],[0.1,0.6],[0.4,0.5]>\},$$
$$e_2=\{<h_1,[0.0,0.2],[0.1,0.7]>,<h_2,[0.0,0.1],[0.4,0.8]>\},$$
$$e_3=\{<h_1,[0.0,0.0],[0.0,0.0]>,<h_2,[0.0,0.0],[0.0,0.0]>\}\}.$$

Then

$$cl(I_C)=\tilde{E}\,\tilde{\cap}\,{G_B}^C={G_B}^C=\{e_1=\{<h_1,[0.2,0.7]>,<h_2,[0.2,0.7],[0.4,0.7]>\},$$
$$e_2=\{<h_1,[0.0,0.3],[0.1,0.8]>,<h_2,[0.0,0.2],[0.4,0.8]>\},$$
$$e_3=\{<h_1,[1.0,1.0],[1.0,1.0]>,<h_2,[1.0,1.0],[1.0,1.0]>\}\}.$$

Proposition3.11. Let (U,E,τ) be a IVHFSTS and F_A, G_B be two IVHFSs over (U,E).
Then the following are true:

(i) $cl(\tilde{\phi})=\tilde{\phi}, cl(\tilde{E})=\tilde{E}$. (ii) $F_A \tilde{\subseteq} cl(F_A)$

(iii) F_A is an interval-valued hesitant fuzzy soft closed set iff $F_A=cl(F_A)$.

(iv) $F_A \tilde{\subseteq} G_B \Rightarrow cl(F_A) \tilde{\subseteq} cl(G_B)$.

(v) $cl(F_A \tilde{\cup} G_B) = cl(F_A) \tilde{\cup} cl(G_B)$.

(vi) $cl(F_A \tilde{\cap} G_B) \tilde{\subseteq} cl(F_A) \tilde{\cap} cl(G_B)$.

(vii) $cl(cl(F_A)) = cl(F_A)$.

Proof. (i) Obvious.

(ii) The proof directly follows from definition.

(iii) Let (U, E, τ) be a IVHFSTS. Let F_A be a IVHFSS over (U, E) such that $F_A = cl(F_A)$. Therefore from definition of interval-valued hesitant fuzzy soft closure, we have $cl(F_A)$ is interval-valued hesitant fuzzy soft closed sets. Hence $cl(F_A)$ is intervalvaluedhesitant fuzzy soft closed and $cl(F_A) = F_A$. i.e. F_A is interval-valued hesitantfuzzy soft closed.

Conversely, let F_A be interval-valued hesitant fuzzy soft closed in (U, E, τ). Therefore from definition of interval-valued hesitant fuzzy soft closure that any interval-valuedhesitant fuzzy soft closed set G_B, $F_A \tilde{\subseteq} G_B \Rightarrow cl(F_A) \tilde{\subseteq} G_B$.

Since $F_A \tilde{\subseteq} F_A \Rightarrow cl(F_A) \tilde{\subseteq} F_A$ and from definition $F_A \tilde{\subseteq} cl(F_A)$.

Hence it follows that $F_A = cl(F_A)$.

(iv) Let $F_A \tilde{\subseteq} G_B$. Since $G_B \tilde{\subseteq} cl(G_B)$. Therefore $F_A \tilde{\subseteq} cl(G_B)$.

Again $cl(F_A)$ is the smallest interval-valued hesitant fuzzy soft closed set containing F_A.

Hence $cl(F_A) \tilde{\subseteq} cl(G_B)$.

(v) From definition of union of IVHFSSs

$F_A \tilde{\subseteq} F_A \tilde{\cup} G_B, G_B \tilde{\subseteq} F_A \tilde{\cup} G_B$.

Therefore $cl(F_A) \tilde{\subseteq} cl(F_A \tilde{\cup} G_B), cl(G_B) \tilde{\subseteq} cl(F_A \tilde{\cup} G_B)$.

$\Rightarrow cl(F_A) \tilde{\cup} cl(G_B) \tilde{\subseteq} cl(F_A \tilde{\cup} G_B)$. ....(A1)

Again $cl(F_A \tilde{\cup} G_B) \tilde{\subseteq} cl(F_A) \tilde{\cup} cl(G_B)$. ....(A2)

Since $cl(F_A \tilde{\cup} G_B)$ is the smallest interval-valued hesitant fuzzy soft closed set containing

$F_A \tilde{\cup} G_B$. Hence from (A1) and (A2),

$cl(F_A \tilde{\cup} G_B) = cl(F_A) \tilde{\cup} cl(G_B)$.

(vi) From definition of intersection of IVHFSSs

$F_A \tilde{\cap} G_B \tilde{\subseteq} F_A, F_A \tilde{\cap} G_B \tilde{\subseteq} G_B$.

Therefore

$cl(F_A \tilde{\cap} G_B) \tilde{\subseteq} cl(F_A), cl(F_A \tilde{\cap} G_B) \tilde{\subseteq} cl(G_B)$

$\Rightarrow cl(F_A \tilde{\cap} G_B) \tilde{\subseteq} cl(F_A) \tilde{\cap} cl(G_B)$.

(vii) If F_A is a interval-valued hesitant fuzzy soft closed set then $F_A = cl(F_A)$. Hence

$cl(cl(F_A)) = cl(F_A)$.

Definition3.12. Let (U, E, τ) be a IVHFSTS. Let F_A be a IVHFSS over (U, E). Theinterval-valued hesitant fuzzy soft interior of F_A is defined as the union of all interval-valuedhesitant fuzzy soft open sets (IVHFSOSs) which contained F_A and is denoted by $\text{int}(F_A) or F_A^{o}$. We write

$\text{int}(F_A) = \tilde{\cup}\{G_B : G_B$ is IVHFSOS and $G_B \tilde{\subseteq} F_A\}$.

Example3.13. From example 3.5, we consider a interval-valued hesitant fuzzy soft set I_C over (U, E) as

$I_C = \{e_1 = \{< h_1, [0.3, 0.8] >, < h_2, [0.3, 0.7], [0.3, 0.8] >\},$
$e_2 = \{< h_1, [0.2, 1.0], [0.7, 1.0] >, < h_2, [0.2, 0.7], [0.8, 1.0] >\},$
$e_3 = \{< h_1, [0.0, 0.0], [0.0, 0.0] >, < h_2, [0.0, 0.0], [0.0, 0.0] >\}\}.$

Therefore

$\text{int}(I_C) = G_B \tilde{\cup} \tilde{\phi} = G_B =$
$\{e_1 = \{< h_1, [0.3, 0.8] >, < h_2, [0.3, 0.6], [0.3, 0.8] >\},$
$e_2 = \{< h_1, [0.2, 0.9], [0.7, 1.0] >, < h_2, [0.2, 0.6], [0.8, 1.0] >\},$
$e_3 = \{< h_1, [0.0, 0.0], [0.0, 0.0] >, < h_2, [0.0, 0.0], [0.0, 0.0] >\}\}.$

Proposition3.14. Let (U, E, τ) be a IVHFSTS and F_A, G_B be two IVHFSs over (U, E).
Then the following are true:
(i) $\text{int}(\tilde{\phi}) = \tilde{\phi}, \text{int}(\tilde{E}) = \tilde{E}$. (ii) $\text{int}(F_A) \tilde{\subseteq} F_A$
(iii) F_A is an interval-valued hesitant fuzzy soft open set iff $F_A = \text{int}(F_A)$.
(iv) $F_A \tilde{\subseteq} G_B \Rightarrow \text{int}(F_A) \tilde{\subseteq} \text{int}(G_B)$.
(v) $\text{int}(F_A) \tilde{\cup} \text{int}(G_B) \tilde{\subseteq} \text{int}(F_A \tilde{\cup} G_B)$.
(vi) $\text{int}(F_A \tilde{\cap} G_B) = \text{int}(F_A) \tilde{\cap} \text{int}(G_B)$.
(vii) $\text{int}(\text{int}(F_A)) = \text{int}(F_A)$.

Proof. (i) Obvious.
(ii) The proof directly follows from definition.
(iii) Let (U, E, τ) be a IVHFSTS. Let F_A be a IVHFSS over (U, E) such that $F_A = \text{int}(F_A)$. Therefore from definition of interval-valued hesitant fuzzy soft interior, we have $\text{int}(F_A)$ is interval-valued hesitant fuzzy soft open sets. Hence $\text{int}(F_A)$ is intervalvalued hesitant fuzzy soft openand $\text{int}(F_A) = F_A$. i.e F_A is interval-valued hesitant fuzzy soft open.
Conversely, let F_A be interval-valued hesitant fuzzy soft open in (U, E, τ). Therefore from definition of interval-valued hesitant fuzzy soft interior that any interval-valued hesitant fuzzy soft open set $G_B \tilde{\subseteq} F_A \Rightarrow G_B \tilde{\subseteq} \text{int}(F_A)$.
Since $F_A \tilde{\subseteq} F_A \Rightarrow F_A \tilde{\subseteq} \text{int}(F_A)$ and from definition $\text{int}(F_A) \tilde{\subseteq} F_A$.
Hence it follows that $F_A = \text{int}(F_A)$.
(iv). Let $F_A \tilde{\subseteq} G_B$ Since $\text{int}(F_A) \tilde{\subseteq} F_A \tilde{\subseteq} G_B$, therefore $\text{int}(F_A)$ be a interval valued hesitant fuzzysoft open subset of G_B. Hence from definition of interval valued hesitant fuzzy softinterior, we have

$F_A \tilde{\subseteq} G_B \Rightarrow \text{int}(F_A) \tilde{\subseteq} \text{int}(G_B)$.

(v) Since

$F_A \tilde{\subseteq} F_A \tilde{\cup} G_B, G_B \tilde{\subseteq} F_A \tilde{\cup} G_B$.

Therefore we have $\text{int}(F_A) \tilde{\subseteq} \text{int}(F_A \tilde{\cup} G_B), \text{int}(G_B) \tilde{\subseteq} \text{int}(F_A \tilde{\cup} G_B)$.

Hence

$\text{int}(F_A) \tilde{\cup} \text{int}(G_B) \tilde{\subseteq} \text{int}(F_A \tilde{\cup} G_B)$.

(vi) Since

$F_A \tilde{\cap} G_B \tilde{\subseteq} F_A, F_A \tilde{\cap} G_B \tilde{\subseteq} G_B$.

These implies that

$\text{int}(F_A \tilde{\cap} G_B) \tilde{\subseteq} \text{int}(F_A), \text{int}(F_A \tilde{\cap} G_B) \tilde{\subseteq} \text{int}(G_B)$.

Therefore

$\text{int}(F_A \tilde{\cap} G_B) \tilde{\subseteq} \text{int}(F_A) \tilde{\cap} \text{int}(G_B)$...........................$(B1)$

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Again we know that

$\text{int}(F_A) \tilde{\subseteq} F_A$ and $\text{int}(G_B) \tilde{\subseteq} G_B$. Therefore

$\text{int}(F_A) \tilde{\cap} \text{int}(G_B) \tilde{\subseteq} F_A \tilde{\cap} G_B$............................$(B2)$

Hence from (B1) and (B2) we get

$\text{int}(F_A \tilde{\cap} G_B) = \text{int}(F_A) \tilde{\cap} \text{int}(G_B)$.

(vii). From (iii), if F_A is an interval-valued hesitant fuzzy soft open set then $\text{int}(F_A) = F_A$. Therefore $\text{int}(\text{int}(F_A)) = \text{int}(F_A)$.

Proposition3.15. If $\{\tau_\lambda : \lambda \tilde{\in} I\}$ is a family of IVHFSTS on (U, E), then $\cap_\lambda \{\tau_\lambda : \lambda \tilde{\in} I\}$ isalso a IVHFST on (U, E).

Proof. Suppose $\{\tau_\lambda : \lambda \tilde{\in} I\}$ be a IVHFSTS. Therefore $\tilde{\phi}, \tilde{E} \tilde{\in} \cap_{\lambda \tilde{\in} I} \{\tau_\lambda\}$. If $F_A, G_B \tilde{\in} \cap_{\lambda \tilde{\in} I} \{\tau_\lambda\}$ then $F_A, G_B \tilde{\in} \tau_\lambda, \forall \lambda \tilde{\in} I$. Therefore $F_A \tilde{\cap} G_B \tilde{\in} \tau_\lambda, \forall \lambda \tilde{\in} I$.

Thus $F_A \tilde{\cap} G_B \tilde{\in} \cap_{\lambda \tilde{\in} I} \{\tau_\lambda\}$.

Let $\{F_\alpha\}_{\alpha \tilde{\in} J} \tilde{\subseteq} \cap_{\lambda \tilde{\in} I} \{\tau_\lambda\}$.

Therefore $F_\alpha \tilde{\subseteq} \cap_{\lambda \tilde{\in} I} \{\tau_\lambda\}, \alpha \tilde{\in} J$.: This implies $F_\alpha \tilde{\in} \tau_\lambda, \forall \lambda \tilde{\in} I, \alpha \tilde{\in} J$. Therefore

$\cup_{\alpha \tilde{\in} J} F_\alpha \tilde{\in} \cap_{\lambda \tilde{\in} I} \{\tau_\lambda\}$.

Definition3.16. Let τ_1 and τ_2 be IVHFSTS on (U, E). We say that τ_1 is coarser (orweaker) than τ_2 or τ_2 is finer (or stronger) than τ_1 if and only if $\tau_1 \tilde{\subseteq} \tau_2$ i.e. every τ_1 intervalvalued hesitant fuzzy soft open set (IVHFSOS) is τ_2 IVHFSOS. Again IVHFST τ_1 and τ_2 are said to be comparable if either $\tau_1 \tilde{\subseteq} \tau_2$ or $\tau_2 \tilde{\subseteq} \tau_1$. If $\tau_1 \tilde{\not\subset} \tau_2$ and $\tau_2 \tilde{\not\subset} \tau_1$, then we say the IVHFST τ_1 and τ_2 are not comparable.

Example3.17. From example 3.5, we consider IVHFST τ_1 and τ_2 on (U, E) as

$\tau_1 = \{\tilde{\phi}, \tilde{E}, \tilde{F}_A\}$, $\tau_2 = \{\tilde{\phi}, \tilde{E}, \tilde{F}_A, \tilde{G}_B\}$.

Therefore $\tau_1 \tilde{\subseteq} \tau_2$ and hence τ_1 is coarser than τ_2 .

Definition3.18. The IVHFSS F_A over (U, E) is called a interval valued hesitant fuzzy softpoint (IVHFSP) in (U, E) is denoted by $e(F_A)$, if for the element $e \tilde{\in} A, \mu_{F(e)} \neq [0,0]$ and $\mu_{F(e')} = [0,0], \forall e' \tilde{\in} A - e$.

Example3.19. Let $U = \{h_1, h_2\}$, $E = \{e_1, e_2, e_3\}$ and $A = \{e_1, e_2\} \tilde{\subseteq} E$. Suppose a IVHFSS F_A over (U, E) as

$F_A = \{e_1 = \{< h_1, [0.0,0.0] >, < h_2, [0.0,0.0], [0.0,0.0] >\}$

$e_2 = \{< h_1, [0.3,0.5], [0.4,0.6] >, < h_2, [0.0,0.6] >\}\}$.

Here $e_2 \tilde{\in} A, \mu_{F(e_2)} \neq [0,0]$ and for $\forall e' \tilde{\in} A - e_2, \mu_{F(e')} = [0,0]$.

Thus F_A is a IVHFSP in (U, E) denoted by $e_2(F_A)$.

Definition3.20. The IVHFSP $e(F_A)$ is said to be in the IVHFSS G_B if $A \tilde{\subseteq} B$ and for theelement $e \tilde{\in} A, \mu_{F(e)} \tilde{\subseteq} \mu_{G(e)}$. We denoted as $e(F_A) \tilde{\in} G_B$.

Example3.21. From example 3.19, consider the IVHFSP $e_2(F_A)$ and an IVHFSS G_B as

$G_B = \{e_1 = \{< h_1, [0.1,0.9], [0.2,0.3] >, < h_2, [0.6,0.9] >\}$,

$e_2 = \{< h_1, [0.3,0.6], [0.5,0.6], [0.5,0.8] >, < h_2, [0.3,0.8] >\}$,

$e_3 = \{< h_1, [0.5,0.8] >, < h_2, [0.2,0.7], [0.3,0.8], [0.1,0.9] >\}\}$.

Here $e_2 \tilde{\in} A, \mu_{F(e_2)} \tilde{\subseteq} \mu_{G(e_2)}$. Hence $e_2(F_A) \tilde{\in} G_B$.

Definition3.22. A IVHFSS I_C in a IVHFSTS (U,E,τ) is called a interval valued hesitantfuzzy soft neighbourhood (IVHFSNBD) of the IVHFSP $e(F_A)\tilde{\in}(U,E)$ if there is a IVHFSOS G_B such that $e(F_A)\tilde{\in}G_B\tilde{\subseteq}I_C$.

Example3.23. From examples 3.19, 3.21, we consider the IVHFST $\tau=\{\tilde{\phi},\tilde{E},\tilde{G}_B\}$. and

IVHFSS I_C as

$I_C=\{e_1=\{<h_1,[0.2,1.0],[0.2,0.3]>,<h_2,[0.6,0.9],[0.6,1.0]>\},$

$e_2=\{<h_1,[0.3,0.6],[0.5,0.8]>,<h_2,[0.3,0.9]>\},$

$e_3=\{<h_1,[0.5,0.8],[0.5,0.9]>,<h_2,[0.3,0.9],[0.3,1.0]>\},$

$e_4=\{<h_1,[0.1,0.6],[0.7,0.9]>,<h_2,[0.2,0.6]>\}\}.$

Where $E=\{e_1,e_2,e_3,e_4\},C=\{e_1,e_2,e_3,e_4\}\tilde{\subseteq}E.$

Therefore $e(F_A)\tilde{\in}G_B\tilde{\subseteq}I_C.$

Hence I_C is a IVHFSNHD of the IVHFSP $e_2(F_A).$

Definition3.24. The family consisting of all neighbourhoods of $e(F_A)\tilde{\in}(U,E)$ neighbourhoodsystem of a fuzzy soft point $e(F_A)$. It is denoted by $N_\tau(e(F_A))$.

Definition3.25. A IVHFSS I_C in a IVHFSTS (U,E,τ) is called a IVHFSNBD of theIVHFSS H_A if there is a IVHFSOS G_B such that $H_A\tilde{\subseteq}G_B\tilde{\subseteq}I_C$.

Example3.26. From examples 3.21, 3.23 and consider the IVHFSS H_A as

$H_A=\{e_1=\{<h_1,[0.1,0.5]>,<h_2,[0.6,0.7],[0.6,0.8]>\},$

$e_3=\{<h_1,[0.5,0.6],[0.4,0.6]>,<h_2,[0.2,0.3]>\}\}.$

Where $A=\{e_1,e_3\}\tilde{\subseteq}E.$

Therefore $H_A\tilde{\subseteq}G_B\tilde{\subseteq}I_C.$

Hence IVHFSS I_C is IVHFSNBD of the IVHFSS H_A.

Proposition3.27. The neighbourhood system $N_\tau(e(F_A))$ at $\forall e(F_A)$ in an IVHFSTS (U,E,τ) has the following properties:

(i) If $G_B\tilde{\in}N_\tau(e(F_A))$ then $e(F_A)\tilde{\in}G_B$.

(ii) If $G_B\tilde{\in}N_\tau(e(F_A))$ and $G_B\tilde{\subseteq}H_C$ then $H_C\tilde{\in}N_\tau(e(F_A))$.

(iii) If $G_B,H_C\tilde{\in}N_\tau(e(F_A))$ then $G_B\tilde{\cap}H_C\tilde{\in}N_\tau(e(F_A))$.

(iv) If $G_B\tilde{\in}N_\tau(e(F_A))$ then there is a $H_C\tilde{\in}N_\tau(e(F_A))$ such that $G_B\tilde{\in}N_\tau(e'(M_D))$ foreach $e'(M_D)\tilde{\in}H_C$.

Proof. (i) If $G_B\tilde{\in}N_\tau(e(F_A))$, then there is a IVHFSOS H_C such that $e(F_A)\tilde{\in}H_C\tilde{\subseteq}G_B$.

Therefore we have $e(F_A)\tilde{\in}G_B$.

(ii) Let $G_B\tilde{\in}N_\tau(e(F_A))$ and $G_B\tilde{\subseteq}H_C$. Then there is a L_D such that $e(F_A)\tilde{\in}L_D\tilde{\subseteq}G_B$ and $e(F_A)\tilde{\in}L_D\tilde{\subseteq}G_B\tilde{\subseteq}H_C$. Therefore $H_C\tilde{\in}N_\tau(e(F_A))$.

(iii) If $G_B,H_C\tilde{\in}N_\tau(e(F_A))$ then there exist IVHFSOSs L_D,M_E such that $e(F_A)\tilde{\in}L_D\tilde{\subseteq}G_B$ and $e(F_A)\tilde{\in}M_E\tilde{\subseteq}H_C$. Thus $e(F_A)\tilde{\in}L_D\tilde{\cap}M_E\tilde{\subseteq}G_B\tilde{\cap}H_C$. Since $L_D\tilde{\cap}M_E\tilde{\in}\tau$. Hence wehave $G_B\tilde{\cap}H_C\tilde{\in}N_\tau(e(F_A))$.

(iv) If $G_B\tilde{\in}N_\tau(e(F_A))$, then there is an IVHFSOS $L_P\tilde{\in}\tau$ such that $e(F_A)\tilde{\in}L_P\tilde{\subseteq}G_B$.

Now put $H_C=L_P$. Then for each $e'(M_D)\tilde{\in}H_C, e'(M_D)\tilde{\in}H_C\tilde{\subseteq}G_B$. This implies

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$G_B \tilde{\in} N_\tau(e^{'}(M_D))$.

References

[1] K.T. Atanassove, Intuitionistic Fuzzy Sets, Springer Physica-Verlag.Heidelberg (1999).
[2] K.T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20(1986) 87-96.
[3] K.V. Babitha, S.J. Johan, Hesitant fuzzy soft sets, J. New Results Sci. 3(2013) 98-107.
[4] M.J. Borah, B. Hazarika, Some operators on interval-valued Hesitant Fuzzy soft sets,arXiv:1604.00902v1, 1 Apr 2016.
[5] S. Broumi and F. Smarandache, New operations over interval valued intuitionistic hesitant fuzzy set,Math. Stat. 2(2) (2014) 62-71.
[6] N. Cagman and N. S. Enginoglu, Soft Set Theory and Uni-int Decision Making, Euro. J. Opera. Research207(2010) 848-855.
[7] C. L. Chang, Fuzzy Topological Spaces, J. Math. Anal. Appl. 24(1)(1968) 182-190.
[8] N.Chen, Z.Xu, M.Xia, Interval-valued hesitant preference relations and their applications to groupdecision making, Knowledge-Based Systems 37(2013) 528-540.
[9] Kuo-Ping Chiao, Fundamental properties of interval vector max-norm, Tamsui Oxford J. Math. 18(2)(2002) 219-233.
[10] P.S.Dwyer, Linear Computation, New York, Wiley, 1951.
[11] H.C. Liao, Z.S. Xu, Subtraction and division operations over hesitant fuzzy sets, J. Intell. Fuzzy Systems27(1)(2014) 65-72.
[12] J. Mahanta and P. K. Das, Results on Fuzzy soft topological Spaces, arXiv:1203.0634v1.
[13] P. K. Maji, R. Biswas, R. Roy, An application of soft sets in a decision making problem, Comput. Math.Appl. 44(2002) 1077-1083.
[14] P. K. Maji, R. Biswas, R. Roy, Soft set theory, Comput. Math. Appl. 45(2003) 555-562.
[15] P. K. Maji, R. Biswas and A. R. Roy, Fuzzy Soft Sets, J. Fuzzy Math. 9(3)(2001) 589-602.
[16] D. A. Molodstov, Soft Set Theory-First Result, Comput. Math. Appl. 37(1999) 19-31.
[17] D. A. Molodtsov, V. Y. Leonov, D.V. Kovkov, Soft sets technique and its application, Nechetkie Sistemyi Myagkie Vychisleniya 1(1)(2006) 8-39.
[18] T. J. Neog, D. K. Sut and G. C. Hazarika, Fuzzy Soft Topological Spaces, Inter. J. Latest Trends Math.2(1)(2012) 54-67.
[19] X. Peng, Y. Yang, Interval-valued Hesitant Fuzzy Soft Sets and their Application in Decision Making,Fundamenta Informaticae 141(1)(2015) 71-93.
[20] S. Ray and T. K. Samanta, A note on Fuzzy Soft Topological Spaces, Annals Fuzzy Math. Inform.3(2)(2012) 305-311.
[21] M. Shabir and M. Naz, On soft topological spaces, Comput. Math. Appl. 61(2012) 412-418.
[22] B. Tanay and M. B. Kandemir, Topological Structure of fuzzy soft sets, Comput. Math. Appl.61(2011) 2952-2957.
[23] V. Torra, Hesitant fuzzy sets, Int. J. Intell. Systems 25(6)(2010) 529-539.
[24] V. Torra and Y. Narukawa, On hesitant fuzzy sets and decision. In proceeding of the 18th IEEE international conference on fuzzy systems, Jeju Island, Republic of Korea (2009) 1378-1382.
[25] R. Verma and B.D.Sharma, New operations over hesitant fuzzy sets, Fuzzy Inform. Eng. 2(2013) 129-146.
[26] J. Wang, X. Li and X. Chen, Hesitant fuzzy soft sets with application in multicrteria group decision making problems, The Scientific World J. 2015(2015) 1-14.
[27] M. Xia, Z. Xu, Hesitant fuzzy information aggregation in decision making, Int. J. Appr. Reasoning52(2011) 395-407.
[28] Z. Xu, Intuitionistic fuzzy aggregation operators, IEEE Transaction on Fuzzy Systems 15(6)(2007)1179-1187.
[29] Z. Xu, Q.L. Da, The uncertain OWA operator, Int. J. Intell. Systems 17 (2002) 569-575.
[30] Z. Xu, M. Xia, Distance and similarity measures for hesitant fuzzy sets, Inform. Sci. 181 (2011) 2128-2138.
[31] L. A. Zadeh, Fuzzy Sets, Inform. Control. 8(1965) 338-353.
[32] H. Zhang, L. Xiong and W. Ma, On interval-Valued Hesitant Fuzzy Soft Sets, Math. Problems Eng.2015(2015), Article ID 254764, 17 pages.