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Fixed Point Results for Certain Contractive Self–Mappings in D–Metric Space

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***Abstract*—** There have been number of generalizations of Metric spaces. D–Metric space is one such generalization initiated by Dhage [1] in 1992 and open new research area. Rhoades [2] generalized Dhage's contractive condition by increasing number of factors and proved fixed point of self–mapping in D–Metric space. Then many authors have obtained, interesting fixed point results in D–Metric space satisfying contractive type condition.The present paper studies some fixed point theorems in D-Metric space and proved new fixed point theorem and its corollary in a bounded D–Metric space for a contractive self – mapping.

Keywords- *D–Metric space, fixed point theorems.*

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I. Introduction

Banach first proved contraction mapping priniciple for self mappings in Metric spaces satisfying contraction condition in 1922, Kannan in 1968 give new turn to Banach fixed point theorem and introduce new class of contraction mapping possessing the unique fixed point. All over the world different authors generalized and extend of above theorem and number of good research done on fixed point theorems in Metric space. Dhage [3] in 1984 introduced the concept of D–Metric space in his Ph.D. thesis in which it has been possible to determine the geometrical nearness i.e. the distance between two or more points of the set under consideration. Geometrically, D–Metric d(a, b, c) represent the perimeter of the triangle with vertices a, b and c. Some examples and few details of D–Metric space observe in [1], In [4] Dhage proved some fixed point results of self mappings of D – Metric space satisfying some contractive conditions. In [4, 5, 6, 7, 8] Dhage explain topological structures of D–Metric space and several fixed point theorems. These works have been the basis for a substantial number of results by various authors.

2. Preliminaries and definitions :

Dhage introduced new structure D–Metric space which is higher dimensional metric space of ordinary metric space.

Definition 2.1 [2] : Let X be a non–empty set. Let function $d : X \times X \times X \to [0, \infty)$ is called a D–Metric if D satisfies, for all $x, y, z, a \in X$

D_1) $d(x, y, z) = 0$ iff $x = y = z$ (coincidence)
D_2) $d(x, y, z) = d(p\{x, y, z\})$ (Symmetry)
When p is a permutation of x, y, z.
D_3) $d(x, y, z) \leq d(x, y, a) + d(x, a, z) + d(a, y, z)$ (Tetrahedral inequality)

The non–empty set X together with D–Metric "d" is called D–Metric space and it is denoted by (X, d).

Geometrically, D–Metric d (x, y, z) is perimeter of a triangle whose vertices are x, y, z.

Example 2.1 : Let $d_1 : X \times X \times X \to [0, \infty)$ define by ,

$d_1(a, b, c) = \max\{\varrho(a, b), \varrho(b,c), \varrho(c, a)\}$ for all $a, b, c \in X$ and $\varrho(a, b) = | a - b |$ is an ordinary metric on X, then(X, d_1) is D–Metric space.

Example 2.2 : Let $d_1 : X \times X \times X \to [0, \infty)$ define by,

$d_1(a, b, c) = \varrho(a, b) + \varrho(b, c) + \varrho(c, a)$ for all $a, b, c \in X$ where $\varrho(a, b)$ is an ordinary metric on X then d_1 is D–Metric on X and (X, d_1) is a D–Metric space.

Example 2.3 : Define $d : R^n \times R^n \times R^n \to [0, \infty)$ by,

$d(a, b, c) = C \max\{\| a - b \|, \| b - c\|, \| c - a \|\}$ for all $a, b, c \in R^n$, $C > 0$

Where $\| \;\|$ is norm in Euclidean space R^n is a D–Metric on R^n. Hence (R^n, d) is a D–Metric space.

Definition 2.2 [4] : Sequence $\{a_n\}$ in D–Metric space (X, d) is said to D–Convergent to $a \in X$ if $\lim_{m,n} d(a_n, a_m, a) = 0$

That is, for a point $a \in X$, if each $\in > 0$ there exist positive integer n_0 such that $d(a_n, a_m, a) < \epsilon$ for all $m, n \geq n_0$.

Definition 2.3[9]: Sequence $\{a_n\}$ in D − Metric space is called D– Cauchy if

$$\lim_{m,n,p} d(a_n, a_m, a_p) = 0$$

Definition 2.4 [4] : Every D-Cauchy sequence converges to a point in D–Metric space is called complete D–Metric space .

Definition 2.5 [4] : Let (X, d) be D–Metric space. A subset U *of* X is said to be bounded if these exist constant s > o such that, $d(a, b, c) < s$, for all $a, b, c \in U$ and s is called D-bound of U.

For a bounded sequence $\{y_n\}$ in D–Metric space (X, d), let $a_n = \delta(\{y_n, y_{n+1}, y_{n+2}, \ldots\})$ for $n \in N$. Then a_n is finite for all $n \in N$ and $\{a_n\}$ is nonincreasing and $a_n \geq 0$

for all $n \in N$ so there is an $a \geq 0$ such that,

$\lim_{n \to \infty} a_n = a$.

Definition 2.6 [9] : Consider (X, d) be D–Metric space and $f : X \to X$. The orbit of f at the point $a \in X$ is the set $O(a) = \{a, fa, f^2 a, \ldots\}$

Definition 2.7 [9] : Consider (X, d) be D–Metric space and $O(a)$ be orbit of $f : X \to X$ is said to be bounded if there exists a constant $C > 0$ such that $d(x, y, z) \leq C$ for all $x, y, z \in O(a)$. The constant C is called D–bound of $O(a)$.

D–Metric space is said to be f–orbitally bounded if $O(a)$ is bounded for each $a \in X$.

Definition 2.8 [9] : An orbit $O(a)$ is said to be f–orbitally complete if every D–Cauchy sequence in $O(a)$ converges to a point in X.

Definition 2.9 [11] : For D–Metric space (X, d), $Y (\neq \emptyset) \subseteq X$ the diameter of Y is defined by,

$\delta_d(Y) = \sup \{ d(a, b, c) \ / \ a, b, c \in Y \}$

For bounded sequence $\{x_n\}$ consider $r_n = \delta_d(\{x_n, x_{n+1}, \ldots\})$ for $n \in N$.

Then r_n is finite for all $n \in N$ and $\{r_n\}$ is decreasing, $r_n \geq 0$ for $n \in N$. Therefore there exists

$r \geq 0$ such that, $\lim_{n \to \infty} r_n = r$.

Let ψ *be class* of all upper semicontinuous function $\emptyset \ R_+^5 \times R_+ \to R_+$ and $\emptyset$ is increasing on R_+^5 satisfying $\emptyset((x, x, x, x, x), y) \geq 0$ implies $y \leq f(x)$

Where $f : R_+ \to R_+$ is increasing upper semi continuous function with $f(0) = 0$ and $f(z) < z$ for $z > 0$.

Example 2.4 : Let $\emptyset_m : R_+^5 \times R_+ \to R_+$ defined by,

$\emptyset_m((x_1, x_2, x_3, x_4, x_5), x_6) = f(\max\{x_1, x_2, x_3, x_4, x_5,\}) - x_6$

Where $f : R_+ \to R_+$ is a increasing upper semi continuous function with $f(0) = 0$ and $f(x) < x$ for $x > 0$.Then $\emptyset_m$ is upper semi continuous Further, $\emptyset_m((x, x, x, x, x), y) \geq 0$ implies $y \leq f(x)$. Thus $\emptyset_m \in \psi$.

3 : Some fixed point theorems on D-Metric Space :

Theorem 3.1 (D–Cauchy Principle) [10]: Let $\{a_n\}$ be a bounded sequence in D–Metric space X with D–bound C satisfying $d(a_n, a_{n+1}, a_m) \leq \alpha^n C$ (1)

For all positive integer $m > n$ and $0 \leq \alpha < 1$

Then $\{a_n\}$ is D–Cauchy.

Rhodes [2]established the following theorem

Theorem 3.2 [2] : Consider X be a complete and bounded D–Metric space and $f : X \to X$ satisfying the following condition

There exist $\alpha \in [0,1)$ such that for all $a, b, c \in X$

$d(fa, f_b, f_c) \leq \alpha \max\{d(a, b, c), d(a, fa, c), d(b, f_b, c), d(a, f_b, c), d(b, fa, c)\}$

Then f has a unique fixed point q in X and f is continuous at q.

Theorem 3.3 [10] : Let (X, d) be a D–Metric space and $f : X \to X$ be self – map. Let $a_o \in X$ such that, $O(a_o)$ is D–bounded and f – orbitally complete . Also f satisfies,

$d(f_a, f_b, f_c) \leq k \max\{d(a, b, c), d(a, f_a, c)\}$ for $a, b, c \in \overline{O(a_o)}$ (2)

For some $0 \leq k \leq 1$. Then f has unique fixed point in X.

Proof : We prove this by induction

From (2) for any m and by definition of D–bound of $O(a_o)$,

$d(a_1, a_2, a_m) \leq k \max\{d(a_o, a_1, a_{m-1}), d(a_o, a_1, a_{m-1})\} \leq kC$ (3)

By (2),

$d(a_2, a_3, a_m) \leq k \max\{d(a_2, a_3, a_{m-1}), d(a_1, a_2, a_{m-1})\}$ (4)

Using (3)

$d(a_2, a_3, a_m) \leq k \max\{d(a_2, a_3, a_{m-1}), kC\}$ (5)

Equation (5) can be a recursion formula in m.

$\therefore d(a_2, a_3, a_m) \leq k \max\{k \max\{d(a_2, a_3, a_{m-2}), kC\}, kC\} \leq k^2 C$ (6)

By induction hypothesis and from (1)

$d(a_{n+1}, a_{n+2}, a_m) \leq k \max\{d(a_{n+1}, a_{n+2}, a_{m-1}), d(a_n, a_{n+1}, a_{m-1})\}$
$\leq k \max\{d(a_{n+1}, a_{n+2}, a_{m-1}), k^n C\}$ (7)

Equation (7) can be regarded recursion formula in m.

$\therefore d(a_{n+1}, a_{n+2}, a_m) \leq k \max\{k \max\{d(a_{n+1}, a_{n+2}, a_{m-2}), k^n C\}, k^n C\}$

$= \max\{k^2 d(a_{n+1}, a_{n+2}, a_{m-2}), k^{n+2} C, k^{n+1} C\}$

$= \max\{k^2 d(a_{n+1}, a_{n+2}, a_{m-2}), k^{n+1} C\}$

$\leq \max\{k^2 . k \max\{d(a_{n+1}, a_{n+2}, a_{m-3}), k^{n+1} C\}, k^{n+1} C\}$

$= \max\{k^3 d(a_{n+1}, a_{n+2}, a_{m-3}), k^{n+1} C\}$ (8)

$\leq \ldots\ldots\ldots\ldots$

$\leq \max\{k^n d(a_{n+1}, a_{n+2}, a_{m-n}), k^{n+1} C\}$

$\leq \max\{k^n k \max\{d(a_{n+1}, a_{n+2}, a_{m-n-1}), k^{n+1} C\}, k^{n+1} C\} = k^{n+1} C.$

and $\{a_n\}$ is D–Cauchy

$\because$ X is a_0–orbitally complete, there exists a $q \in X$ with $\lim a_n = q$.

In (2) set $a = a_n$, $c = q$ to obtain

$d(a_{n+1}, a_{n+2}, fq) \leq k \max\{d(a_n, a_n a), d(a_n, a_{n+1}, q)\}$ (9)

Taking limit of (9) as $n \to \infty$ gives d (q, q, fq) $\leq$ k d (q, q, q) = 0 and q = fq.

Now, for prove uniqueness, let p is also fixed point of f

Then from (2)

d (q, q, p) = d (fq, fq, fq) $\leq$ k max {d (q, q, p), d (q, fq, p)}

= k d (q, q, p) (10)

Gives p = q

Theorem 3.4 [11] : Let X be complete bounded D–Metric space and f be self map of X such that,

$\emptyset$((d (a, b, c), d (a, f_a, c), d (b, f_b, c) , d (a, f_b, c), d (b, f_a, c)), d (f_a, f_b, f_c)) ≥ 0 (11)

Then f has unique fixed point q in X and f is continuous at q.

Proof : Consider $a_0 \in$ X and $f_{a_n} = a_{n+1}$.

Then the orbit { a_n} is bounded.

Let $x_n = \delta_d$ ({a_n, a_{n+1}, a_{n+2}, …… }), n ϵ N

Then $\lim_{n\to\infty} x_n$ = x for some x ≥ 0.

If $a_n = a_{n+1}$ for some n ϵ N, then f has a fixed point say q ϵ X

∴ assume that $a_n \neq a_{n+1}$ for n ϵ N.

Let, k ϵ N be fixed.

Taking $a = a_{n-1}$, b = b_{n+m-1} and c = $a_{n+m+\ell-1}$ in (11)

Where n $\geq$ k and m, ℓ ϵ N.

∴ $\emptyset$ ((d (a_{n-1}, a_{n+m-1}, $a_{n+m+\ell-1}$),d (a_{n-1}, fa_{n-1}, $a_{n+m+\ell-1}$), d(a_{n+m-1}, fa_{n+m-1}, $a_{n+m+\ell-1}$), d(a_{n-1},fa_{n+m-1},$a_{n+m+\ell-1}$), d(a_{n+m-1},fa_{n-1},$a_{n+m+\ell-1}$),d(fa_{n-1},fa_{n+m-1}, $fa_{n+m+\ell-1}$)) = $\emptyset$ ((d(a_{n-1}, a_{n+m-1}, $a_{n+m+\ell-1}$), d(a_{n-1}, a_n, $a_{n+m+\ell-1}$), d(a_{n+m-1}, a_{n+m} , $a_{n+m+\ell-1}$), d (a_{n-1}, a_{n+m}, $a_{n+m+\ell-1}$), d (a_{n+m-1} , a_n , $a_{n+m+\ell-1}$)) , d (a_n, a_{n+m} , $a_{n+m+\ell}$)) ≥ 0

∴ We get,

$\emptyset$((x_{n-1}, x_{n-1}, x_{n+m-1}, x_{n-1}, x_{n+m-1}),d(a_n,a_{n+m}, $a_{n+m+\ell}$)) ≥ 0

∵ $\emptyset$ is increasing on R_+^5 and {x_n} is decreasing we have,

$\emptyset$ ((x_{k-1}, x_{k-1}, x_{k-1}, x_{k-1}, x_{k-1}), d (a_n, a_{n+m}, $a_{n+m+\ell}$)) ≥ 0

gives,

d (a_n, a_{n+m}, $a_{n+m+\ell}$) $\leq \psi$ (x_{k-1})

Taking limit sup over n $\geq$ k, we get, $x_k \leq \psi$ (x_{k-1}). As k → ∞, we have x $\leq \psi$ (x).

If x > 0, then x $\leq \psi$ (x) < x, which is a contradiction.

∴ $x = 0$ *and hences* $\lim_{n\to\infty} x_n = 0$.

Thus given ϵ > 0, there is M ϵ N such that, x_M < ϵ. Then we have for n $\geq$ M and m, ℓ ϵ N,

d (a_n, a_{n+m}, $a_{n+m+\ell}$) < ϵ

∴ { a_n} is a D–Cauchy sequence in X.

∴ $\lim_{n\to\infty} a_n = q$ *since X is complete*

Hence $\lim_{n\to\infty} f_{a_n}$ = q

Taking $a = a_{n-1}$, b = b_{n+m-1} and c = q in (11) we get,

$\emptyset$((d(a_{n-1}, a_{n+m-1}, q), d(a_{n-1},fa_{n-1} ,q), d(a_{n+m-1}, $f_{a_{n+m-1}}$, q), d(a_{n-1}, $f_{a_{n+m-1}}$, q), d(a_{n+m-1}, $f_{a_{n-1}}$, q)), d($f_{a_{n-1}}$, $f_{a_{n+m-1}}$, f_q) ≥ 0

As n → ∞, we get,

$\emptyset$ ((d (q, q, q), d (q, q, q), d (q, q, q), d (q, q, q), d(q, q, q)), d (q, q, f_q)) $\geq$ 0

Gives d(q, q, f_q) $\leq \psi$ (d(q, q, q)) = ψ(0) = 0

Hence f_q = q.

Now for uniqueness, let q and p be fixed points of f.

Let a = q, b = q and c = p in (11) we have,

$\emptyset$((d(q, q, p),d(q, fq, p), d (q, fq, p), d (q, fq , p), d (q, fq, p)), d (fq, fq, fp))

= $\emptyset$ ((d(q, q, p), d(q, q, p) , d(q, q, p), d (q, q, p), d (q, q, p)), d (q, q, p)) $\geq$ 0

Given d (q, q, p) $\leq \psi$ (d (q, q, p)) < d (q, q, p), which is a contradiction,

∴ q = p

Now, to prove f is continuous at q.

Consider {b_n} be sequence in X and $\lim_{n\to\infty} b_n = q$

Taking a = q, b = q, and c = b_n in (11) we get,

$\emptyset$((d(q, q, b_n), d(q, f_q,b_n), d (q, f_q, b_n), d (q, f_q, b_n) d (q, f_q, b_n)), d (f_q, f_q, f_{b_n}))

= $\emptyset$((d(q, q, b_n), d(q, q, b_n), d(q, q, b_n), d(q, q, b_n), d (q, q, b_n)), d (q, q, f_{b_n})) ≥ 0

⟹ d (q, q, f_{b_n}) $\leq \psi$ (d (q, q, b_n))

Taking lim sup, we get,

$\overline{\lim}$ d (q, q, f_{b_n}) $\leq \overline{\lim}\ \psi$ (d (q, q, b_n)) $\leq \psi$ (0) = 0

Hences lim f_{b_n} = q = f_q and hence f is continuous at q.

Corollary 3.1 : Consider X be a bounded complete D–Metric space, n ϵ N and f be self mapping of X such that for a,b, c ϵ X

$\emptyset$((d(a, b, c), d(a, f_a^n, c), d(b, f_b^n, c), d (a, f_b^n, c) d (b, f_a^n, c)), d (f_a^n, f_b^n, f_c^n)) ≥ 0

Then f has a unique fixed point q in X and f^n is continuous at q

Proof : By theorem 3.4., f^n has a unique fixed point q in X and f^n is continuous at q.

∵ $f_q = ff^n{}_q = f^n f_q$, f_q is also a fixed point of f^n

By the uniqueness, f_q = q

4. Main Result :

Theorem 4.1 : Let X be complete bounded D – Metric space and f be self – map on X such that,

$d(f_x,f_y,f_z) \leq \alpha \max\{d(x,y,z) + d(x,fy,z),\ d(x,fx,z) + d(x,fy,z),\ d(x,y,z) + d(y,fx,z)\}$ (12)

For all x, y, z ϵ X, $0 \leq \alpha < \frac{1}{2}$. Then f has a unique fixed point.

Proof : Let $x_0 \epsilon X$ and define $f_{x_n} = x_{n+1}$

If $x_{n+1} = x_n$ for some n, then f has a unique fixed point. So assume that, $x_{n+1} \neq x_n$ for each n, setting $x = x_0$, $y = x_1$, $z = x_{m-1}$, $m > 1$ we have,

$d(fx, fy, fz) = d(x_1, x_2, x_m)$

$\leq \alpha \max\{d(x_0, x_1, x_{m-1}) + d(x_0, x_2, x_{m-1}),\ d(x_0, x_1, x_{m-1}) + d(x_0, x_2, x_{m-1}),\ d(x_0, x_1, x_{m-1}) + d(x_1, x_1, x_{m-1})\}$

$\leq \max_{a,b,c} (d(x_a, x_b, x_c) + d(x_a, x_b, x_c))$

where $0 \leq a \leq 1$, $1 \leq b \leq 2$, and $1 \leq c \leq m$

$\leq 2\alpha \max_{a,b,c} d(x_a, x_b, x_c) \leq 2\alpha k$,

where k is D–bound of $\{x_n\}$

Again getting $x = x_1$, $y = x_2$, $z = x_{m-1}$, $m > 2$ we get,

$d(x_2, x_3, x_m)$

$\leq \alpha \max\{d(x_1, x_2, x_{m-1}) + d(x_1, x_3, x_{m-1}),\ d(x_1, x_2, x_{m-1}) + d(x_1, x_3, x_{m-1}),\ d(x_1, x_2, x_{m-1}) + d(x_2, x_2, x_{m-1})\}$

$\leq \alpha^2 \max_{a,b,c} (d(x_a, x_b, x_c) + d(x_a, x_b, x_c))$

Where $0 \leq a \leq 2$, $1 \leq b \leq 3$ and $2 \leq c \leq m$

$\leq 2\alpha^2 \max_{a,b,c} d(x_a, x_b, x_c)$

$\leq 2\alpha^2 k$

By induction we get,

$d(x_n, x_{n+1}, x_m) \leq 2\alpha^n \max_{a,b,c} d(x_a, x_b, x_c) \leq 2\alpha^n k$.

Where $0 \leq a \leq n$, $1 \leq b \leq n + 1$ and $n \leq c \leq m$, for all $m > n \epsilon N$

∴ By D–Cauchy principle, $\{x_n\}$ is D–Cauchy.

∵ X is complete, $\{x_n\}$ converges. Call the limit q.

From equation (12),

$d(x_n, x_{n+1}, f_q) \leq \alpha \max\{d(x_{n-1}, x_{n+2}, q) + d(x_{n-1}, x_{n+1}, q),\ d(x_n, x_{n+1}, q) + d(x_n, x_{n+2}, q),\ d(x_{n-1}, x_{n+2}, q) + d(x_n, x_{n+1}, q)\}$

Taking limit as $n \to \infty$ gives, $d(q, q\ f_q) \leq 0$.

∴ $f_q = q$

∴ f has a fixed point.

Now, to prove uniqueness, assume that $w \neq q$ is also fixed point of f

Form equation (12)

$d(f_q, f_w, f_q) = d(q, w, q)$

$\leq \alpha \max\{d(q, w, q) + d(q, w, q),\ d(q, q, q) + d(q, w, q),\ d(q, w, q) + d(w, q, q)\}$

$\leq 2\alpha\, d(q, w, q)$

Which is a contradiction

∴ w = q

∴ f has a unique fixed point.

Corollary 4.1 : Let X be complete bounded D – Metric space, m a positive integer, and f be self – maps on X satisfying,

$d(f_x^m, f_y^m, f_z^m) \leq \alpha \max\{d(x, y, z) + d(x, f_y^m, z),\ d(x, f_x^m, z) + d(x, f_y^m, z),\ d(x, y, z) + d(y, f_x^m, z)\}$ (13)

for all x, y, z ϵ X, $0 \leq \alpha < \frac{1}{2}$. Then f has a unique fixed point.

Proof : Define $T = f^m$ then (13) reduces to (12) and T has a unique fixed point p.

∵ $p = T_P = f_p^m$.

Thus, $f_P = f_p^{m+1} = T(f_P)$ and f_P is also a fixed point of T.

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