 ISSN: 2321-8169
431 – 434

On (2,2)-Domination in Hexagonal Mesh Pyramid

D. Antony Xavier and Elizabeth Thomas
Department of Mathematics, Loyola College, Chennai, India
e-mail: elizathomas.25@gmail.com

Abstract- Network topology plays a key role in designing an interconnection network. Various topologies for interconnection networks have been proposed in the literature of which pyramid network is extensively used as a base for both software data structure and hardware design. The pyramid networks can efficiently handle the communication requirements of various problems in graph theory due to its inherent hierarchy at each level. Domination problems are one of the classical types of problems in graph theory with vast application in computer networks and distributed computing. In this paper, we obtain the bounds for a variant of the domination problem namely (2,2)-domination for a pyramid network called Hexagonal mesh pyramid**.**

Keywords- *Domination, (2,2)- domination, Hexagonal mesh, Hexagonal mesh pyramid.*

*****

I. INTRODUCTION

Interconnection networks are currently being used for many different applications ranging from inter-ip connections in VLSI circuits to wide area computer networks [8]. An interconnection network can be modeled by a graph where a processor is represented by a vertex and a communication channel between two processing vertices is represented by an edge. Various topologies for interconnection networks have been proposed in the literature: these include cubic networks (e.g meshes, tori, k–ary n-cubes, hypercubes, folded cubes and hypermeshes), hierarchical networks (e.g pyramids, trees), and recursive networks (e.g RTCC networks, OTIS networks, WK recursive networks and star graphs) that have been widely studied in the literature for topological properties.

A famous network topology that has been used as the base of both hardware architectures and software structures is the pyramid. By exploiting the inherent hierarchy at each level, pyramid structures can efficiently handle the communication requirements of various problems in graph theory, digital geometry, machine vision, and image processing [4].The main problems with traditional pyramids are hardware scalability and poor network connectivity and bandwidth. To address these problems, in [1] a new pyramidal network called as hexagonal mesh pyramid is proposed. The new network preserves many desirable properties of traditional pyramid network.

Domination is a rapidly developing area of research in graph theory, and its various applications to ad-hoc networks, distributed computing, social networks and web graphs partly explain the increased interest. A dominating set of a graph G is a vertex subset with the property that every vertex of the graph is either in the dominating set or adjacent to a vertex in the dominating set. The minimum size of the dominating set is called the domination number and is denoted by $\gamma(G)$ .Let $G = (V(G), E(G))$be a simple graph. For $u, v \in V(G)$, the distance $d_G(u,v)$ between u and v is the length of the shortest uv-paths in G .The diameter of G is $d(G) = max\{d_G(u,v): u,v \in V(G)\}$. For an integer $k \geq 1$ and $v \in V(G)$, the open k – neighborhood of v is $N_k(v) = \{u \in V(G): 0 < d_G(u,v) \leq k\}$, and the closed k – neighborhood of v is $N_k[v] = N_k(v) \cup \{v\}$. For a vertex $v \in V(G)$ we define the degree of v as $d(v) = |N(v)|$. The minimum degree of G is denoted by $\delta(G) = \min\{d(v) \mid v \in V(G)\}$.

For two positive integers k and r, a subset S of the vertex of a graph G is (k,r)-dominating set of G if every vertex $v \in V(G) / S$ is within distance k to atleast r vertices in S. The parameter $\gamma_{k,r}(G)$ denotes the minimum cardinality of a (k,r)-dominating set of G and is called the (k,r)-dominating number.

This dominating concept is a generalization of the two concepts distance domination and r-domination of a graph. So the study of (k,r)-domination of a graph is more interesting and has received the attention of many researchers. Some results and bounds of (k,r)-domination and (2,2)-domination number of a graph that are used in the sequel are given below;

Theorem 1.1[3] Let G be a graph. Then $k \leq \gamma_{k,r}(G) \leq n$ and these bounds are sharp.

Theorem 1.2[3] If $r = d(G)$, then $\gamma_{k,r}(G) = k$.

Theorem 1.3[9] If G is a spider with n vertices, then $\gamma_{2,2}(G) = (n+1)/2$.

Corollary 1.1[9] If G is a connected graph on $n \geq 3$ vertices, then $\gamma_{2,2}(G) \leq (n+1)/2$ with equality if and only if G is a spider.

In this paper we focus on (2,2)-domination of hexagonal mesh pyramid and give an improved upper bound for it.

 ISSN: 2321-8169
431 – 434

II. HEXAGONAL MESH AND HEXAGONAL MESH PYRAMID

In this section we briefly describe the interconnection networks, hexagonal mesh and hexagonal mesh pyramid. We also recall some of its topological properties.

A. Hexagonal Mesh

Triangular, square and hexagon are the three existing regular plane tessellations which are composed of the same kind of regular polygons. To design direct interconnection networks we use any one of this, also these types of interconnection networks are highly competitive in overall performance. The triangular tessellation is used to define hexagonal network and this type of hexagons are widely studied in [6].

An n-dimensional hexagonal mesh HX(n) has $3n^2 - 3n + 1$ vertices and $9n^2 - 15n + 6$ edges where n is the number of vertices on one side of the hexagon. The diameter of $HX(n)$ is $2(n-1)$. In this structure, there are 6 vertices with degree 3 which we call as corner vertices. The vertex, which is at $n-1$ distance from the corner vertices, is called the centre vertex and it is unique. See Fig.1. This particular structure has wide application in the field of computer graphics, cellular phone base stations, image processing and in chemistry as the representation of benzenoid hydrocarbons.

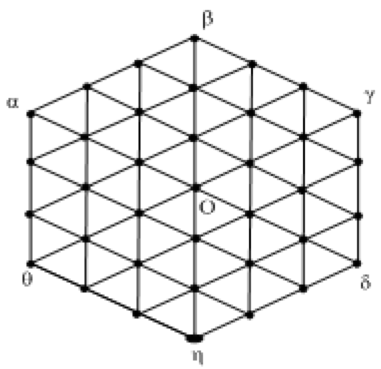

Fig.1. A hexagonal mesh of dimension five, $HX(5)$.

B. Hexagonal Mesh Pyramid

A hexagonal mesh pyramid of n levels denoted as $HXP(n)$ consists of a set of vertices, arranged in n levels of a hexagonal mesh. A vertex is addressed as $(k,(x,y,z))$ and is said to be a vertex at level k. The part (x,y,z) of the address determines the address of a vertex within the layer k, of the hexagonal network. The vertices at level k, form a network $HX(n)$. A vertex with the address $(k,(x,y,z))$ placed at level k, of the hexagonal network is connected to all its adjacent vertices. This vertex is also connected to all the vertices of the hexagon with centre $(k+1(x,y,z))$See Fig.2.

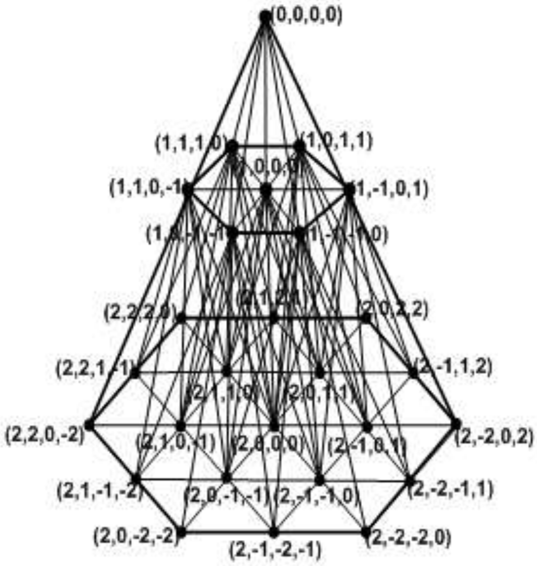

Fig.1. A hexagonal mesh pyramid of dimension 3, $HXP(3)$.

The number of vertices and edges of $HXP(n)$ is n^3 and $3n^2(n-1) + 6(n-1)^3$ respectively. The diameter of $HXP(n)$ is $2(n-1)$ is same as the diameter of the hexagonal mesh network. Furthermore, it is Hamiltonian and pancyclic.

III. UPPER BOUND FOR (2,2)-DOMINATION NUMBER OF HEXAGONAL MESH PYRAMID

In this section we deduce an upper bound for (2,2)-domination number of a n-dimensional hexagonal mesh pyramid. Before proceeding to the main theorems, we need the following results.

Proposition3.1.[2] For a 2-dimensional hexagonal mesh $HX(2)$,$\gamma_{2,2}(HX(2)) = 2$.

Theorem 3.1.[2] For any n-dimensional hexagonal mesh $HX(n)$ $(n > 2)$, if n is even then there exists (2,2)-dominating set D such that $|D| = (n^2 - 2)/2$.

Theorem 3.2.[2] For any n-dimensional hexagonal mesh $HX(n)$ $(n > 2)$, if n is odd then there exists (2,2)-dominating set D such that $|D| = (n^2 + 1)/2$.

Theorem3.2. $\gamma_{2,2}(HXP(2)) = 2$ and $\gamma_{2,2}(HXP(3)) = 2$.

Proof: Since the diameter of $HXP(2)$is 2, from theorem 1.2, $\gamma_{2,2}(HXP(2)) = 2$.

For $XP(3)$, clearly the single vertex in level 0 and the centre vertex of the HX_2 form a (2,2)-dominating set. Hence $\gamma_{2,2}(HXP(3)) = 2$.

Theorem 3.3. For any n-dimensional hexagonal mesh $HXP(n)$ $(n > 3)$, if n is even then there exists (2,2)-dominating set D such that $|D| = (n^3 + 3n^2 - 4n + 12)/12$.

Proof: For any given n-dimensional hexagonal mesh pyramid $HXP(n)$ (n > 3), we will choose a subset D of the vertex set V such that it consists of only the vertices of the (2,2) −dominating set of each $HX(k)$ (refer [2]) , in the level $k-1$, where $2 \leq k \leq n$ and k is even.

We claim that D is a (2,2)-dominating set such that $|D| = (n^3 + 3n^2 - 4n + 12)/12$.

The proof is by induction on the dimension (where $n > 3$ and n is even.If $n=4$, then the set D will consist of the vertices of the (2,2)-dominating set of $HX(2)$ and $HX(4)$, in the levels 1 and 3 respectively.The (2,2)-dominating set of $HX(2)$ consists of the vertices with label $(1,1,0,-1)$ and $(-1,0,1)$. Clearly the top most vertex (i.e) the vertex with label $(0,0,0,0)$ will be at distance one from these two vertices. Also all the vertices in level 2 will be at a distance atmost two from at least one of these vertices. Now it enough to show that all the vertices in level 2 will be at a distance at most two from at least one of the vertices in the (2,2)-dominating set of $HX(4)$. We know that every vertex in level 2 will generate a hexagonal mesh of dimension two in the next level. Clearly all the vertices in each of these hexagonal meshes are at distance one to the vertex that generates it. We observe that the hexagonal meshes either consists on one point from the (2,2)-dominating set of $HX(4)$ or has a point from the (2,2)-dominating set of $HX(4)$ in its 1-neighbourhood. Hence all the vertices all the vertices in level 2 will be at a distance at most two from at least one of the vertices in the (2,2)-dominating set of $HX(4)$. Hence D is (2,2) -dominating set of $HXP(4)$. Clearly $|D| = 2 + 7 = 9$. Hence in this case $|D| = (n^3 + 3n^2 - 4n + 12)/12$.

Assume that the result is true for a hexagonal mesh dimension n. Consider a hexagonal mesh of dimension $n+2$. The subset D' of the vertex set V is chosen such that it consists of only the vertices of the (2,2)-dominating set of each $HX(k)$, in the level $k-1$, where $2 \leq k \leq n+2$ and k is even.

Now it remains to show that D' is a (2,2)-dominating set of $HXP(n+2)$. By induction, there exist a (2,2)-dominating set of $HXP(n)$, say D_n such that $|D_n| = \frac{n^3+3n^2-4n+12}{12}$.

Hence it is enough to prove that all the vertices in level n will be at a distance at most two from at least two of the vertices in D'. Cleary all the vertices in level n will be at a distance at most two from at least one of the vertices in the (2,2)-dominating set of the hexagonal mesh of the previous level i.e.$HX(n)$. Also we know that every vertex in level n will generate a hexagonal mesh of dimension two in the next level. Clearly all the vertices in each of these hexagonal meshes are at distance one to the vertex that generates it. We observe that the hexagonal meshes either consists on one point from the (2,2)-dominating set of $HX(n+2)$ or has a point from the (2,2)-dominating set of $HX(n+2)$ in its 1-neighbourhood. Hence all the vertices all the vertices in level n will be at a distance atmost two from at least one of the vertices in the (2,2)-dominating set of $HX(n+2)$. Hence D' is (2,2) -dominating set of $HXP(n+2)$. Clearly D' is the union of D_n and the (2,2)-dominating set of $HX(n+2)$, say S_{n+2} . By theorem 3.1, $|S_{n+2}| = ((n+2)^2 - 2)/$. Hence, $|D'| = |D_n| + |S_{n+2}|$.

$$\Rightarrow |D'| = (n+2)^3 + 3(n+2)^2 - 4(n+2) + 12)/12.\square$$

Theorem 3.4. For any n-dimensional hexagonal mesh $HXP(n)$ $(n > 3)$, if n is odd then there exists (2,2) -dominating set D such that $|D| = (n^3 + 3n^2 + 5n + 3)/12$.

Proof: For any given n-dimensional hexagonal mesh pyramid $HXP(n)$ $(n > 3)$, we will choose a subset D of the vertex set V such that it consists of only the vertices of the (2,2)-dominating set of each $HX(k)$, in the level $k-1$, where $3 \leq k \leq n$ (k is odd) and the top most vetex (i.e) at level 0. We need to prove that D is a $(2,2)$-dominating set such that $|D| = n^3 + 3n^2 + 5n + 3/12$. The proof is similar to the previous theorem.□

As a consequence of the above theorems we have the following;

Corollary3.1. For any n-dimensional hexagonal mesh pyramid, $HXP(n)$ $(n > 3)$,

$$\gamma_{2,2}(HXP(n)) \leq \begin{cases} \dfrac{n^3 + 3n^2 - 4n + 12}{12}, \text{when } n \text{ is even.} \\ \dfrac{n^3 + 3n^2 + 5n + 3}{12}, \text{when } n \text{ is odd.} \end{cases}$$

Remark: We observe that the bound is reduced when compared to the general upper bound for any connected graph obtained in corollary 1.1. The newly obtained upper bound for (2,2) -domination number of hexagonal mesh pyramid is significantly smaller when compared to the general upper bound for any connected graph.

IV. Conclusion

In this paper we have present an upper bound for the (2,2)-domination number of hexagonal mesh pyramid. We observe that in the case of hexagonal mesh pyramid we get a reduced upper bound when compared to the general upper bound for any connected graphs. This work could be further extended to other interconnection networks.

Acknowledgment

This work is supported by Maulana Azad Fellowship F1-17.1/2013-14/MANF-CHR-TAM-28120 of the University Grants Commission, New Delhi, India.

ISSN: 2321-8169
431 – 434

REFERENCES

[1] D Antony Xavier, DeeniC.J, "Hexagonal mesh pyramid: Some topological properties," Mathematical Scs. International Research Journal,Vol 03,Issue 1,2014, 266-271.

[2] D Antony Xavier, Deeni C.J, Elizabeth Thomas, S.Kulandai Therese, "(2,2)-Domination InHexagonal mesh," Proc. of International Conference on Advances In Applied Probability, Graph theory and Fuzzy Mathematics, pp.179-180 , St.Peter's College, Kolenchery, Kerala India, January 11 – 14, 2014.

[3] G.Joythilakshmi, A.P.Pushpalatha, S.Suganthi, V.Swaminathan, "(k,r) – domination in graphs," Int.J.Contemp.Math.Sciences, Vol.6, 2011, no.29, 1439-1446.

[4] H.Sarbazi-Azad, M.Quld-Khaoua, L.Mackenzie, "Algorithemic constructions of Hamiltonians in pyramid networks," Inf.proc.letters 80, 2001, 75-79.

[5] M.Fischermann, L.Volkmann, "A remark on a conjecture for the (k,p) domination number," Utilitas Mathematica, 67, 2005, 223 – 227.

[6] M.S.Chen, K.G.Shin, DilipD.Kandlur, "Addressing, Routing, Broadcasting in Hexagonal Mesh Multiprocessors". IEEE transactions on Computers, 39, 1990, 10-18

[7] T.J. Bean, M.A Henning, H.C Swart, "On the integrity of distance domination in graphs," Australas J. Comb, 10, 1994, 29-43.

[8] T.W Hanynes, S.T Hedetnieni, P.J Slater, Fundamentals of domination in graphs. Marcel Dekker, Inc. New York, 1998.

[9] You Lu, XinminHou, Jun – Ming Xu, "On the (2,2) – domination number of trees," DiscussionesMathematicae Graph Theory, 30, 2010 185-199.