Miscellaneous Properties Of Full Graphs

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Abstract. In this paper, we stablish miscellaneous properties of the full graph of a graph. We obtain characterizations of this graph. Also, we prove that for any connected graph G, the full graph of G is not separable.

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1. Introduction

All graphs considered here are finite, undirected without loops and multiple edges. For all definitions and notations not given in this paper, we refer to [4].

A graph G with p vertices and q edges is called a (p, q) graph, the number p is referred to as the order of a graph G and q is referred to as the size of a graph G [4].

If B = {u1, u2, u3, ..., ur, r≤2} is a block of a graph G, then we say that vertex u1 and block B are incident with each other, as are u2 and block B and so on. If B = {e1, e2, e3, ..., es, s≤1} is a block of a graph G, then we say that edge e1 and block B are incident with each other, as are e2 and B and so on. If two distinct blocks B1 and B2 are incident with a common cut vertex then, they are adjacent blocks [6]. The vertices, edges and blocks of a graph are called its members. A cut vertex of a connected graph G is a vertex whose removal increases the number of components of G [4].

A set D of vertices in a graph G is called dominating set of G if every vertex in V − D is adjacent to some vertex in D, the minimum cardinality of a dominating set in G is called the domination number $\gamma(G)$ of a graph G [5]. A dominating set D is a total dominating set if the induced subgraph D has no isolated vertices, the minimum cardinality of a total dominating set in G is called the total domination number $\gamma t(G)$ of G [5].

The girth of a graph G, denoted girth(G) is size of the smallest cycle in G. The chromatic number $\chi(G)$ of a graph G is the minimum number of colors required to assign to the vertices of G in such a way that no two adjacent vertices of G receive the same color [4].

A graph G is called a null graph (or empty graph) if E(G) is empty, a null graph of order p is denoted by Np [9]. A graph G is called planer if it can be drawn in the plane without any intersecting edges [3].

The eccentricity e(v) of a vertex v in a connected graph G is defined as $e(v)=\max\{d(v, u)|u \in V(G)\}$ where d(v, u) is the distance between u and v. The radius r(G) is the minimum eccentricity among the vertices of G, and the diameter d(G) is the maximum eccentricity [4].

A wheel graph Wp of order p, sometimes simply called a p-wheel is a graph that contains a cycle of order p − 1, and for which every vertex in the cycle is joined to one other vertex in the center [4]. A cycle passing through all the vertices of a graph is called a hamiltonian cycle. A graph containing a hamiltonian cycle is called a hamil- tonian graph [4].

The line graph L(G) of a graph G is the graph whose vertex set is the set of edges of G and two vertices are adjacent if the corresponding edges are adjacent in G [4].

The block graph B(G) of a graph G is the graph whose vertex set is the set of blocks of G and two vertices are adjacent if the corresponding blocks are incident with a cut vertex in G [4].

The full graph F(G) of a graph G is the graph whose set of vertices is the union of the set of vertices, edges and blocks of G, in which two vertices are adjacent if the corresponding members of G are adjacent or one corresponds to a vertex and the other to an edge incident with it or one corresponds to a block B and other to a vertex v of G and v is in B or one corresponds to a block B and the other to an edge e of G and e is in B [7]. Many other graph valued functions in graph theory were studied, for example, in [2, 4, 8, 10].

The following results will be useful in the proof of our results.

Theorem 1.1. [4] A vertex v of a connected graph G with at least three vertices is a cut vertex of G if and only if there exist vertices u and w of G distinct from v such that v is in every u − w path in G. Theorem 1.2. [4] A nontrivial graph is bipartite if and only if all its cycles are even.

2. Miscellaneous properties of full graph

Remark 2.1. For any graph G, the block graph B(G) of a graph G is a subgraph of F(G).

Remark 2.2. For any graph G, the line graph of G is a subgraph of F(G).

Theorem 2.1. For any cycle C_p, the wheel W_{2p+1} is a spanning subgraph of $F(C_p)$.

Proof. Suppose G is a cycle C_p, then the number of blocks is 1, then in $F(C_p)$ there is a vertex v corresponding to this block and by definition of $F(C_p)$ this vertex must be joined to all vertices corre- sponding to vertices and edges of C_p. So, v will be the center vertex adjacent to all vertices in $F(C_p)$, and the other vertices of $F(C_p)$ form a cycle since any vertex corresponding to edge of C_p is adjacent to two vertices corresponding to vertices of C_p and so on, this establishes a wheel W_{2p+1}.

Corollary 2.1. For any cycle C_p, $r(F(C_p)) = d(F(C_p)) = 2$.

Corollary 2.2. For any cycle C_p, $F(C_p)$ is hamiltonian graph.

Proposition 2.1. Let G = (p, q) be a graph which is a block. Then the maximum degree in F(G) is the degree of the vertex corresponding to the block of G which (p + q).

Proof. Let G be a graph which is a block, then in F(G) there is a vertex v corresponding to this block and v is adjacent to all vertices in F(G) that corresponds to vertices and edges of G, thus the degree of v is (p + q), which is the maximum degree in F(G) since $|V(F(G))| = p + q + 1$.

Lemma 2.1. If v is a vertex of cycle C_p, and v' is the corresponding vertex in $F(C_p)$, then $deg(v') = 5$.

Proof. Let G be a cycle Cp, then deg(v) = 2 for any vertex v in Cp, let v' be the corresponding vertex in F (Cp), v' is adjacent to two vertices corresponding to two vertices in C_p, v' is adjacent to two vertices which corresponds to two edges in C_p, also v' is adjacent to a vertex corresponding to a block in C_p, then $deg(v') = 5$.

Theorem 2.2. For any connected graph G, F(G) has no cut ver- tices.

Proof. Let G be a connected graph, first let v be any vertex in F(G) which corresponds to a vertex or an edge of G, and let u, w be any two distinct vertices of F(G) which are diff erent from v, if u, w correspond to adjacent blocks in G then u, w are adjacent, otherwise there exists a u − w path starting from u passing through some vertices which correspond to blocks of G, this means v is not on this path. By Theorem 1.1, v is not a cut vertex of F(G). Next, let v be any vertex in F(G) which corresponds to a block of G, if G has one block then F(G) − v is total graph of G which is connected since G is connected. Now if G has more than one block then in F(G) − v there is a vertices which correspond to the blocks of G and the total graph (vertices which corresponding to vertices and edges of G), since G is connected then total graph of G is connected, also each one of the vertices corresponding to the blocks of G is adjacent with at least one vertex from the vertices which correspond to a vertices of G, this means that F(G) − v is connected, then v isn't a cut vertex of G, therefore F(G) has no cut vertex.

Corollary 2.3. For any connected graph G, F(G) is not separable.

Theorem 2.3. For any tree T, if v is a cut vertex, and deg(v) = n, then in F(T), deg(v) = 3n, where v is the vertex corresponding to v in F(T).

Proof. Let v be a cut vertex in a tree T, and deg(v) = n, this means that v is adjacent to n vertices and incident with n edges in T, since v is a cut vertex and every edge in T is a block, v lies in n blocks, therefore by the definition of F(T), v is adjacent to n vertices corresponding to vertices of T, and to n vertices corresponding to edges of T, and to n vertices corresponding to blocks of T, then deg(v) = 3n.

Theorem 2.4. For any edge in a graph G, with edge degree n, the degree of the corresponding vertex in F(G) is n + 3.

Proof. If an edge e in G is of edge degree n. Then e is adjacent to n edges, say, $e_1, e_2, , e_n$. Let v be a vertex in F (G) which is corresponding to e clearly v is adjacent to n vertices in F(G) corre- sponding to, $e_1, e_2, \ldots, e_n$. Also v is adjacent to two vertices that correspond to vertices incident with e in G, and v is adjacent to one vertex which is corresponding to a block B in G and e is in B. This means v is adjacent to n + 3 vertices in F(G).

Theorem 2.5. For any connected graph G = (p, q), F(G) is tree if and only if $G \cong K_1$.

Proof. Suppose a connected graph G is K_1, then in F(G) there are two vertices one corresponding to the vertex of G and the other one to a block of G, by definition of F(G), both vertices are adjacent, this establishes P2, then F(G) is a tree. Conversely, suppose F(G) is tree, we now prove that G is K_1. On the contrary, assume that connected graph G is not K_1, clearly p≥2. Then G has a block B. Let vertices u, v ∈ B. Since u, v are incident with B, it implies that the vertices corresponding to B, u, v, form a cycle C3 in F (G), this means F (G) is not tree, a contradiction. We conclude that $G \cong K_1$.

Theorem 2.6. If graph G has the path P_2 as a subgraph, then F (G) is not a bipartite graph.

Proof. Let G be a graph having the path P_2 as a subgraph. Then G has a block B. Let vertices u, v ∈ B. Since u, v are incident with B, it implies that the vertices corresponding to B, u, v, form a cycle C_3 in F(G). By Theorem 1.2, F (G) is not a bipartite graph.

Theorem 2.7. If G has the path P_2 as a subgraph, girth(F(G)) = 3
.

Proof. Let G be a graph having the path P_2 as a subgraph, then by the proof of Theorem 2.7, F(G) contains a cycle C_3 which is the shortest cycle, then girth(F (G)) = 3.

Proposition 2.2. If G has the path P_2 as a subgraph, then F(G) has K_4 as an induced subgraph.

Proof. Let G be a graph having the path P_2 as a subgraph, then G has a block B which contains one edge e and two vertices say, v, u. Since e, v, u, are incident with B, and v, u, are incident with e, then the vertices corresponding to u, v, e, B, form a complete graph K_4 in F(G).

Corollary 2.4. If G has the path P_2 as a subgraph, χ(F (G)) ≥ 4.

Theorem 2.8. For any connected graph G, γ(F(G)) ≤ |B| where
|B| is the number of blocks of G.

Proof. Let G be a connected graph. The following cases are considered.

Case1. G has one block, then in F (G) the vertex corresponding to this block is adjacent to all other vertices of F(G). By the definition of F(G), in this case γ(F(G)) = 1 = |B|.

Case2. G has more than one blocks, then in F(G) every vertex cor- responding to a block of G is adjacent to all vertices corresponding to vertices and edges of G that lie in this block. Also the vertices corre- sponding to blocks of G are connected since G is connected, this means that the vertices corresponding to blocks of G together are adjacent to all vertices of F(G), then they are dominating set of F(G), therefore γ(F(G)) ≤ |B|.

Corollary 2.5. For any connected graph G, γ_t(F(G)) ≤|B|.

Theorem 2.9. For any connected graph G with |B|≤ 2, $\gamma(\overline{F(G)})$ =2.

Proof. Suppose that G is a connected graph with |B| ≥ 2, then there are at least two blocks say, B_1, B_2, and there are two distinct vertices v ∈ B_1, u ∈ B_2, such that v and u are not cut vertices. Let v', u', be the vertices in F (G) corresponding to v, u, respectively. Clearly by the definition of F (G), v' is adjacent only to vertices corresponding to elements of G lying in B_1 and the same is to u' in B_2. Now in ($\overline{F(G)}$), v' is djacent to all vertices which corresponds to elements of G that are not in B_1, and u' is adjacent to all vertices corresponding to elements of G that are not in B_2. This means v', u' are dominating ($\overline{F(G)}$), then $\gamma(\overline{F(G)})$ = 2.

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