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85 – 87

A Study of G-Fuzzy Congruence Relations

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Abstract:- Many authors studied on the fuzzy counterpart of Group Theory, since Rosenfeld. But, there is little literature on Fuzzy Semigroups. Here the authors try to extend the fuzzy congruence relations to Semi groups.

Keywords:- Fuzzy Set, Fuzzy Relation, G-fuzzy equivalence Relation, Semigroup.

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1. Introduction

The theory of 'fuzzy semigroup' is a part of fuzzy algebra, which is an important branch of fuzzy mathematics. Many important results in crisp algebra are not yet generalized to fuzzy algebra, not because of the impossibility or the difficulty of such tasks, but because of the insufficiency of fuzzy algebraic tool. In [4], Rosenfeld formulated the notion of fuzzy subgroups and showed how some basic notions of group theory can be extended in an elementary manner to fuzzy subgroups. Since then, fuzzy sets with many other kinds of algebraic structure have received attention in the literature[1,2,3, 5]. These have included 'Semigroups' also.

The notion of a fuzzy congruence on a semigroup is studied by many authors. The set of alternatives X can be realized as a semigroup, by introducing a semigroup structure in X. We make use of the fuzzy weak preference relation R on X for this purpose. A quotient semigroup of X, induced by the fuzzy indifference relation I is introduced, and the Homomorphism Theorem with respect to the relation is presented.

2. Fuzzy Set

A function A from a set X to the closed unit interval $[0,1]$ in $\mathbb{R}$ is called a *fuzzy set* in X.ie., $A: X \to [0,1]$. For every $x \in X$, $A(x)$ is called a *membership grade* of x in A[7]

3. Fuzzy Relation

A fuzzy relation μ in a set X is a fuzzy subset of $X \times X$.

ie., $\mu: X \times X \to [0,1]$

The value $\mu(x,y)$ represents the strength of the relation of x to y where $x, y \in X$.

Composition of Fuzzy Relations

The composition $\lambda \circ \mu$ of two fuzzy relations λ and μ in X is the fuzzy subset of $X \times X$ definedby:

$$(\lambda \circ \mu)(x,y) = \sup_{z \in X} \min\{\lambda(x,z), \mu(z,y)\}$$

Reflexivity and Symmetry of Fuzzy Relations

A fuzzy relation μ is said to be:

i. Reflexive if $\mu(x,x) = 1;\ \forall\, x \in X$

ii. Irreflexive if $\mu(x,x) = 0;\ \forall\, x \in X$

iii. Weakly Reflexive if $\mu(x,x) > 0; \forall\, x \in X$

or, if $\mu(x,x) > \mu(x,y);\ \forall\, x, y \in X$

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62 – 68

iv. Symmetricif $\mu(x,y)=\mu(y,x); \forall\, x,y \in X$

v. Antisymmetricif

$$\min\{\mu(x,y), \quad \mu(y,x)\} = 0; \;\; \forall\, x,y \in X$$

Transitivity of a Fuzzy Relations

A fuzzy relation μ is said to be max-min transitive if:

$$\mu(x,y) > 0,\; \mu(y,z) > 0 \;\Rightarrow\; \mu(x,z) \geq \min\{\mu(x,y), \quad \mu(y,z)\}\,\forall\, x,y,z \in X$$

Or, $\mu \circ \mu \subseteq \mu$

Inverse of a Fuzzy Relation

The inverse $\mu^{-1}$of a fuzzy relation μ in $X$is a fuzzy relation in Xdefined as, $\mu^{-1}(x,y) = \mu(y,x);\;\; \forall\, x,y \in X$

For any two fuzzy relations λ andμ,

$$(\mu^{-1})^{-1} = \mu \text{and} (\lambda \circ \mu)^{-1} = \mu^{-1} \circ \lambda^{-1}$$

Fuzzy Equivalence Relation

A fuzzy relation μ on X is said to be a fuzzy equivalence relation on$X$ if it is reflexive, symmetric and transitive.

G-Reflexive Fuzzy Relations

A fuzzy relation μ is said to be G-Reflexive if

$\mu(x,x) > 0$and$\inf_{t\in X} \mu(t,t) \geq \mu(x,y);\;\; \forall\, x,y \in X$ such that $x \neq y$.

G-Fuzzy Equivalence Relation

A fuzzy relation μ on X is said to be a G-fuzzy equivalence relation on X if it is G-reflexive, symmetric and transitive.

4. Semigroup

A Semigroup S is a nonempty set equipped with a binary operation $\cdot$, which is associative[6].

Monoid

A monoid is a semigroup with an identity element.

Idempotent

An element $i$of a semigroup is called an idempotent, if $i \cdot i = i$.

Monoid of Fuzzy Relations

Let $\mathcal{F}_X$ be the set of all fuzzy relations in a set X. The composition $\circ$ is an associative binary operation in $\mathcal{F}_X$. Thus$\mathcal{F}_X$is a monoid under the operation of composition and a G-fuzzy equivalence relation is an idempotent element of $\mathcal{F}_X$.

Fuzzy Compatible Relations

A fuzzy binary relation μ on a semigroup $X$is said to be

i. fuzzy left compatible if $\mu(x,y) \leq \mu(zx,zy);\;\; \forall\, x,y,z \in X$

ii. fuzzy right compatible if $\mu(x,y) \leq \mu(xz,yz);\;\; \forall\, x,y,z \in X$

iii. fuzzy compatible if it is both left and right compatible.

A G-fuzzy equivalence relation on X is called a G-fuzzy left congruence if it is fuzzy left compatible and a G-fuzzy right congruence if it is fuzzy right compatible. A G-fuzzy equivalence relation on X is a G-fuzzy congruence if it is fuzzy compatible.

Result-1 and Result-2 given below follow immediately.

Result-1

Let μ be a fuzzy relation on a set X.

i. Then $\cup_{n=1}^{\infty} \mu^n$ is the smallest transitive fuzzy relation on X containing μ, where, $\mu^n = \mu \circ \mu \circ \ldots \circ \mu$.

ii. If μ is symmetric, then so is $\cup_{n=1}^{\infty} \mu^n$.

Result-2

If μ is a fuzzy compatible relation on a semigroup S, then so is $\cup_{n=1}^{\infty} \mu^n$

Result-3

If μ is a G-reflexive fuzzy relation on a set X, then $\mu^{n+1}(x,y) \geq \mu^n(x,y)$, for all natural numbers n and all $x, y \in X$.

Proof:

$$\mu^2(x,y) = (\mu \circ \mu)(x,y) = \sup_{z \in X} \min\{\mu(x,z), \quad \mu(z,y)\}$$

$$\geq \min\{\mu(x,x), \quad \mu(x,y)\} = \mu(x,y)$$

Suppose that $\mu^{k+1}(x,y) \geq \mu^k(x,y),\ \forall\, x, y \in X$

Then $\mu^{k+2}(x,y) = (\mu \circ \mu^{k+1})(x,y) = \sup_{z \in X} \min\{\mu(x,z),\ \mu^{k+1}(z,y)\}$

$$\geq \sup_{z \in X} \min\{\mu(x,z), \quad \mu^k(z,y)\}$$

$$= (\mu \circ \mu^k)(x,y) = \mu^{k+1}(x,y)$$

Hence the result follows by Mathematical Induction.

The following result follows immediately.

Result-4

If μ and each ν_i; $\forall\, i \in I$, are fuzzy relations in a set X, then, $\mu \circ (\cap_{i \in I} \nu_i) \subseteq \cap_{i \in I} (\mu \circ \nu_i)$

G-fuzzy Congruences on Semigroups

Result-5

If μ is a G-reflexive fuzzy relation on a set S, then, so is $\cup_{n=1}^{\infty} \mu^n$

Proof:

The result is true for $n = 1$.

Let it be true for $n = k$. That is, μ^k is G-reflexive.

$$\mu^{k+1}(x,x) = (\mu \circ \mu^k)(x,x)$$
$$= \sup_{z \in X} \min\{\mu(x,z), \quad \mu^k(z,x)\}$$

$$\geq \min\{\mu(x,x), \quad \mu^k(x,x)\} > 0$$

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