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Matching Domination of Lexicograph Product of Two Graphs

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Abstract—The paper concentrates on the theory of domination in graphs. In this paper we define a new parameter on domination called matching domination set, matching domination number and we have investigated some properties on matching domination of Lexicograph product of two graphs. The following are the results:

- $N_G(u_i, v_j) = \{N_{G_1}(u_i) X V_2\} \cup \{(u_i) X N_{G_2}(v_j)\}$
- $deg_G(u_i, v_j) = | N_{G_1}(u_1) || V_2 | \cup | N_{G_2}(v_j)$
- $deg_G(u_i, v_j) = 0$ if and only if $deg_{G_1}(u_i) = 0$ and $deg_{G_2}(v_j) = 0$
- If G_1, G_2 are simple finite graphs without isolated vertices then $G_1(L)G_2$ is a finite graph without isolated vertices.
- If G_1, G_2 are any two graphs without isolated vertices then $\gamma_m | G_1(L)G_2 | = \gamma_m(G_1)$

Keywords - Lexicograph product of graphs, Domination Set, Domination number, finite graphs, Isolated vertices, degree, regular graphs, Neighbourhood graph(N_G).

I. Introduction

The study on dominating sets was initiated as a problem in the game of chess in 1850. It is about the placement of the minimum number of Queens/rooks/horses, in the game of chess so as to cover every square in the chess board. However a precise notion of a dominating set is said to be given by Konig [12], Berge [13] and Ore [7], Vizing [14] were the first to derive some interesting results on dominating sets. Since then a number of graph theorists Konig [15], Ore [7], Bauer Harary [16], Laskar [5], Berge [13], Cockayne [l7], Hedetniemi [10], Alavi[18], Allan [19], Chartrand [18], Kulli [3], Sampathkumar [3], Walikar[20], Arumugam [21], Acharya [22], Neeralgi [23], Nagaraja Rao [15] and many others have done very interesting and significant work in the domination numbers and other related topics. Cockayne [17] and Hedetniemi [10] gave an exhaustive survey of research on the theory of dominating sets in 1975 and it was updated in 1978 by Cockayne [17]. A survey on the topics on domination was also done by Hedetniemi and Laskar recently.

A domination number is defined to be the minimum cardinality of all dominating sets in the graph G and a set $S \subseteq V$ is said to be a dominating set in a graph, if every vertex in V/S is adjacent to some vertex in S.

In this paper, we have defined two new domination parameters viz., matching domination set and matching domination number.

The matching domination is defined as follows:

Let $G :< V, E >$ be a finite graph without isolated vertices. Let $S \subseteq V$. A dominating set S or G is called a matching dominating set if the induced subgraph $< S >$ admits a perfect matching. The cardinality of a minimum matching dominating set is called the matching domination number.

We have obtained the matching domination of the product of two graphs G_1 and G_2 in cartesian product graphs and obtained an expression for this number in terms of matching domination number of $G_1$and G_2. While obtaining these results, we have obtained several other interesting results on matching domination on Lexicograph product of two graphs .

II. LEXICOGRAPH PRODUCT OF GRAPHS

Definition 2.1

If G_1, G_2 are simple graphs with their vertex sets as $V_1 : \{u_1, u_2, ...\}$ and $V_2 : \{v_1, v_2 .....\}$ respectively, then the Lexicograph product is a graph with its vertex set as $V_1 x V_2 : \{w_1, w_2, ...\}$ and if $w_1 = (u_1, v_1)$, $w_2 = (u_2, v_2)$ then w_1, w_2 is an edge in this product graph if and only if either (i) $u_1, u_2 \epsilon E(G_1)$ or (ii) $u_1 = u_2$ and $v_1, v_2 \epsilon E(G_2)$. This product graph is called Lexicograph product graph and is denoted by $G_1(L)G_2$.

Illustration follows

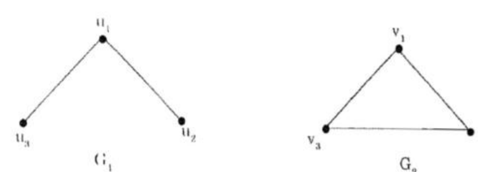

Fig. 1. .

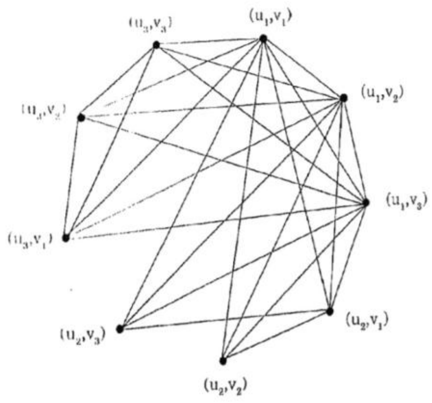

Fig. 2. $G_1(L)G_2$

In this product graph also, it can be proved that if G_1 and G_2 are simple finite graphs without isolated vertices then $G_1(L)G_2$ is also finite graph without isolated vertices.
To establish this, we first obtain an expression for $N_{G_1(L)G_2}(u_i, v_j)$ Where $N_G(u)$ denotes the neighbourhood set of u in the graph G.
Theorem 2.2

$$N_G(u_i,v_j) = \{N_{G_1}(u_i)XV_2\} \cup \{(u_i)XN_{G_2}(v_j)\}$$

Proof :

Suppose

$$N_{G_1}(u_i) = \{u_1,u_2, \ldots\ldots u_r\},$$

$$V_1 = \{v_1,v_2, \ldots\ldots v_{m_1}\};$$

and

$$N_{G_2}(v_j) = \{v_1,v_2, \ldots\ldots v_s\},$$

$$V_2 = \{v_1,v_2, \ldots\ldots v_{m_2}\};$$

Then vertex (u_i, v_j) is adjacent to the vertices

$$\{(u_1, v_1)\ (u_1, v_2)\ \ldots\ (u_1, v_{dm2})$$
$$(u_2, v_1)\ (u_2, v_2)\ \ldots\ (u_2, v_{dm2})$$
$$(u_r, v_1)\ (u_r, v_2)\ \ldots\ (u_r, v_{dm2})\}$$

The vertex (u_i, v_j) is adjacent with any vertex (u_m, v), if $u_m \epsilon N_{G_1}(u_1)XV_2$
Also (u_i, v_j) is adjacent with all the vertices of the set $\{u_i\}XN_{G_2}(v_j)$, for if (u_i, v_j) is any element in the set $\{u_i\}XN_{G_2}(v_j)$ where $v_j \epsilon N_{G_2}(v_j)$, then (u_i, v_j) is adjacent with (u_i, v_t) since v_j , v_t are adjacent. Thus

$$\{N_{G_1}(u_i)XV_2\} \cup \{u_j\, XN_{G_1}(v_j)\} \subset N(u_i, v_j)$$

Conversely, if $(u_x, v_y) \subset N_G(u_i, v_j) \Rightarrow (u_i, v_j)$ is adjacent with (u_x, v_y) (By definition 2.1), this is possible only if u_i is adjacent with u_x i.e., $u_x \epsilon N_{G_1}(u_i)$ or if $u_i = u_x$ and v_j is adjacent with v_y i.e.,$v_y \epsilon N_{G_2}(v_j)$.

$$\Rightarrow (u_i, v_j) \epsilon N_{G_1}(u_i)XV_2$$

or

$$(u_x, v_j) \epsilon \{u_i\}XN_{G_2}(v_j)$$

Thus

$$(u_x, v_i) \epsilon \{N_{G_1}(u_i)XV_2\} \cup \{\{u_i\}XN_G(v_j)\}$$

Hence

$$N_G(u_i,v_j) \subset \{N_{G_1}(x_i)XV_2\} \cup \{\{u_i\}XN_{G_2}(v_j)\}$$

From above equations the theorem follows.
To obtain an expression for the matching domination number of $G_1(L)G_2$. We require the following result.

Theorem 2.3

$$deg_G(u_i, v_j) =| N_{G_1}(u_1) || V_2 | \cup | N_{G_2}(v_j)$$

Proof :

From the theorem 2.2,

$$N_G(u_i,v_j) = \{N_{G_1}(u_1)XV_2\} \cup \{\{u_i\}XN_{G_2}(v_j)\}$$

The two cartesian product sets on the RHS are disjoint sets;Since any element in the cartesian product $\{u_i\}XN_{G_2}(v_j)$ is of the form (u_i, v_x), where as any element in the cartesian product $N_{G_1}(u_i)XV_2$ is of the form (u_t, v) since i = t i.e., $u_i = u_t$.
Hence

$$deg(u_i, v_j) =| N_{G_1}(u_i, v_j) |$$
$$=| N_{G_1}(u_i)XV_2 | + | N_{G_2}(v_j) |$$

Corollary 2.4

$deg_G(u_i, v_j) = 0$ if and only if $deg_{G_1}(u_i) = 0$ and $deg_{G_2}(v_j) = 0$

Proof :

If

$$deg_G(u_i, v_j) = 0;$$

by the pervious theorem,

$$| N_{G_1}(u_i)XV_2 | + | N_{G_2}(v_j) |= 0$$

$$\Rightarrow | N_{G_1}(u_i)XV_2 |= 0$$

and

$$| N_{G_2}(v_j) |= 0$$

$$\Rightarrow N_{G_1}(u_i) = 0, N_{G_2}(v_j) = 0$$

$$\Rightarrow deg_{G_1}(u_i) = 0,\ deg_{G_2}(v_j) = 0$$

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Conversely, if $deg_{G_1}(u_i) = 0$ and $deg_{G_2}(v_j) = 0$, by retracing the above steps, we get $deg_G(u_i, v_j) = 0$

Now the following result is an immediate consequence.

theorem 2.5

If G_1, G_2 are simple finite graphs without isolated vertices then $G_1(L)G_2$ is a finite graph without isolated vertices.

Proof :

Suppose $G_1(L)G_2$ is a finish graph follows by the definition 2.1. Further G_1, G_2 are graphs without isolated vetices. i.e., for any i,

$$deg_{G_1}(u_i) = 0$$

for any j,

$$deg_{G_2}(v_j) = 0$$

Hence from corollary 2.4, $deg_{G_1(L)G_2}(u_i, v_j) = 0$ for any i, j.

It can also be established that $G_1(L)G_2$ is a complete graph if and only if G_1 and G_2 are complete graphs.

III. MATCHING DOMINATION NUMBER

Definition 3.1

A set $S \subseteq V$ is said to be a dominating set in a graph G if every vertex in V/S is adjacent to some vertex in S and the domination number γ of G is defined to be the minimum cardinality of all dominating sets in G.

We have introduced a new parameter called the matching domination set of a graph.

Definition 3.2

A dominating set of a graph G is said to be matching dominating set if the induced subgraph < D > admits a perfect matching.

The cardinality of the smallest matching dominating set is called matching domination number and is denoted by γ_m

Illustration

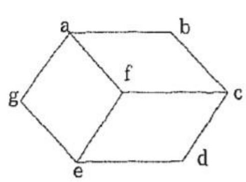

Fig. 3.

In this graph {a, b, c, f, e, g } is a matching domination set, since this is a dominating set and the induced subgraph {a, b, c, e, f, g } has perfect matching formed by the edges af, bc, eg, {a,b,e,f } is also matching dominating set. Similarly {a,b,c,g} is a matching dominating set where the induced subgraph of this set admits a perfect matching given by the edges be,ag.

However there are no matching dominating sets of lower cardinality and it follows that the matching domination number of the graph in figure 3 is 4. Thus a graph can have many matching dominating sets of minimal cardinality. We make the following observations as an immediate consequence.

(a) Not all dominating sets are matching domination sets. For example in figure 3, {a,c,e } is a dominating set but it is not a matching dominating set.

(b) The cardinality of matching dominating set is always even. The matching dominating set D of a graph requires the admission of a perfect matching by the induced subgraph < D >. Thus it is necessary that D has even number of vertices for admitting a perfect matching.

(c) Not all dominating sets with even number of vertices are matching dominating sets. For example in figure 3,{b,d,g,f } is a dominating set containing even number of vertices, but induced subgraph formed by these four vertices does not have a perfect matching.

(d) The necessary condition for a graph G to have matching dominating set is that G is a graph without isolated vertices. The matching domination number of the graph G (figure 3) is 4, where as the domination number is 2 ; {a,d } being a minimal dominating set. If G is a graph with isolated vertices then any dominating set should include these isolated vertices and consequently the induced subgraph of this set containing isolated vertices will not admit a perfect matching.

It is interesting to see that in this type of product graphs the matching domination number of Lexicograph product graph G_1 and G_2 is same as the matching domination number of the graph G_1.

Theorem 3.3

If G_1, G_2 are any two graphs without isolated vertices then $\gamma_m \mid G_1(L)G_2 \mid = \gamma_m(G_1)$

Proof:

Let D_1, D_2 be the matching dominating sets of minimum cardinality of G_1 and G_2 respectively.

Let

$$\left.\begin{array}{l} D_1 = \{u_1, u_2, \ldots\ldots, u_{2r}\} \\ D_2 = \{v_1, v_2, \ldots\ldots\ldots, v_{2s}\} \end{array}\right\} \ldots\ldots \quad (1)$$

Consider the set $D = \{(u_1, v_1), (u_2, v_2), \ldots\ldots\ldots(u_{2r}, v_{2r})\}$ (if $r \leq s$) or consider the set $D = \{(u_1, v_1), (u_2, v_2), \ldots\ldots\ldots(u_{2s}, v_{2s}), (u_{2s+1}, v_1), (u_{2s+2}, v_1), \ldots\ldots(u_{2r-1}, v_1), (u_{2r}, v_1)\}$ (if $r > s$)

D will be a matching dominating set, Further D is of minimum cardinality for if we remove any of the vertices in D, D is not a matching dominating set any more in view of (1), and from the definition of Lexicograph product.

Thus D is a matching dominating set of minimum cardinality.

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Hence,

$$\gamma_m(G_1(L)G_2) = \gamma_m(G_1).$$

Illustrations follows.

Illustration

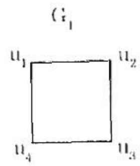

Fig. 4. matching dominating set : $\{u_1, u_2\}\gamma_m(G_1) = 2$

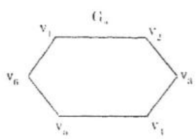

Fig. 5. matching domination set $\{v_1, v_2, v_4, v_5\}$, $\gamma_m(G_2) = 4$

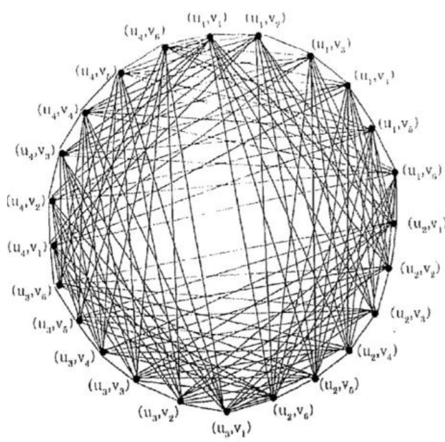

Fig. 6. matching domination set $\{(u_1, v_1), (u_2, v_2)\}$, $\gamma_m(G_1(L)G_2) = 2$

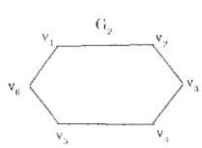

Fig. 7. matching dominating set : $\{u_1, u_2\}\gamma_m(G_1) = 2$

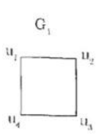

Fig. 8. matching domination set $\{v_1, v_2, v_4, v_5\}$, $\gamma_m(G_2) = 4$

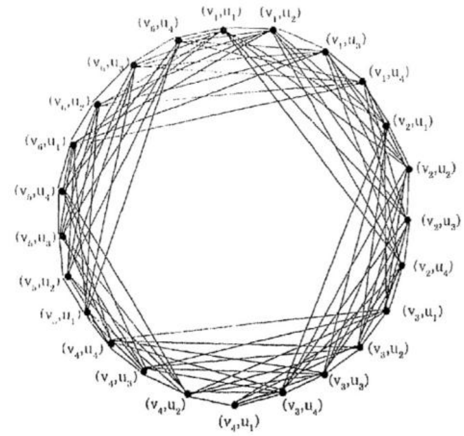

Fig. 9. matching domination set $\{< v_1, u_1 >, < v_2, u_2 >, < v_4, u_1 >, < v_5, u_1 >\}$, $\gamma_m[G_2(L)G_1] = 4$

IV. Conclusion

The Theory of domination has been the nucleus research activity in graph theory in recent times. This is largely due to a variety of new parameters that can developed from the basic definition of domination. The study of Lexicograph product graphs,the matching domination of Lexicograph product graphs has been providing us sufficient stimulation for obtaining some in-depth knowledge of the various properties of the graphs. It is hoped that encouragement provided by this study of these product graphs will be a good straight point for further research.

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